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INVARIANT DIFFERENTIAL OPERATORS ON A SEMISIMPLE SYMMETRIC SPACE AND FINITE MULTIPLICITIES IN A PLANCHEREL FORMULA

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We investigate some properties of the algebra $\mathbb{D}(G/H)$ of invariant differential operators on a semisimple symmetric space G/H. Our main results are that the action of $\mathbb{D}(G/H)$ diagonalizes over the discrete part of $L^2(G/H)$, and that the irreducible constituents of an abstract Plancherel formula for $L^2(G/H)$ occur with finite multiplicities. In particular this implies that discrete series representations occur with finite multiplicities in $L^2(G/H)$.

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0. Introduction.

Let G be a connected real semisimple Lie group with finite centre, and let τ be an involutive automorphism of G. Put $G^\tau=\{x\in G:\tau(x)=x\}$, and let H be a closed subgroup of G with $(G^\tau)_e\subset H\subset G^\tau$; here $(G^\tau)_e$ denotes the identity component of G^τ.

In this paper we investigate some properties of the algebra $\mathbb{D}(X)$ of invariant differential operators on the semisimple symmetric space $X=G/H$. Our main results are that the action of $\mathbb{D}(X)$ diagonalizes over the discrete part of $L^2(X)$ (Theorem 1.5), and that the irreducible constituents of an abstract Plancherel formula for X occur with finite multiplicities (Theorem 3.1). Both results are proved by using techniques of Harish-Chandra adapted to the situation at hand.

1. The action of $\mathbb{D}(X)$.

Let dx be a choice of left-invariant measure on X. Then by the left regular representation L, G acts unitarily on $L^2(X)=L^2(X,dx)$. An irreducible subrepresentation of L is called a discrete series representation of X. The closure $L^2_d(X)$ of the linear span of such irreducible subrepresentations is called the discrete part of $L^2(X)$.

Let K be a τ-stable maximal compact subgroup of G. Then by [5] the space $L^2_d(X)$ is non-trivial if rank (G/H)=rank $(K/K\cap H)$. In [15] it is proved that this rank condition is also necessary for the existence of discrete series. In the proof, the assertion that every discrete series representation can be realized in an eigenspace for $\mathbb{D}(X)$ is basic. This assertion is related to [17, Remark following Lemma 9], where it is claimed that every formally self-adjoint operator in $\mathbb{D}(X)$ is essentially self-adjoint as an unbounded operator in $L^2(X)$. However the proof given in [17] is incomplete. The missing ingredients are provided by Lemmas 1.1 and 1.2 below.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively, $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$'s complexification $\mathfrak{g}_c$, and $\mathfrak{Z}$ the centre of $U(\mathfrak{g})$. Given $u\in U(\mathfrak{g})$, we write $L_u=L(u)$ (resp. $R_u=R(u)$) for the infinitesimal action of u on $C^\infty(G)$, induced by the left-(right-) regular representaton L (resp. R) of G. If u lies in the space $U(\mathfrak{g})^H$ of $Ad_G(H)$-invariant elements of $U(\mathfrak{g})$, then R_u leaves the space $C^\infty(G/H)$ invariant, and thus determines an element of $\mathbb{D}(X)$, which we also denote by R_u. As is well known the map $u\mapsto R_u$ induces an isomorphism of $U(\mathfrak{g})^H/(U(\mathfrak{g})^H\cap U(\mathfrak{g})\mathfrak{h})$ onto $\mathbb{D}(X)$ (cf. [9]). Moreover, $\mathbb{D}(X)$ is commutative and finitely generated as a $\mathfrak{Z}$-module (cf. [9] and [20, Thm. 2.2.1.1]); if G is classical we even have $\mathbb{D}(X)=R(\mathfrak{Z})$ (cf. [10]).

Let $L^2(X)^\infty=\{f\in C^\infty(X);\ L_uf\in L^2(X) \text{ for all } u\in U(\mathfrak{g})\}$ be equipped with the topology induced by the seminorms

$$p_u: f\mapsto \|L_uf\|_{L^2(X)} \qquad (u\in U(\mathfrak{g})). \tag{1}$$

Then we have the following lemmas.

Lemma 1.1. $\mathbb{D}(X)$ maps $L^2(X)^\infty$ continuously into itself.

Lemma 1.2. $C_c^\infty(X)$ is dense in $L^2(X)^\infty$.

We shall prove Lemma 1.1 at the end of this section, and postpone the proof of Lemma 1.2 to the next. But first we derive the result we set out for. If $D\in\mathbb{D}(X)$, we define the differential operator D^*, called the formal adjoint of D, by

$$(Df,g)=(f,D^*g) \qquad (f,g\in C_c^\infty(X)).$$

By G-invariance of D and $(\cdot,\cdot)$ it follows that $D^*\in\mathbb{D}(X)$. The following lemma is a straightforward consequence of Lemmas 1.1 and 1.2.

Lemma 1.3. If $f,g \in L^2(X)^\infty$, $D \in \mathbb{D}(X)$, then $(Df,g) = (f,D^* g)$.

The above lemma completes the proof of [17, Lemma 9], so that we have

Lemma 1.4. If $D \in \mathbb{D}(X)$, $D = D^*$, then D is an essentially self adjoint operator in $L^2(X)$ with operator core $L^2(X)^\infty$.

Let $\mathbb{D}_s(X) = \{D \in \mathbb{D}(X);\ D = D^*\}$. If $D \in \mathbb{D}(X)$, then $D + D^*$ and $i(D - D^*)$ belong to $\mathbb{D}_s(X)$, so that the real subalgebra $\mathbb{D}_s(X)$ spans $\mathbb{D}(X)$ over $\mathbb{C}$.

Remark. In view of [13, Cor 9.2], the elements of $\mathbb{D}_s(X)$ have mutually commuting spectral resolutions.

Theorem 1.5. $L_d^2(X)$ admits an orthogonal decomposition $L_d^2(X) = \Sigma_{i=1}^\infty V_i$ (Hilbert sum) into irreducible closed G-invariant subspaces, such that $\mathbb{D}(X)$ acts by scalars on every V_i.

Proof. Let $V \subset L_d^2(X)$ be a non-zero irreducible closed G-invariant subspace, and write V_K for the subspace of K-finite vectors in V. Then $\mathfrak{Z}$ acts by scalars on $V_K \subset L^2(X)^\infty$. By Lemma 1.1 the elements of $\mathbb{D}(X)$ act as $(\mathfrak{g},K)$-homomorphisms on $L^2(X)^\infty$, so that $U = \mathbb{D}(X)V_K$ is a $(\mathfrak{g},K)$-submodule of $L^2(X)^\infty$. It is a finite direct sum of copies of V_K because $\mathbb{D}(X)$ is a finite $\mathfrak{Z}$-module. Thus if W is the closure of U in $L^2(X)$, then $W_K = U$. Select a K-type $\delta \in \hat{K}$ occurring in V. Then $\mathbb{D}(X)$ leaves the subspace $W(\delta)$ of K-finite vectors of isotopy type δ invariant. Moreover, by Lemma 1.3, the elements of $\mathbb{D}_s(X)$ act as self-adjoint operators on the finite dimensional space $W(\delta)$. Since $\mathbb{D}_s(X)$ is commutative there exist distinct homomorphisms $\chi_j : \mathbb{D}_s(X) \to \mathbb{R}$ $(1 \leqslant j \leqslant m)$, and non-trivial subspaces $W(\delta)_j$ $(1 \leqslant j \leqslant m)$ of $W(\delta)$, such that $W(\delta) = \oplus_{j=1}^m W(\delta)_j$ and every $D \in \mathbb{D}_s(X)$ acts by the scalar $\chi_j(D)$ on $W(\delta)_j$. Put $U_j = U(\mathfrak{g})W(\delta)_j$, $W_j = cl(U_j)$; then $(W_j)_K = U_j$. Moreover, every $D \in \mathbb{D}_s(X)$ acts as $\chi_j(D).I$ on U_j. The χ_j being distinct, one easily sees that $U_i \perp U_j$ if $i \neq j$. Hence $W_i \perp W_j$ $(i \neq j)$. Every U_i is a finite multiple of V_K, hence every W_i is a finite orthogonal direct sum of copies of V (cf. [6, Theorem 8]). It follows that V is contained in a finite orthogonal direct sum $\Sigma_{i=1}^n V_i$ where V_i are irreducible closed G-invariant subspaces of $L^2(X)$, all equivalent to V, and such that $\mathbb{D}(X)$ acts by scalars on V_i $(1 \leqslant i \leqslant n)$. The theorem now follows by an easy application of Zorn's lemma; the ultimate decomposition is countable because $L^2(X)$ is separable.

Let us denote the infinitesimal involution corresponding to τ: $G \to G$ by the same symbol. Thus $\mathfrak{h}$, the Lie algebra of H, equals the $+1$ eigenspace of τ: $\mathfrak{g} \to \mathfrak{g}$. The Cartan involution θ, associated with K, commutes with τ, and we have a direct sum decomposition

$$\mathfrak{g} = (\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{q}) \oplus (\mathfrak{p} \cap \mathfrak{h}), \tag{2}$$

where $\mathfrak{p}$ and $\mathfrak{q}$ are the -1 eigenspaces of θ and τ respectively. Fix a maximal abelian subspace $\mathfrak{a}_{pq}$ of $\mathfrak{p} \cap \mathfrak{q}$, and let $\Delta = \Delta(\mathfrak{g},\mathfrak{a}_{pq})$ be its restricted root system. Then Δ is a (possibly non-reduced) root system (cf. [18]). If $\alpha \in \Delta$, we write $\mathfrak{g}^\alpha$ for the corresponding root space. Select a system Δ^+ of positive roots in Δ, and put:

$$\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}^\alpha, \qquad \bar{\mathfrak{n}} = \sum_{\alpha \in \Delta^+} \mathfrak{g}^{-\alpha}, \tag{3}$$

Since τ and θ both leave $\mathfrak{a}_{pq}$ invariant, the centralizer $\mathfrak{l}$ of $\mathfrak{a}_{pq}$ in $\mathfrak{g}$ admits the decomposition

$$\mathfrak{l} = \mathfrak{l}_{kq} \oplus \mathfrak{l}_{kh} \oplus \mathfrak{a}_{pq} \oplus \mathfrak{l}_{ph} \tag{4}$$

subordinate to (2). We will frequently use notations like $\mathfrak{l}_h = \mathfrak{l} \cap \mathfrak{h}$, etc. Since $\tau\theta = I$ on $\mathfrak{a}_{pq}$, $\tau\theta$ leaves every root space $\mathfrak{g}^\alpha$ $(\alpha \in \Delta)$ invariant, and we have corresponding decompositions

$$\mathfrak{g}^\alpha = \mathfrak{g}_+^\alpha \oplus \mathfrak{g}_-^\alpha$$

in $+1$ and -1 eigenspaces. We put

$$\Delta_+ = \{ \alpha\in\Delta;\ \mathfrak{g}_+^\alpha \neq 0 \}. \tag{5}$$

Thus, if $\Delta_+ = \varnothing$, then $\mathfrak{a}_{pq}$ is central in the reductive subalgebra $\mathfrak{g}_+ = \mathfrak{g}^{\tau\theta}$ of $\mathfrak{g}$. If $\Delta_+ \neq \varnothing$, one has the obvious identifications $\Delta_+ = \Delta(\mathfrak{g}_+,\mathfrak{a}_{pq})$, $\mathfrak{g}_+^\alpha = \mathfrak{g}^\alpha \cap \mathfrak{g}_+$ $(\alpha\in\Delta_+)$. Now put

$$\mathfrak{a}_{pq}^+ = \{Y \in \mathfrak{a}_{pq};\ \alpha(Y) > 0 \ \text{ for all } \alpha \in \Delta_+ \cap \Delta^+\},$$

and $A_{pq}^+ = \exp(\mathfrak{a}_{pq}^+)$. If $\Delta_+ = \varnothing$ this should be interpreted as $\mathfrak{a}_{pq}^+ = \mathfrak{a}_{pq}$.

Let $\mathcal{P}$ be the collection of positive systems P for Δ, satisfying $P \cap \Delta_+ = \Delta^+ \cap \Delta_+$. If $P \in \mathcal{P}$, then $\mathfrak{a}_{pq}^+(P) = \{Y \in \mathfrak{a}_{pq};\ \alpha(Y)>0 \ \text{ for all } \alpha \in P\}$ is contained in $\mathfrak{a}_{pq}^+$, and

$$cl(\mathfrak{a}_{pq}^+) = \bigcup_{P \in \mathcal{P}} cl(\mathfrak{a}_{pq}^+(P)). \tag{6}$$

Moreover, we put $\bar{\mathfrak{n}}(P) = \Sigma_{\alpha\in P}\, \mathfrak{g}^{-\alpha}$, and write $\mathcal{R}(P)$ for the ring of functions $A_{pq} \to \mathbb{R}$ generated by 1 and

$$a^{-\alpha} = e^{-\alpha \log a} \quad (\alpha\in P).$$

Clearly, the elements of $\mathcal{R}(P)$ are bounded on $A_{pq}^+(P) = \exp(\mathfrak{a}_{pq}^+(P))$.

Given any subset $\mathfrak{s}$ of $\mathfrak{g}$, we let $U(\mathfrak{s})$ (resp. $S(\mathfrak{s})$) denote the complex subalgebra generated by 1 and $\mathfrak{s}$ of $U(\mathfrak{g})$ (resp. of the symmetric algebra $S(\mathfrak{s})$ of $\mathfrak{g}_c$).

Proposition 1.6. Let $D \in U(\mathfrak{g})$, $P \in \mathcal{P}$. Then there exist $f_i \in \mathcal{R}(P)$, $\xi_i \in U(\bar{\mathfrak{n}}(P))$, $u_i \in U(\mathfrak{l})$, $\eta_i \in U(\mathfrak{h})$ $(1\leqslant i\leqslant I)$, such that for all $a \in A_{pq}$ we have:

$$D = \sum_{i=1}^{I} f_i(a)\, \xi_i^{a^{-1}}\, u_i \eta_i\, .$$

Proof. One easily checks that $\mathfrak{g}$ admits the direct sum decomposition

$$\mathfrak{g} = \bar{\mathfrak{n}}(P) \oplus \mathfrak{l}_q \oplus \mathfrak{h}.$$

Hence, by the Poincaré-Birkhoff-Witt theorem, $U(\mathfrak{g}) = U(\bar{\mathfrak{n}}(P))U(\mathfrak{l})U(\mathfrak{h})$. The assertion follows from this decomposition and the observation that $X_{-\alpha} = a^{-\alpha} Ad(a^{-1})(X_{-\alpha})$, for $X_{-\alpha} \in \mathfrak{g}^{-\alpha}$ $(\alpha\in P)$ and $a \in A_{pq}$.

Proposition 1.7. Let $\tilde{D} \in \mathbb{D}(X)$. Then there exist a constant $C>0$ and $v_j \in U(\mathfrak{g})$ $(1\leqslant j\leqslant J)$, such that for all $\phi \in C^\infty(X)$ we have:

$$|\, \tilde{D}\phi(x)\, | \leqslant C. \max_{1\leqslant j\leqslant J} |\, L_{v_j}\phi(x)\, | \qquad (x \in X). \tag{7}$$

Proof. We have $\tilde{D} = R_D$ for some $D \in U(\mathfrak{g})^H$. By the Cartan decomposition

$$G = K\ cl(A_{pq}^+)\, H$$

(cf. [4, Theorem 4.1]) it suffices to prove (7) for $x \in K\ cl(A_{pq}^+)$, and since (6) is a finite union, it even suffices to prove (7) for $x \in K\ cl(A_{pq}^+(P))$, with $P \in \mathcal{P}$ fixed. By Proposition 1.6 there exist $w_n \in U(\mathfrak{g})$, and $f_n \in \mathcal{R}(P)$ $(1\leqslant n\leqslant N)$, such that

$$R_D\phi(a) = \sum_{n=1}^{N} f_n(a)\, [L(w_n)\phi](a)\, ,$$

for all $\phi \in C^{\infty}(G/H)$, $a \in A_{pq}$. Let $v_1, \dots, v_J$ be a basis for the linear subspace of $U(\mathfrak{g})$ spanned by $\{ (w_n)^k;\ 1 \leqslant n \leqslant N,\ k \in K\}$, and define functions $m_n^j : K \to \mathbb{C}$ by

$$(w_n)^k = \sum_{j=1}^{J} m_n^j(k) v_j.$$

Then

$$R_D\phi(ka) = L(k^{-1})(R_D\phi)(a) = R_D(L(k^{-1})\phi)(a)$$
$$= \sum_{n=1}^{N} f_n(a)\,[L(w_n)L(k^{-1})\phi](a),$$

whence

$$R_D\phi(ka) = \sum_{n=1}^{N}\sum_{j=1}^{J} f_n(a)\, m_n^j(k)\,[\,L(v_j)\phi\,](ka).$$

Now the m_n^j are bounded on K, whereas the f_n are bounded on $cl(A_{pq}^{+}(P))$. This proves (7).

Lemma 1.1 now follows easily from Proposition 1.7 and the fact that $L_u\tilde{D} = \tilde{D}L_u$ for all $u \in U(\mathfrak{g})$.

2. Density of $C_c^{\infty}(X)$ in $L^2(X)^{\infty}$.

In this section we prove Lemma 1.2, following closely the ideas of Harish-Chandra [8, §13] (cf. also [19, p.342]). Let $\sigma_G : G \to [0,\infty)$ be the function defined by

$$\sigma_G(k\exp Y) = \|Y\| = [-\,B(Y,\theta Y)]^{\frac{1}{2}}.$$

for $k \in K$, $Y \in \mathfrak{p}$. Recall that σ_G is bi-K-invariant and continuous; $\sigma_G(e) = 0$, $\sigma_G(x)>0$ for $x \notin K$, and if $x,y \in G$, then $\sigma_G(x) = \sigma_G(x^{-1})$ and:

$$\sigma_G(xy) \leqslant \sigma_G(x) + \sigma_G(y)$$

(cf. [19, p.320]).

The map $K \times (\mathfrak{p}\cap\mathfrak{q}) \times (\mathfrak{p}\cap\mathfrak{h}) \to G, (k,Y,Z) \mapsto k\exp Y \exp Z$ is a diffeomorphism ([4, Proof of Thm. 4.1]). We define σ_X: $G \to [0,\infty)$ by $\sigma_X(k \exp Y \exp Z) = \|Y\|$ ($k \in K$, $Y \in \mathfrak{p}\cap\mathfrak{q}$, $Z \in \mathfrak{p}\cap\mathfrak{h}$). From the Cartan decomposition $H = (H\cap K)\exp(\mathfrak{p}\cap\mathfrak{h})$, one easily deduces that

$$\sigma_X(kah) = \|\log a\|,$$

for $k \in K$, $a \in A_{pq}$, $h \in H$.

Proposition 2.1. The function σ_X is continuous, and left K- and right H-invariant; $\sigma_X(e) = 0$, $\sigma_X(x)>0$ if $x \notin KH$, and if $x \in G$, $y \in G$, then

$$\sigma_X(\tau x) = \sigma_X(x),$$
$$\sigma_X(xy) \leqslant \sigma_G(x) + \sigma_X(y). \tag{8}$$

Proof. The first assertions are obvious by what we said above. The first formula follows from the fact that the decomposition $G = K\exp(\mathfrak{p}\cap\mathfrak{q})\exp(\mathfrak{p}\cap\mathfrak{h})$ is τ-invariant, whereas τ acts as $-I$ on $\mathfrak{p}\cap\mathfrak{q}$. Formula (8) follows from a reasoning similar to the one in [14, Lemma 2.31]. We give it for the sake of completeness.

Fix a maximal abelian subspace $\mathfrak{a}_{ph}$ of $\mathfrak{l}_{ph}$ and put $\mathfrak{a}_p = \mathfrak{a}_{pq} \oplus \mathfrak{a}_{ph}$, $A_p = \exp\mathfrak{a}_p$. Let $x \in KaK$, $y \in KbH$ ($a \in A_p$, $b \in A_{pq}$). Then $\sigma_G(x) = \|\log a\|$, $\sigma_X(y) = \|\log b\|$, and $xy \in KaKbH$. Also $xy \in KcH$ for some $c \in A_{pq}$. It folows that $ch = k_1ak_2b$ for certain $h \in H$, $k_1,k_2 \in K$. Hence

$h = c^{-1}k_1ak_2b = ck_1^{\tau}a^{\tau}k_2^{\tau}b^{-1}$, so that $c^2 = k_1ak_2b^2(k_2^{\tau})^{-1}(a^{\tau})^{-1}(k_1^{\tau})^{-1}$. Hence $2\|\log c\| = \sigma_G(c^2) = \sigma_G(ak_2b^2(k_2^{\tau})^{-1}(a^{\tau})^{-1}) \leqslant \sigma_G(a) + \sigma_G(a^{\tau}) + 2\|\log b\|$. The estimate (8) now follows from the obvious fact that $\|\log a^{\tau}\| = \|\log a\|$.

We also view σ_X as a function on X, and for $t>0$ we define $B_X(t) = \{ x \in X;\ \sigma_X(x) \leqslant t\}$. Then $B_X(t)$ is compact in X, for every $t>0$.

Lemma 2.2. Let $\epsilon>0$. Then there exist left K-invariant functions $\psi_t \in C_c^{\infty}(X)$, such that:

(i) $0 \leqslant \psi_t(x) \leqslant 1 \qquad (t>0, x \in X)$,
(ii) $\psi_t = 1$ on $B_X(t)$ and $\operatorname{supp}(\psi_t) \subseteq B_X(t+\epsilon) \quad (t>0)$,
(iii) if $u \in U(\mathfrak{g})$, then there exists a $C_u>0$ such that:

$$\sup_X |L_u\psi_t| \leqslant C_u \qquad (\text{all } t>0).$$

Proof. Fix $\psi \in C_c^{\infty}(K \backslash G / K)$ such that $\operatorname{supp}\psi \subseteq B_G(\epsilon/4) = \{x \in G;\ \sigma_G(x) \leqslant \epsilon/4\}$, such that $\psi(x) = \psi(x^{-1})$ for all $x \in G$, and such that $\int_G \psi(g)dg = 1$ (where some choice of Haar measure for G has been made). Moreover, let χ_t be the characteristic function of the set $B_X(t+\frac{1}{2}\epsilon)$, and put $\psi_t = \psi * \chi_t$, i.e. $\psi_t(x) = \int_G \psi(g)\chi_t(g^{-1}x)dg \ (x \in X)$. Then the ψ_t satisfy the assertions. In fact, (i) is obvious, (ii) follows from $B_G(\frac{1}{4}\epsilon)\, B_X(t+\frac{1}{2}\epsilon) \subseteq B_X(t+\frac{3}{4}\epsilon)$ (cf (8)). Finally (iii) follows from $L_u\psi_t = (R_u\psi) * \chi_t$.

Proof of Lemma 1.2. Fix a seminorm p_u $(u \in U(\mathfrak{g}))$, and let $\{\psi_t\}$ be as in Lemma 2.2. Then just as in [19, Thm 2, p.343] it follows that $p_u(\psi_t f - f) \to 0$ as $t \to +\infty$, for every $f \in L^2(X)^{\infty}$.

3. Finite multiplicity theorems.

Since G is of type I (cf. [6]), the left regular representation L of G on $L^2(X)$ has a direct integral decomposition

$$\int_{\hat{G}}^{\oplus} \pi^{\alpha}\, d\mu(\alpha), \tag{9}$$

where $d\mu$ is some Borel measure on the unitary dual $\hat{G}$ of G, equipped with its usual Borel structure (cf. e.g. [12]). The π^{α} are multiples of $\alpha \in \hat{G}$ of possibly infinite multiplicity $m(\alpha,\pi^{\alpha})$. The main result of this section is:

Theorem 3.1. For almost every $a \in \hat{G}$ we have $m(\alpha,\pi^{\alpha})<\infty$.

Remark. In particular this implies that every discrete series representation of G/H occurs with finite multiplicity in $L_d^2(G/H)$.

In order to prove Theorem 3.1 we need some results of [16], which we now briefly describe.

If π is a unitary representation of G in a separable Hilbert space $\mathcal{H} = \mathcal{H}_{\pi}$, we write $\mathcal{H}^{\infty}$ for the space of C^{∞}-vectors in $\mathcal{H}$, equipped with its usual Sobolev topology (i.e. the topology defined by seminorms as in (1)). An element δ of the topological dual $\mathcal{H}^{-\infty}$ of $\mathcal{H}^{\infty}$ is said to be a generalized cyclic vector if $\phi=0$ is the only element of $\mathcal{H}^{\infty}$ satisfying $\delta(\pi(g)\phi)=0$ for all $g \in G$. Thus, the Dirac measure δ_{eH} of

$X = G/H$ at eH is a generalized cyclic vector for $(L, L^2(X))$. The decomposition (9) induces a decomposition

$$\delta_{eH} = \int_{\hat{G}}^{\oplus} \delta^\alpha \, d\mu(\alpha)$$

in the sense of [16, Corollary C.I.]. Here the δ^α are generalized cyclic vectors in $\mathcal{H}^\alpha = \mathcal{H}_{\pi^\alpha}$. They are uniquely determined for almost every $\alpha \in \hat{G}$; since δ_{eH} is H-invariant, the δ^α must therefore be H-invariant for almost every α.

A unitary representation π together with a generalized cyclic vector ϵ is called a cyclic pair. Such a cyclic pair has a canonical realization on a left G-invariant Hilbert subspace V_π of the space $\mathcal{D}'(G)$ of distributions on G, with the G-action induced by the left regular representation of G on $C_c^\infty(G)$. The isomorphism $T: \mathcal{H}_\pi \to V_\pi$ is defined by

$$Tu\ (\phi) = \epsilon(\pi(\phi^\vee)u),$$

for $u \in \mathcal{H}_\pi$, $\phi \in C_c^\infty(G)$. Here $\phi^\vee(x) = \phi(x^{-1})$. Obviously ϵ is H-invariant iff $V_\pi \subset \mathcal{D}'(G/H)$. We conclude:

Proposition 3.2. For almost every $\alpha \in \hat{G}$, π^α has a canonical realization on a Hilbert subspace V^α of $\mathcal{D}'(G/H)$.

Proof of Theorem 3.1. Let $\chi^\alpha: \mathfrak{Z} \to \mathbb{C}$ be the infinitesimal character of $\alpha \in \hat{G}$, and let $\epsilon \in \hat{K}$ be a K-type occurring in α. Then the space $V^\alpha(\epsilon)$ of K-finite vectors of type ϵ in V^α is contained in $\mathcal{D}_\epsilon'(G/H;\chi^\alpha) = \{u \in \mathcal{D}'(G/H)(\epsilon);\ L_Z u = \chi^\alpha(Z)u \text{ for all } Z \in \mathfrak{Z}\}$. By an application of the elliptic regularity theorem as in [19, Proof of Thm. 7.8, p.310] it follows that the elements of $\mathcal{D}_\epsilon'(G/H;\chi^\alpha)$ are real analytic functions. Therefore this space will also be denoted by $A_\epsilon(G/H;\chi^\alpha)$. In the remainder of this section we will prove that $dim_{\mathbb{C}} A_\epsilon(G/H;\chi^\alpha)$ is bounded by a finite number $dim(\epsilon)^2[W(\Phi):W(\Phi_0)]$ involving the index of one Weyl group in another (Corollary 3.10). Hence

$$m(\alpha,\pi^\alpha) \leq dim(\epsilon)^2 \cdot [W(\Phi):W(\Phi_0)],$$

for almost all $\alpha \in \hat{G}$.

For the sake of completeness we list the following lemma which is proved along similar lines.

Lemma 3.3. Let π be an irreducible unitary representation of G in a Hilbert space $\mathcal{H}$. Then the space $(\mathcal{H}^{-\infty})^H$ of H-fixed distribution vectors has finite dimension over $\mathbb{C}$.

Remark. For other results concerning H-fixed distribution vectors related to the Plancherel formula we refer the reader to [2, 3, 11].

The remainder of this section is devoted to the proof of Corollary 3.8. Recall the definitions (3) and (4) of $\bar{\mathfrak{n}}$ and $\mathfrak{l}_{kq}$.

Proposition 3.4. The algebra $\mathfrak{g}$ splits into a direct sum of vector subspaces

$$\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{l}_{kq} \oplus \mathfrak{a}_{pq} \oplus \mathfrak{h}. \tag{10}$$

Proof. If $\alpha \in \Delta = \Delta(\mathfrak{g}, \mathfrak{a}_{pq})$, then $\tau(\mathfrak{g}^\alpha) = \mathfrak{g}^{-\alpha}$. It is easily seen that the map $\mathfrak{l}_h \times \mathfrak{n} \to \mathfrak{h}$, $(X,Y) \mapsto X + Y + \tau Y$ is bijective and so $\mathfrak{n}^- \oplus \mathfrak{l}_h \oplus \mathfrak{n} = \mathfrak{n}^- \oplus \mathfrak{h}$. The assertion now follows from the obvious decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{l}_{kq} \oplus \mathfrak{a}_{pq} \oplus \mathfrak{l}_h \oplus \mathfrak{n}$.

Extend $\mathfrak{a}_{pq}$ to a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, and let $\Phi = \Delta(\mathfrak{g}_c, \mathfrak{a}_c)$. Then restriction of $\tilde{\Phi} = \{ \alpha \in \Phi;\ \alpha|\mathfrak{a}_{pq} \neq 0\}$ to $\mathfrak{a}_{pq}$ gives all of Δ, and we may select a system Φ^+ of positive roots for Φ which is compatible with Δ^+.

If $\alpha \in \Delta_+ \cap \Delta^+$ (cf. (5)), we define $f^{\alpha}_{+}, g^{\alpha}_{+} : A^{+}_{pq} \to \mathbb{R}$ by

$$f^{\alpha}_{+}(a) = (a^{\alpha} - a^{-\alpha})^{-1}, \quad g^{\alpha}_{+} = -a^{-\alpha} f^{\alpha}_{+}(a).$$

Moreover, if $\alpha \in \Delta^+$, $\mathfrak{g}^{\alpha}_{-} \neq 0$, we put

$$f^{\alpha}_{-}(a) = (a^{\alpha} + a^{-\alpha})^{-1}, \quad g^{\alpha}_{-}(a) = -a^{-\alpha} f^{\alpha}_{-}(a),$$

for $a \in A_{pq}$. Let $\mathcal{F}^+$ be the algebra of functions $A^{+}_{pq} \to \mathbb{R}$ generated by $f^{\alpha}_{+}, g^{\alpha}_{+}, f^{\beta}_{-}, g^{\beta}_{-}$ ($\alpha \in \Delta_+ \cap \Delta^+$; $\beta \in \Delta^+$, $\mathfrak{g}^{\beta}_{-} \neq 0$), and let $\mathcal{F}$ be the ring generated by 1 and $\mathcal{F}^+$.

Proposition 3.5. Let $X_\alpha \in \mathfrak{g}^{\alpha}_{+}$ (or $\in \mathfrak{g}^{\alpha}_{-}$). Then there exist $f_1, f_2 \in \mathcal{F}^+$, such that for all $a \in A^{+}_{pq}$ one has:

$$\theta X_\alpha = f_1(a)(X_\alpha + \theta X_\alpha)^{a^{-1}} + f_2(a)(X_\alpha + \tau X_\alpha). \tag{11}$$

Proof. If $X_\alpha \in \mathfrak{g}^{\alpha}_{+}$, then $\tau X_\alpha = \theta X_\alpha$ and one easily checks (11) to hold with $f_1 = f^{\alpha}_{+}, f_2 = g^{\alpha}_{+}$. On the other hand, if $X_\alpha \in \mathfrak{g}^{\alpha}_{-}$, then $\tau X_\alpha = -\theta X_\alpha$ and (11) holds with $f_1 = f^{\alpha}_{-}, f_2 = g^{\alpha}_{-}$.

Let $\Phi_0 = \{\alpha \in \Phi;\ \alpha|\mathfrak{a}_{pq} = 0\}$. Then $\Phi_0 = \Delta(\mathfrak{l}_c, \mathfrak{a}_c)$, and $\Phi_0^+ = \Phi_0 \cap \Phi^+$ is a system of positive roots for Φ_0. Put $\rho(\Phi) = \frac{1}{2}\Sigma_\alpha \alpha$ (summation over Φ^+), $\rho(\Phi_0) = \frac{1}{2}\Sigma_\alpha \alpha$ ($\alpha \in \Phi_0^+$), $\mathfrak{n}_c(\Phi) = \Sigma_\alpha \mathfrak{g}^{\alpha}_c$ ($\alpha \in \Phi^+$), $\mathfrak{n}_c(\Phi_0) = \Sigma_\alpha \mathfrak{g}^{\alpha}_c$ ($\alpha \in \Phi_0^+$), $\bar{\mathfrak{n}}_c(\Phi) = \Sigma_\alpha \mathfrak{g}_c^{-\alpha}$ ($\alpha \in \Phi^+$), etc. By the Poincaré-Birkhoff-Witt theorem we have direct sum decompositions

$$U(\mathfrak{g}) = \{\bar{\mathfrak{n}}_c(\Phi)U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_c(\Phi)\} \oplus S(\mathfrak{a}),$$
$$U(\mathfrak{l}) = \{\bar{\mathfrak{n}}_c(\Phi_0)U(\mathfrak{l}) + U(\mathfrak{l})\mathfrak{n}_c(\Phi_0)\} \oplus S(\mathfrak{a}).$$

Let $\tilde{\gamma}$ and $\tilde{\gamma}_0$ be the corresponding projections $U(\mathfrak{g}) \to S(\mathfrak{a})$ and $U(\mathfrak{l}) \to S(\mathfrak{a})$. Given $\lambda \in \mathfrak{a}_c^*$, let T_λ denote the automorphism of $S(\mathfrak{a})$ determined by

$$T_\lambda(H) = H - \lambda(H) \qquad (H \in \mathfrak{a}_c),$$

and put $\gamma = T_{\rho(\Phi)} \circ \tilde{\gamma}|\mathfrak{Z}$, $\gamma_0 = T_{\rho(\Phi_0)} \circ \tilde{\gamma}_0|\mathfrak{Z}(\mathfrak{l})$; here $\mathfrak{Z}(\mathfrak{l})$ denotes the centre of $U(\mathfrak{l})$. Thus γ is Harish-Chandra's canonical isomorphism of $\mathfrak{Z}$ onto the algebra $I(\mathfrak{a})$ of elements in $S(\mathfrak{a})$ which are invariant under the Weyl group $W(\Phi)$ of the root system Φ. Similarly, γ_0 is the canonical isomorphism of $\mathfrak{Z}(\mathfrak{l})$ onto the algebra $I_0(\mathfrak{a})$ of $W(\Phi_0)$-invariant elements in $S(\mathfrak{a})$. Let $\tilde{\mu}$: $U(\mathfrak{g}) \to U(\mathfrak{l})$ be the projection corresponding to the decomposition

$$U(\mathfrak{g}) = (\bar{\mathfrak{n}}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}) \oplus U(\mathfrak{l}).$$

One easily checks that $\tilde{\gamma}_0 \circ \tilde{\mu} = \tilde{\gamma}$, and that $\tilde{\mu}|\mathfrak{Z}$ is an algebra homomorphism of $\mathfrak{Z}$ into $\mathfrak{Z}(\mathfrak{l})$.

Proposition 3.6. If $Z \in \mathfrak{Z}$, then

$$Z - \tilde{\mu}(Z) \in \bar{\mathfrak{n}}U(\mathfrak{g}).$$

Proof. Let $Z \in \mathfrak{Z}$. Then $Z - \tilde{\mu}(Z)$ is contained in the centralizer of $\mathfrak{a}_{pq}$ in $\bar{\mathfrak{n}}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}$, which by the Poincaré-Birkhoff-Witt theorem must be contained in $\bar{\mathfrak{n}}U(\mathfrak{g})\mathfrak{n}$.

Now let $v \mapsto {}^{`}v$ be the automorphism of $U(\mathfrak{l})$ determined by ${}^{`}X = X - \frac{1}{2}\,\mathrm{tr}(ad(X)|\mathfrak{n})$ for $X \in \mathfrak{l}$, and let $v \mapsto v'$ denote its inverse. One easily checks that $\tilde{\gamma}_0({}^{`}Z) = {}^{`}\tilde{\gamma}_0(Z)$ for $Z \in \mathfrak{Z}(\mathfrak{l})$. Defining $\mu \colon \mathfrak{Z} \mapsto \mathfrak{Z}(\mathfrak{l})$ by $\mu(Z) = {}^{`}\tilde{\mu}(Z)$ for $Z \in \mathfrak{Z}$, we thus obtain a commutative diagram

$$\begin{tikzcd}
I(\mathfrak{a}) \arrow[r, hook] & I_0(\mathfrak{a}) \\
\mathfrak{Z} \arrow[u, "\gamma"] \arrow[r, "\mu"] & \mathfrak{Z}(\mathfrak{l}) \arrow[u, "\gamma_0"']
\end{tikzcd}$$

In particular μ maps $\mathfrak{Z}$ isomorphically into $\mathfrak{Z}(\mathfrak{l})$, and $\mathfrak{Z}(\mathfrak{l})$ becomes a $\mathfrak{Z}$-module in this way. By transportation of [20, Thm. 2.1.3.6] we obtain the following well known version of [7, Lemma 5]. Put $r = [W(\Phi) \colon W(\Phi_0)]$.

Lemma 3.7 . There exist r elements $v_1 = 1, v_2, \ldots, v_r$ of $\mathfrak{Z}(\mathfrak{l})$ such that the $\gamma_0(v_j)$ $(1 \leqslant j \leqslant r)$ are homogeneous, and such that every element $v \in \mathfrak{Z}(\mathfrak{l})$ can be written uniquely in the form

$$v = \sum_{1 \leqslant j \leqslant r} \mu(Z_j) v_j$$

with $Z_j \in \mathfrak{Z}$. Moreover, $\deg(v) = \deg(Z_j) + \deg(v_j)$ $(1 \leqslant j \leqslant r)$.

Lemma 3.8. Let $D \in U(\mathfrak{g})$. Then there exist a $D_0 \in U(\mathfrak{k} \cap \mathfrak{l})(\Sigma_{1 \leqslant j \leqslant r} \mathfrak{Z} v'_j) U(\mathfrak{h})$ and finitely many $f_i \in \mathfrak{F}^+$, $\xi_i \in U(\mathfrak{k})$, $\eta_i \in (\Sigma_{1 \leqslant j \leqslant r} \mathfrak{Z} v'_j) U(\mathfrak{h})$ $(1 \leqslant i \leqslant I)$, such that

(i) $D = D_0 + \sum_{1 \leqslant i \leqslant I} f_i(a) \xi_i^{a^{-1}} \eta_i$ for all $a \in A_{pq}^+$;

(ii) $\deg(D_0) \leqslant \deg(D)$, $\deg(\xi_i) + \deg(\eta_i) \leqslant \deg(D)$ $(1 \leqslant i \leqslant I)$;

(iii) $D \equiv D_0 \bmod \bar{\mathfrak{n}} U(\mathfrak{g})$.

Proof. We prove the lemma by induction on $\deg(D)$. For $\deg(D) = 0$ the lemma is trivial. Thus, let $\deg(D) = m > 0$, and assume that the lemma has been proved already for $\deg(D) < m$. From (10) it follows that there exists a $D^* \in U(\mathfrak{l}_{kq}) U(\mathfrak{a}_{pq}) U(\mathfrak{h}) \cap U(\mathfrak{g})_m$ (where $U(\mathfrak{g})_m$ denotes the set of elements of degree $\leqslant$m), such that

$$D - D^* \in \bar{\mathfrak{n}}\, U(\mathfrak{g})_{m-1}. \tag{12}$$

Now put

$$D^* = \sum_{n=1}^{N} Q_n H_n W_n, \tag{13}$$

with $Q_n \in U(\mathfrak{l}_{kq})$, $H_n \in U(\mathfrak{a}_{pq})$, $W_n \in U(\mathfrak{h})$, $\deg(Q_n)+\deg(H_n)+\deg(W_n) \leqslant m$ $(1 \leqslant n \leqslant N)$. Since $H_n \in \mathfrak{Z}(\mathfrak{l})$, we may apply Lemma 3.7 to H_n and thus obtain an expression

$$H_n = \sum_{j=1}^{r} \tilde{\mu}(Z_{n,j})v'_j, \tag{14}$$

with $Z_{n,j} \in \mathfrak{Z}$, $\deg(Z_{n,j})+\deg(v'_j)=\deg(Z_{n,j})+\deg(v_j)=\deg(H_n)=\deg(H_n)$. Now fix n,j for the moment, put $d=\deg(Z_{n,j})$ and consider the expression

$$Q_n\,(Z_{n,j} - \tilde{\mu}(Z_{n,j}))v'_j\, W_n. \tag{15}$$

Here $Z_{n,j}-\tilde{\mu}(Z_{n,j}) \in \bar{\mathfrak{n}}U(\mathfrak{g})_{d-1}$. Since $\mathfrak{l}_{kq}$ normalizes $\bar{\mathfrak{n}}$, we have $Q_n\bar{\mathfrak{n}}U(\mathfrak{g})_{d-1} \subset \bar{\mathfrak{n}}U(\mathfrak{g})_s$ with $s=\deg(Q_n)+d-1$, and so (15) belongs to $\bar{\mathfrak{n}}U(\mathfrak{g})_{m-1}$. Hence by (12), (13) en (14), the element

$$D_0 = \sum_{n=1}^{N} Q_n Z_{n,j} v'_j W_n$$

satisfies the requirement (iii). Moreover, clearly $\deg(D_0) \leqslant \deg(D)$ and $D_0 \in U(\mathfrak{k}\cap\mathfrak{l})(\Sigma_{i=1}^{I}\mathfrak{Z}v'_i)U(\mathfrak{h})$. Thus it suffices to prove the lemma with $D_0=0$ for $D \in \bar{\mathfrak{n}}U(\mathfrak{g})_{m-1}$, and without loss of generality we may further assume that $D=\theta(X_\alpha)\tilde{D}$ with $\tilde{D} \in U(\mathfrak{g})_{m-1}$, $\alpha \in \Delta^+$ and $X_\alpha \in \mathfrak{g}_+^\alpha$ or $X_\alpha \in \mathfrak{g}_-^\alpha$. Using the decomposition (11) we then obtain

$$D = f_1(a)(X_\alpha + \theta X_\alpha)^{a^{-1}}\tilde{D} + f_2(a)\{\tilde{D}(X_\alpha + \tau X_\alpha) + \bar{D}\},$$

with $\bar{D}=[X_\alpha+\tau X_\alpha, \tilde{D}] \in U(\mathfrak{g})_{m-1}$. Applying the induction hypothesis to $\tilde{D}$ and $\bar{D}$ and keeping in mind that $\mathfrak{F}^+$ is an ideal in $\mathfrak{F}$ and that A_{pq} centralizes $\mathfrak{k}\cap\mathfrak{l}$ we obtain the desired result.

Given a finite dimensional representation μ of K in a vector space E, we write $C(G,E,\mu)$ for the space of continuous functions ϕ: $G \to E$ that are left μ-spherical, i.e.

$$\phi(kx) = \mu(k)\phi(x),$$

for all $x \in G$, $k \in K$. If χ: $\mathfrak{Z} \to \mathbb{C}$ is an infinitesimal character, we write $A(G/H,E,\mu,\chi)$ for the space of real analytic right H-invariant functions $\phi \in C(G,E,\mu)$ satisfying

$$L_Z\phi = \chi(Z)\phi \tag{16}$$

for all $Z \in \mathfrak{Z}$. By an application of the elliptic regularity theorem as in [19, Thm. 7.8, p. 310] it follows that the elements of $A(G/H,E,\mu,\chi)$ are in fact real analytic.

Lemma 3.9. Let μ be a finite dimensional representation of K in E, and let χ: $\mathfrak{Z} \to \mathbb{C}$ be an infinitesimal character. Then

$$\dim_{\mathbb{C}} A(G/H,E,\mu,\chi) \leqslant \dim(E).[W(\Phi) : W(\Phi_0)].$$

Proof. Fix $a \in A_{pq}^+$ and define the linear map $\mathcal{V}$: $A(G/H,E,\mu,\chi) \to E^r$ $(r=[W(\Phi) : W(\Phi_0)])$ by $\mathcal{V}(\phi)=([R(v'_i)\phi](a))_{i=1}^r$. The lemma will follow once we have shown that $\mathcal{V}$ is injective. Thus, let $\phi \in A(G/H,E,\mu,\chi)$, and suppose $\mathcal{V}(\phi)=0$. By Lemma 3.8, every $D \in U(\mathfrak{g})$ can be written as

$$D \equiv \sum_{i=1}^{I}\sum_{j=1}^{r} f_{ij}(a)\xi_i^{a^{-1}}Z_{ij}v'_j \mod U(\mathfrak{g})\mathfrak{h},$$

where $f_{ij} \in \mathfrak{F}$, $\xi_{ij} \in U(\mathfrak{k})$, $Z_{ij} \in \mathfrak{Z}$. Thus

$$(R_D\phi)(a) = \Sigma_{i,j} f_{ij}(a)\,\chi(Z_{ij}^{\vee})\mu(\xi_{ij}^{\vee})[R(v'_j)\phi](a)=0.$$

By analyticity of ϕ this implies $\phi=0$.

Remark. Of course by essentially the same proof an analogous result holds for $\mathfrak{Z}$-finite, (μ_1,μ_2)-spherical functions $G \to E$, if μ_1,μ_2 are commuting representations of K and H respectively in a finite dimensional vector space E (cf. also [7, Lemma 8]).

Given a finite dimensional irreducible representation $\epsilon \in \hat{K}$, and an infinitesimal character χ, we write $A(G/H,\chi)$ for the space of right H-invariant real analytic functions $G \to \mathbb{C}$ satisfying (16), and $A_\epsilon(G/H,\chi)$ for the subspace of K-finite elements of type ϵ.

Corollary 3.10. If $\epsilon \in \hat{K}$, χ an infinitesimal character, then

$$\dim_{\mathbb{C}} A_\epsilon(G/H,\chi) \leqslant \dim(\epsilon)^2 . [W(\Phi) : W(\Phi_0)]. \tag{17}$$

Proof. Let E be the space of the left K-finite functions of type ϵ in $L^2(K)$, and let μ be right regular representation of K restricted to E. Then there exists a natural bijective linear map ν: $A_\epsilon(G/H,\chi) \to A(G/H,E,\mu,\chi)$; if $\phi \in A_\epsilon(G/H,\chi)$, then $\nu(\phi)$ is given by $\nu(\phi)(x)(k) = \phi(kx)$ $(x \in G/H,\ k \in K)$. Hence (17) follows from Lemma 3.9 and the fact that $\dim(E) = \dim(\epsilon)^2$.

Some final remarks. Let π be an irreducible unitary representation of G in a Hilbert space $\mathcal{H}$, and let $\phi \in (\mathcal{H}^{-\infty})^H$. Given a K-type $\epsilon \in \hat{K}$ occuring in $\mathcal{H}$, and $u \in \mathcal{H}(\epsilon)$, we may form the matrix coefficient

$$m_{\phi,u} = \phi(\pi(x^{-1})u).$$

One easily checks that $m_{\phi,u}$ satisfies the system (16), where χ is the infinitesimal character of π; hence the associated spherical function $f = \nu(m_{\phi,u})$ does. Now in [7] it is shown that from a result like Lemma 3.8 one may derive a system of differential equations for $F = (f, R(v'_2)f, \ldots, R(v'_r)f)$ on $A^+_{pq}(P)$ $(P \in \mathcal{P}$, cf. (6)), which has simple singularities in the sense of [1, Appendix]. Therefore the $m_{\phi,u}$ have converging series expansions very similar to those for K-finite matrix coefficients of admissible representations. In another paper we will discuss such results in more detail.

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