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A note on equivalent random theory

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A NOTE ON EQUIVALENT RANDOM THEORY

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The purpose of this note is to provide proofs for two fundamental results in the context of equivalent random theory that are generally accepted as valid, but for which no proofs are available in the literature.

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1. Introduction

Let $x \geqslant 0$, $a > 0$ and

$$B(x,a) = \{a \int_0^\infty e^{-ay}(1+y)^x dy\}^{-1}, \tag{1}$$

$$m(x,a) = aB(x,a), \tag{2}$$

$$z(x,a) = 1 - m(x,a) + a / (x+1+m(x,a)-a). \tag{3}$$

If x is an integer then $B(x,a)$ equals the Erlang loss function:

$$B(x,a) = (a^x / x!) / \sum_{i=0}^{x} a^i / i!, \tag{4}$$

while $m(x,a)$ and $z(x,a)$ are the mean and peakedness factor, respectively, of the overflow traffic resulting when Poisson traffic with intensity a is offered to a trunk group of size x [2], [4]. In several contexts, particularly that of equivalent random theory, the need arose to extend the definition of the Erlang loss function to nonintegral values of x. It has been customary to do this by use of the integral formula (1) ascribed to Fortet (cf. [3]).

The purpose of this note is to provide proofs for two fundamental results that are generally accepted as valid, but for which no proofs are available in the literature. The first result is that $z(x,a) > 1$ if $x > 0$, $a > 0$, and the second that for given $m > 0$, $z \geqslant 1$, there is exactly one solution $a > 0$, $x \geqslant 0$ to the system of equations $m = m(x,a)$, $z = z(x,a)$.

2. Results.

Lemma 1.

(i) If $a > 0$, then

$$m(0,a) = a, \quad z(0,a) = 1 \tag{5}$$

(ii) If $x > 0$ and $a > 0$, then

$$\max(0, a-x) < m(x,a) < a \tag{6}$$

and

$$1 < m(x,a) + z(x,a) < a+1 \tag{7}$$

Proof. The equalities (5) are evident.

From (1) and (2) it follows that $m(x,a)$ is positive and strictly decreasing in x for $x > 0$. Hence $0 < m(x,a) < m(0,a) = a$. Next suppose that $a > x$, then

$$1/m(x,a) = \int_0^\infty e^{-ay}(1+y)^x dy =$$

$$= \int_0^\infty e^{-(a-x)y}\{e^{-y}(1+y)\}^x dy < \int_0^\infty e^{-(a-x)y} dy = 1/(a-x),$$

which establishes (6). From (6) we see that $1 < x+1+m(x,a)-a$, so that (7) follows by (3). □

According to the previous lemma, for any $a > 0$, $x \geqslant 0$ one has $(m(x,a), z(x,a)) \in P \equiv \{(m,z)\colon m > 0,\ m+z > 1\}$. Now let (m,z) be a fixed but otherwise arbitrary element of P. We shall show that the number of solutions (x,a) : $x \geqslant 0$, $a > 0$, of the system

$$m = aB(x,a) \tag{8}$$

$$z = 1 - m + a / (x+1+m-a) \tag{9}$$

is zero if $z<1$ and one if $z \geqslant 1$. Once we have done this, we shall have proven the following theorems.

Theorem 1. If $x>0$ and $a>0$, then $z(x,a)>1$.

Theorem 2. If $m>0$ and $z \geqslant 1$, then there is exactly one pair (x,a): $x \geqslant 0$, $a>0$, such that $m = m(x,a)$ and $z = z(x,a)$.

We proceed by defining

$$p=(m+z)/(m+z-1), \quad q=(m+1)/p. \tag{10}$$

We clearly have

$$p>1, \tag{11}$$

while some simple algebra yields that

$$z \underset{<}{\overset{>}{=}} 1 \Leftrightarrow q \underset{<}{\overset{>}{=}} m\ . \tag{12}$$

Solving for x in (9) and substituting the result in (8), we can reformulate our problem as: what is the number of solutions of the equation $m=aB(ap-m-1,a)$, or , equivalently, of the equation

$$f(a) \equiv \int_0^\infty e^{-ay}(1+y)^{p(a-q)}dy \ = 1/m \tag{13}$$

with a in the interval $[q,\infty)$? In settling this question we shall use the following properties of $f(a)$.

Lemma 2.

(i) $f(q)=1/q$,

(ii) $f(a)$ is a convex function of a for $a>0$,

(iii) $f(a) \to \infty$ *as* $a \to \infty$.

Proof.

(i) This is evident.

(ii) Convexity follows from the fact that

$$f''(a)=\int_0^\infty e^{-ay}(1+y)^{p(a-q)}(-y+p\ln(1+y))^2dy>0.$$

(iii) Consider the integrand in (13)

$$g(a,y) \ \equiv \ e^{-ay}(1+y)^{p(a-q)}, \quad y \geqslant 0. \tag{14}$$

In view of (11) there exists a point $y^*>0$ such that

$$-y+p\ln(1+y) \underset{>}{\overset{<}{=}} 0 \ \Leftrightarrow \ y \underset{<}{\overset{>}{=}} y^* \tag{15}$$

As a consequence, the derivative

$$\frac{\partial}{\partial a}g(a,y)=(1+y)^{-pq}(-y+p\ln(1+y))e^{-a(y-p\ln(1+y))} \tag{16}$$

is positive and increasing to infinity as $a \to \infty$ uniformly in y on any closed subset of the interval $(0,y^*)$, and negative and increasing to zero as $a \to \infty$ for $y \in (y^*,\infty)$. It follows that

$$f'(a)=\int_0^{y^*}\frac{\partial}{\partial a}g(a,y)dy + \int_{y^*}^{\infty}\frac{\partial}{\partial a}g(a,y)dy>0$$

for a sufficiently large. This, in combination with the convexity of f, implies (iii). □

The two types of possible behaviour for f that are allowed by Lemma 2 are depicted in Figure 1. It is clear now that if $z>1$, then, by (12), $1/m>1/q$, so irrespective of the sign of $f'(q)$, there is exactly one solution to (13) in $[q,\infty)$. To prove that for $z=1$ (hence $1/m=1/q$) there is one solution to (13), and for $z<1$ (hence $1/m<1/q$) there is no solution to (13) in the interval $[q,\infty)$, it is sufficient to show that $f(a)$ is increasing at $a = q$ if $z\leqslant 1$. This we shall do next.

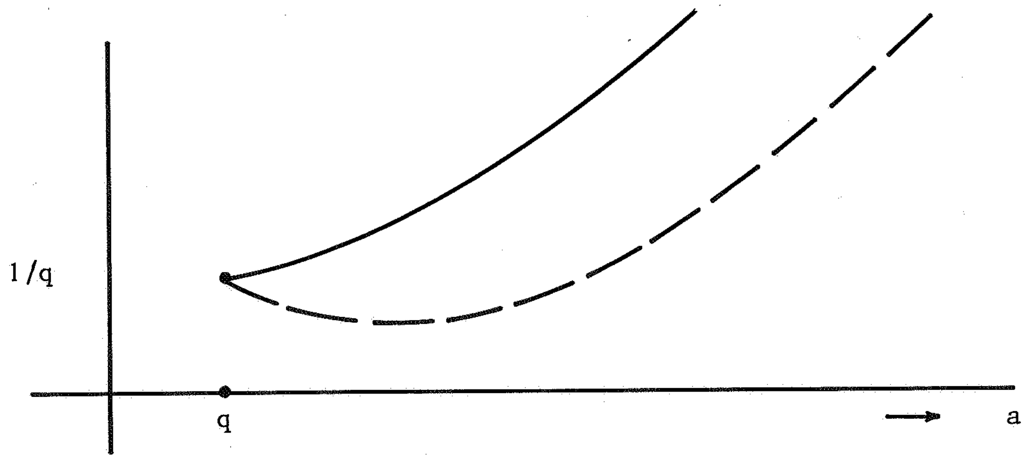

Figure 1: Possible behaviour of $f(a)$.

From (10), (13), (14) and (16) we obtain

$$f'(q)=\int_0^\infty e^{-qy}(-y+p\ln(1+y))dy, \tag{17}$$

which after integration by parts and substitution of $z=(1+y)q$ reduces to

$$q^2f'(q) = -1+pqe^q\int_q^\infty(e^{-z}/z)dz. \tag{18}$$

Since

$$e^q\int_q^\infty(e^{-z}/z)dz>1/(q+1) \tag{19}$$

[1, (5.1.19)], it follows with (10) that

$$q^2f'(q) > (m-q)/(q+1). \tag{20}$$

Recalling (12), we can conclude from (20) that

$$z\leqslant 1 \;\Rightarrow\; f'(q)>0, \tag{21}$$

which finally settles our problem.

Remark. The bound (19) can be sharpened, viz.

$$e^q\int_q^\infty(e^{-z}/z)dz>\tfrac{1}{2}\ln(1+2/q) \tag{22}$$

[1, (5.1.20)]. Using this result in (18) reveals after some calculations that

$$z\leqslant\tfrac{4}{3} \;\Rightarrow\; f'(q)>0. \tag{23}$$

On the other hand, it can be shown that, given m, $f'(q)<0$ for z sufficiently large.

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