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ON ERGODIC ACTIONS WHOSE SELF-JOININGS ARE GRAPHS

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We call an ergodic measure-preserving action of a locally compact group G on a probability space simple if every ergodic joining of it to itself is either product measure or is supported on a graph, and a similar condition holds for multiple self-joinings. This generalizes Rudolph's notion of minimal self-joinings and Veech's property S.
Main results: The joinings of a simple action with an arbitrary ergodic action can be explicitly described. A weakly-mixing group extension of an action with minimal self-joinings is simple. The action of a closed, normal, co-compact subgroup in a weakly-mixing simple action is again simple. Some corollaries: Two simple actions with no common factors are disjoint. The time-one flow of a weakly-mixing flow with minimal self-joinings is prime. Distinct positive times in a $\mathbb{Z}$-action with minimal self-joinings are disjoint.

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0. INTRODUCTION AND DEFINITONS

The notion of minimal self-joinings for $\mathbb{Z}$-actions was introduced in [Ru2] as a source of counter-examples. In this paper we generalize this notion to what we call simple group actions and develop some general theory for these actions. This allows us to broaden the repertoire of actions displaying this sort of behaviour. We deal with actions of fairly general groups because it is convenient for our purposes and not much more difficult, but the main interest lies in $\mathbb{Z}$ and $\mathbb{R}$-actions. Most of our results are new even within the setting of $\mathbb{Z}$-actions.

We consider a standard Borel space (X,B), that is there exists a complete separable metric on X such that $B = B(X)$ is the σ-algebra of Borel sets generated by the corresponding topology on X. Suppose that X is equipped with a Borel probability measure μ and that G is a locally-compact group. By a (left) action of G on X we mean a Borel map $G \times X \to X$ denoted $(g,x) \mapsto gx$ such that

$$(hg)x = h(gx) \qquad \forall h,\ g \in G,\ x \in X$$

and

$$ex = x \qquad \forall x \in X$$

where e denotes the identity element of G and $x \mapsto gx$ is a measure-preserving map for each $g \in G$. We then say that $\mathcal{X} = (X,B(X),\mu,G)$ is a G-action or a G-space. We will often shorten this to $\mathcal{X} = (X,\mu,G)$ or (X,G). For convenience, throughout this paper $\mathcal{X}$ always represents $(X,B(X),\mu,G)$ and $\mathcal{Y}$ represents $(Y,B(Y), \nu,G)$. We require that all our actions be ergodic, that is all (everywhere) invariant Borel sets have measure 0 or 1. Equivalently, $gA = A$ a.e. implies $\mu(A) = 0$ or 1. (see Theorem 3 of [Ma1]).

If $\mathcal{X}_i = (X_i,B(X_i), \mu_i, G)$ $i = 1,\dots,k$ are G-actions by a joining of $\mathcal{X}_1,\dots,\mathcal{X}_k$ we mean a Borel measure λ on $X_1\times\dots\times X_k$ which is invariant under the natural diagonal G-action $g(x_1,\dots,x_k) = (gx_1,\dots,gx_k)$ and whose marginal (projection) on each X_i is μ_i. Thus $(X_1\times\dots\times X_n,\lambda,G)$ is an action and we will frequently identify the joining with the corresponding action. When we need to emphasize the role of G we will speak of a G-joining. By a k-joining of the single G-action $\mathcal{X}$ we mean a joining of k copies of $\mathcal{X}$. We denote by

$\mathcal{J}(X_1,\ldots,X_n)$ the space of joinings of $X_1,\ldots,X_n$.

We denote by $C(X)$ the centralizer of the action X, that is the semigroup of (equivalence classes of) measure-preserving maps commuting a.e. with the action of each $g \in G$. For $S \in C(X)$ we denote by μ_S the Borel measure on $X\times X$ which is the image under the map $x \mapsto (x,Sx)$ of the measure μ. Thus

$$\mu_S(A\times B) = \mu(A\cap S^{-1}B),$$

which makes it clear that μ_S does not depend on the choice of representative of S. μ may also be defined as $(id\times S)\mu_\Delta$ where $\mu_\Delta = \mu_{id}$ is the diagonal measure on $X\times X$.

μ_S is a 2-joining of X: it's marginals are μ because S is measure-preserving and it is G-invariant because S commutes with the G-action. The corresponding action is isomorphic to X via the map $x \mapsto (x,Sx)$ so μ_S is ergodic because of our standing assumption of ergodicity. We will call joinings of the form μ_S off-diagonal. $\mu\times\mu$ is also a 2-joining which is ergodic precisely if X is weak-mixing. (We may take this as the definition of weak-mixing.) We shall say X is 2-simple if every ergodic 2-joining is either product measure $\mu\times\mu$ or an off-diagonal. (This does not mean that X is weak-mixing!) For the case of $\mathbb{Z}$-actions this notion is due to Veech who called it property S. If in addition each $S \in C(X)$ agrees a.e. with the action of some $g \in G$ then we say X has 2-fold minimal self-joinings (MSJ).

If $T \in C(X)$ denote by $\tilde{\mu}_T$ the image of μ under $x \mapsto (Tx,x)$. Then $\tilde{\mu}_T$ is an ergodic 2-joining, so if X is 2-simple $\tilde{\mu}_T = \mu_S$ for some $S \in C(X)$. Evaluating this equation on the rectangle $S^{-1}B\times B$ we obtain

$$\mu(T^{-1}S^{-1}B\cap B) = \mu(S^{-1}B\cap B^{-1}B) = \mu(B)$$

so $T^{-1}S^{-1}B = B$ a.e. for $B \in \mathcal{B}(X)$. As is well known in a standard Borel space this implies that $ST = id$ a.e. Similarily $TS = id$ a.e. Thus 2-simplicity forces $C(X)$ to be a group. This removes the evident asymmetry in the definition of 2-simplicity: an equivalent definition is that $C(X)$ is a group and every ergodic 2-joining of X is either product measure or a measure of the form $(S_1\times S_2)\mu_\Delta$, $S_1, S_2 \in C(X)$.

We now want to make a definition which restricts in a similar way the k-joinings of X to the obvious ones. What are the obvious ones? If $S_1,\ldots,S_k \in C(X)$ then we call the image of μ under the map $x \mapsto (S_1x,\ldots,S_kx)$ an off-diagonal measure. An off-diagonal measure is evidently an ergodic k-joining. By a product of off-diagonals (POOD) on X^k we mean that the index set $(1,\ldots,k)$ has been split into subsets $k_1,\ldots,k_r$, on each X^{k_i} we put an off-diagonal measure and then take the product of these off-diagonal measures. A POOD is evidently a self-joining of X. Note that product measure is itself a POOD - an off-diagonal may sit on a single factor of X^k. We say that X is simple if $C(X)$ is a group and for every k each ergodic k-joining of X is a POOD. If in addition each $S \in C(X)$ agrees a.e. with the action of some $g \in G$ then we say X has MSJ.

Some comments about our terminology are in order. In [Ju1] (and, following [Ju1], in [Ve]) the term simple was (unhappily, we now feel) used to mean 2-fold minimal self-joinings for $\mathbb{Z}$- actions. For $\mathbb{Z}$- actions generated by a map T our definition of minimal self-joinings restricts only the joinings of T with itself whereas [Ru1] also restricts joinings of unequal powers of T. We feel that the present terminology is apter: the term self-joining should refer only to joinings of T with itself. Moreover, as we shall see later (section 6) the present definition is almost equivalent to the stronger one. Finally, simplicity generalizes Veech's property S ([Ve]) which is 2-fold simplicity (Veech works only with $\mathbb{Z}$-actions).

We now briefly describe our results. Section 1 reviews some background on G-actions, group extensions and joinings. In section 2 we show that when a locally compact group acts ergodically on a compact group by left translations then the action is simple and the centralizer consists of all the right translations. This is in some sense the trivial case and we show that every non weak mixing simple action must be of this type. The main interest lies in the weak mixing case.

VEECH [Ve] has shown that a simple $\mathbb{Z}$-action is a group extension of any non-trivial factor. In section 3 we reprove this in the general setting. We go on to characterise joinings of factors of a given simple action and determine when a factor of a simple action is again simple.

Section 4 contains our main result, a characterization of joinings of a simple G-action X with an arbitrary ergodic G-action Y. Just as a simple G-action has only the obvious joinings with itself it turns out that it has only the "obvious" joinings with other actions. For $\mathbb{Z}$-actions with MSJ S.Glasner [Gl] has given a different proof of this result. Glasner does not actually describe all the joinings but rather characterizes those Y which are not disjoint from X. A corollary of our result is that two simple actions with no common factors are disjoint.

In section 5 we show that a weak mixing group extension of an action with MSJ is simple. We present an example due to Glasner which shows that a weal-mixing group extension of a simple action need not be simple. For the proof of the result on group extensions we introduce the auxiliary notion of a pairwise determined action - one for which any self-joining which is pairwise independent must be independent - a notion which we think is of independent interest.

In section 6 we show that in a weakly-mixing simple action of a group G any closed, normal cocompact subgroup H acts simply and that it's centralizer is the centralizer of the full action. A corollary of this is that in a weakly-mixing flow with MSJ the time one map is prime, i.e. it has only the trivial invariant σ-algebras. Moreover if X and Y are weakly-mixing simple G-actions such that any ergodic joining of X and Y is weak mixing then any H-joining of X and Y is a G-joining. We make some further applications to flows and show that our definition of minimal self-joinings in the case of $\mathbb{Z}$-actions is almost as strong as the original one in [Ru1]. We conclude with some open problems.

As we have already stated our main interest lies in weakly mixing, simple or MSJ, $\mathbb{Z}$ or $\mathbb{R}$ actions. Examples of such actions with MSJ are already available; see [Ru1], [JRS], [Ju1] for $\mathbb{Z}$-examples, [Ra], [J,P] for $\mathbb{R}$-examples. Theorems 5.4 and 6.1 show how to obtain simple actions from actions with MSJ. The construction in [Ru2] can probably be modified to obtain a simple map with a Bernoulli shift in its centralizer. All these examples of simplicity depend on a very explicite knowledge of the centralizer. Elsewhere we will construct a completely different sort of example: a weakly mixing simple prime $\mathbb{Z}$-action which is also rigid, that is there exists a sequence of powers of the map which converges weakly to the identity. As is well-known this forces the centralizer to be uncountable.

We owe a large debt of gratitude to S. Glasner. He was the first to formulate a theorem like our theorem 4.1, in the case of a $\mathbb{Z}$-action

with MSJ, which in addition has a strong condition on generic points. (This condition was established for the Chacōn example in [J,K].) We then realized that we could prove theorem 4.1 for any $\mathbb{Z}$-action with MSJ and Glasner independently found a different proof [Gl1]. Then we extended the result to arbitrary groups and simple actions. Several of our examples and proofs have also been simplified by suggestions of Glasner's.

Section 1.

As described in p. 138 of [Ma2] the only possible cardenalities for a standard Borel space are finite, countable and the cardinality of the continuum, and any two standard Borel spaces of the same cardinality are Borel isomorphic. Since the topology of X plays no role in our results we can and shall assume whenever convenient that X is compact metric (for example either the unit interval or a finite or closed countable subset thereof).

1.1 Boolean G-spaces

Suppose that X is a G-space. The action of G may also be viewed as an action on sets, that is, as a Boolean G-action as defined by Mackey [Ma1]. We denote by $B(\mu)$ the σ-Boolean algebra of Borel subsets of X, two subsets being identified when they differ by a null set. The measure μ is well-defined on $B(\mu)$ and again denoted μ. $B(\mu)$ is a complete metric space under the metric $\mu(E\Delta F)$. We say $B(\mu)$ is a Boolean G-space if G acts on $B(\mu)$ by measure-preserving σ-Boolean algebra automorphisms and for $E \in B(\mu)$ the map $g \mapsto gE$ is Borel. (We have followed the definition in [Ram] which is easily seen to be equivalent to Mackey's. Note also that we do not consider abstract Boolean G-space.) If X is a G-space then $B(\mu)$ becomes a Boolean G-space under the natural G-action as is shown in Lemma 1 of [Ma1]. Boolean G-spaces arise in another natural way. Let $G(\mu)$ denote the group of all measure-preserving invertible Borel maps of X, two maps being identified when they agree a.e. $G(\mu)$ is a complete separable metric group under the weak topology ($S_n \to S \iff S_n(A) \to S(A)$ in $B(\mu)$ $\forall A \in B(\mu)$). If G is a locally compact subgroup of $G(\mu)$ then the natural G-action on $B(\mu)$ is a Boolean G-action since $g \mapsto gA$ is continuous by definition of the weak topology.

Two Boolean G-spaces $B(\mu)$ and $B(\nu)$ are said to be isomorphic if there is a measure-preserving Boolean algebra isomorphism $B(\mu) \to B(\nu)$ which is G-equivariant. $B(\nu)$ is said to be a factor of $B(\mu)$ if there is a G-equivariant measure-preserving Boolean algebra homomorphism of $B(\nu)$ into $B(\mu)$.

1.2 Isomorphism of G-actions

If X and Y are G-actions by an isomorphism $\phi: X \to Y$ we mean a measure-preserving Borel isomorphism ϕ between G-invariant Borel co-null subsets X^* and Y^* of X and Y which is also G-equivariant. Suppose that ϕ' is a measure-preserving Borel isomorphism between Borel co-null subsets of X and Y such that for each $g \in G$ $\phi'(gx) = g(\phi'x)$ for a.a. x (the null set may depend on x!). Then there is an isomorphism $\phi: X \to Y$ which agrees with ϕ' a.e. : ϕ' induces an isomorphism of the Boolean G-spaces associated with X and Y and by theorem 2 of [Ma1] this Boolean isomorphism is induced by an isomorphism of X and Y. Since ϕ and ϕ' induce the same Boolean map they must be equal a.e.

Suppose that $\phi_i : X_i \to Y_i$ $i = 1,\ldots,k$ are G-space isomorphisms defined between G-invariant Borel co-null subsets X_i^* and Y_i^*. Then $\phi = \phi_1 \times\ldots\times\phi_k$ is a Borel isomorphism between $X_1^* \times\ldots\times X_k^* = U$ and $Y_1^* \times\ldots\times Y_k^* = V$. If $\lambda \in J(X_1,\ldots,X_n)$ then λ is supported on U so $\phi(\lambda|_U)$ is a Borel probability measure on V. This measure may be regarded as a Borel measure on $Y_1\times\ldots\times Y_n$ which is evidently a joining. Thus the notion of joining is preserved under isomorphism and in particular simplicity and MSJ are preserved under isomorphism.

1.3 Factor maps

By a factor map $\phi : X \to Y$ we mean a measure-preserving Borel map from a G-invariant Borel co-null subset X^* of X to Y which is also G-equivariant. If ϕ' is a measure-preserving Borel map from a Borel co-null subset X^* of X to Y such that $\phi'(gx) = g(\phi'x)$ a.e. then there exists a factor map $\phi : X \to Y$ which agrees with ϕ' a.e. This is proved in the course of the proof of Proposition 2.1 of [Zi1].

If $\phi : X \to Y$ is a factor map $\phi^{-1}(B(Y))$ is a G-invariant sub σ-algebra of $B(X^*)$ which can be extended in a natural way to the G-invariant sub-σ-algebra G of B(X) consisting of all Borel sets agreeing a.e. with some set in $\phi^{-1}B(Y)$. Evidently G is unchanged if ϕ is replaced by a factor map $\phi' = \phi$ a.e. We write $G = \phi^{-1}(Y)$ and call it the factor algebra generated by ϕ. In general we call any G-invariant sub-σ-algebra of B(X) which contains all the null sets a factor algebra of X. Proposition 2.1 of [Zi1] guarantees that every factor algebra of X is generated by a factor map. Note further that if $\phi : X \to Y$ and $\phi : X \to Y'$ are factor maps generating the same factor algebra then Y and Y' are isomorphic, since the associated Boolean G-actions are isomorphic. Thus a factor algebra of X gives rise to a factor Y which is unique up to isomorphism.

If $\phi_i : X_i \to Y_i$, $i = 1,\dots,n$, are factor maps generating factor algebras G_i there is a natural correspondence between $J(Y_1,\dots,Y_n)$ and G-invariant measures on $G_1\times\dots\times G_n$ with marginals μ. On the one hand any such measure λ on $G_1\times\dots\times G_n$ clearly projects under $\phi = \phi_1 \times\dots\times\phi_n$ to a joining $\bar{\lambda}$ of $Y_1,\dots, Y_n$ just as we discussed in the case of isomorphisms, if we note in addition that the domain of ϕ is a product of co-null subsets, hence belongs to $G_1 \times\dots\times G_n$. On the other hand if $\bar{\lambda} \in J(Y_1,\dots,Y_n)$ it lifts under ϕ to a measure λ on the σ-algebra $\phi^{-1}(B(Y_1\times\dots\times Y_n))$ $= \phi_1^{-1} B(Y_1) \times\dots\times \phi_n^{-1} B(Y_n)$ in $X_1^*\times\dots\times X_n^*$. Because λ has marginals $\mu_i|_{X_i^*}$ λ may be extended to a measure on $G_1\times\dots\times G_n$.

1.4 Integration of measures

If $(X,B(X))$ is a standard Borel space we denote by $M(X)$ the space of Borel probability measures on X. We give $M(X)$ the Borel structure generated by all the functions $\mu \mapsto \mu(f) = \int_X f d\mu$ for bounded Borel f. Since such an f is an increasing pointwise limit of simple functions f_n and $\mu(f_n) \to \mu(f)$ for each μ, this Borel structure is also generated by the functions $\mu \mapsto \mu(1_A)$, A Borel.

Since X is compact metric M(X) is a compact metric space with respect to the weak $-*$ topology as a set of linear functionals on C(X). We claim that this topology generates the Borel structure described above. The class F of bounded measurable functions f for which $\mu \mapsto \mu(f)$ is weak$-*$ measurable is evidently closed under monotone pointwise limits and contains C(X). For A closed in X, 1_A is a decreasing limit of continuous functions hence $1_A \in F$. Similarily if A is an F_σ, $1_A \in F$. In a complete metric space each open set is an F_σ whence it follows easily that the algebra generated by the closed sets consists of F_σ's. Thus by the monotone class theorem, $1_A \in F$ for A Borel, which establishes the claim.

If (Y,G) is a measurable space a measurable map $Y \to M(X)$, denoted

$y \mapsto \mu_y$ will be called a measurable field of measures. If $y \mapsto \mu_y$ is a measurable field then $y \mapsto \mu_y(A)$ is a measurable function for each Borel A. Thus if ν is a probability measure on (Y,G) we may integrate μ_y to obtain a measure $\mu = \int \mu_y \, d\nu(y)$

defined by

$$\mu(A) = \int_Y \mu_y(A) \, d\nu(y).$$

Approximating by simple functions one sees that

$$\mu(f) = \int_Y \mu_y(f) \, d\nu(y)$$

for each bounded measurable f.

If Y is complete metric and $G = B(Y)$ then $y \mapsto \delta_y$ is a continuous map into $M(Y)$. Since $(\sigma,\tau) \mapsto \sigma\times\tau$ is a continuous map from $M(Y) \times M(X) \mapsto M(Y\times X)$ we conclude that $y \mapsto \delta_y \times \mu_y$ is a measurable field. Thus whenever Y is a standard Borel space we may define the direct integral measure

$$\lambda = \int^{\oplus} \mu_y \, d\nu(y) \text{ on } Y\times X$$

by

$$\lambda = \int_Y \delta_y \times \mu_y \, d\nu(y).$$

In other words

$$\lambda(A) = \int_Y \mu_y \, (A\cap\{y\}\times X)$$

for A Borel in $Y\times X$.

1.5 Disintegration of measures

We continue to suppose that X and Y are standard. Suppose that μ and ν are Borel probabilities on X and Y and that $\phi : X \to Y$ is a measure-preserving Borel map. Then λ may be disintegrated over the fibres of ϕ, that is there is a measurable field $y \mapsto \mu_y$ such that μ_y is supported on $\phi^{-1}\{y\}$ and

$$\mu = \int_Y \mu_y \, d\nu(y).$$

Moreover μ_y is ν-essentially unique. (See Theorem 5.8 of [Ful].) As a special case, if λ is a Borel probability on $Y \times X$ projecting onto the measure ν on Y then the disintegration takes the form

$$\lambda = \int^{\oplus} \lambda_y \, d\nu(y)$$

where the λ_y are measures on X. If X and Y are also G-spaces and ϕ is a G-factor map then the invariance of λ together with uniqueness of the disintegration yields that for $g \in G$ $g\mu_y = \mu_{gy}$ for ν a.a.y (the exceptional null set may depend on g).

1.6 Relatively independent extension and relative product

Suppose now that $\phi_i : X_i \to Y_i$, $i = 1,\ldots,n$ are factor maps of G-actions and that μ_i has the disintegration

$$\mu_i = \int_{Y_i} \mu_{iy} \, d\nu_i(y).$$

If $\lambda \in J(Y_1,\ldots,Y_n)$ we define its relatively independent extension $\hat{\lambda} \in J(X_1,\ldots,X_n)$ by

$$\hat{\lambda} = \int_{Y_1 \times \ldots \times Y_n} \mu_{1y} \times \ldots \times \mu_{ny} \, d\lambda(y).$$

It is easy to check that $\hat{\lambda}$ is indeed a joining.

We shall mainly use a special case of this construction, the relative product. If $\phi_i : X_i \to Y$, $i = 1,\ldots,n$ are factor maps of G-actions we define the Y-relatively independent product of $X_1,\ldots,X_n$ $\lambda \in J(X_1,\ldots,X_n)$ by

$$\lambda = \int_Y \mu_{1y} \times \ldots \times \mu_{ny} \, d\nu(y),$$

in other words λ is the relatively independent extension of the diagonal n-joining of Y.

1.7 Ergodic decompositions

Suppose we have a Borel action of a locally compact group G on the standard Borel space X. We denote by $M_G(X)$ the space of G-invariant Borel probability measures on X and by $M_G^e(X)$ the space of ergodic G-invariant probabilities on X. $M_G(X)$ is compact and convex and, as is well-known and easy to prove $\mathrm{ext}(M_G(X)) = M^e(X)$.

THEOREM 1.7. *$M_G(X)$ and $M_G^e(X)$ are Borel subsets of $M(X)$. For each $\mu \in M_G(X)$ there is a unique Borel measure ν on $M_G^e(X)$ such that $\mu = \int_{M_G^e(X)} \sigma \, d\nu(\sigma)$.*

REMARK: The meaning of the integral is, of course, that $\mu(A) = \int \sigma(A) d\nu(\sigma)$ for each Borel A.

PROOF. By Theorem 8.7 of [Va] we may assume that G acts in a jointly continuous manner on a compact metric space $\tilde{X}$, that X is a G-invariant Borel subset of $\tilde{X}$ and that the action of G on X is the restriction of the action on $\tilde{X}$. M(X) may be identified with a subset of $M(\tilde{X})$ (namely those measures which are supported on X), this subset is Borel in $M(\tilde{X})$ and the Borel structure of M(X) is the same as its Borel structure as a subset of $M(\tilde{X})$. Now $M_G(\tilde{X})$ is compact in $M(\tilde{X})$ because G acts by continuous maps. Also $M_G^e(\tilde{X})$ $= \mathrm{ext}\, M_G(\tilde{X})$ is a G_δ by proposition 1.3 of [Ph], since $M_G(\tilde{X})$ is compact metric. Thus $M_G(X) = M(X) \cap M_G(\tilde{X})$ is Borel. Moreover $M_G^e(X) = M(X) \cap M_G^e(\tilde{X})$, (ergodicity is independent of embedding) so $M_G^e(X)$ is also Borel.

Now if $\mu \in M_G(X)$, by the theorem on p. 82 of [Ph] there is a unique Borel probability ν on $M_G^e(\tilde{X})$ such that $\mu(f) = \int_{M_G^e(\tilde{X})} \sigma(f) d\nu(\sigma)$ for $f \in C(\tilde{X})$. Then the measure $\int \sigma \, d\nu(\sigma)$ as defined in 1.4 ($\sigma \mapsto \sigma$ is a measurable field!) agrees with μ on continuous functions hence $\mu = \int \sigma \, d\nu(\sigma)$.
In particular

$$1 = \mu(X) = \int \sigma(X) \, d\nu(\sigma)$$

so $\nu\{\sigma : \sigma(X) = 1$, that is $\nu(M(X)) = 1$. Since $\nu(M_G^e(\tilde{X})) = 1$ we get $\nu(M_G^e(X)) = \nu(M(X) \cap M_G^e(\tilde{X})) = 1$. Finally since a representing measure ν on $M_G^e(\tilde{X})$ is unique a representing measure on $M_G^e(X)$ is a fortiori unique. □

We will mainly use the ergodic decomposition for joinings. If X and Y are, as always, ergodic and $\lambda \in J(X,Y)$ then λ has an ergodic decomposition

$$\lambda = \int_{M_G^e(X \times Y)} \sigma \, d\nu(\sigma)$$

where ν is a Borel probability on the space of G-invariant ergodic Borel probabilities on X×Y. Denoting projection on X by π we have.

$$\mu = \pi\lambda = \int_{M_G^e(X\times Y)} \pi(\sigma)d\nu(\sigma).$$

By extremality (ergodicity) of μ we conclude that $\nu\{\sigma : \pi(\sigma) = \mu\} = 1$. (Note that $\{\sigma : \pi\sigma = \mu\}$ is Borel.) Similarily ν-a.a. σ have marginal ν on Y, so we have established the following.

COROLLARY 1.7: *If* $\lambda \in J(X,Y)$ then there is a unique Borel probability on $J^e(X,Y)$ the (Borel) set of ergodic joinings of X and Y, such that

$$\lambda = \int_{J^e(X,Y)} \sigma\, d\nu(\sigma).$$

1.8 Group extensions

Let X be a G-action and K a compact metric group. Suppose that a: $G\times X \to K$ is Borel and that the prescription

$$g(x,k) = (gx, a(g,x)k)$$

defines an action of G on X×K. This amounts to requiring that a satisfy the cocycle equation

$$a(g_2g_1,x) = a(g_2,g_1x)\ a(g_1,x)$$

but we will have no occasion to use this. The action is evidently Borel and preserves the measure $\nu = \mu\times dk$ (dk denotes normalized Haar measure on K). We denote this G-action by $G\times_a K$ and refer to it as a group extension (or K-extension) of X. K acts on the right on X×K by $(x,k)k_0 = (x,kk_0)$ and the action of K commutes with that of G. We denote the action of $k \in K$ by R_k. Finally the projection $X\times K \to X$ is a G-factor map and the σ-algebra it generates is the σ-algebra of (a.e) K-invariant Borel sets in X×K.

Denote now $Y = X\times_a K$ and assume further that Y is ergodic. (The interested reader may refer to Corollary 3.8 of [Zi1] for a necessary and sufficient condition for ergodicity.) Then we have, abstractly, the following setup:

$\mathcal{Y}$ is an ergodic G-space and K is a subgroup of $C(\mathcal{Y})$ which is compact in the weak topology. The following well-known result asserts that all such abstract situations arise as K-extensions. Whenever we have a left action of a group K on a space W we have an associated right action defined by $wk = k^{-1}w$.

THEOREM 1.8. *Suppose* $\mathcal{Y}$ *is an ergodic* G*-action and* K *is a compact subgroup of* $C(\mathcal{Y})$. *Then there is an isomorphism of* $\mathcal{Y}$ *with a* K*-extension* $\mathcal{Y}' = X \times_a K$ *which is also an isomorphism of right Boolean* K*-spaces.* ($\mathcal{Y}$ *is a (left) Boolean* K*-space via the natural action of* K *as a subgroup of* $G(\nu)$.)

We include a proof of theorem 1.8 because we have not found it explicitly in the literature and because the following lemma, which contains most of the work, has another application (see §4). Let us say that a K-space $\mathcal{Y}$ (K locally compact) is free if, for each $y \in Y$, $I_y = \{k \in K : ky = y\}$ is the trivial subgroup of K. We call a Boolean K-space $B(\nu)$ free if $kA = A \;\; \forall A \in B(\mu)$ implies $k = id$. Evidently the natural Boolean K-action of a subgroup K of $G(\nu)$ is free.

LEMMA 1.8. (i) *Suppose* $\mathcal{Y}$ *is an ergodic* G*-space and* K *is a locally compact subgroup of* $C(\mathcal{Y})$. *Then there is a* G×K*-space* $\mathcal{Y}'$ *and an isomorphism of the* G*-spaces* $\mathcal{Y}$ *and* $\mathcal{Y}'$ *which is also an isomorphsm of Boolean* K*-spaces. In particular, as a Boolean* K*-space,* $\mathcal{Y}$ *is free.*

(ii) Suppose $\mathcal{Y}$ *is a* G×K *space which is free as a Boolean* K*-space. If we set for* $y \in \mathcal{Y}$

$$I_y = \{k \in K : ky = y\}$$

then $I_y = \{e\}$ *for all* y *in a co-null* G*- and* K*-invariant Borel set in* Y (e *the edentity element of* K).

REMARK. When $\mathcal{Y}$ is a G×K space it is a G-space via $g(y) = (g,e)y$ and similarily a K-space.

PROOF. (i): We have commuting actions of G and K on $B(\nu)$ hence G×K acts on $B(\nu)$ Moreover for $E \in B(\nu)$ the map $(g,k) \mapsto gkE$ is the composition of the Borel map

$$(g,k) \longmapsto (gE,k)$$

and the map

$$(E,k) \longmapsto kE .$$

This last map is continuous:

$$\nu(k'E'\Delta kE) \le \nu(E'\Delta E) + \nu(k'E\Delta kE),$$

so $(g,k) \mapsto gkE$ is Borel, that is $B(\nu)$ is a Boolean $G\times K$ - space. By Theorem 3.3 of [Ram] (see also Theorem 1 of [Ma1]) the Boolean $G\times K$ space is isomorphic to the Boolean $G\times K$ - space associated with a $G\times K$ - action $\mathcal{Y}'$ via a map $\phi : B(\nu) \mapsto B(\nu')$. (We are making the obvious adjustment of Ramsey's result to the measure-preserving case: he provides a quasi-invariant measure, say λ, on $\mathcal{Y}'$ but ϕ carries ν to an invariant measure ν' on $B(\lambda)$, which is nothing more than an invariant measure (also denoted ν') on Y equivalent to λ.) Regarding ϕ for the moment only as an isomorphism of Boolean G-spaces, Theorem 2 of [Ma1] implies that ϕ is implemented by an isomorphism α of the G-spaces $\mathcal{Y}$ and $\mathcal{Y}'$. α is the desired isomorphsm.

(ii) By Theorem 8.7 of [Va] we may assume at the outset that there is a separable metric on Y generating the Borel structure of Y such that the K action on Y is jointly continuous. This implies that the I_y are closed subgroups. For $g \in G$ we have $I_{gy} = I_y$ for all y, since the G and K actions commute. We now want to conclude that I_y is constant a.e. by considering a suitable Borel structure on the space 2^K of closed subsets of K which makes $y \mapsto I_y$ Borel.

To this end recall that K is locally compact metric with metric d, say, choose a countable dense subset $\{k_i\}$ of K and define functions f_i on 2^K by $f_i(E) = d(k_i,E)$. Observe that the family $\{f_i\}$ separates the points of 2^K hence generates a Borel structure $\mathcal{A}$ on 2^K which is countably separated. To show $y \mapsto I_y$ is Borel we must show that $y \mapsto d(k_i,I_y)$ is Borel. Choose compact subsets $C_1 \subset C_2 \subset \ldots \subset K$ with $\cup C_i = K$. Since $d(k_i,I_y) = \inf_j d(k_i,I_y \cap C_j)$ it suffices to show that $y \mapsto d(k_i,I_y \cap C_j)$ is Borel. We claim that it is in fact lower semi-continuous (with respect to the metric on Y). If this were not so there would exist $y_n \to y$ and $\varepsilon > 0$ such that

$$d(k_i, I_{y_n} \cap C_j) < d(k_i, I_y \cap C_j) - \varepsilon.$$

Thus we could find $h_n \in I_{y_n} \cap C_j$ with

$$d(k_i, h_n) < d(k_i, I_y \cap C_j) - \varepsilon.$$

Passing to a subsequence we may assume that $h_n \to h \in C_j$. Since $h_n y_n = y_n$ and $y_n \to y$, joint continuity gives $hy = y$, that is $h \in I_y \cap C_j$.
Thus

$$d(k_i, I_y \cap C_j) - \varepsilon \geq \lim_n \; d(k_i, h_n)$$

$$= d(k_i, h) \geq d(k_i, I_y \cap C_j),$$

a contradiction.

Thus I_y is a G-invariant Borel function into a countably separated Borel space and hence is a.e. constant. If $I_y = K$ ν a.e. then for $k \in K$ $ky = y$ ν a.e. so $k = e$ by freeness of the Boolean K-action. To complete the proof note that $\{y: I_y = \{e\}\}$ is G-invariant and also K-invariant since $I_{ky} = k\, I_y\, k^{-1}$.

Proof of Theorem 1.8.

By Lemma 1.8' we may assume that there is a free action of K which commutes everywhere with the G-action and whose induced Boolean action on $B(\nu)$ is the natural action of K (as a subgroup of $G(\nu)$). We work with the associated right action of K. Since it commutes with the G-action we may unambiguously write $g y k$ for $g \in G$, $y \in Y$, $k \in K$.

Denote by Y/K the space of K orbits with the quotient Borel structure and $\pi : Y \to Y/K$ the quotient map. We claim that Y/K is standard and that π has a Borel cross-section $\sigma: Y/K \to Y$. To see this we may, forgetting for the moment about the G-action, assume by Theorem 8.7 of [Va] that there is a compact metric (right) K-space Z, that the K-action on Z is jointly continuous and that the K-space Y is a K-invariant Borel subset of Z. We denote by π also the projection from Z to Z/K. Now for $z \in Z$, zK is compact and for E closed in Z, EK is Borel (in fact also compact). Thus by Theorem 4, Section 6.8 of [Bo] there is a Borel subset S of Z meeting each orbit in exactly one point. By Theorem 5.2 of [Ma2] it follows that Z/K is standard. Moreover $\pi|_S$ is 1-1 and onto Z/K, so by Theorem 3.2 of [Ma2] it is a Borel isomorphism onto Z/K. Thus $Y/K = \pi|_S(S \cap Y)$ is standard and $\sigma = (\pi|_S)^{-1}|_{Y/K}$ is the desired cross-section.

G acts on Y/K and the action is Borel since it may be defined by $g\bar{y} = \pi\, g(\sigma\bar{y})$ for $\bar{y} \in Y/K$. Moreover the action preserves the quotient measure $\bar{\nu}$ on Y/K.

Observe that the map $(y,k) \mapsto (y,yk)$ from $Y \times K$ to $Y \times Y$ is Borel and 1-1, in view of freeness. It follows that its range $\mathcal{O}$, the orbit relation in $Y \times Y$, is standard and that the inverse map is also Borel. In particular the map $\theta: \mathcal{O} \to K$ defined by $\theta(y,yk) = k$ is Borel.

We now define $\Psi : Y \to (Y/K) \times K$ by

$$\Psi(y) = (\pi(y),\ \theta(\sigma(\pi(y)),\ y)), \quad \text{that is}$$

$$\Psi(y) = (\pi(y),k) \text{ where } y = \sigma(\pi(y))\, k.$$

Ψ is Borel, 1-1 and surjective and

$$\Psi^{-1}(\bar{y},k) = \sigma(\bar{y})k.$$

Now set $a(g,\bar{y}) = \theta(\sigma(g\bar{y}),\ g(\sigma\bar{y}))$ and calculate

$$\begin{aligned} \Psi\, g\, \Psi^{-1}(\bar{y},k) &= \Psi\, g\, (\sigma(\bar{y})k) \\ &= \Psi[(g\,\sigma(y))k] \\ &= \Psi[(\sigma(g\bar{y})a(g,\bar{y}))k] \\ &= (g\bar{y},\ a(g,\bar{y})k). \end{aligned}$$

Thus Ψ conjugates the G-action on Y to the action

$$g(y,k) = (g\bar{y}, a(g,\bar{y})k).$$

Moreover Ψ evidently conjugates the right K-actions on Y and $(Y/K)\times K$. It remains only to show that $\Psi(\nu) = \bar{\nu}\times dk$. But $\Psi(\nu)$ evidently projects on $\bar{\nu}$ and since $\Psi(\nu)$ is K-invariant, disintegrating it over Y/K we immediately see that ν-a.a. fibre measures are Haar measure on K that is $\Psi(\nu) = \bar{\nu}\times dk$. □

We wish to thank Neil Falkner for reference [Bo] and a useful suggestion for the above argument.

2. THE NON-WEAKLY-MIXING CASE

Let K be a compact group with normalized Haar measure dk, G a locally compact group and $\phi : G \to K$ a Borel homomorphism onto a dense subgroup of K. K is then a G-space under the action $gk = \phi(g)k$.

THEOREM 2.1. *The* G-*space described above is simple with centralizer consisting of all the right multiplications by elements of* K. *It has* MSJ *if and only if* $\phi(G) = K$ *and* K *is abelian.*

PROOF. We denote left and right multiplication by $k \in K$ by L_k and R_k respectively. Let λ be a 2-joining of the G-space, that is a measure on $K\times K$ with marginals dk which is invariant under left multiplication by (k,k) for each $k \in \phi(G)$, and hence for each $k \in K$ by continuity. Consider the Borel map $\theta: (k_1,k_2) \mapsto k_1^{-1}k_2$ of $K\times K$ to K. Evidently

$$\theta[(k,k)(k_1,k_2)] = \theta(k_1,k_2).$$

It follows that $\theta\lambda$ is a Borel measure on K invariant and ergodic under the identity map. Thus $\theta\lambda$ is a point mass, say δ_{k_0}. This means that λ is supported on the graph of R_{k_0}, and since it has marginals μ it must be $\mu_{R_{k_0}}$. Thus the G-action is 2-simple and it's centralizer is $\{R_k : k \in K\}$. We leave it as an exercise to show that n-joinings are also off-diagonals (or it can simply be deduced from the case n = 2).

If $\phi(G) = K$ and K is abelian then the action evidently has MSJ. On the other hand if it has MSJ then for each $k \in K$, R_k agrees a.e. with an $L_{\phi(g)}$ and hence by continuity agrees everywhere with $L_{\phi(g)}$. In particular

$$\phi(g) = L_{\phi(g)}e = R_k e = k.$$

Thus $\phi(G) = K$ and $R_k = L_k$, that is K is abelian. □

THEOREM 2.2. *If a simple action X is not weakly-mixing then it is isomorphic to an example as in Theorem* 2.1.

PROOF: We denote $K = C(X)$. Since X is simple and not weak-mixing

$$J^e(X,X) = \{\mu_k : k \in C(X)\}.$$

We claim that the identification $\mu_k \mapsto k$ is a Borel isomorphism of $J^e(X,X)$ with $C(X)$. To see this first note that the Borel structure on $C(X)$ is generated by the maps $k \mapsto \mu(A \cap k^{-1}B)$ $A, B \in B(X)$, because

$$\mu(A \Delta k^{-1}B) = \mu(A) + \mu(B) - 2\,\mu(A \cap k^{-1}B).$$

Also the Borel structure of $M(X\times X)$ is generated by the maps $\lambda \to \lambda(A\times B)$ by a simple monotone class argument similar to that in §1.4. Since the identification $\mu_k \Longleftrightarrow k$ carries the one family of maps to the other, the claim follows.

Now by Corollary 1.7', $\mu\times\mu$ may be written as an integral over the ergodic joinings and by the above remarks this may be replaced by an integral

$$\mu\times\mu = \int_K \mu_k \, d\tau(k),$$

where τ is a Borel probability on K. This implies that K acts ergodically on $B(\mu)$: if $kA = A$ a.e. for all $k \in K$ then $\mu_k(A\times A^c) = 0$ for all k whence $(\mu\times\mu)(A\times A^c) = 0$.

For $k_0 \in K$

$$\begin{aligned} \mu\times\mu &= (id\times k_0)\,\mu\times\mu \\ &= \int_K \mu_{k_0 k}\, d\,\tau(k) \\ &= \int_K \mu_k\, d\,\tau'(k) \end{aligned}$$

where τ' denotes the left translate of τ by k_0.

By uniqueness of ergodic decomposition we conclude that $\tau' = \tau$. Thus τ is a left invariant Borel probability on K. By the argument in Proposition 4.5 of [Ve] this forces K to be compact.

By Lemma 1.8' we may now assume that there is a free right K-action, commuting everywhere with the G-action, which induces the natural right Boolean K-action. The projection $\pi : X \to X/K$ is a K-invariant function into a standard, hence countably separated, Borel space (see the proof of Theorem 1.8). Since K acts ergodically we conclude that π is a.e. constant, that is there is a co-null K-orbit. Such a K-orbit must also be G-invariant so we may further assume that there is an x_0 with $x_0K = X$.

Now $\theta : k \mapsto x_0k$ maps K onto X in a 1-1 Borel fashion so θ^{-1} is Borel. θ evidently conjugates the right actions of K on K and on X and it is measure-preserving by uniqueness of Haar measure. For $g \in G$, θ^{-1} conjugates the action of g on X to a map of K commuting with all right translations, hence to the left translation by an element of K which we denote $\phi(g)$. Thus the pairing

$$(g,k) \longmapsto \phi(g)k$$

makes K into an ergodic G-space, whence ϕ is necessarily a homomorphism. Moreover the composition

$$g \longmapsto (g,e) \longmapsto \phi(g)e = \phi(g)$$

is Borel.

To complete the proof we need only observe that ergodicity of the G-action forces $\overline{\phi(G)} = K$. Indeed if $\overline{\phi(G)}$ is a strict subgroup consider the right coset space $\overline{\phi(G)}\backslash K$ (that is the space of orbits for the action of $\overline{\phi(G)}$ by left multiplication) and let $\pi: K \longrightarrow \overline{\phi(G)}\backslash K$ denote the projection. Now $\pi(dk)$ is not a point mass: this would mean that dk is supported on a single coset. Since $\overline{\phi(G)}\backslash K$ is a standard Borel space (as in the proof of theorem 1.8, for example) and hence countably separated it follows that there is a Borel set E in $\overline{\phi(G)}\backslash K$ whose measure is neither 0 nor 1. Then $\pi^{-1}(E)$ is a non-trivial set in K which is invariant under left multiplication by $\overline{\phi(G)}$, contradicting ergodicity.

3. FACTORS OF SIMPLE ACTIONS

If X is a G-action and H is a subgroup of $C(X)$ the σ-algebra

$$G(H) \doteq \{A: hA = A \text{ a.e. } \forall h \in H\}$$

is a factor algebra of X. Conversely if G is a factor algebra we define the subgroup

$$H(G) = \{h \in C(X): hA = A \text{ a.e. } \forall A \in G\}.$$

For the case of $\mathbb{Z}$ actions, statement (a) of the following theorem is due to Veech (Theorem 1.2 of [Ve]).

THEOREM 3.1. *Suppose* X *is a* 2-*fold simple* G-*action.*

(a) *If* G *is a non-trivial factor algebra of* X *then* $H(G)$ *is compact and* $G = G(H(G))$.

(b) *If* K *is a compact subgroup of* $C(X)$ *then* $K = H(G(K))$.

PROOF. Let λ denote the G-relatively independent product of X with itself, that is the Borel measure on X^2 which is defined on rectangles by

$$\lambda(A\times B) = \int_X P(A|G)\, P(B|G)\, d\mu.$$

It is not hard to show by elementary means, using the topology of X, that there is such a measure (see for example the proof of Proposition 5.1 of [Ju2]). Alternately one may choose a factor map $\phi: X \to Z$ generating G and let $\lambda = \mu\times_Z\mu$.

As in the proof of Theorem 2.2, λ may be written as an integral of ergodic joinings:

$$\lambda = c(\mu\times\mu) + \int_{C(X)} \mu_k\, d\tau(k)$$

where τ is a Borel measure on $C(X)$ of mass $1-c$.

For $A \in G$ we have

$$0 = \lambda(A\times A^c) = c\mu(A)\mu(A^c) + \int_{C(X)} \mu(A\cap k^{-1}A^c)\,d\tau(k).$$

Since G is non-trivial we can find $A \in G$ with $\mu(A)\mu(A^c) \neq 0$, so we have $c = 0$. Moreover we conclude that for each $A \in G$, $\mu(A\cap k^{-1}A^c) = 0$ for τ-a.a.k. Choosing a countable family $\{A_i\}$ dense in G we can conclude that the closed subgroup

$$\{k : \mu(A_i\cap k^{-1}A_i^c) = 0\ \forall i\}$$

$$=\{k : \mu(A\cap k^{-1}A^c) = 0\ \forall A \in G\}$$

$$= H(G)$$

is τ-co-null, that is τ is supported on $H(G)$.

Now for $k_0 \in H(G)$

$$\lambda(A\times k_0^{-1}B) = \int P(A|G)\ P(k_0^{-1}B|G)\,d\mu$$

$$= \int P(A|G)\ P(B|G)\,d\mu = \lambda(A\times B),$$

so $\qquad (id\times k_0)\ \lambda = \lambda.$ But

$$(id\times k_0)\ \lambda = \int_{H(G)} (id\times k_0)\ \mu_k\,d\tau(k)$$

$$= \int_{H(G)} \mu_{k_0k}\,d\tau(k) = \lambda = \int_{H(G)} \mu_k\,d\tau(k).$$

By uniqueness of ergodic decompositions we conclude, as in the proof of theorem 2.2, that τ is left invariant on $H(G)$ and $H(G)$ is compact.

It remains to show that $G = G(H(G))$. Evidently $G \subset G(H(G))$. On the other hand if $A \in G(H(G))$

$$\lambda(A\times A^c) = \int_{H(G)} \mu(A\cap k^{-1}A^c)\,d\tau(k) = 0.$$

It is an exercise that $\lambda(A\times A^c) = 0$ implies $A \in G$.

For the proof of the second statement first observe that $H(\mathcal{G}(K))$ is always compact. If $\mathcal{G}(K)$ is non-trivial this is a consequence of (a). If $\mathcal{G}(K)$ is trivial, that is K is a compact ergodic subgroup of the centralizer, then as in the proof of theorem 2.2, $\mathcal{X}$ is isomorphic to an action by left translations on K, via an isomorphism which is also an isomorphism of the right Boolean K-spaces $\mathcal{X}$ and K. (Note that the proof of 2.2 did not use the fact that $K = C(\mathcal{X})$, only that K commutes with the G-action.) Since the right translations are the full centralizer of the G-action on K we conclude that $K = C(\mathcal{X})$ so $H(\mathcal{G}(K)) = C(\mathcal{X}) = K$ is compact

To complete the proof let $K' = H(\mathcal{G}(K))$ and observe that $K' \supseteq K$ and $\mathcal{G}(K') = \mathcal{G}(K)$ by part (a). Suppose that $K' \supsetneq K$. Then, as we showed in the proof of theorem 2.2 K' has a Borel subset E which is invariant under right translation by each $k \in K$ but has Haar measure neither 0 nor 1. Representing $\mathcal{X}$ as a K' extension $\mathcal{Y} \times_a K'$ (Theorem 1.8) we see that $Y \times E$ is K-invariant but not K'-invariant, contradicting $\mathcal{G}(K) = \mathcal{G}(K')$. □

<u>REMARK</u>. $\mathcal{X}$ is called prime if its only factor algebras are the algebra of Borel null or co-null sets and the full Borel algebra $B(X)$. It follows from Theorem 3.1 that a simple weak mixing $\mathcal{X}$ is prime if and only if $C(\mathcal{X})$ has no compact subgroups other than {id}. The "if" direction is clear. For the "only if" if $K \neq \{id\}$ is a compact subgroup of $C(\mathcal{X})$ then $\mathcal{G}(K)$ is not $B(X)$, so $\mathcal{G}(K)$ is the null, co-null algebra, that is, K acts ergodically. As we showed in the proof of theorem 3.1(b) this implies that $\mathcal{X}$ is isomorphic to an action by left translations on a compact group and hence is not weak mixing.

Alternately a simple weak-mixing $\mathcal{X}$ is prime if and only if each $S \in C(\mathcal{X})$ such that $S \neq id$ is ergodic. On the one hand if $S \in C(\mathcal{X})$ is not ergodic $\{A:SA = A\}$ is a non-trivial factor algebra. On the other hand if $\mathcal{X}$ is not prime then $C(\mathcal{X})$ has a non-trivial compact subgroup K. If K were ergodic then as we observed above $\mathcal{X}$ is not weak-mixing. Thus K is non-ergodic and hence contains a non-trivial non-ergodic map S. (One can also see directly from the proof of Theorem 3.1 that the ergodicity condition on $C(\mathcal{X})$ implies primality.)

We now fix for the remainder of this section for each compact subgroup K of $C(X)$ a factor map generating $\mathcal{G}(K)$ and denote it $\phi_K: X \to X/K$. (Note that here X/K is not the space of K orbits - K has no orbits.) We will identify the Boolean G-space corresponding to $\mathcal{G}(K)$ with X/K, via the map ϕ_K.

Now suppose $K_1,\dots,K_k$ are compact subgroups of $C(X)$ and $S_1,\dots,S_k \in C(X)$. The off-diagonal k-joining $(S_1 \times\dots\times S_k)\mu_\Delta$ of X projects onto a joining of $X/K_1,\dots,X/K_k$. We call such a joining rigid. A rigid joining need not be off-diagonal as will be clarified by the following results.

Another way to describe the above joining λ is as follows. $\phi_{K_i} S_i$ is a factor map generating the factor algebra $S_i^{-1}\mathcal{G}(K_i) = \mathcal{G}(S_i^{-1}K_iS_i)$ and λ is evidently the joining induced by the imbeddings via $\phi_{K_i} S_i$ of the actions X/K_i in X (as the Boolean G-space corresponding to the factor algebra $\mathcal{G}(S_i^{-1}K_iS_i)$). From this point of view the results we are about to describe bear an interesting formal similarity to Ratner's results on joinings of horocycle flows, [Ra].

<u>THEOREM 3.2</u>. *If X is a simple action and $K_1,\dots,K_k$ are compact subgroups of $C(X)$ then every ergodic joining of $X/K_1\dots X/K_k$ is a product of rigid joinings.*

<u>PROOF</u>. If λ is an ergodic joining of $X/K_1,\dots,X/K_k$ denote by $\hat{\lambda}$ the relatively independent extension (§1.6) of λ to a k-joining of X. $\hat{\lambda}$ may be decomposed as an integral $\hat{\lambda} = \int \tau d\sigma(\tau)$ of ergodic k-joinings of X, which, by simplicity, are all POOD's. Denoting the map $\phi_{K_1}\times\dots\times\phi_{K_k}$ by π we have

$$\lambda = \int \pi\tau \, d\sigma(\tau).$$

By extremality of λ we have $\pi\tau = \lambda$ for σ-a.a. τ. In particular there is at least one POOD τ such that $\pi\tau = \lambda$ so λ is a product of rigid joinings. □

If $K_1 \subset K_2$ are compact subgroups of $C(X)$ then $\mathcal{G}(K_1) \supset \mathcal{G}(K_2)$ so there is a natural factor map between the corresponding Boolean G-spaces (namely the identity map). Via ϕ_{K_1} and ϕ_{K_2} this translates to a Boolean factor map and hence a (point) factor map $X/K_1 \to X/K_2$. More generally if $S \in C(X)$ and $S^{-1}K_1S \subset K_2$ then S induces a Boolean isomorphism from $\mathcal{G}(K_1)$ to $S^{-1}\mathcal{G}(K_1) = \mathcal{G}(S^{-1}K_1S) \supset \mathcal{G}(K_2)$ and thus a factor map denoted $S_{K_1,K_2}: X/K_1 \to X/K_2$.

COROLLARY 3.3. *If* K_1 *and* K_2 *are compact subgroups of* $C(X)$ *each factor map* $X/K_1 \to X/K_2$ is an S_{K_1,K_2} *for some* $S \in C(X)$ *such that* $S^{-1}K_1S \subset K_2$. *If the factor map is an isomorphism then* $S^{-1}K_1S = K_2$. $C(X/K_1) = \{S_{K_1,K_1} : S^{-1}K_1S = K_1\}$.

PROOF. If ϕ: $X/K_1 \to X/K_2$ is a factor map the corresponding joining λ of X/K_1 and X/K_2 is, by theorem 3.2, the projection of a rigid joining. Lifting ϕ to a Boolean map $\hat{\phi}^{-1}: G(K_2) \to G(K_1)$ we thus have that there is an $S \in C(X)$ such that

$$\mu(A \cap \hat{\phi}^{-1}B) = \mu(A \cap S B)$$

for $A \in G(K_1)$, $B \in G(K_2)$. Taking $A = S B$ shows that $\phi^{-1}(B) = S(B)$ (a.e.) for each B. In particular $SG(K_2) = G(S K_2 S^{-1}) \subset G(K_1)$ whence, by Theorem 3.1(b), $SK_2S^{-1} \supset K_1$ so $S^{-1}K_1S \subset K_2$. If ϕ is an isomorphism we have $\hat{\phi}^{-1}G(K_2) = G(K_1)$ so $G(SK_2 S^{-1} = G(K_1)$ and $S^{-1}K_1S = K_2$. The last statement is an immediate consequence since $C(X/K_1)$ is the automorphism group of X/K_1.

The following corollary will be technically useful.

COROLLARY 3.4. *Suppose* $Y_1 = X/K_1$ *and* $Y_2 = X/K_2$ *are factors of the simple* G-*action* X *and* λ *is an ergodic joining of* X/K_1 *and* X/K_2 *which is not product measure. Then the extension*

$$(Y_1 \times Y_2, \lambda) \longrightarrow (Y_1, \nu_1)$$

(we omit the G's) *has relatively discrete spectrum in the sense of* [Zi1].

PROOF: Since λ is not product measure, by theorem 3.2 it is the projection of a rigid 2-joining μ_S, $S \in C(X)$. Now the extension $X \longrightarrow X/K_1$ has relatively discrete spectrum by theorem 1.8 and Example 4.1 of [Zi1]. It is isomorphic as an extension to the extension $(X\times X, \mu_S) \longrightarrow X/K_1$ so this extension also has relatively discrete spectrum. But this extension is the composition

$$(X\times X, \mu_S) \longrightarrow (Y_1\times Y_2, \lambda) \longrightarrow (Y_1, \nu_1).$$

Thus the second extension must also have relatively discrete spectrum by the lemma below.

LEMMA 3.5: *If $X \to Z$ is an extension with relatively discrete spectrum, which factors as $X \to Y \to Z$ then $Y \to Z$ also has relatively discrete spectrum.*

PROOF. We adopt the notation and terminology of [Zi1] and [Zi2]. $L^2(X)$ is in a natural way a Hilbert bundle on Z, $L^2(X) = \int_Z^{\oplus} L^2(F_z)$. Denote by α the natural cocycle representation of Z on $\{L^2(F_z)\}$. Because $X \to Z$ has relatively discrete spectrum, $L^2(X)$ is a direct sum of subspaces $E_0 \oplus E_1 \oplus \ldots$ where the E_i are finite-dimensional invariant sub-bundles $E_i = \int^{\oplus} E_{iz}$ of $L^2(X)$. We can and do assume that the E_i are minimal invariant sub-bundles. We take E_0 to be the subbundle $L_2(Z) \subset L_2(X)$. Denoting by α_i the restriction of α to E_i we claim that α_i and α_j are inequivalent for $i \neq j$. To see this we may, by Lemma 7.7 of [Zi2] assume that $i,j \neq 0$. If α_i and α_j were equivalent then we could find Borel functions $f_m(x) \in E_i$, $h_n(x) \in E_j$ $(1 \le m,n \le k)$ and $\alpha_{mn}(z,g)$ as in the beginning of the proof of Theorem 7.8 of [Zi2]. Then defining $\theta(x) = \sum_{i=1}^{n} f_i(x)h_i(x)$ ones sees that $\theta \in L_\infty(X)$ and $\theta(x)$ is a.e. invariant - the argument is formally the same as in the proof just cited. Moreover since $f_m \perp h_n$, $\theta \perp 1$ which contradicts ergodicity. Thus by the comments at the top of p. 385 of [Zi1] and minimality there is no non-trivial intertwining of α_i and α_j.

Now $= L_2(Y) = \int_Z^{\oplus} E_z$ is in a natural way an invariant subbundle of $L_2(X)$. Suppose that for some i, $P_{E_i}(E)$ is non-trivial. Then $\{P_{E_{i,z}} | z\}$ is an intertwining field for $\alpha|_E$ and α_i, hence by p. 385 of [Zi1] and minimality there is a sub-bundle E_i' of E such that $\alpha|_{E_i'}$ and α_i are equivalent. For each $j \neq i$ $P_{E_{j,z}}|E_{i,z}'$ is an intertwining field for $\alpha|_{E_i'}$ and $\alpha|_{E_j}$ so by the above remarks it must be trivial, that is $E_i' \perp E_j$ for $j \neq i$. Thus $E_i' = E_i$. In summary, whenever $P_{E_i} E \neq 0$, $E_i \subset E$. It follows that E is the sum of a subset of $\{E_i\}$ so $\alpha|_E$ has discrete spectrum. □

COROLLARY 3.6. *If X is simple and* K *is a compact subgroup of* C(X) *then X/K is simple if and only if* K *is normal in* C(X).

PROOF. Suppose K is normal. By Theorem 3.2 it will suffice to show that each rigid k-joining of X/K is off-diagonal. Such a joining is the projection of an off-diagonal k-joining $(S_1 \times \ldots \times S_k)\mu_\Delta$ of X. Since $S_i^{-1}KS_i = K, S_{iK,K} \in C(X/K)$ (Corollary 3.3)and the joining in question is the off-diagonal

$(S_{1K,K} \times \ldots \times S_{kK,K})\,\bar{\mu}_\Delta$, where $\bar{\mu}_\Delta$ is the diagonal k-joining of X/K.

Now suppose X/K is simple and consider the 2-joining λ of X/K which is the projection of the off-diagonal 2-joining μ_S of X, $S \in C(X)$. If X/K is not weakly-mixing then λ is an off-diagonal. On the other hand if X/K is weakly-mixing then product measure is a weakly-mixing extension of X/K while λ, by the proof of Corollary 3.4, is an extension of X/K with relatively discrete spectrum which is incompatible with weak-mixing by Lemma 8.11 and Theorem 8.7 of [Zi2] (X/K is not trivial!). Thus in the weakly-mixing case we can also conclude that λ is an off-diagonal. This means, by Corollary 3.3 that there is a $T \in C(X)$ such that $T^{-1}G(K) = G(K)$ and

$$\mu_T(A\times B) = \mu_S(A\times B)$$

for $A,B \in G(K)$. Taking $B = TA \in G(K)$ we have

$$\mu(A\cap T^{-1}TA) = \mu(A\cap S^{-1}TA)$$

whence $TA = SA$ for each $A \in G(K)$. In particular $S^{-1}G(K) = G(K)$ so, as in the proof of Corollary 3.3, $S^{-1}KS = K$. As S was arbitrary, K is normal.

We mention here that it is now possible to carry over much of the analysis in [Ru1] of a map with MSJ to the case of a general simple group action, at least to the extent that the results in [Ru1] deal with constant powers of T. Denoting by X^k the cartesian product action (X^k,μ^k,G), we state the following result as a sample.

PROPOSITION 3.7: *If X is weakly-mixing and simple then* $C(X^k)$ *is generated by the maps* $S_1\times\ldots\times S_k$, $S_i \in C(X)$, *and the co-ordinate permutations. Moreover if G is a factor algebra of* X^k *which is not contained in any of the σ-algebras generated by a strict subset of the co-ordinate projections, then* $H(G)$ *is compact in* $C(X^k)$ *and* $G = G(H(G))$.

4. JOININGS OF A SIMPLE ACTION WITH ANOTHER ACTION

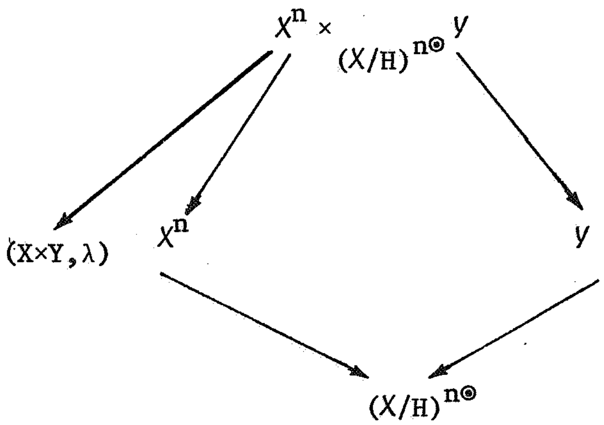
In this section we study joinings of a simple action $\mathcal{X}$ with an arbitrary (ergodic) action $\mathcal{Y}$. When are two such actions not disjoint, that is when does there exist a joining other than product measure? One possibility is that $\mathcal{X}$ and $\mathcal{Y}$ have a non-trivial common factor - then the relatively independent joining over that factor is not product measure.

To describe the other possibility we need the notion of a symmetric product. We denote by $\mathcal{X}^n$ the action $(X^n, \mu^n, \;)$. The symmetric group S_n acts on X^n in a natural way by co-ordinate permutation. We denote by $X^{n\odot}$ the quotient space X^n/S_n, which is a standard Borel space. The quotient map $\pi : X^n \to X^{n\odot}$ is equivariant with respect to the action of G so we have a factor map

$$\pi\colon \mathcal{X}^n \to \mathcal{X}^{n\odot}$$

where $\mathcal{X}^{n\odot}$ denotes the quotient G-action equipped with the quotient Borel structure and the projection of μ^n.

Now suppose that K is a compact subgroup of $C(\mathcal{X})$ and $\phi\colon \mathcal{Y} \to (\mathcal{X}/K)^{n\odot}$ is a factor map. $(\mathcal{X}/K)^{n\odot}$ is also a factor of $\mathcal{X}^n$ so we may form the relatively independent joining of $\mathcal{X}^n$ and $\mathcal{Y}$ over $(\mathcal{X}/K)^{n\odot}$. Restricting to X×Y we get a joining λ of $\mathcal{X}$ and $\mathcal{Y}$. (It is easy to see that this joining does not depend on which of the n copies of X is chosen.) In pictures



As a simple example of this construction when K is trivial and $Y = X^{2\odot},\lambda$ is the joining arising from the common embedding of X and $X^{2\odot}$ in X^2 via the natural factor maps. When X is a $\mathbb{Z}$-action with MSJ this gives an example [Ru1] of two $\mathbb{Z}$-actions without common factors which are not disjoint.

The main result of this section is (almost) that every ergodic joining of a simple X with an arbitrary Y arises in this way. However we need to make a further assumption about X. Let us say X is regular if there is a complete separable metric group C (not necessarily locally compact) and a Borel measure-preserving action of C on (X,μ) which commutes everywhere with the G-action, is everywhere free (that is, all the stabilizer subgroups $\{c : cy = y\}$ are trivial) and such that each $S \in C(X)$ agrees everywhere with the action of some $c \in C$.

Regularity is a technical condition (which seems to be) necessary for the proof of theorem 4.1. By Lemma 1.8' if $C(X)$ is locally compact then X is isomorphic to a regular action. For virtually all the examples we know of simple X, $C(X)$ is locally compact and indeed usually $C(X)$ is obviously a commuting free action of a locally compact group, because the simplicity arises from a very explicit knowledge of the centralizer. Anticipating results in later sections the examples we have in mind are these. If X has MSJ $C(X)$ corresponds to a closed subgroup of G, so is locally compact. If X is a weakly-mixing group extension $Y \times_a K$ of a Y with MSJ then $C(X)$ is $C(Y) \times K$. In the case of the action of a closed, normal, co-compact subgroup in a weakly-mixing simple action the centralizer is the same as the centralizer of the full action. The only possible exception to regularity of simple actions that we know of is our forthcoming example, mentioned in the introduction, of a simple weakly-mixing, rigid $\mathbb{Z}$-action where, at any rate, the centralizer appears not to be locally compact.

THEOREM 4.1. *If X is a simple regular action and Y is any action then every ergodic joining of X and Y which is not product measure arises as described above, namely it is the projection on* $X \times Y$ *of the relatively independent joining of X^n and Y over $(X/K)^{n\odot}$ for some compact subgroup K of $C(X)$ and some factor map $\phi : Y \to (X/K)^{n\odot}$. If X is not weakly-mixing n may be taken to be* 1.

REMARK: The conclusion of the theorem is valid for any X' isomorphic to X, hence the hypotheses on X may be weakened to the requirement that X is simple and isomorphic to a regular action.

For the proof of theorem 4.1 we will need several lemmas.

LEMMA 4.2: *Suppose given a free Borel action of a complete separable metric group* C *on the standard Borel space* X. *Then for each Borel* $F \subset C$ *the function* $(\tau,x) \mapsto \tau(Ex)$ *mapping* $M(X)\times X \to \mathbb{R}$ *is Borel.*

PROOF. The set $\{(x,ex) : x \in X,\ e \in E\} \subset X\times X$ is the image of $X\times E$ under the 1-1 Borel map $(x,c) \mapsto (x,cx)$ from $X\times C$ to $X\times X$, (both standard Borel spaces), hence it is Borel. Thus it suffices to show that for any Borel $A \subset X\times X$, $(\tau,x) \mapsto \tau(A\cap\{x\}\times X)$ is a Borel function. This is an extension of Fubini's theorem and the proof is similar. Denote by $\mathcal{A}$ the class of sets A for which $(\tau,x) \mapsto \tau(A\cap\{x\}\times X)$ is Borel. If $A = A_1\times A_2$ is a rectangle then

$$(\tau,x) \mapsto \tau(A\cap(\{x\}\times X)) = 1_{A_1}(x)\ \tau(A_2)$$

is Borel since $\tau(A_2)$ is a Borel function of τ. A standard argument shows that $\mathcal{A}$ is a monotone class and hence contains the Borel σ-algebra in $X\times X$. □

LEMMA 4.3: (a) *Suppose that* C *is a complete separable metric group,* σ *is a Borel probability on* C *and* F *is a Borel set in* C. *Then* $c \mapsto \sigma(F_c)$ *is a Borel function on* C. *When* F *is open this function is lower semi-continuous.*

(b) *Let* $\mathcal{F}$ *denote the class of all finite unions of some countable basis for the topology of* C. *Suppose that for each* $F \in \mathcal{F}$, *the function* $c \mapsto \sigma(Fc)$ *is* σ*-a.e. constant on* C. *Then* σ *is a right translate of Haar measure on some compact subgroup* K *of* C.

(c) *The subgroup* K *depends only on the* σ*-a.e. constant value of* $\sigma(Fc)$, $F \in \mathcal{F}$.

PROOF: (a) The first part of (a) is a special case of Lemma 4.2, where C acts on C by right multiplication. Now suppose F is open and suppose $c_n \to c \in C$. Then

$$\underline{\lim}\ 1_{Fc_n} \geq 1_{Fc}\ .$$

Thus by Fatou's lemma

$$\underline{\lim}\ \sigma(Fc_n) = \underline{\lim} \int 1_{Fc_n}\, d\sigma$$

$$\geq \int \underline{\lim}\ 1_{Fc_n}\, d\sigma \geq \int 1_{Fc}\, d\sigma = \sigma(Fc).$$

(b) First note that the hypotheses actually imply that $\sigma(F_c)$ is constant a.e. - σ for each Borel F. To see this let $\mathcal{A}$ denote the class of sets F having this property. Then $\mathcal{A}$ is clearly a monotone class: if $F_n \to F$ monotonically then $\sigma(F_n c) \to \sigma\ (Fc)$ for each c, so $\sigma(Fc)$ is σ - a.e. constant if each $\sigma(F_n c)$ is. Since $\mathcal{A}$ contains $\mathcal{F}$ it then contains all open sets and hence all G_δ's. Now in a metric space a closed set is a G_δ so it is easy to see that the algebra generated by the open sets consists of G_δ's. Thus $\mathcal{A}$ contains this algebra and hence, by the monotone class theorem, the Borel sets.

Now the property of σ in question holds for any right translate $R_{c_0}\sigma$ of σ: if $\sigma(Fc) = k$ for $c \in C'$ with $\sigma(C') = 1$ then

$$(R_{c_0}\sigma)(Fc) = \sigma(Fcc_0^{-1}) = k$$

for $c \in C'c_0$ and $(R_{c_0}\sigma)\ C'c_0) = 1$. Thus we can and do assume that e, the identity element of C, belongs to the support of σ and our aim is now to prove that σ is actually Haar measure on a compact subgroup.

For each Borel F let $\tau(F)$ denote the σ- a.e. constant value of $\sigma(Fc)$. τ is a measure on the Borel sets because if F_i i = 1,2,... are disjoint and $F = \cup F_i$ we can find a single c such that $\tau(F_i) = \sigma(F_i c)$ and $\tau(F) = \sigma(Fc)$. We claim that in fact $\tau = \sigma$. To see this set

$$K = \{c \in C\colon \sigma(Fc) = \tau(F) \quad \forall F \in \mathcal{F}\}$$

Evidently $\sigma(K) = 1$ so $e \in \bar{K}$. Taking $c_i \in K$ such that $c_i \to e$ we have for any $F \in \mathcal{F}$

$$\tau(F) = \underline{\lim}\ \sigma(Fc_i) \geq \sigma(F),$$

by lower semi-continuity. The inequality $\tau(F) \geq \sigma(F)$ evidently persists under monotone limits so it persists for G_δ sets F, hence by the monotone class theorem for all Borel sets. Since τ and σ are both probabilities we conclude $\tau = \sigma$. Thus

$$K = \{c : \sigma(Fc) = \sigma(F) \quad \forall F \in \mathcal{J}\}$$

$$= \{c : \sigma(Fc) = \sigma(F) \quad \forall \text{ Borel } F\},$$

since the measures $R_c\sigma$ and σ agree if they agree on $\mathcal{F}$. This makes it clear that K is a group. It is also closed: if $c_i \in K$ and $c_i \to c$ then for open F

$$\sigma(F) = \underline{\lim}\ \sigma(Fc_i) \geq \sigma(Fc),$$

so as we argued before $\sigma(F) = \sigma(Fc)$ for all Borel F. Thus K is a closed subgroup supporting σ and σ is right invariant on K. By the argument in Proposition 4.5 of [Ve], K must be compact.

(c) This is implicit in the proof of (b): if σ' is another measure with the same property and $\tau'(F)$, the a.e. constant value of $\sigma'(Fc)$, is the same as $\tau(F)$ then $\sigma' = \tau' = \tau = \sigma$ so

$$K = \{c : \sigma(Fc) = \sigma(F)\}$$

$$= \{c : \sigma'(Fc) = \sigma'(F)\} = K'. \quad \square$$

<u>LEMMA 4.4</u> (i) *For* $\nu \in M(X)$ *let* ν^a *denote the atomic part of* ν. *Then the map* $\nu \longmapsto \nu^a$ *is Borel as a map into the space of sub-probability measures on* X, *with Borel structure generated by the functions* $\mu \longmapsto \mu(A)$,

(ii) *For* $\nu \in M(X)$ *and* $\alpha > 0$ *let* ν^α *denote the trace of* ν *on the union of all its atoms of measure* $\geq \alpha$. *Then* $\nu \longrightarrow \nu^\alpha$ *is a Borel map.*

(iii) *Denote the equivalence class of* $(x_1,\ldots,x_n)$ *in* $X^{n\circledS}$ *by* $[x_1,\ldots,x_n]$ *and let* $\delta_{[x_1,\ldots,x_n]}$ *denote the measure on* X *which gives mass* 1/n *to each* x_i *(some* x_i*'s may be equal). Suppose* Y *is a measurable space* $\phi: Y \to X^{n\circledS}$ *and the field* $y \longmapsto \delta_{\phi(y)}$ *is measurable. Then* ϕ *is measurable.*

PROOF. (i),(ii): We take X to be compact metric. Denoting by $\underline{M}(X)$ the space of Borel sub-probabilities on X, $\underline{M}(X)$ is compact metric in the weak* topology and this topology generates the Borel structure of $\underline{M}(X)$. Now choose a refining sequence $\{P_n\}$ of finite Borel partitions of X which separates the points of X. For fixed n, α set, for $\nu \in M(X)$, $A_\nu = \cup\{p \in P_n : \nu(p) > \alpha\}$ and $\nu_{n,\alpha} = \nu|_{A_\nu}$, the trace of ν on A_ν.

We claim $f:\nu \mapsto \nu_{n,\alpha}$ is measurable. To see this, for each E which is a union of atoms of P_n set $E^* = \{\nu \in M(X): A_\nu = E\}$. E^* is Borel as it is the intersection of the Borel sets $\{\nu:\nu(p) > \alpha\}$ for $E \supset p \in P_n$. Now f may be expressed as

$$f(\nu) = \sum_E 1_{E^*}(\nu)\nu|_E.$$

It is immediate that $\nu|_E$ is a Borel function of ν, hence so is it's restriction $1_{E*}(\nu)\,\nu|_E$ to the Borel set E^*. Finally, a finite sum of Borel functions into $\underline{M}(X)$ is easily seen to be Borel, whence the claim follows.

Now for each ν, as $n \to \infty$ $\nu_{n,\alpha} \longrightarrow \nu^\alpha$ weak - * (indeeed even in norm). Thus $\nu \longmapsto \nu^\alpha$ is Borel, which is (ii). As $\alpha \to 0$ $\nu^\alpha \to \nu^a$, which completes the proof of (i).

PROOF. (iii): Let $\pi: X^n \longrightarrow X^{n\circledS}$ denote the canonical projection. Denote by F_Σ the σ-algebra of symmetric Borel sets in X^n, and by B the class of symmetrized rectangles in X^n (sets of the form $\cup_{\sigma\in S_n}\sigma(R)$ R a rectangle). As is well known B generates F_Σ, in other words $B(X^{n\circledS})$ is generated by $\{\pi(R):R$ a rectangle$\}$. Thus to check the measurability of ϕ it suffices to show that for $A_1,\ldots,A_n$ Borel in X, $\phi^{-1}\pi\,(A_1\times\ldots\times A_n)$ is measurable in Y. Now $[x_1,\ldots,x_n] \in X^{n\circledS}$ belongs to $\pi(A_1\times\ldots\times A_n)$ if and only if $\delta_{[x_1,\ldots,x_n]}(A_{i_1}\cup\ldots\cup A_{i_k}) \geq k/n$ for all $i_1,\ldots,i_k$: this is an application of the marriage lemma. Thus

$$\phi^{-1}\pi(A_1\times\ldots\times A_n) = \bigcap_{i_1,\ldots,i_k}\{y: \delta_{\phi(y)}(A_{i_1}\cup\ldots\cup A_{i_k}) \geq k/n\}$$

is measurable. □

PROOF OF THEOREM 4.1.

Let λ be a joining of $\mathcal{X}$ and $\mathcal{Y}$ and

$$\lambda = \int_Y^{\oplus} \lambda_y d\nu(y)$$

its disintegration over Y. We are now going to distinguish two possibilities for this disintegration according to which λ will be either product measure or not.

By regularity of X there is a free action of a complete separable metric group C which commutes everywhere with the G-action and realizes C(X). We denote by $\mathcal{O}$ the orbit relation in X×X for the action C. $\mathcal{O}$ is Borel, as in the proof of Lemma 4.2. Now $y \mapsto \lambda_y \times \lambda_y$ is a measurable field so

$$Y' = \{y: \lambda_y \times \lambda_y(\mathcal{O}) = 0\}$$

is Borel. Since $\mathcal{O}$ is G-invariant in X×X and $\lambda_{gy} \times \lambda_{gy} = g(\lambda_y \times \lambda_y)$ a.e. ν for each g we conclude that $gY' = Y'$ a.e. for each $g \in G$ so $\nu(Y') = 0$ or 1 by ergodicity of $\mathcal{Y}$.

Consider first the case $\nu(Y') = 1$. We shall show that then $\lambda = \mu \times \nu$. To this end consider the $\mathcal{Y}$-relatively independent product of λ with itself. In the present situation this may be described as the measure $\hat{\lambda}$ on X×X×Y given by the disintegration

$$\hat{\lambda} = \int_Y^{\oplus} \lambda_y \times \lambda_y \, d\nu(y).$$

$\hat{\lambda}$ is a joining of $(\mathcal{X},\mathcal{X},\mathcal{Y})$. Projecting on $\mathcal{X}\times\mathcal{X}$ we obtain a 2-joining $\bar{\lambda}$ of X. $\bar{\lambda}$ is of course the average

$$\bar{\lambda} = \int_Y \lambda_y \times \lambda_y \, d\nu(y),$$

so if $\nu(Y') = 1$ then $\bar{\lambda}(\mathcal{O}) = 0$.

Now since X is simple the joining $\bar{\lambda}$ may be written, as in the proof of theorem 3.1,

$$\overline{\lambda} = c(\mu\times\mu) + \int_{C(X)} \mu_S d\tau(S)$$

for some Borel measure τ on C(X). We then calculate

$$0 = \overline{\lambda}(\mathcal{O}) = c(\mu\times\mu)(\mathcal{O}) + \int_{C(X)} \mu_S(\mathcal{O})d\tau(S).$$

If $S \in C(X)$ agrees (a.e.) with the action of $c \in C$ then μ_S is supported on $\{(x, cx): x \in X\} \subset \mathcal{O}$ so $\mu_S(\mathcal{O}) = 1$. Thus we have

$$0 = c(\mu\times\mu)(\mathcal{O}) + \tau(C(X))$$

so $\tau = 0$, that is $\overline{\lambda} = \mu\times\mu$.

Since λ is a joining we have $\int_Y \lambda_y d\nu(y) = \mu$ so we may write

$$\overline{\lambda} = \mu\times\mu = \int_Y \lambda_y d\nu(y) \times \int_Y \lambda_y d\nu(y)$$

$$= \int_Y (\lambda_y\times\lambda_y) d\nu(y).$$

Thus for any Borel $A \subset X$ we have

$$\left(\int_Y \lambda_y(A) d\nu(y)\right)^2 = \int_Y \lambda_y(A)^2 d\nu(y),$$

so by strict convexity of the function x^2 we have that $\lambda_y(A)$ is constant ν a.e. It follows that $\lambda = \mu\times\nu$.

We now turn to the case where $\nu(Y') = 0$. For each Borel $E \subset C$ the function F_E defined on $X\times Y$ by

$$F_E(x,y) = \lambda_y(Ex),$$

is Borel by Lemma 4.2. Moreover for a given $g \in G$

$$\lambda_{gy}(Egx) = \lambda_{gy}(gEx) = \lambda_y(Ex),$$

as long as $g\lambda_y = \lambda_{gy}$. Since this is so for ν a.a. y we conclude that $F_E(gx,gy) = F_E(x,y)$ λ-a.e. Thus by ergodicity of λ $F_E(x,y)$ is a constant, say $\tau(E)$, λ-a.e. Letting F denote the class of finite unions of sets in a countable basis for the topology of C we may now find a co-null Borel set $Y^* \subset Y$ such that for $y \in Y^*$ and $E \in F$, $F_E(x,y) = \tau(E)$ λ_y-a.e.

Now since $\nu(Y') = 0$, for a.a.y there is an $x(y)$ with $\lambda_y(C\,x(y)) > 0$. (We make no claim that $x(y)$ is a measurable function.) If $y \in Y^*$ and $\lambda_y(C\,x(y)) = 0$ then mapping $C\,x(y)$ to C via the Borel map $c\,x(y) \longmapsto c$ we obtain a non-zero measure σ which, after normalization, satisfies the hypotheses of Lemma 4.3(b). Thus, adjusting the choice of $x(y)$ if necessary, we conclude that there is a compact subgroup K of C, so that the trace of λ_y on $C\,x(y)$ is a non-zero multiple of Haar measure on $K\,x(y)$. K does not depend on y by Lemma 4.3(c).

Now let π: $X \to X/K$ be the canonical projection. We also write π for the map $(x,y) \mapsto (\pi(x),y)$ and let $\bar{\lambda} = \pi\lambda$. $\bar{\lambda}$ is an ergodic joining of X/K and Y. Evidently $\bar{\lambda} = \int^{\oplus} \bar{\lambda}_y \, d\nu(y)$ where $\bar{\lambda}_y = \pi\lambda_y$. Moreover by our above remarks $\bar{\lambda}_y$ has at least one atom for ν-a.a. y. Now let $\bar{\lambda}_y^a$ denote the atomic part of $\bar{\lambda}_y$. $\bar{\lambda}_y^a$ is a measurable field by Lemma 4.4(i), so we may define

$$\bar{\lambda}^a = \int^{\oplus} \bar{\lambda}_y^a \, d\nu(y).$$

Since $g\bar{\lambda}_y^a = \bar{\lambda}_{gy}^a$ for ν a.a.y, $\bar{\lambda}^a$ is a G-invariant Borel measure on $X/K \times Y$ which is almost continuous with respect to $\bar{\lambda}$. By ergodicity of $\bar{\lambda}$ we conclude $\bar{\lambda}^a = c\bar{\lambda}$ which of course means that $\bar{\lambda}^a = \bar{\lambda}$.

Next we claim that that for ν-a.a. y $\bar{\lambda}_y^a = \bar{\lambda}_y$ consists of n point masses of equal weight. Indeed, for each $\alpha > 0$, the trace $\bar{\lambda}_y^\alpha$ of $\bar{\lambda}_y$ on its atoms of weight greater than α is a measurable field by Lemma 4.4(ii) so

$$\bar{\lambda}^\alpha = \int^{\oplus} \bar{\lambda}_y^\alpha \, d\nu(y)$$

is a G-invariant measure almost continuous with respect to $\bar{\lambda}$. As before we conclude that for each α, $\bar{\lambda}^\alpha = \bar{\lambda}$ or 0. Taking $\alpha = \sup\{\alpha: \bar{\lambda}^\alpha = \bar{\lambda}\}$ it is easy to see that $\alpha = 1/n$ and for almost all y, $\bar{\lambda}_y$ consists of n point masses of weight $1/n$.

In other words, for a.a.$y \in Y$ there is a $\phi(y) \in (X/K)^{n\circleddash}$ such that $\phi(y)$ consists of n distinct points and $\overline{\lambda}_y = \delta_{\phi(y)}$ (notation as in Lemma 4.4 (iii)). ϕ is measurable by Lemma 4.4 (iii) and a.e. G-equivariant but we no not yet know that it is a measure-preserving map onto the space $(X/K)^{n\circleddash}$. Moreover by our previous discussion if $\overline{\lambda}_y = \delta_{[\overline{x}_1,\ldots,\overline{x}_n]}$ then λ_y is $\frac{1}{n}\Sigma_{i=1}\lambda_{\overline{x}_i}$ where $\lambda_{\overline{x}_i}$ denotes normalized Haar measure on the K-orbit $\overline{x}_i$, and the n orbits $\overline{x}_1,\ldots,\overline{x}_n$ lie in distinct C-orbits.

Now define a measure σ on $X^n \times Y$ as follows. For $\overline{x} \in X/K$ denote by $\lambda_{\overline{x}}$ normalized Haar measure on the K-orbit $\overline{x}$. For $\xi = [\overline{x}_1,\ldots,\overline{x}_n] \in (X/K)^{n\circleddash}$ with $\overline{x}_1,\ldots,\overline{x}_n$ distinct denote by σ_ξ the probability on X^n defined by

$$\sigma_\xi = \frac{1}{n!}\sum_{\tau\in S_n} \lambda_{\overline{x}_{\tau(1)}} \times\cdots\times \lambda_{\overline{x}_{\tau(n)}} .$$

We take a moment to check that σ_ξ is a measurable field. Since the Borel structure of $(X/K)^{n\circleddash}$ is the quotient Borel structure for the map $(X/K)^n \to (X/K)^{n\circleddash}$ this amounts to showing that the composed map

$$(\overline{x}_1,\ldots,\overline{x}_n) \longmapsto \frac{1}{n!}\sum_{\tau\in S_n} \lambda_{\overline{x}_{\tau(1)}} \times\cdots\times\lambda_{\tau(n)}$$

from $(X/K)^n$ to $M(X^n)$ is measurable. Thus it suffices to show

$$(\overline{x}_1,\ldots,\overline{x}_n) \longmapsto \lambda_{\overline{x}_1}\times\cdots\times\lambda_{\overline{x}_n}$$

is measurable and since $(\lambda_1,\ldots,\lambda_n) \longmapsto \lambda_1\times\cdots\times\lambda_n$ is continuous it suffices to show that $\overline{x} \longmapsto \lambda_{\overline{x}}$ is a measurable map from X/K to M(X). This amounts to showing that $x \longmapsto \lambda_{Kx}$ is a measurable map from X to M(X). But for A Borel in X

$$\lambda_{Kx}(A) = \int_K 1_A(kx)\,dk$$

is a measurable function of x by Fubini's theorem. Thus σ_ξ is indeed a measurable field and we observe that σ_ξ is the fibre measure over ξ for the canonical factor map $X^n \to (X/K)^{n\circleddash}$. Now set

$$\sigma = \int_Y^{\oplus} \sigma_{\phi(y)}\,d\nu(y).$$

Note that σ is a G-invariant measure on $X^n \times Y$. (If we knew that ϕ was measure-preserving then we would know that σ was the relatively independent joining of X^n and Y over $(X/K)^{n\odot}$. We shall see in a moment that this is so.)

Note that if $\phi(y) = [x_1,\dots,x_n]$ then in the sum defining $\sigma_{\phi(y)}$ each $\lambda_{\bar{x}_i}$ occurs as the first factor in a fraction $1/n$ of the terms. Thus the projection of $\sigma_{\phi(y)}$ on the first co-ordinate (or similarily, on any other) is $\frac{1}{n}\Sigma_i\lambda_{\bar{x}}$, that is, λ_y. It follows that the projection of σ on any factor $X \times Y$ in $X^n \times Y$ is

$$\int^{\oplus} \lambda_y \, d\nu(y) = \lambda.$$

In particular the projection of σ on any factor X in $X^n \times Y$ is μ.

To finish the proof it remains only to show that $\phi: Y \to (X/K)^{n\odot}$ is measure-preserving, or, what is the same thing, that the projection $\bar{\sigma}$ of σ onto X^n is μ^n. We already know that it is an n-joining of X and so is an average of POOD's. If it is not product measure then there must be two co-ordinates, say the first two, for example, so that this average gives positive weight to POOD's which link those two co-ordinates. This means that the projection $\hat{\sigma}$ of $\bar{\sigma}$ on the first two co-ordinates gives positive mass to the C-orbit relation $\mathcal{O} \subset X \times X$. But

$$\hat{\sigma} = \int \frac{1}{n!} \sum_{\tau \in S_n} \left(\lambda_{\bar{x}_{\tau(1)}} \times \lambda_{\bar{x}_{\tau(2)}}\right) d\nu(y)$$

and each $\lambda_{\bar{x}_{\tau(1)}} \times \lambda_{\bar{x}_{\tau(2)}}$ appearing in this average gives mass 0 to $\mathcal{O}$, since the K-orbits $\bar{x}_{\tau(1)}$ and $\bar{x}_{\tau(2)}$ lie in different C-orbits. This conflict means that $\bar{\sigma}$ is μ^n and completes the proof.

In the case where X is not weak-mixing then, as in the proof of Theorem 2.2, C may be taken to be a compact group acting transitively, and the above argument evidently gives $n = 1$. □

COROLLARY 4.5 *If X and Y are simple, any ergodic joining of X and Y is given as in theorem 4.1, but with* $n = 1$.

PROOF. If X is not weak-mixing we are done by Theorem 4.1, so we suppose X is weak-mixing. By theorem 4.1 it now suffices to show that for $n > 1$, $(X/K)^{n\odot}$ cannot be a factor of the simple action Y. We set $Z = X/K$ and all we shall use about Z is that it is weak-mixing and non-trivial. We show that $Z^{n\odot}$ cannot be a factor of a simple action by exhibiting an ergodic 2-joining of $Z^{n\odot}$ which is not product measure but which does not have relatively discrete spectrum over $Z^{n\odot}$ (see Corollary 3.4).

Consider the 2-joining σ of Z^n obtained by linking the first co-ordinates in each of the copies of Z^n diagonally to each other. Precisely

$$\sigma(A_1\times\ldots\times A_n\times B_1\times\ldots\times B_n) = \lambda(A_1\cap B_1)\ \lambda(A_2)\ldots\lambda(A_n)\lambda(B_2)\ldots\lambda(B_n),$$

where λ denotes the measure on Z. We also denote by σ the projection of σ on $Z^{n\odot}\times Z^{n\odot}$. We remark that the extension

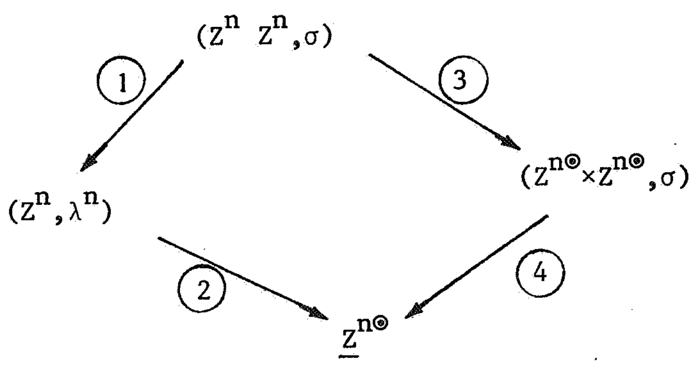
$$(Z^n\times Z^n,\sigma)\rightarrow(Z^n,\lambda^n)$$

is naturally isomorphic to the extension

$$(Z^{2n-1},\lambda^{2n-1})\rightarrow(Z^n,\lambda^n).$$

This last extension is a direct product and hence is a weakly mixing extension (in the sense of Definition 7.9 of [Zi2]) by Corollary 7.11 of [Zi2], since Z is weak mixing.

Now consider the following commutative diagram of extensions.



(1) is a weakly mixing extension as we have just remarked. Suppose that (4) had relatively discrete spectrum. Since the finite extension (3) certainly has discrete spectrum we would conclude that the composition of (3) and (4)

or equivalently (1) and (2) has generalized discrete spectrum (Definition 8.4 of [Zi2]). It follows that (1) would also have generalized discrete spectrum (use the equivalence of generalized discrete spectrum and a relatively separating sieve, together with Proposition 8.6 of [Zi2]). This is incompatible with the weak-mixing of (1) (Lemma 8.11 of [Zi2]). Thus (4) cannot have discrete spectrum, completing the proof.

COROLLARY 4.6. *If* X *and* Y *are simple* G-*actions with no common factor then* X *and* Y *are disjoint.*

PROOF. This is immediate from Corollary 4.5.

5. WEAKLY-MIXING GROUP EXTENSIONS

Our main aim in this section is to prove that a weakly-mixing group extension of an action with MSJ is simple. The following general lemma, which is similar to Proposition 3.10 of [Fu], will be our main tool.

LEMMA 5.1. *Let* $Y = X \times_a K$ *be an ergodic group extension. Let* λ *be any* G-*invariant measure on* $Y = X{\times}K$ *which projects onto* μ. *Then* $\lambda = \mu{\times}dk$ *where* dk *denotes normalized Haar measure.*

PROOF. We denote the action of $k \in K$ on $X{\times}K$ by right translation by R_k. For A Borel in Y, $(R_k\lambda)$ A is measurable function of k (we leave the proof as an exercise). Thus we may define

$$\bar{\lambda} = \int_K R_k \lambda \, dk.$$

Evidently $R_k \bar{\lambda} = \bar{\lambda}$ for each $k \in K$ and $\bar{\lambda}$ projects on μ. By disintegrating $\bar{\lambda}$ over X it follows immediately that $\bar{\lambda} = \mu \times dk$. Since each $R_k \lambda$ is G-invariant, ergodicity of $\bar{\lambda}$ gives that $R_k \lambda = \bar{\lambda}$ for a.a. k and hence for all k by continuity. In particular $\lambda = \bar{\lambda} = \mu{\times}dk$. □

Before proceeding to the main result we introduce an auxiliary concept which is of some interest in its own right. Let us say an action *X* is pairwise independently determined (PID) if for all n any n-joining of X which is pairwise independent (that is, its projection on the product of any two copies of X in X^n is product measure) must be product measure μ^n. Note that for weakly-mixing X it suffices to require this for ergodic n-joinings: if λ is an arbitrary pairwise independent joining then almost all of its ergodic components must also be pairwise independent because $\mu\times\mu$ is G-ergodic.

We observe that a weakly-mixing X is simple (has MSJ) iff it is 2-fold simple (has 2-fold MSJ) and is PID. Indeed if X is 2-fold simple and PID and λ is any ergodic n-joining, split X^n as a product of maximal factors on which λ is off-diagonal. Each of these factors is isomorphic to X itself and any two of them are independent since they are not off-diagonally linked and X is 2-fold simple. Thus these factors are jointly independent and λ is a POOD.

LEMMA 5.2. *A weakly-mixing group extension* $Y = X\times_a K$ of a PID action is again PID.

PROOF. Let λ be an n-joining of Y which is pairwise independent. The projection of λ on X^n is an n-joining of X which is again pairwise independent and hence must be product measure. Now Y^n is a group extension of X^n by the group K^n, which is ergodic since Y is weakly mixing. Moreover λ is a G-invariant measure on $X^n\times K^n$ which projects on μ^n as we have just seen. Thus by Lemma 5.1 $\lambda = \mu^n\times(dk)^n = (\mu\times dk)^n$. □

The following lemma says that joinings of different PID processes obey the same rule as joinings of a single PID process: pairwise independence implies independence. We will use this result in section 6.

PROPOSITION 5.3. *Let* λ *be a joining of the PID processes* $X_1,\ldots,X_k$. *If* λ *is pairwise independent then* λ *is the product joining* $\mu_1\times\ldots\times\mu_k$.

PROOF. First we establish a special case, namely $X_1 = X$, $X_2 =\ldots= X_k = Y$. We form the relatively independent product of λ with itself over X_1, namely

$$\hat{\lambda} = \int^{\oplus}_{X_1} \lambda_{x_1} \times \lambda_{x_1}\, d\mu(x_1),$$

where $\lambda = \int_{X_1}^{\oplus} \lambda_{x_1} \, d\mu(x_1)$ is the disintegration of λ over X_1. $\overline{\lambda}$ is a joining of $X_1, X_2, \ldots, X_k, X_2, \ldots, X_k$ and with respect to $\hat{\lambda}$ any single factor Y is independent of X_1, since $\hat{\lambda}$ projects on λ which is pairwise independent. For the same reason any two copies of Y both coming from the first group $Y_2, \ldots, X_k$, or both from the second are independent. If we consider copies of Y taken from the first and second groups respectively they are also independent, because they are independent conditionally on X_1 (definition of $\hat{\lambda}$) and each is independent of X_1. Thus the projection of $\hat{\lambda}$ on $X_2 \times \ldots \times X_k \times X_2 \times \ldots \times X_k$ is pairwise indepent, hence independent, since Y is PID. It follows by the convexity argument used in the proof of Theorem 4.1 that λ on $X \times Y^{k-1}$ is the product of its projection on X and on Y^{k-1}. But the projection of λ on Y^{k-1} is product measure since Y is PID. This completes the proof of the special case.

For the general case we proceed by induction on the number of distinct actions among $X_1, \ldots, X_k$. We may as well assume, for simplicity of notation, that each action occurs the same number of times, say r, in $X_1, \ldots, X_k$. If λ is a pairwise independent joining of $X_1, \ldots, X_k$ all the copies of a fixed system sit jointly independent, so gathering together like copies and relabelling we may assume λ is a joining of $X_1^r, \ldots, X_k^r$ in which any pair X_i, X_j sit independently. Moreover by our special case each $X_i, i > 1$ is independent of X_1^r. Form the relatively independent product $\hat{\lambda}$ of λ with itself over X_1^r, considered as a measure on $X_1^r \times X_2^r \times \ldots \times X_k^r \times X_2^r \times \ldots \times X_k^r$.

We claim that with respect to $\hat{\lambda}$ any single factor X_i is independent of any other single factor X_j (i and j may be identical). To see this we need only consider the case where X_i comes from the first group $X_2^r \times \ldots \times X_k^r$ and X_j from the second for on each group λ is product measure by our induction hypothesis. But then X_i and X_j are conditionally independent given X_1^r and each is independent of X_1^r, so they are independent of each other.

Now projecting $\hat{\lambda}$ on $X_2^r \times \ldots \times X_2^r \times X_2^r \ldots \times X_k^r$ we have a pairwise independent joining $\overline{\lambda}$ of copies of only k-1 distinct systems. By induction (and when k=2 by the PID property of X_2) we conclude that $\overline{\lambda}$ is product measure. As we have already seen this implies that λ is product measure.

THEOREM 5.4. *Suppose* X *is a* G-*action with* MSJ, *and the associated Boolean* G-*space is free. Suppose further that* $Y = X \times_a K$ *is a weakly-mixing group extension. Then* Y *is simple and* $C(Y)$ *is the group generated by the natural action of* K, *together with the action of those* $g \in G$ *whose action on* X *belongs to* $C(X)$. *Moreover, the natural action of* $G \times K$ *on* $X \times K$ *has* MSJ.

PROOF. By Lemma 5.2 it suffices to show that any ergodic two-joining of Y is off-diagonal or product measure. Let λ be an ergodic 2-joining of Y, that is a G-invariant measure on $X \times K \times X \times K$ whose projection on each $X \times K$ is $\mu \times dk$. The projection $\bar{\lambda}$ of λ on $X \times X$ is an ergodic 2-joining of X, hence it is product measure or an off-diagonal. If $\bar{\lambda}$ is $\mu \times \mu$ then as in the proof of Lemma 5.2, λ is $(\mu \times dk)^2$.

Suppose now that $\bar{\lambda}$ is an off-diagonal

$$\mu_g(A \times B) = \mu(g^{-1} A \cap B)$$

for some $g \in G$ (recall that X has MSJ). Because the action of g belongs to $C(X)$ freeness implies that g belongs to $C(G)$, the centralizer of G. Let λ' be the image of λ under the map $id \times g^{-1}$ of $Y \times Y$ to itself. (By abuse of notation we identify g^{-1} with its action on Y.) Then λ' is again a G-joining (because $g \in C(G)$) and its projection on $X \times X$ is diagonal measure. We may thus naturally identify λ' with a measure $\tilde{\lambda}$ on $X \times K \times K$ invariant under the action

$$g(x,k_1,k_2) = (gx, a(g,x)k_1, a(g,x)k_2).$$

Now define $\theta : X \times K \times K \to K$ by

$$\theta(x,k_1,k_2) = k_1^{-1} k_2$$

and observe that the following diagram commutes

$$\begin{array}{ccc}
(x,k_1,k_2) & \xrightarrow{g} & (gx, a(x,g)k_1,\ a(x,g)k_2) \\
\theta \big\downarrow & & \big\downarrow \theta \\
k_1^{-1}k_2 & \xrightarrow{id} & k_1^{-1}k_2 .
\end{array}$$

Thus $\theta\,\tilde{\lambda}$ is an ergodic measure for the trivial G-action on K, so $\theta\,\tilde{\lambda}$ is a point mass δ_{k_0}, some $k_0 \in K$. Thus $\tilde{\lambda}$ is supported on $\{(x,k,kk_0): x \in X, k \in K\}$ and its projection on X×K is $\mu \times dk$. It follows that λ' is $R_{k_0}\,\nu_\Delta$ where $\nu = \mu \times dk$ and ν_Δ is the diagonal measure over ν on X×K×X×K. This implies that λ is the off-diagonal $(id \times g)R_{k_0}\,\nu_\Delta$. Thus we have shown that Y is simple and the centralizer is as claimed.

If G×K acts on X×K then the map $(id \times g)R_{k_0}$ belongs to the action so the action has MSJ. □

We are grateful to S. Glasner for suggesting the use of the map θ above, which simplified our original argument.

REMARK. It is perhaps worth highlighting why simplicity of $\mathcal{X}$ would not suffice for the proof of theorem 5.4. We assumed that λ projected onto $(id \times g)\,\mu_\Delta$ and then worked with $(id \times g^{-1})\lambda$. If λ projected on $(id \times S)\mu_\Delta$, $S \in C(\mathcal{X})$, S need not extend to a map belonging to $C(\mathcal{Y})$. If it did the proof would go through.

EXAMPLE 5.5. Theorem 5.4 and Corollary 3.6 allow us to find an example of a simple X with a non-simple factor. It suffices to let $\mathcal{X} = \mathcal{Y} \times_a K$ where $\mathcal{X}$ is free with MSJ, $\mathcal{Y}$ is weakly-mixing and K is a compact group with a closed non-normal subgroup K'. Since K' is not normal in K it is a fortiori non-normal as a subgroup of $C(\mathcal{Y})$, so $\mathcal{Y}/K'$ is not simple. (We remark that it is well-known that for an arbitrary weakly-mixing $\mathcal{X}$ and compact group K there is an abundance of cocycles a such that $\mathcal{X} \times_a K$ is again weakly-mixing.)

It is natural to ask whether the assumption of MSJ in theorem 5.4 can be weakened to simplicity. The following counterexample is due to S.Glasner (Proposition 1.7 of [G]). It replaces our original more complicated construction. The fact that it is not simple is implicit in Proposition 1.7 of [G] but we sketch a proof here without using the language of quasifactors.

EXAMPLE 5.6. A weakly mixing group extension of a simple $\mathbb{Z}$-action which is not simple.

When X is a $\mathbb{Z}$-action we write $X = T$ where T is the map generating the action. Let T be any weakly-mixing map with MSJ and ϕ a cocycle into the circle group K such that $S = T\times_\phi K$ is weakly-mixing. (We will identify ϕ with the function $\phi(1,\cdot)$.) S is simple by Theorem 5.4. Now define a K-extension R of S by

$$R(x,k_1,k_2) = (S(x,k_1),\ k_1k_2)$$

$$= (Tx,\phi(x)\,k_1,\ k_1k_2).$$

R is weakly-mixing by Proposition 1.7 of [G].

We exhibit an ergodic 2-joining of R which is neither product measure nor an off-diagonal. Consider the measures λ_1 and λ_2 on $(X\times K\times K)^2$ defined by the disintegrations

$$\lambda_1 = \int_{X\times K\times K}^{\oplus} \delta_{(x,-k_1,k_2)}\, d\mu(x)\,dk_1dk_2$$

$$\lambda_2 = \int_{X\times K\times K}^{\oplus} \delta_{(x,-k_1,-k_2)}\, d\mu(x)\,dk_1dk_2.$$

λ_1 and λ_2 each have both marginals on X×K×K equal to $d\mu\, dk_1\, dk_2$, but they are not 2-joinings of R. Indeed

$$(R\times R)((x,k_1,k_2),(x,-k_1,k_2))$$

$$= ((Tx,a(x)k_1,k_1k_2),\ (Tx,-a(x)k_1,-k_1k_2)),$$

whence it follows that $(R\times R)\lambda_1 = \lambda_2$ and similarily $(R\times R)\lambda_2 = \lambda_1$. Thus $\frac{1}{2}(\lambda_1+\lambda_2)$ is a 2-joining of R which is not product measure and not off-diagonal (it has 2-point fibres over X×K×K). Moreover it is ergodic (but not weak-mixing) as it is isomorphic in an obvious way to R×f, where f denotes the interchange map on $\{-1,1\}$.

We conclude this section with some general remarks about PID $\mathbb{Z}$-actions. It is easy to see that Bernoulli shifts are not PID. Since every positive entropy map has a Bernoulli factor it follows, via relatively independent extension, that positive entropy maps are not PID. On the other hand it is not hard to see that translation by $e^{2\pi i\alpha}$, α irrational, on the circle group is not PID. The translation by 1 on $\mathbb{Z}/m\mathbb{Z}$ is also not PID. It follows, again by extension, that any non-weakly mixing map is not PID. However we know of no weakly mixing 0-entropy counterexample. It is not hard to see that if a map is 2-mixing but not 3-mixing then it is not PID so a proof that 0-entropy weak-mixing implies PID will not be easily found. A more specific problem is: does 2-fold simplicity (MSJ) imply simplicity (MSJ)?

Passing to $\mathbb{Z}^2$-actions we observe that Ledrappier's example ([Le]) of a 2-mixing but not 3-mixing action furnishes an example of a non-PID, mixing, 0-entropy $\mathbb{Z}_2$-action. This example is also not 2-fold simple: there is a natural 2-1 factor map from it to itself.

6. THE ACTION OF A CO-COMPACT SUBGROUP

<u>THEOREm 6.1.</u> *Let X be a weakly-mixing simple* G-*action and* H *a closed, normal, co-compact subgroup of* G. *Then* H *acts simply and* $C(X,H) = C(X,G)$.

<u>PROOF</u>. First we show that the action of H is weakly-mixing. As is well-known this is the case if and only if the only functions $f \in L_2(X)$ such that Hf is precompact in the norm topology of $L_2(X)$ are the constants. (See, for example [Zi2], Theorem 7.1 and Theorem 7.8 specialized to the case where Y is trivial.) Suppose, then, that Hf is precompact and choose $\{h_1,\dots h_n\} \subset H$ such that $\{h_1 f,\dots,h_n f\}$ is ε-dense in Hf.

Given any $g_0 \in G$ let

$$S = \{Hg : d(Hgf, Hg_0 f) < \varepsilon\} \subset H\backslash G = G/H.$$

We claim S is open in $H\backslash G$, that is $\cup S$ is open in G. Indeed, if $hg \in \cup S$ $(h \in H)$ there are $h', h'' \in H$ such that $\|h'g f - h'' g_0 f\|_2 < \varepsilon$. Now if $\bar{g}$ is sufficiently close to hg, $\|\bar{g} f - hg f\| < \delta$ since the action of G on $L_2(X)$ is continuous. But

$$\| hgf - h(h')^{-1}h''g_0 f\|_2 < \varepsilon$$

so if δ is sufficiently small

$$\| \bar{g} f - h(h')^{-1}h''g_0 f\|_2 < \varepsilon.$$

Thus $d(H\bar{g}f, Hg_0 f) < \varepsilon$ whence $\bar{g} \in \cup S$ and $\cup S$ is open as claimed.

Thus by compactness of $H\backslash G$ we can find $g_1,\ldots,g_n \in G$ such that for each $g \in G$ there is a g_i with

$$d(Hgf, Hg_i f) = d(gHf, g_i Hf) < \varepsilon.$$

For such a g and g_i there is an $h \in H$ such that

$$\| gf - g_i hf\|_2 < \varepsilon.$$

Then we may choose an h_j such that $\| hf - h_j f\| < \varepsilon$.
Putting this together we have

$$\| gf - g_i h_j f\| < 2\,\varepsilon.$$

Since g was arbitrary we've shown $\{g_i h_j f\}$ is 2ε-dense in Gf. Since ε was arbitrary, Gf is precompact whence f is constant by weak-mixing of the G-action.

Now let $\pi: G \to G/H$ denote the canonical projection, choose a Borel cross-section $\sigma: G/H \to G$ (Theorem 8.11 of [Va]). We denote normalized Haar measure on G/H by $d\zeta$.

Suppose that λ is an ergodic k-joining of (X,H). Note that the field of measures $\{g\lambda\}_{g\in G}$ is measurable: for A Borel in X^k

$$(g\lambda)(A) = \lambda(g^{-1}A) = \int_{X^k} 1_A(gx)\,d\lambda$$

is a measurable function of g by Fubini's theorem. Thus $\{\sigma(\xi)\lambda\}_{\xi\in G/H}$ is also a measurable field so we may define

$$\bar{\lambda} = \int_{G/H} \sigma(\xi)\, \lambda d\xi.$$

$\bar{\lambda}$ has marginals μ, since each $\sigma(\xi)\lambda$ has marginals μ. Moreover for $g_0 \in G$

$$g_0\bar{\lambda} = \int_{G/H} (g_0\sigma(\xi)\lambda\, d\,\xi.$$

Now $g_0\ \sigma(\xi)$ and $\sigma\ (g_0\xi)$ both belong to the coset $g_0\xi$, so they differ by multiplication on the right by an element of H. Since λ is H-invariant, $(g_0\sigma(\xi))\ \lambda\ = \sigma(g_0\xi)\ \lambda$. Thus

$$g_0\bar{\lambda} = \int_{G/H} \sigma(g_0\xi)\ \lambda\, d\xi$$

$$= \int_{G/H} \sigma(\xi)\lambda\, d\xi = \bar{\lambda},$$

by invariance of $d\xi$. We have shown $\bar{\lambda}$ is a G-joining. We claim $\bar{\lambda}$ is also G-ergodic. Indeed if $A \subset X^n$ is G-invariant (literally, not $\bar{\lambda}$ -a.e.) then $\lambda(A) = 0$ or by H-ergodicity of λ. Since A is G-invariant $\lambda((\sigma(\xi))^{-1}A) = \lambda(A)$ so $\bar{\lambda}(A) = 0$ or 1 according as $\lambda(A) = 0$ or 1.

Thus by simplicity of (X,G) we now have that $\bar{\lambda}$ is a POOD (with respect to C(X,G)). Since (X,H) is weak-mixing a POOD is also ergodic with respect to H. Thus $\bar{\lambda}$ is an H-ergodic average of the H-invariant measures $\sigma(\xi)\lambda$ so by extremality we conclude that $\sigma(\xi)\ \lambda\ = \bar{\lambda}$ for a.a. $\xi \in G/H$. Since $d\xi$ has full support we can find cosets $g_nH \to H$ such that $\sigma(g_nH)\ \lambda = \bar{\lambda}$. Since $g_nH \to H$ we can find $h_n \in H$ such that $g_nh_n^{-1} \to \text{id}$. Then

$$\bar{\lambda} = \sigma(g_nH)\lambda = g_nh_n^{-1}\ \lambda\,.$$

Now if $A_1,\ldots,A_k$ are Borel subsets of X

$$g_nh_n^{-1}\ \lambda\ (A_1\times\ldots\times A_k) = \lambda(g_nh^{-1}A_1\times\ldots\times g_nh_n^{-1}A_k).$$

Now using the fact that $\mu(g_n^{-1}h_nA_i\Delta A_i) \to 0$ (continuity of the G-action on $L_1(X)$) together with the fact that λ has marginals μ it follows that

$$\lambda(g_nh_n^{-1}A_1\times\ldots\times g_nh_n^{-1}A_k) \to \lambda(A_1\times\ldots\times A_k)$$

so we conclude $\bar{\lambda}$ and λ agree on rectangles in X^k so $\lambda = \bar{\lambda}$. Thus we have shown that λ is a POOD with respcct to C(X,G) which completes the proof. □

COROLLARY 6.2. *With the hypotheses of Theorem* 6.1 *every* H*-invariant factor algebra of* X *is* G*-invariant. If* (X,G) *is prime so is* (X,H).

PROOF. Follows immediately from Theorems 3.1 and 6.1.

PROPOSITION 6.3. *Suppose that* H *is a closed, normal, co-compact subgroup of* G *and that* X *and* Y *are weakly-mixing simple* G*-actions such that every ergodic* G*-joining of* X *and* Y *is weakly mixing. (For example this is true if either* X *or* Y *is prime by Corollary* 4.5.) *Then any* H*-joining of* X *and* Y *is a* G*-joining. In particular any* H*-factor map* $X \to Y$ *is a* G*-factor map.*

PROOF. It suffices to prove this for an ergodic H-joining λ. Then, as in the proof of Theorem 6.1, we form the G-invariant and ergodic joining

$$\bar{\lambda} = \int_{G/H} \sigma(\xi)\lambda\, d\xi.$$

By hypothesis $\bar{\lambda}$ is weakly mixing as a G-action, hence as in the proof of theorem 6.1, also weakly mixing as an H-action, hence also H-ergodic. One concludes as in 6.1 that $\lambda = \bar{\lambda}$. □

We now apply the above results to $\mathbb{Z}$- and $\mathbb{R}$-actions.

COROLLARY 6.4. *If* $\{T_t\}$ *is a weak mixing simple prime flow then* T_a *is a prime map for* $a \neq 0$. *If* $a,b \neq 0$ *then* T_a *and* T_b *are either disjoint or isomorphic.* T_1 *and* T_a *are isomorphic if and only if the flows* $\{T_t\}_{t\in\mathbb{R}}$ *and* $\{T_{at}\}_{t\in\mathbb{R}}$ *are isomorphic. More generally, if* $\{T_t\}$ *and* $\{S_t\}$ *are weakly-mixing simple prime flows then the maps* T_1 *and* S_1 *are either disjoint or isomorphic according as* $\{T_t\}$ *and* $\{S_t\}$ *are disjoint or isomorphic.*

PROOF. This follows from Corollary 6.2, Proposition 6.3 and Corollary 4.5.

This is a good place to observe that a weak mixing flow with MSJ (examples are provided by [J,P] and [Ra]) is prime, so that the above results apply. In particular its time one map is simple and prime.

PROPOSITION 6.5. *A weak mixing flow with minimal self-joinings is prime.*

PROOF. Since each non-zero time in a weakly-mixing flow is again weakly-mixing and a fortiori ergodic this follows immediately from the remark following the proof of theorem 3.1.

For $\mathbb{Z}$-actions with MSJ Corollary 6.4 may be sharpened.

COROLLARY 6.5. *If* T *is a weak-mixing map with* MSJ *and* $|n| > |m| > 0$ *then* T^n *and* T^m *are disjoint.*

PROOF. It suffices by Corollary 6.4 to show that T^n and T^m are not isomorphic. We claim that T^m has no n^{th} root (while T^n, of course, does). Indeed if S were an n^{th} root of T^m then $S \in C(T^m) = C(T)$ (Theorem 6.1) so $S = T^{\ell}$. Thus $T^{\ell n} = T^m$ *and* $\ell n = m$ which is impossible when $|n| > |m|$. □

We observe that Proposition 6.4 cannot be similarily strengthened for flows with MSJ. Indeed it is shown in [Ra] that certain horocycle flows $\{T_t\}$ have MSJ, providing examples where T_a and T_b are isomorphic for all $a,b > 0$.

We are now in a position to clarify the relation between our definition of minimal self-joinings in the Case of $\mathbb{Z}$-actions and the original apparently much stronger one used in [Ru1]. Let's say that a map T has minimal power joinings (MPJ) if any ergodic joining of possibly different non-zero powers of T is a POOD (with respect to T). (Warning: not all POOD's are joinings now. Off-diagonal links can occur only between co-ordinates which are acted on by the same power of T.) This is what was called minimal self-joinings in [Ru1].

PROPOSITION 6.7. *A weak mixing map* T *has* MPJ *if and only if it has* MSJ *and* T *and* T^{-1} *are not isomorphic.*

PROOF. The "only if" direction is obvious. Suppose that T is weak-mixing with MSJ. First observe that T^m and T^{-m} are disjoint by Theorem 6.1 and Corollary 6.4 applied to the simple $\mathbb{Z}$-actions T^m and T^{-m}. Combining this with 6.6 we have that any two non-zero powers of T are disjoint.

Suppose now that λ is a joining of powers of T (with multiplicities). Grouping together co-ordinates on which like powers of T act we have that on any group the marginal of λ on that group is a POOD (for T) because T^n is simple and $C(T^n) = C(T)$. Furthermore on any group λ is isomorphic to a cartesian power of T^n, since an off-diagonal factor is isomorphic to T^n. Thus we may assume that λ is a joining of copies of T^n (for various n) in which any two like copies sit independently. Copies of different powers automatically sit independently since they are disjoint. Since each copy is weak-mixing and simple, and hence PID, Proposition 5.3 implies that λ is the product joining which completes the proof. □

We remark that there are weak mixing maps T with MSj such that T and T^{-1} are isomorphic, as in the examples of [Ju1]. A symmetrized version of Chacón's example (see [JRS]) gives easier examples. (Use, for example the substitution $0 \to 00100$, $1 \to 1$.) Even when T and T^{-1} are isomorphic one can explicitly describe all ergodic joinings of powers of T. For if $\phi T = T^{-1}\phi$ then ϕ may be used to replace negative powers of T by positive ones. After the relabelling the joining is a POOD which means that the original joining is a product of off-diagonals "skewed" by ϕ. By a skewed off-diagonal we mean a joining of copies of T^n and T^{-n} (for a fixed n) where any two T^n's are linked by a T^{ℓ} and a T^n is linked with a T^{-n} by a $T^{\ell}\phi$. Note moreover that ϕ must be an involution, $\phi^2 = 1$.

Say a flow $\{T_t\}$ has minimal re-scaling joinings (MRJ) if for all k and $a_1, \dots, a_k \in \mathbb{R} - \{0\}$ every ergodic joining of the flows $\{T_{a_1 t}\}_{t \in \mathbb{R}}, \dots, \{T_{a_k t}\}_{t \in \mathbb{R}}$ is a POOD. The following result was applid in [J,P] to conclude that the weak mixing flow with MSJ constructed in that paper actually has MRJ. The proof is similar to the proof of Proposition 6.7.

PROPOSITION 6.8. *A weakly-mixing flow has* MRJ *if and only if it has* MSJ *and for all* $a \in \mathbb{R} - \{1\}$, $\{T_t\}$ *and* $\{T_{at}\}$ *are non-isomorphic.*

The following example of weakly-mixing simple maps S and $\bar{S}$ such that S^2 and $\bar{S}^2$ are isomorphic but S and $\bar{S}$ are not shows that 6.3 may fail when the actions in question have ergodic joinings which are not weak mixing.

EXAMPLE 6.9. Let T be a weak mixing map with MSJ, $K = \{-1,1\}$ and ϕ a cocycle into K such that $S = T\times_\phi K$ is weak mixing. We claim that weak mixing of S is equivalent to the requirement that ϕ not be cohomologous to a constant function, that is the equation

$$(1) \qquad \phi(x) = b(Tx)b(x)^{-1}k_0 \quad \text{a.e.}$$

has no measurable solution $b: X \to K$ for $k_0 = -1$ or 1. (As usual we identify ϕ with the function $\phi(1,\cdot)$.) On the one hand if (1) is satisfied then $f(x,k) = b(x)k$ defines a non-constant eigenfunction of S with eigenvalue k_0.

On the other hand if $f(x,k)$ is a non-constant eigenfunction of S with eigenvalue λ write

$$f(x,k) = g_1(x) + g_2(x)k.$$

Then
$$f(Tx,\phi(x)k) = g_1(Tx) + g_2(Tx)\,\phi(x)k$$
$$= \lambda\, g_1(x) + \lambda\, g_2(x)k.$$

By weak mixing of T we conclude that $g_1(x)$ is constant a.e. Since f is not constant a.e. g_2 is not a.e. zero. Now

$$g_2(Tx)\phi(x) = \lambda\, g_2(x)$$

so

$$g_2^2(Tx) = \lambda^2 g_2^2(x).$$

By weak mixing of T (since g_2^2 is not a.e. zero) $\lambda^2 = 1$ and $g_2^2 = 1$ a.e. Thus $\lambda = k_0 \in K$, g_2 takes values in K and

$$\phi(x) = k_0 g_2(Tx) g_2(x)^{-1}$$

is the desired conclusion.

It follows that $\bar{S} = T\times_{-\phi}K$ is also weak mixing. Both S and $\bar{S}$ are simple by Theorem 5.4. Note that the relatively independent joining of S and $\bar{S}$ over the common factor T is in a natural way isomorphic to $T\times_{\phi\times-\phi}K\times K$ where $(\phi\times-\phi)(x) = (\phi(x),-\phi(x))$. Defining $\theta(x,k_1,k_2) = (x,k_1,k_1k_2)$ we have the following diagram

$$\begin{array}{ccc} (x,k_1,k_2) & \xrightarrow{T\times_{\phi\times-\phi}K\times K} & (Tx,\phi(x)k_1,\ -\phi(x)k_2) \\ \theta\downarrow & & \downarrow\theta \\ (x,k_1,k_1k_2) & \xrightarrow{T\times_{\phi\times-1}K\times K} & (Tx,\phi(x)k_1,\ -k_2) \end{array}$$

Thus $T\times_{\phi\times-\phi}K\times K$ is isomorphic to $T\times_{\phi\times-1}K\times K$ which is ergodic but not weak mixing.

Now $S^2 = \bar{S}^2$ so id: $X\times K \to X\times K$ is an isomorphism of S^2 and $\bar{S}^2$ but not of S and $\bar{S}$. Moreover it is easy to see that S and $\bar{S}$ are non-isomorphic. Indeed an isomorphism ϕ would also have to be an isomorphism of S^2 and $\bar{S}^2$, that is $\phi \in C(S^2)$. By theorem 6.1 $C(S^2) = C(S)$ so ϕ would commute with S, a contradiction.

We conclude by mentioning a few open problems, restricting ourselves for the most part to $\mathbb{Z}$-actions. It is natural to ask how prevalent the class of simple maps is. While we now have a fairly wide variety of examples, they are all of a very special nature. For one thing they are all constructed from something with MSJ, either by group extension or by taking a non-zero time in a $\mathbb{Z}$-or $\mathbb{R}$-action with MSJ. The class of maps with MSJ is small in the precise sense that it is meagre in the weak topology. This is because in general (that is, for a residual set) a map is rigid, that is $\exists\, n_i \to \infty$ such that $T^{n_i} \to \text{id}$, which implies that C(T) has the cardinality of the continuum.

Elsewhere we will show how Chacõn's map can be modified to give a rigid simple prime map. This is a step in the right direction as it shows there is at least one simple prime map in the rigid class, which is generic. Moreover the construction has nothing to do with MSJ. Of course it leaves open the question of whether the simple maps form a residual class. One may ask the same question about the prime maps. It is interesting to note that every example of primality so far known derives more or less directly from simplicity. Is there an essentially different sufficient condition for primality?

We have been unable to answer the following question: does every weak mixing simple map have a non-trivial prime factor? The only examples we have of such maps are either themselves prime or group extension of maps with MSJ.

Can one say something about joinings of $X_1,\dots,X_k$, X_i simple, in the spirit of Corollary 4.5? If the X_i are all prime then they are pairwise disjoint or isomorphic so Corollary 4.5 and Proposition 5.3 describe all joinings of $X_1,\dots,X_k$.

In section 5 we already raised the questions: Does 0 entropy weak-mixing imply PID? Does 2-fold MSJ imply 3-fold MSJ?

Many of the results of this paper can almost surely be relativized. The natural definition of relative 2-fold simplicity of the extension $X \to Y$ has already been given by Veech (for $\mathbb{Z}$-actions): every ergodic 2-joining of X which is diagonal on Y is either the Y-relative product or an off-diagonal. He has shown (Theorem 4.8 of [Ve]) that if $X \to Y$ is relatively simple and the Y-relative centralizer of X (namely those $S \in C(X)$ which fix each set in the factor algebra G corresponding to Y) has no non-trivial compact subgroups then $X \to Y$ is relatively prime, that is there are no factor algebras strictly between G and $B(X)$.

The group $SL_2(\mathbb{Z})$ acts as automorphisms of the 2-torus. We conjecture that this action has MSJ and that its centralizer is trivial so the only 2-joinings are product measure and diagonal measure: minimal self-joinings in the strongest possible sense!

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