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A Stochastic Model of Traffic Flow on Freeways

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A stochastic model of traffic flow on freeways is presented, which consists of two equations. The first one is a conservation equation. The second equation is a stochastic evolution equation for the velocity. This evolution equation, which is an example of a stochastic partial differential equation (SPDE), contains a noise term, that is modeled an infinite-dimensional Brownian motion.

Key words: stochastic partial differential equations, freeway traffic flow.
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PREFACE

In this paper an attempt is undertaken to develop a stochastic model for traffic flow on freeways. Such a model can contribute to the understanding of the fundamental properties of traffic flows which is a necessary ingredient for traffic control and freeway-capacity calculations.

The model presented in this paper can be seen as a stochastic modification of the model that H.J. Payne proposed in 1971. It consists of two partial differential equations, one of which contains a noise term that is modeled as an infinite dimensional Brownian motion.

Chapter one presents a small survey of the various types of models, which have been suggested in the last two decades. At the same time the choice made in this paper is motivated.

The second chapter contains the necessary mathematical background.

In the third chapter we give a rigorous formulation of the model and discuss how to solve the equations.

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Chapter 1

A Survey of Some Models for Traffic Flow

1.1. Introduction

This chapter contains a brief survey of some models for traffic flow on freeways.

We will restrict our attention to an arbitrary stretch of a freeway without entrances or exits. All the models to be discussed deal with multilane traffic flow in one of the two directions.

There are two important criteria used to distinguish between models:

a) is the model microscopic or macroscopic? (i.e. do individual vehicles play a role in the model or does the model only deal with aggregate variables?)

b) does the model explicitly contain stochastic components?

On the basis of these two criteria we have four classes of models (see Table 1).

	deterministic	stochastic
microscopic	car-following-models [1]	headway-models [2] /simulation-models [3]
macroscopic	continuum-models [4,5]	stochastic continuum-models

TABLE 1. *Classification of traffic-flow models.*

A class of models that does not fit into this table in the class of 'Boltzmann-models', models based on a kind of Boltzmann-equation [6].

In the following sections we will concentrate on the macroscopic continuum-models and on the Boltzmann-models.

1.2. Models based on a Boltzmann-equation

The central idea is to describe the traffic along the stretch of the freeway by a distribution-function $f(x,v,t)$: at time t the number of vehicle present at a location between x and $x+dx$ and having velocity between v and $v+dv$ equals $f(x,v,t)dxdv$.

Change of $f(x,v,t)$ is partly due to the fact that drivers increase or decrease their velocity. At the same time f changes even without velocity-changes as a consequence of the fact that the traffic moves. This last effect will be called 'convection'.

Prigogine [6] assumed two reasons for velocity-change of individual drivers:
a) car-drivers react to each other (interaction)
b) car-drivers wish to drive at some desired velocity (relaxation).
These basic considerations give the following 'Prigogine-Boltzmann equation'

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = \left[\frac{\partial f}{\partial t}\right]_{\text{rel}} + \left[\frac{\partial f}{\partial t}\right]_{\text{int}} \tag{1}$$

where d/dt is the total time derivative w.r.t. a moving observer, which decomposes into the true time derivative plus the convection-term, mentioned above. So the resulting evolution-equation for the distribution function f is

$$\frac{\partial f}{\partial t} = \left[\frac{\partial f}{\partial t}\right]_{\text{rel}} + \left[\frac{\partial f}{\partial t}\right]_{\text{int}} - v\frac{\partial f}{\partial x} \tag{1b}$$

Obviously the terms of the RHS of (1b) bear the names: relaxation-term interaction-term and convection-term respectively.

Much now depends on the choice of the relaxation and interaction terms in equation (1b). There has been extensive discussion on this issue. For this I refer to the literature. [7]

Important in the present context is that this class of models has limited validity for high densities. This conclusion is connected with the intermediate position of the model between the microscopic and the macroscopic viewpoint. By the wish to retain a role for the individual driver one is forced to impose rather restrictive assumptions in order to keep calculations tractable. More or less the same is true for the fully microscopic car-following models; these models apply mainly to single lane traffic. A promising alternative on the microscopic level seems to be simulation, but then simulation yields little insight in the crucial properties of a traffic stream.

For all these reasons we turn to the macroscopic models.

1.3. Deterministic continuum-models

Continuum-models deal with traffic streams in terms of some aggregate variables. This macroscopic approach results in a limited number of equations which are relatively easy to understand. Since continuum-models view the traffic as a continuous stream, they are obviously especially suitable for high-density traffic. Of course these models are not rigorously built from microscopic 'principles' (i.e. information on individual vehicle-behaviour). On the other hand they contain empirical information which is available from field-measurements.

We leave the general properties of continuum-models and have a closer look at the deterministic continuum-models.

There are three basic variables, viz.: the flow q (vehicles p.h.), density ρ (vehicles p. km.) and velocity (km. p.h.). In the continuous approach we have two exact relations:

$$q = \rho \cdot v \tag{2}$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial q}{\partial x}. \tag{3}$$

Equation (2) is obvious; equation (3) states 'the conservation of vehicles'. We assume that ρ and q are differentiable. In general ρ, q and v are functions in x and t. The range of x and t is usually specified afterwards (these specifications are necessary for solving a set of equations, once they have been determined).

To complete the model we need extra information. They are two possibilities:
a) assume that q is a function of ρ i.e. $q = Q(\rho)$
b) design a third equation describing the evolution of the velocity.
The first possibility yields, writing $dQ/d\rho = c(\rho)$,

$$\frac{\partial \rho}{\partial t}+c(\rho)\frac{\partial \rho}{\partial x} = 0. \tag{4}$$

This non-linear first-order partial differential equation can be solved. One will get shock-waves as solutions [4]. These shock-waves bear resemblance to phenomena observed in traffic streams. Assuming that q is a function of ρ is equivalent to assuming that v is a function of ρ: $v = V(\rho)=Q(\rho)/\rho$.

Now the second possibility to complete the continuum-model exists in assuming a more complicated relation between v and ρ. Several evolution-equations for the velocity have been suggested. I will mention two of them. Payne [8] proposed the following equation:

$$\frac{dv}{dt} = \frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x} = \lambda(V(\rho)-v)-\frac{K}{\rho}\frac{\partial \rho}{\partial x} \tag{5}$$

where d/dt is the total time derivative for a moving observer; $v\,\partial v/\partial x$ is the convection term; $\lambda(V(\rho)-v)$ is the relaxation term and $\frac{K}{\rho}\frac{\partial \rho}{\partial x}$ is called the anticipation term (K being a constant). Note the similarity with equation (1)!

The anticipation-term can be subjected to criticism [9]. I propose the following alternative:

$$\frac{dv}{dt} = \frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x} = \lambda(V(\rho)-v)+\mu\frac{\partial v}{\partial x} \tag{6}$$

i.e.

$$\frac{\partial v}{\partial t} = \lambda(V(\rho)-v)+(\mu-v)\frac{\partial v}{\partial x}$$

where μ is some constant. The reasons for choosing this anticipation term are twofold: Firstly drivers in a traffic stream will more readily anticipate velocity changes than density-changes, as the first are more directly creating dangerous situations. The second reason for this choice is that it simplifies the equations somewhat.

Let us summarize the equations that make up the continuum-model in its alternative form:

$$q = \rho \cdot v \tag{2}$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial q}{\partial x} = 0 \tag{3}$$

$$\frac{\partial v}{\partial t} = \lambda(V(\rho)-v)+(\mu-v)\frac{\partial v}{\partial x}. \tag{6}$$

Equation (2) and (3) are often combined into one equation, so that in fact two equations remain. Of course if one wants to solve these equations ranges for x and t have to be specified and correspondingly, initial and boundary conditions.

A stability analysis shows that the model is stable if $\mu+V'(\rho_0)\rho_0>0$ [cf. 4] for the constant solution $\rho=\rho_0$ and $v=v_0=V(\rho_0)$.

1.4. Stochastic continuum-model

There are several reasons for introducing some form of randomness into the deterministic model.

The first is that all measurements exhibit a lack of 'regularity' in spite of certain global effects. If we are right to believe that the deterministic content of the equations can account for these overall-effect, perhaps an extra noise term might be suitable to represent the small scale iregularity. A second consideration concerns the level of aggregation. The amount of 'particles' per unit of distance or time is relatively small, compared with e.q. fluid-flows (in fact, much of the terminology used here stems from this partial analogy). This implies that the influence of individual particles (vehicles) is still felt; even more so because the vehicles are not identical (in contrast with particles in fluid flow). Finally we have from car-following-theories the notion of 'acceleration-noise', the uncertainly of each individual driver in controlling his speed.

As a first try it seems adequate to supplement the velocity (evolution-) equation with a noise term. At the same time we will also add the smoothing term $K\frac{d^2v}{dx^2}$ to the velocity equation. There are two reasons for doing this. The first reason is simply a computational one, which will become clear in the second and third chapter of this paper. The second notivation is that so far nothing in the model accounts for the fact that two vehicles, having finite length, cannot be at the same place at the same time. It seems reasonable to assume that the second derivative will more or less take care of this effect, as the smoothing term will prevent the appearance of very rapid change in the velocity. We will however take K to be (very) small because for quiet flows the effect ought to be negligable.

After all these remarks let us write down the stochastic (continuum-) model in its first tentative form

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0, \qquad \text{where} \qquad q = \rho \cdot v \tag{3}$$

$$\frac{\partial v}{\partial t} = K\frac{\partial^2 v}{\partial x^2} + \lambda(V(\rho) - v) + (\mu - v)\frac{\partial v}{\partial x} + r(t,x) \tag{7}$$

where $r(t,x)$ denotes for the time being the noise term. Equation (7) needs a lot of interpretation, which will be developed in chapter two. In chapter three, then we shall present the final version of the model.

Chapter 2

Stochastic Integration in Hilbert Spaces

2.1. Short introduction to Itô's stochastic integral in one-dimension [10]

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a right-continuous increasing family $\{\mathcal{F}_t, t \geqslant 0\}$ of such σ-algebra's each containing all P-null sets. We say that the set-up $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ satisfies the usual conditions.

By $\{b(t), t \geqslant 0\}$ or simply b we will denote a (standard) one-dimensional Brownian motion, which is characterized by

(i) $\{b(t), t \geqslant 0\}$ is a real-valued process for $t \geqslant 0$ and $b(0) = 0$

ii) b is $\{\mathcal{F}_t\}$-adapted, i.e. $b(t)$ is $\mathcal{F}_t$-measurable for every t

(iii) for all $t \geqslant s \geqslant 0$ we have that $b(t) - b(s)$ is independent of $\mathcal{F}_s$ (i.e. we have independent increments) and $b(t) - b(s)$ is a Gaussian random variable with mean 0 and variance $t - s$.

It can be shown that b has a sample-continuous version. In the following therefore we will assume that b has continuous paths.

Following Ikeda and Watanabe, let $L_2([0, \infty) \times \Omega)$ denote the space of all real measurable processes ϕ (measurable w.r.t. $B([0, \infty)) \times \mathcal{F}$) that are adapted to $\{\mathcal{F}_t, t \geqslant 0\}$ and satisfy: $E[\int_0^T \phi^2(t, \omega) dt] < \infty$ for all T.

Suppose ϕ is a stepfunction:

$$\phi(t, \omega) = f_0(\omega) 1_{\{t=0\}}(t) + \sum_i f_i(\omega) I_{(t_i, t_{i+1}]}(t),$$

where $\{t_i\}$ is a partition of $[0, \infty)$ such that $t_0 = 0$.

Assuming that $T=t_m$ for some m we define

$$\int_0^T \phi(t,\omega)db(t) \equiv \sum_{i=0}^{m-1} f_i[b(t_{i+1})-b(t_i)].$$

We can calculate that

$$E[\int_0^T \phi(t,\omega)db(t)]^2 = \sum_i E[f_i^2(t_{i+1}-t_i)] = E[\int_0^T \phi^2(t,\omega)dt].$$

Note that this calculation owes its simplicity to the fact that by definition the increment $b(t_{i+1})-b(t_i)$ is 'ahead of' $\Phi(t_i)$, so that all cross-terms disappear when taking the expectation. For the same reason: $E[\int_0^T \phi(t,\omega)db(t)]=0$. For general $\phi\in L_2([0,\infty)\times\Omega)$ we can set up an approximation argument.

2.2. Stochastic integrals in a Hilbert-space [11,12]

As before the set-up $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}, P)$ satisfies the usual conditions. H will denote a real separable Hilbert-space with inner product $<\cdot,\cdot>$ and norm $\|\cdot\|$.

DEFINITION 2.1. A family of random linear functionals $\{B_t, t\geqslant 0\}$ on H (i.e. B_t: $H\rightarrow L_0(\Omega)$ linearly) is called a cylindrical Brownian motion on H if it satisfies the following condition:

For every $x\in H$ $(x\neq o)$ $B_t(x)/\|x\|$ is a standard one-dimensional Brownian motion.

DEFINITION 2.2. For every H-valued $\mathfrak{F}_t$-adapted measurable function $f(t,\omega)$ satisfying $E[\int_0^T \|f(t,\omega)\|^2 dt]<\infty$ we define

$$\int_0^T <f(t), dB_t> \equiv \sum_{n=1}^{\infty} <f(t),\phi_n>dB_t(\phi_n)$$

where $\{\phi_n\}$ is a complete orthonormal system in H.

REMARK. From definition 2.1 we have for $x=\Sigma_i\alpha_i\phi_i$

$$B_t(x) = \sum_i \alpha_i B_t(\phi_i)$$

so

$$E\left[\frac{B_t(x)}{\|x\|}\right]^2 = E(\sum_i \alpha_i B_t(\phi_i))^2/\sum_i \alpha_i^2$$

but also: $E(B_t(x)/\|x\|)^2 = t$. Thus we have the equality

$$E(\sum_i \alpha_i B_t(\phi_i))^2 = t\sum_i \alpha_i^2$$

where, by definition, each $\{B_t(\phi_i)\, t\geqslant 0\}$ is a one-dimensional (standard) Brownian motion. Noting that the $\{B_t(\phi_i), i\geqslant 1\}$ are jointly normally distributed, we can conclude that all $B_t(\phi_i)$ are mutually independent (e.g. the choice $\alpha_1=\alpha_2=1$ and $\alpha_i=0$ for $i\geqslant 3$ leads to: $E(B_t(\phi_1)B_t(\phi_2))=0$.

The above remark implies

$$E|\int_0^T <f(t), dB_t>|^2 = \sum_n E(\int_0^T <f(t),\phi_n>dB_t(\phi_n))^2 = \sum_n E\int_0^T <f(t),\phi_n>^2 dt = E\int_0^T \|f(t)\|^2 dt.$$

The remark also provides us with a general representation for a cylindrical Brownian motion:

$B_t(x)=\Sigma_{n=1}^{\infty} b_n(t)<\phi_n, x>$ where all $b_n(t)$ are independent standard one-dimensional Brownian motions.

DEFINITION 2.3. Let $\mathfrak{L}_2(H)$ denote the Banach space of Hilbert-Schmidt operators on H with norm $\|\cdot\|_{HS}$. For an $\mathfrak{L}_2(H)$-valued $\mathfrak{F}_t$-adapted, measurable function $F(t, \omega)$ satisfying $E\int_0^T \|F(t)\|_{HS}^2 dt<\infty$, we define an H-valued stochastic integral $\int_0^T F(t)dB_t$ by the following equality:

$$<\int_0^T F(t)dB_t, x> = \int_0^T <F^*(t)x, dB_t>, \quad \forall x \in H$$

where F^* is the adjoint operator of F. By linearity it is enough to take for x the sequence $\{\phi_n\}$.

We have immediately

$$E\|\int_0^T F(t)dB_t\|^2 = E\sum_n <\int_0^T F(t)dB_t, \phi_n>^2 = E\sum_n <\int_0^T F^*(t)\phi_n, dB_t>^2$$

$$= \sum_n E\int_0^T \|F^*(t)\phi_n\|^2 dt = E\int_0^T \sum_n \|F^*(t)\phi_n\|^2 dt = E\int_0^T \|F(t)\|_{HS}^2 dt$$

Let us now return to the definition of B_t. B_t is a linear random functional on H. Now one might ask whether it is possible to view B_t as an infinite-dimensional random-variable. It is indeed possible to construct a Hilbert-space V, into which H can be densely embedded, such that B_t is V-valued.

To be specific: $V \equiv l^2\{a_n, N\}$ [12] where $\forall n$, $a_n>0$ and $\Sigma_n a_n<\infty$, so every element x of V is a sequence $(h_1, h_2, \cdots)$, h_1, h_2 etc. being real numbers, such that $\Sigma_{n=1}^{\infty} a_n h_n^2<\infty$. Let $y=(g_1, g_2, \cdots)$, then the inner-product $<x, y>_V$ is defined as

$$<x, y>_V = \sum_{n=1}^{\infty} a_n h_n g_n.$$

The dense embedding is u: $H \rightarrow V$ and $u(x=\Sigma_n h_n \phi_n)=(h_1, h_2, \cdots)$.

Now we can represent $B_t(\omega)$ as an element of V in the following form:

$$B_t(\omega) = (b_i(t, \omega), b_2(t, \omega), \cdots)$$

where $b_n \equiv B_t(\phi_n)$. Indeed, $E\|B_t\|_V^2 = E\Sigma_n b_n^2(t)a_n = t\Sigma_n a_n<\infty$.

This explains why it makes sense to write $B_t=\Sigma_n b_n(t)\phi_n$ and we will do this sometimes, even when V is not explicitly mentioned (note that V is to some extent arbitrary).

FINAL REMARK. Of course it is also possible to define an infinite-dimensional stochastic integral directly, i.e. without having recourse to the one-dimensional case.

Suppose $F(t)$ is as above and furthermore piecewise constant so that we can define

$$\int_0^T F(t)dB_t = \sum_{i=1}^{n} F_i[B(t_{i+1})-B(t_i)].$$

Careful calculation leads to the following result:

$$E\|\int_0^T F(t)dB_t\|^2 = \sum_i E\|F_i\|_{HS}^2 \Delta t_i = \int_0^T E\|F(t)\|_{HS}^2 dt.$$

For general $F(t)$ an approximation argument is used [13].

2.3. Some properties of the stochastic integral $\int F(t)dB_t$ [11,14]

a) $\int_0^T F(t)dB_t$ is a martingale, i.e.

$$E(<\int_0^t F(s)dB_s, \phi_n>|\mathcal{F}_{t_0}) = <\int_0^{t_0} F(s)dB_s, \phi_n> \quad \forall n \quad \text{and} \quad t \geq t_0,$$

$<\int_0^t F(s)dB_s, \phi_n>$ is integrable $\forall n$ and $<\int_0^t F(s)dB_s, \phi_n>$ is adapted (again $\forall n$). In other words: $<\int_0^t F(s)dB_s, \phi_n>$ is a one dimensional martingale for each n;

b) $E(\sup_{0\leq t\leq T} \|\int_0^t F(s)dB_s\|^2) \leq 4E(\|\int_0^T F(s)dB_s\|^2) = 4E\int_0^T \|F(s)\|^2_{HS}ds$;

c) an infinite-dimensional Itô-formula is valid;

d) if $\int_0^T (E\|F(t)\|^{2p}_{HS})^{1/p}dt < \infty$, then for $p=1,2,\cdots$ there exists a positive constant $C=C(p)$ such that
$E(\|\int_0^T F(t)dB_t\|^{2p}) \leq C\{\int_0^T (E\|F(t)\|^{2p}_{HS})^{1/p}dt\}^p$.

2.4. An application: the heat equation driven by white noise

In this section we set $H=L^2(0,1)$ and we consider the following (formal) stochastic partial differential equation (SPDE):

$$dX_t = \frac{d^2}{dx^2}X_t dt + dB_t, \quad t>0$$

where: X_0 is fixed ($X_0 \in L^2(0,1)$), X_t should be zero at the boundaries ($x=0$, $x=1$), i.e. the 'temperature' is kept fixed at the boundaries (X_t is a 'relative temperature').

B_t is the cylindrical Brownian motion; dB_t is also called white noise, because $B_{t_2}(1_{(x_1,x_2]})-B_{t_1}(1_{(x_1,x_2]})$ and $B_{s_2}(1_{(y_1,y_2]})-B_{s_1}(1_{(y_1,y_2]})$ are independent Gaussian random variables if $(t_1,t_2]$ and $(s_1,s_2]$ or $(y_1,y_2]$ and $(x_1,x_2]$ are disjunct.

Solving such an equation means in the first place determining in what (function) space a solution will be sought. This issue is not a trivial matter. However, the semigroup approach is very helpful in this case.

Suppose that the noise is absent. Then the equation reduces to

$$\begin{cases} dX_t = \dfrac{d^2}{dx^2}X_t dt, & t>0 \\ X_0 \in L^2(0,1) & \\ X_t(0) = X_t(1) = 0, & t>0. \end{cases}$$

Note that the equation is now in fact deterministic. We have written d^2/dx^2 instead of $\partial^2/\partial x^2$ to indicate that we are viewing d^2/dx^2 ($\equiv A$) as an operator on $L^2(0,1)$. As this operator is unbounded, it is very important to specify what is its domain. In agreement with the boundary conditions we choose: $D(A)=\{f\in L^2(0,1): f''\in L^2(0,1) \text{ and } f(0)=f(1)=0\}$. Provided with this domain A is a closed operator with eigenfunctions $\{\phi_i\}$ ($\phi_i=\sqrt{2}\sin(\pi i x)$) and eigenvalues $\{-\lambda_i\}$ ($\lambda_i=\pi^2 i^2$). Furthermore A generates a semigroup of operators on $L^2(0,1)$, U_t, such that: $\partial/\partial t U_t f = AU_t f$, $t>0$, $f\in L^2(0,1)$. Thus we have that $U_t X_0$ is a solution for the partial differential equation (PDE). Without further explanation we mention that $U_t X_0$ is also the unique solution. We have for the semigroup the following explicit representation:

$$U_t X_0 = U_t \sum_i <X_0, \phi_i>\phi_i = \sum_i <X_0, \phi_i> e^{-\lambda_i t}\phi_i.$$

Note that from this representation it follows at once that $U_tX_0 \in D(A)$ for $t>0$. For more detailed information on the semigroup theory I refer to the literature [15].

Next, suppose, we would have the P.D.E.

$$dX_t = \frac{d^2}{dx^2}X_t dt + f(t, x)dt$$

(where $f(t, \cdot) \in L^2(0,1)$), then the solution (w.r.t. the same boundary and initial conditions as before) would be

$$X_t = U_tX_0 + \int_0^t U_{t-s} f(s, \cdot)ds.$$

This fact suggests that we should have a look at the integral equation

$$X_t = U_tX_0 + \int_0^t U_{t-s} dB_s.$$

This appears to be a nice equation, as U_{t-s} has finite HS-norm for all $0 \leqslant s < t$, and furthermore

$$E\int_0^t \|U_{t-s}\|_{HS}^2 ds = \int_0^t \|U_{t-s}\|_{HS}^2 ds = \int_0^t (\sum_i e^{-2\lambda_i(t-s)})ds \leqslant \sum_i \frac{1}{2\lambda_i} < \infty.$$

Thus we have

$$E\|\int_0^t U_{t-s} dB_s\|^2 = E\int_0^t \|U_{t-s}\|_{HS}^2 ds \leqslant \sum_i \frac{1}{2\lambda_i} < \infty.$$

(We could also have calculated

$$E\|\int_0^t U_{t-s} dB_s\|^2 = E\|\int_0^t U_{t-s} \sum_i db_i(s)\phi_i\|^2 = E\sum_i (\int_0^t e^{-\lambda_i(t-s)} db_i(s))^2 = \sum_i \int_0^t e^{-2\lambda_i(t-s)} ds).$$

What we have, then, is an X_t that is $L^2(0,1)$-valued. Is this process also in some sense a solution for the original equation? There are two answers:

1) yes, by definition
2) yes, when we consider a Hilbert-space of distributions that is large enough to contain B_t and d^2X_t/dx^2, it is possible to show that X_t as a process in this 'large' space satisfies the original equation.

Chapter 3

A Stochastic Model for Traffic Flow

3.1. General remarks

In this chapter we will present a rigorous formulation of the stochastic continuum model of which a first draft was given in chapter one, section 1.4. Using the theory summarized in chapter two we will solve the velocity-evolution-equation in a suitable Hilbert-space, which will turn out to be a subspace of $L^2(0,1)$.

In our present approach it is necessary to have a noise term which is smooth in the space-direction (e.g. the cylindrical Brownian motion on the Hilbert-subspace of $L^2(0,1)$, mentioned above, will do). For smoothness (in space direction) of the noise term implies smoothness of the velocity, e.g. differentiability of the velocity in space direction, which is needed to be able to interpret the conservation equation.

From a practical point of view this means that we assume that the disturbances of the flow are not strictly local, but have some nonzero, though small, range. This assumption seems reasonable, except possibly in case of an accident.

It is important to notice that we are dealing here with a continuous model, i.e. the relevant variables (mean speed and density) are continuous in space and time. In practice measurements are performed using nonzero space- and time-increments (the space-increments are the freeway sections). Thus a model corresponding directly to these measurements is discrete. Therefore a continuous model is to be seen as a limit of (a series of) discrete models.

The continuous model described in this paper results from the idealization that the time- and space-increments can be taken increasingly small, in the limit even infinitesimally small.

Another way of arriving at a continuous model might be the rescaling of the space- and time-coordinates. In this case more and more time- and space-increments (of constant length!) are rescaled onto fixed intervals. Probably this procedure would allow us to use ordinary white noise (i.e. without correlation in space-direction).

3.2. Formulation of the model

In order to express the fact that we are dealing with stochastic variables, we will write for the stochastic density: $X(t, x)$ and for the stochastic velocity: $W(t, x)$.

Thus the conservation-equation or density-equation becomes

$$\frac{\partial X}{\partial t} = -\frac{\partial}{\partial x}(XW). \tag{1}$$

This equation is supposed to be valid for each realization separately.

To formulate the velocity equation in a regorous way we will first specify in what Hilbert-function-space we are going to use to solve the equation. All functions will be defined on the interval [0,1], which represents an arbitrary stretch of a freeway. So for each t in the time-interval $[0, T]$ $X(t, \cdot)$ and $W(t, \cdot)$ will be functions on [0,1].

Because the function-space we will use depends on the exact form of the boundary conditions for $W(t, x)$, let us first list the boundary conditions:

$$X(t, 0) = X_{\text{in}}(t)$$

$$W(t, 0) = W_{\text{in}}(t)$$

$$\frac{\partial W}{\partial x}(t, 1) = 0$$

where X_{in} and W_{in} are positive, differentiable functions. We take W_{in} to be constant in order to simply the calculations, although this assumption is not essential.

Now let $\mathcal{D}[0,1]$ be the space of all functions on the interval [0,1] that can be extended to infinitely often differentiable functions on $\mathbb{R}$ vanishing outside of [0,1]. We provide this vector space with the norm: $\|f\|_\alpha = (\Sigma_{i=1}^{\infty} <f, \phi_i>^2 \lambda_i^{2\alpha})^{1/2}$ where $<\cdot,\cdot>$ is the $L^2(0,1)$ inner product, $\phi_i = \sqrt{2}\sin\pi(i-½)x$ and $\lambda_i = \pi^2(i-½)^2K$. The reason for choosing this orthonormal basis (ONB) $\{\phi_i\}$ on $L^2(0,1)$ is that $\{\phi_i\}$ is the set of eigenfunctions of the operator $(-K\, d^2/dX^2)$, which is consistent with the boundary conditions; $\{\lambda_i\}$ are the corresponding eigenvalues. Finally we define $\mathcal{D}_\alpha$ as the closure of $\mathcal{D}[0,1]$ in $L^2(0,1)$ w.r.t. $\|\cdot\|_\alpha$; in this paper $\alpha>0$, even $1 \leq \alpha < 1\frac{1}{4}$, but in general α may also be negative in which case $\mathcal{D}_\alpha$ is larger than $L^2(0,1)$ and contains generalized functions. For $\alpha=0$ we have $\mathcal{D}_0 = L^2(0,1)$. [13]

Let us now write down the velocity-equation for W_t (we will write W_t instead of $W(t, \cdot)$ to stress that we view W_t as a random variable taking values in $\mathcal{D}_\alpha$):

$$dW_t = (K\frac{d^2}{dx^2}W_t + \lambda(V(X_t) - W_t) + \frac{d}{dx}G(W_t))dt + \sigma(W_t)dB_t \tag{2}$$

where we write d^2/dx^2 instead of $\partial^2/\partial x^2$, as differentiation is now seen as an operator. Other shifts of viewpoint are: V is a mapping from $L^2(0,1)$ to $L^2(0,1)$, X_t being a $L^2(0,1)$-valued random variable; if we denote the 'old' function by $\tilde{V}$, then $V(f)(x) = \tilde{V}(f(x))$ for every $f \in L^2(0,1)$. Similarly, if $\tilde{G}(x) = \mu x - \frac{1}{2}x^2$ then

$$G(f)(x) = \tilde{G}(f(x)) = \mu f(x) - \frac{1}{2}f(x)^2$$

(a general function $\tilde{G}$ gives some flexibility for future modification of the model, if desired).

So far these comments merely contained some changes in point of view and corresponding notation. As to the noise term something new enters. Recall that we could write the white noise on $L^2(0,1)$ as $B_t = \Sigma_i b_i(t)\phi_i$ (sections 2.2 and 2.4) for an arbitrary ONB $\{\phi_i\}$. Let us choose here $\phi_i = \sqrt{2}\sin\pi(i-½)x$, in accordance with the boundary conditions. The noise term that we want to use can be represented as $B_t = \Sigma_{i=1}^{\infty}\gamma_i b_i(t)\phi_i$ where γ_i are positive scalars such that $\gamma_i \leq c/i^2$ (c is a constant). (see [16])

Clearly the function of the scalars$\{\gamma_i\}$ is to smooth the noise term in the space direction - we will therefore speak of 'correlated white noise' (i.e.: noise correlated in space, white in time). But why this specific condition: $\gamma_i \leq \frac{c}{i^2}$? To see this, choose $\gamma_i = 1/\lambda_i$ ($\lambda_i = \pi^2(i-½)^2K$). Next, note that an ONB for $\mathcal{D}_1$ (i.e. $\alpha=1$) is $\{\phi_i^{(1)}\} = \{\frac{1}{\lambda_i}\phi_i\}$. But then we can write $B_t = \Sigma_i b_i(t)\phi_i^{(1)}$ or, in other words, B_t is, for this special choice of γ_i, the cylindrical Brownian motion on $\mathcal{D}_1$. In this way the noise term we are going to use corresponds to the choice of the boundary condition and function space.

$\sigma(W_t)$ is meant to control the amplitude of the noise term, σ is a multiplication operator (on $\mathcal{D}_1$), i.e. $\sigma(f)h(x)=\sigma(f)(x)\cdot h(x)=\tilde{\sigma}(f(x))\cdot h(x)$ for $h\in\mathfrak{D}_1$ and $\tilde{\sigma}$ a function from $\mathbb{R}$ into $\mathbb{R}$.

Finally we will interpret equation (2) using the semigroup-generating-properties of the operator $\frac{d^2}{dx^2}$ in the following way (cf. section 2.4):

$$W_t = W_{t,1}+\int_0^t U_{t-s}\sigma(W_s)dB_s+\lambda\int_0^t U_{t-s}(V(X_s)-W_s)ds+\int_0^t U_{t-s}\frac{d}{dx}G(W_s)ds; \tag{3}$$

here $W_{t,1}$ is the solution of the equation: $dW_t=K\frac{d^2}{dx^2}W_t dt$ for W_0 given and boundary conditions as above. If $W_{\text{in}}(t)$ is indeed constant, then: $W_{t,1}=U_t(W_0-W_{\text{in}})+W_{\text{in}}$. We assume that $W_0(\cdot)$ is deterministic, bounded and positive, such that $W_0-W_{\text{in}}\in\mathfrak{D}_\alpha$ for some $\alpha>1\frac{1}{4}$!

To complete the description of the initial conditions we assume that X_0 is a bounded, positive (deterministic) function such that $\|\frac{d}{dx}X_0\|<\infty$ ($\|\cdot\|$ is the $L^2(0,1)$-norm).

3.3. Some assumptions and their consequences

As pointed out in the previous section σ, V and G act on functions, e.g. σ maps (certain) functions into $\mathfrak{L}(\mathfrak{D}_1)$, the space of bounded linear operators on $\mathfrak{D}_1$. σ will act on functions of the type $f\in\mathfrak{D}_1+W_{\text{in}}$, i.e. $(f-W_{\text{in}})\in\mathfrak{D}_1$, where W_{in} is a constant (cf. section 3.2). Note that all functions $g\in\mathfrak{D}_1$ have the property that $g(0)=0$, but σ is to act on functions which do not necessarily have the value 0 at 0.

Nevertheless the meaning and interpretation of these mappings are based on their original form in the deterministic model. Therefore we will impose some assumptions on the functions $\tilde{\sigma}$, $\tilde{V}$ and $\tilde{G}$ (where $V(f)(x)=\tilde{V}(f(x))$ etc.) and not directly on σ, V and G themselves. Let us list these assumptions:

assumptions on $\tilde{\sigma}$ ($\tilde{\sigma}$: $\mathbb{R}\to\mathbb{R}$)
- $\tilde{\sigma}$ is twice differentiable
- $\tilde{\sigma}''$ is locally Lipschitz-continuous (i.e.: $|\tilde{\sigma}''(x)-\tilde{\sigma}''(y)|\leq C_N|x-y|$ for $|x|$, $|y|\leq N$)
- $\tilde{\sigma}(x)=0$ for $x=0$.

assumptions on $\tilde{V}$ ($\tilde{V}$: $\mathbb{R}\to\mathbb{R}^+$)
- $\tilde{V}$ is differentiable and bounded
- $\tilde{V}'$ is locally bounded

assumption on G ($\tilde{G}$: $\mathbb{R}\to\mathbb{R}$)
- $\tilde{G}$ is twice differentiable
- $\tilde{G}''$ is locally Lipschitz-continuous.

REMARK. Eventually (in section 3.6) we will impose further assumptions on $\tilde{\sigma}$ and $\tilde{G}$ in order to obtain a 'global' solution for the model.

We will state some consequences of the foregoing assumptions in a proposition.

PROPOSITION 3.1. *Given the assumptions on* $\tilde{\sigma}$, $\tilde{V}$ *and* $\tilde{G}$ *we can make the following statements:*

(i) *Let* $\sigma^{(L)}$ *be defined as follows:*

$$\sigma^{(L)}: \mathfrak{D}_1+W_{\text{in}}\to\mathfrak{L}(\mathfrak{D}_1)$$

$$(\sigma^{(L)}(f)h)(x)\equiv\int_0^x\frac{d}{dy}(\sigma^{(L)}(f)h)(y)dy$$

$$\frac{d}{dy}(\sigma^{(L)}(f)h)(y)\equiv-\int_y^1\frac{d^2}{dx^2}(\sigma^{(L)}(f)h)(x)dx$$

and

$$\frac{d^2}{dx^2}(\sigma^{(L)}(f)h)(x) \equiv [\tilde{\sigma}''(-L\vee\bar{f}(x)\wedge L)(-L\vee f'(x)\wedge L)^2 h(x)$$

$$+\tilde{\sigma}'(-L\vee\bar{f}(x)\wedge L)(-L\vee f''(x)\wedge L)h(x)$$

$$+2\tilde{\sigma}'(-L\vee\bar{f}(x)\wedge L)(-L\vee f'(x)\wedge L)h'(x)$$

$$+\tilde{\sigma}(-L\vee\bar{f}(x)\wedge L)h''(x)]$$

where $f\in\mathfrak{D}_1+W_{\text{in}}$, $\bar{f}(x)=f(x)-W_{\text{in}}$ $(\bar{f}\in\mathfrak{D}_1)$ *and* $h\in\mathfrak{D}_1$. *Note that if*

$$\text{ess sup}_x|f''(x)|\leqslant L$$

then

$$\frac{d^2}{dx^2}(\sigma^{(L)}(f)h)(x) = \frac{d^2}{dx^2}\tilde{\sigma}(f(x))\cdot h(x) \quad (\textit{almost everywhere}).$$

For $\sigma^{(L)}$ *we have, (L arbitrary, but fixed):*

$$\|\sigma^{(L)}(f)h\|_1^2 = \int_0^1 (\frac{d^2}{dx^2}(\sigma^{(L)}(f)h)(x))^2 dx \leqslant C_1$$

(C_1 depending on L, but not on f) and;

$$\|\sigma^{(L)}(f)-\sigma^{(L)}(g)\|_1^2 \leqslant C_2\|f-g\|_1^2$$

(again C_2 depending only on L)

(ii) *For* $\sigma^{(L)}$ *as above we have:*

$$\|U_{t-s}\sigma^{(L)}(f)\|_{HS,1\to\alpha} \leqslant C_3 \quad (\sum_j e^{-2\lambda_j(t-s)}\lambda_j^{2(\alpha-1)})$$

where '$1\to\alpha$' means that we consider $U_{t-s}\sigma^{(L)}(f)$ *as an operator from* $\mathfrak{D}_1$ *into* $\mathfrak{D}_\alpha$ $(f\in\mathfrak{D}_1+W_{\text{in}})$, *and*

$$\|U_{t-s}(\sigma^{(L)}(f)-\sigma^{(L)}(g))\|_{HS,1\to\alpha} \leqslant C_4\|f-g\|_1^2 \quad (\sum_i e^{-2\lambda_j(t-s)}\lambda_j^{2(\alpha-1)})$$

where $f,g\in\mathfrak{D}_1+W_{\text{in}}$ *and* $1\leqslant\alpha<1\frac{1}{4}$

(iii) *If* $\sup_x\|X_s(x)\|\leqslant C_1$ *and* X_s *is differentiable, then*

$$|<V(X_s),\phi_i>| \leqslant (C_2+C_3\|\frac{d}{dx}X_s\|)/i$$

(iv) *Let* $G^{(N)}$ *be defined as*

$$G^{(N)}:\mathfrak{D}_n+W_{\text{in}}\to L^2(0,1)$$

$$G^{(N)}(f)(x)\equiv\int_0^x\frac{d}{dy}(G^{(N)}(f))(y)dy,\quad f\in\mathfrak{D}_1+W_{\text{in}}$$

$$\frac{d}{dy}G^{(N)}(f)(y)\equiv-\int_y^1\frac{d^2}{dx^2}(G^{(N)}(f))(x)dx$$

and

$$\frac{d^2}{dx^2}(G^{(N)}(f))(x)\equiv[\tilde{G}''(-N\vee\bar{f}(x)\wedge N)(-N\vee f'(X)\wedge N)^2$$

$$+\tilde{G}'(-N\vee\bar{f}(x)\wedge N)(-N\vee f''(x)\wedge N)]$$

(again: $\bar{f}=f-W_{\text{in}}\in\mathcal{D}_1$).

If ess $\sup_x|f''(x)|\leqslant N$ *then*

$$\frac{d^2}{dx^2}(G^{(N)}(f)(x) = \frac{d^2}{dx^2}\tilde{G}(f(x)) \quad \textit{(almost everywhere).}$$

For $G^{(N)}$ we can calculate

$$|<\frac{d}{dx}G^{(N)}(f),\phi_i>| \leqslant (C_1+C_2\|f-W_{\text{in}}\|_1)/i$$

and

$$|<\frac{d}{dx}G^{(N)}(f)-\frac{d}{dx}G^{(N)}(g),\phi_i>| \leqslant C_3\|f-g\|_1/i$$

where $f,g\in\mathcal{D}_1+W_{\text{in}}$.

REMARK. $\{C_i\}$ are generic constants, depending only on N or L. We may call the constants L and N *truncation-constants,* as their function is to provide the bounds that are listed above.

PROOF. Calculations are straightforward. □

3.4. Solving the velocity-equation

In this section we will describe the solution of the velocity-equation, assuming that X is a fixed element of R^p such that $\sup_{0\leqslant s\leqslant T}E\|\frac{d}{dx}X_s\|^{2p}<\infty$, and $\sup_{\omega,x,s}|X_s(x)|\leqslant C_1$. R^p is defined as: {all measurable, adapted, $L^2(0,1)$-valued processes, defined on $[0,T]\text{x}\Omega$ such that $|||X|||\equiv\{\sup_{0\leqslant s\leqslant T}E\|X_s\|^{2p}\}^{1/2p}<\infty$}.

REMARK. We assume a complete probability space plus filtration $(\Omega,\mathcal{F},\{\mathcal{F}_t\},P)$ to be given in connection with the infinite-dimensional Brownian-motion B_t; adapted then means adapted w.r.t. $\{\mathcal{F}_t,t\geqslant0\}$.

DEFINITION 3.1. R^p_α is the Banach-space of all measurable, adapted, $\mathcal{D}_\alpha$-valued processes W_s defined on $[0,T]\text{x}\Omega$ such that $|||W|||\equiv(\sup_{0\leqslant s\leqslant T}E\|W_s\|^{2p}_\alpha)^{1/2p}<\infty$.

THEOREM 3.1. *Under the above-mentioned conditions on $\tilde{\sigma}$, B_t, $\tilde{V}$, $\tilde{G}$ and X equation (3) (the velocity-equation) has a unique solution, which, in a sense to be explained below, belongs 'locally' to R^p_α $(1\leqslant\alpha<1\frac{1}{4})$.*

PROOF. During this proof we consider a kind of 'normalized' velocity, namely W_t-W_{in} instead of W_t itself. In this way we can solve the velocity equation in R^p_α. Let us define the mapping $\xi_{L,N}$: $R^p_\alpha\rightarrow R^p_\alpha$

$$\xi_{L,N}(W)_t = U_t(W_0-W_{\text{in}})+\int_0^t U_{t-s}\sigma^{(L)}(W_s+W_{\text{in}})dB_s+\lambda\int_0^t U_{t-s}(V(X_s)-W_s-W_{\text{in}})ds$$
$$+\int_0^t U_{t-s}\frac{d}{dx}G^{(N)}(W_s+W_{\text{in}})ds$$

where W_0, U_t, $\sigma^{(L)}$, V, $G^{(N)}$ are as explained in 3.2 and 3.3.

In order to apply a fixed-point theorem we calculate for Y, $Z\in R^p_\alpha$ with $Y_0=Z_0=W_0-W_{\text{in}}$

$$E\|\xi_{L,N}(Y)_t-\xi_{L,N}(Z)_t\|_\alpha^{2p} \leqslant 3^{2p-1}E\|\int_0^t U_{t-s}(\sigma^{(L)}(Y_s+W_{\text{in}})-\sigma^{(L)}(Z_s+W_{\text{in}}))dB_s\|_\alpha^{2p}$$

$$+3^{2p-1}E\|\lambda\int_0^t U_{t-s}(Y_s-Z_s)ds\|_\alpha^{2p}$$

$$+3^{2p-1}E\|\int_0^t U_{t-s}(\frac{d}{dx}[G^{(N)}(Y_s+W_{\text{in}})-G^{(N)}(Z_s+W_{\text{in}})])ds\|_\alpha^{2p}.$$

Let us consider the three terms separately:

$$E\|\int_0^t U_{t-s}(\sigma(Y_s+W_{\text{in}})-\sigma(Z_s+W_{\text{in}}))dB_s\|_\alpha^{2p}$$

$$\leqslant C_1\{\int_0^t(E\|U_{t-s}(\sigma(Y_s+W_{\text{in}})-\sigma(Z_s+W_{\text{in}}))_{1\to\alpha}\|_{HS}^{2p})^{1/p}ds\}^p$$

$$\leqslant C_2\{\int_0^t(E[\|Y_s-Z_s\|_\alpha^2\sum_j\lambda_j^{2(\alpha-1)}e^{-2\lambda_j(t-s)}]^p)^{1/p}ds\}^p$$

$$\leqslant C_3\int_0^t(\sum_j\lambda_j^{2(\alpha-1)}e^{-2\lambda_j(t-s)})E\|Y_s-Z_s\|_\alpha^{2p}ds= C_3\int_0^t q(t-s)E\|Y_s-Z_s\|_\alpha^{2p}ds$$

where $q(t-s)\equiv\Sigma_j\lambda_j^{2(\alpha-1)}e^{-\lambda_j(t-s)}$.

The above estimates are obtained using a result on stochastic integration w.r.t. cylindrical Brownian motion (e.g. [11] p. 134), using the estimate of Proposition 3.1 and applying the Cauchy-Schwarz-inequality.

For the second term we have

$$E\|\int_0^t U_{t-s}(Y_s-Z_s)ds\|_\alpha^{2p} = E\|A^\alpha\int_0^t\sum_i U_{t-s}<Y_s-Z_s,\phi_i>\phi_i ds\|^{2p}$$

$$= E[\sum_i(\int_0^t\lambda_i^\alpha e^{-\lambda_i(t-s)}<Y_s-Z_s,\phi_i>ds)^2]^p$$

$$\leqslant E\sum_i(\int_0^t\lambda_i^\alpha e^{-\lambda_i(t-s)}<Y_s-Z_s,\phi_i)ds)^{2p}i^{ap}\cdot(\sum_i(\frac{1}{i^a})^{p/p-1})^{p-1} \quad \text{for} \quad a>\frac{p-1}{p}$$

$$\leqslant C_1E\sum_i\int_0^t(e^{-\beta_1\lambda_i(t-s)}<Y_s-Z_s,\phi_i>)^{2p}ds\lambda_i^{2\alpha p}\cdot(\int_0^t(e^{-\beta_2\lambda_i(t-s)})^{\frac{2p}{2p-1}}ds)^{2p-1}i^{ap}$$

where $\beta_1=\frac{1}{2p}$, $\beta_2=1-\frac{1}{2p}=\frac{2p-1}{2p}$

$$\leqslant C_2E\sum_i\int_0^t e^{-\lambda_i(t-s)}<Y_s-Z_s,\phi_i>^{2p}ds\lambda_i^{2\alpha p+1-2p}i^{ap}$$

$$\leqslant C_3\sum_i\int_0^t e^{-\lambda_i(t-s)}E\|Y_s-Z_s\|_{\alpha/2}^{2p}ds\lambda_i^{2\alpha p+1-2p}i^{ap-2p}$$

$$\leqslant C_3\int_0^t(\sum_i e^{-\lambda_i(t-s)}\lambda_i^{2(\alpha-1)})E\|Y_s-Z_s\|_\alpha^{2p}ds$$

if we choose $a=2(3-2\alpha)(p-1)/p$, which is allowed as $2(3-2\alpha)>1$.

For the third term we can calculate in a similar way

$$E\|\int_0^t U_{t-s}\frac{d}{dx}(G^{(N)}(Y_s+W_{\text{in}})-G^{(N)}(Z_s+W_{\text{in}}))ds\|_a^{2p}$$

$$\leqslant C_1\sum_i\int_0^t e^{-\lambda_i(t-s)}E<\frac{d}{dx}(G^{(N)}(Y_s+W_{\text{in}})-G^{(N)}(Z_s+W_{\text{in}})),\phi_i>^{2p}ds\lambda_i^{2ap+1-2p}i^{ap}$$

$$\leqslant C_2\int_0^t\sum_i e^{-\lambda_i(t-s)}E\|Y_s-Z_s\|_l^{2p}ds\lambda_i^{2ap+1-2p}i^{ap-2p}$$

$$\leqslant C_2\int_0^t(\sum_i e^{-\lambda_i(t-s)}\lambda_i^{2(a-1)})E\|Y_s-Z_s\|_a^{2p}ds$$

choosing a as above. From the above calculations I conclude that

$$E\|\xi_{L,N}(Y)_t-\xi_{L,N}(Z)_t\|_a^{2p}\leqslant C\int_0^t q(t-s)E\|Y_s-Z_s\|_a^{2p}ds$$

where $q(s)$ is defined as

$$q(s)=\begin{cases}\sum_i e^{-\lambda_i s}\ \lambda_i^{2a-1}, & s>0\\ 0, & s\leqslant 0\end{cases}$$

At this point we can use a technique analogous to the one used by Dawson ([14], p. 26-28) to obtain a fixed point theorem.

So, finally, we conclude that $\xi_{L,N}$ has a fixed point. Adding W_{in} to this fixed point we obtain the 'unnormalized solution' that will be denoted by $W^{(LN)}$.

We will call this $W^{LN)}$ a 'local' solution, because, if we define

$$\tau\equiv\inf\{t,\ \sup_x|\frac{\partial^2 W^{(LN)}}{\partial x^2}|\leqslant L\wedge N\}$$

and $W_t\equiv W_t^{(LN)}$, for $0\leqslant t\leqslant\tau$ then W_t is the (unique) solution of equation (3) for $0\leqslant t\leqslant\tau$. So 'local' means: up to the stopping time τ. What happens if L and/or N go to infinity will be discussed in section 3. □

We will need the following two propositions.

PROPOSITION 3.2. $\dfrac{\partial^2 W^{(LN)}}{\partial x^2}$ *is Hölder-continuous w.r.t. t and x.*

SKETCH OF PROOF. We have for $W^{(LN)}$

$$\frac{d^2}{dx^2}W_t^{(LN)}=\frac{d^2}{dx^2}U_t(W_0-W_{\text{in}})+\frac{d^2}{dx^2}\int_0^t U_{t-s}\sigma^{(L)}(W_s^{(LN)})dB_s$$

$$+\lambda\frac{d^2}{dx^2}\int_0^t U_{t-s}(V(X_s)-W_s^{(LN)})ds-\frac{d^2}{dx^2}\int_0^t U_{t-s}\frac{d}{dx}G^{(N)}(W_s)ds$$

$$=U_t\frac{d^2}{dx^2}(W_0-W_{\text{in}})+\int_0^t\frac{d^2}{dx^2}U_{t-s}\sigma^{(L)}(W_s^{(LN)})dB_s$$

$$+\frac{\lambda}{K}(V(X_t)-W_t^{(LN)})-\frac{\lambda}{K}U_t(V(X_0)-W_0))+\frac{1}{K}(\frac{d}{dx}G^{(N)}(W_t^{(LN)}))-\frac{1}{K}U_t(\frac{d}{dx}G^{(N)}(W_0))$$

where K comes from equation (2) section 3.2. For all terms separately we can verify the Hölder-continuity:

— first term: see the assumptions on W_0

— second term: it can be shown that

$$E|SI(t_1,x_1)-SI(t_2,x_2)|^{2p} \leq C_1|t_1-t_2|^{\frac{1}{2}p}+C_2|x_1-x_2|^p,$$

where $SI(t, x)$ denotes:

$$\frac{d^2}{dx^2}\int_0^t U_{t-s}\sigma^{(L)}(W_s^{(LN)})dB_s(x).$$

Kolmogorovs continuous-version-theorem now yields Hölder-continuity for a.s. each realization (see [17] p. 273)

— third up to sixth term: it follows from the properties of $W_t^{(LN)}$ and $\frac{d}{dx}G^{(N)}(W_t^{(LN)})$. □

Proposition 3.3.

(i) *Suppose we have two solutions,* $W^{(LN)}$ *and* $\overline{W}^{(LN)}$, *corresponding to* X *and* $\overline{X}$, *respectively. Then we have:*

$$E\|W_t^{(LN)}-\overline{W}_t^{(LN)}\|_{F/2}^{2p} \leq c\int_0^t(\sum_i e^{-\lambda_i(t-s)})E\|X_s-\overline{X}_s\|^{2p}ds$$

(ii) *Because* $\tilde{\sigma}$ *is such that* $\tilde{\sigma}(x)=0$ *for* $x=0$, *we have* $W^{(LN)}(t,x)\geq 0$ $\forall x\in[0,1]$, $t\in[0,T]$ *and* $\omega\in Q_W^{N\wedge L}$ *where*

$$Q_W^M \equiv \{\omega: \sup_{t,x}|\frac{\partial^2 W}{\partial x^2}|\leq M\}.$$

Proof. (i) Analogous to previous calculations (ii) Starting from the positive W_0 the process $W^{(LN)}$ might at some moment reach the x-as for the first time. Because the stochastic driving term vanishes at this 'contact point' (let us call it 'z'), we have at z $\frac{\partial W}{\partial t}>0$ (note that at z

$$\frac{\partial^2 W}{\partial x^2}>0, \quad \frac{\partial W}{\partial x}=0, \quad W=0;$$

furthermore $\tilde{V}$ is positive). So the process will leave the x axis in an upward direction, the x-axis is a reflecting barrier. □

3.5. *The conservation equation*

We will turn our attention to the conservation-equation (or density-equation):

$$\frac{\partial X}{\partial t} = -\frac{\partial}{\partial x}(XW)=-W\frac{\partial X}{\partial x}-X\frac{\partial W}{\partial x}.$$

This equation is in fact deterministic and, therefore, we will find a solution for each realization separately.

For W we take a variant of the $W^{(LN)}$ obtained in the previous section, viz.

$$W = \begin{cases} W^{(LN)} & \text{for } \omega\in Q_W^M \text{ (definition, see section 3.4)} \\ 0 & \text{otherwise.} \end{cases}$$

We will always deal with a continuous version of $W^{(LN)}$. So, if $\omega\in Q_W^M$, we have for W:

(i) $W(t, x)\geq 0$ for all t and x if $M<(N\wedge L)$.

(ii) $\sup_x W(t, x) \leqslant M + W_{in}$

(iii) $\sup_x |\frac{\partial W}{\partial x}(t, x)| \leqslant M, \forall t$

(iv) $\sup_x |\frac{\partial^2 W}{\partial x^2}(t, x)| \leqslant M, \forall t$

(v) $\frac{\partial^2 W}{\partial x^2}$ is Hölder continuous in t and x

(vi) from (iv) we obtain that $\frac{\partial W}{\partial x}$ is Lipschitz-continuous in x. (with coefficient $\leqslant M$).

Next, we will find the (local) solution for this equation via the method of 'characteristic traces'. Define $x = x(t)$ the following differential equation:

$$\frac{dx}{dt} = W(t, x) = W(t, x(t)), \quad x(0) = s. \tag{4}$$

This equation has a unique solution for $0 \leqslant t \leqslant \min(T, t_{exit})$, where t_{exit} is $\inf\{t;\, x(t) > 1\}$. (For details see [18] p.33)

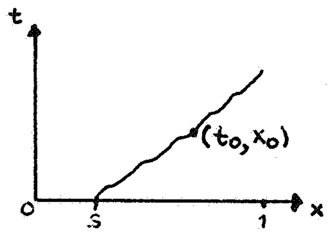

FIGURE 1. *An example of a possible characteristic trace.*

On x we have for the total derivative of X:

$$\frac{dX}{dt} = -\frac{\partial W}{\partial x} X. \tag{5}$$

Equation (5) has as its solution

$$X(t) = X(0, s)\exp(-\int_0^t \frac{\partial W}{\partial x}(t', x(t', s)dt')$$

where we have explicitly included the dependence of X and x on s (the 'starting point'). In terms of the original coordinates we have:

$$X(t_0, x_0) = X(0, s(t_0, x_0))\exp(-\int_0^{t_0} \frac{\partial W}{\partial x}(t, x(t))dt)$$

where

$$X(0, s) = X_0(s)$$

$$x(t) = x_0 + \int_{t_0}^{t} W(t, x(t))dt$$

$$s(t_0, x_0) = x(0).$$

So we have the following theorem:

THEOREM 3.2. *The density equation is equivalent to equations (4) and (5), for which the (unique) solution is implicitly given.*

PROOF. See above. □

At this point we will for technical reasons, change the formula for X somewhat. We put:

$$\begin{cases} X(t_0, x_0) = X(0, s(t_0, x_0))\exp(-\int_0^{t_0} \frac{\partial W}{\partial x}(t, x(t))dt) \\ X(0, s) = X_0(s) \\ X(t) = X_0 + \int_{t_0}^{t} (W(t', x(t')) \vee 0)dt' \\ s(t_0, x_0) = x(0) \end{cases} \tag{6}$$

By doing this we take into account that W can be negative, if $M > (N \wedge L)$.

PROPOSITION 3.4. *Let two velocities, W and $\overline{W}$ be given, then we have for the corresponding densities X and $\overline{X}$:*

$$|X(t_0, x_0) - \overline{X}(t_0, x_0)| \leq C_1 \int_0^{t_0} |W(t, \overline{x}) - \overline{W}(t, \overline{x})| dt + C_2 |\int_0^{t_0} \frac{\partial W}{\partial x}(t, \overline{x}) - \frac{\partial \overline{W}}{\partial x}(t, \overline{x}) dt|$$

where $\overline{x}$ is defined via

$$\begin{cases} \frac{d\overline{x}}{dt} = \overline{W}(t, \overline{x}(t)) \\ \overline{x}(t_0) = x_0. \end{cases}$$

PROOF. The calculation is rather straightforward;

$$|X(t_0, x_0) - \overline{X}(t_0, x_0)) \leq |X(0, s) - \overline{X}(0, \overline{s})| \exp(-\int_0^{t_0} \frac{\partial W}{\partial x}(t, x(t))dt)$$

$$+ |\overline{X}(0, \overline{s})| \, |\exp(-\int \frac{\partial W}{\partial x}(t, x)dt) - \exp(-\int \frac{\partial \overline{W}}{\partial x}(t, x)dt)|.$$

We will consider the terms separately.

For term 1 we have:

$$|X(0, s) - \overline{X}(0, \overline{s})| \exp(-\int_0^{t_0} \frac{\partial W}{\partial x}(t, x(t))dt) \leq C_1 |X(0, s) - \overline{X}(0, \overline{s})|$$

$$\leq C_2 |s - \overline{s}| \qquad \text{as } X(0, s) = \overline{X}(0, s)$$

$$\text{and } X_0 \text{ Lipschitz-continuous}$$

$$= C_2 |\int_{t_0}^{0} W(t, x(t)) - \overline{W}(t, \overline{x}(t))dt| \leq C_3 |\int_{t_0}^{0} W(t, \overline{x}(t)) - \overline{W}(t, \overline{x}(t))dt|$$

by Lemma 3.1. (see below)

For the second term we calculate:

$$|\overline{X}(0,\overline{s})|\,|\exp(-\int\frac{\partial W}{\partial x}(t,x)dt)-\exp(-\int\frac{\partial \overline{W}}{\partial x}(t,\overline{x})dt)|$$

$$\leqslant C_1\exp[\max(-\int\frac{\partial W}{\partial x}dt,\ -\int\frac{\partial \overline{W}}{\partial x}dt)]\,|\int_0^{t_0}\frac{\partial W}{\partial x}(t,x(t))-\frac{\partial \overline{W}}{\partial x}(t,\overline{x}(t))dt|$$

$$\leqslant C_2|\int_0^{t_0}\frac{\partial W}{\partial x}(t,x(t))-\frac{\partial W}{\partial x}(t,\overline{x}(t))dt|+C_2|\int_0^{t_0}\frac{\partial W}{\partial x}(t,\overline{x}(t))-\frac{\partial \overline{W}}{\partial x}(t,\overline{x}(t))dt|$$

$$\leqslant C_3\int_0^{t_0}|x(t)-\overline{x}(t)|dt+C_2|\int_0^{t_0}\frac{\partial W}{\partial x}(t,\overline{x})-\frac{\partial \overline{W}}{\partial x}(t,\overline{x})dt|$$

$$=C_3\int_0^{t_0}|\int_t^{t_0}W(t',x(t'))-\overline{W}(t',\overline{x}(t'))dt'|dt+C_2|\int_0^{t_0}\frac{\partial W}{\partial x}(t,\overline{x})-\frac{\partial \overline{W}}{\partial x}(t,\overline{x})dt|$$

$$\leqslant C_4\int_0^{t_0}\int_t^{t_0}|W(t',x(t'))-\overline{W}(t',\overline{x}(t'))|dt'dt+C_2|\int_0^{t_0}\frac{\partial W}{\partial x}(t,\overline{x})-\frac{\partial \overline{W}}{\partial x}(t,\overline{x})dt|$$

(using again the lemma). Combining these two estimates we get the desired result. □

Along the way we used the following lemma:

LEMMA 3.1. *For*

$$v(u)\equiv|\int_{t_0-u}^{t_0}W(t,x)-\overline{W}(t,\overline{x})dt|,\quad 0\leqslant u\leqslant t_0$$

we have

$$v(u)\leqslant a(u)+\int_0^u a(t)\exp\{C_1(u-t)\}dt$$

where

$$a(u)\equiv|\int_{t_0-u}^{t_0}W(t,\overline{x})-\overline{W}(t,\overline{x})dt|.$$

PROOF.

$$v(u)\leqslant\int_{t_0-u}^{t_0}|W(t,x)-W(t,\overline{x})|dt+|\int_{t_0-u}^{t_0}W(t,\overline{x})-\overline{W}(t,\overline{x})dt|\leqslant C_1\int_{t_0-u}^{t_0}|x(t)-\overline{x}(t)|dt+a(u)$$

$$=C_1\int_{t_0-u}^{t_0}|\int_0^{t_0}W(t',x(t'))-\overline{W}(t',\overline{x}(t'))dt'|dt+a(u)$$

$$=C_1\int_0^{u}|\int_{t_0-u}^{t_0}W(t',x(t'))-\overline{W}(t',\overline{x}(t'))dt'|dt+a(u)=a(u)+C_1\int_0^u v(t)dt.$$

Now a Gronwall-inequality gives the desired result. □

Next we need to find an upperbound for the L^2-norm of $X-\overline{X}$, because this will enable us to relate the density-equation to the velocity-equation.

PROPOSITION 3.5.

$$\|(X-\overline{X})_{t_0}\|^2 \leq C_1\int_0^{t_0}\|W_t-\overline{W}_t\|^2dt+C_2\int_0^{t_0}\|\frac{\partial W_t}{\partial x}-\frac{\partial \overline{W}_t}{\partial x}\|^2dt.$$

PROOF. From the previous proposition we get:

$$\|(X-\overline{X})_{t_0}\|^2 \leq C_1\int_0^1\int_0^{t_0}|W(t,\overline{x})-\overline{W}(t,\overline{x})|^2dt\,dx_0+C_2\int_0^1\int_0^{t_0}|\frac{\partial W}{\partial x}(t,\overline{x})-\frac{\partial \overline{W}}{\partial x}(t,\overline{x})|^2dt\,dx_0$$

$$= C_1\int_0^{t_0}\int_0^1|W(t,\overline{x})-\overline{W}(t,\overline{x})|^2dx_0dt+C_2\int_0^{t_0}\int_0^1|\frac{\partial W}{\partial x}(t,\overline{x})-\frac{\partial \overline{W}}{\partial x}(t,\overline{x})|^2dx_0dt$$

$$\leq C_1\int_0^{t_0}\int_0^1|W(t,\overline{x})-\overline{W}(t,\overline{x})|^2\frac{\partial x_0}{\partial \overline{x}(t)}d\overline{x}dt+C_2\int_0^{t_0}\int_0^1|\frac{\partial W}{\partial x}(t,\overline{x})-\frac{\partial \overline{W}}{\partial x}(t,\overline{x})|^2\frac{\partial x_0}{\partial \overline{x}(t)}d\overline{x}dt$$

where the inequality originates from choosing the integration-interval for $\overline{x}(t)$ (too) large.

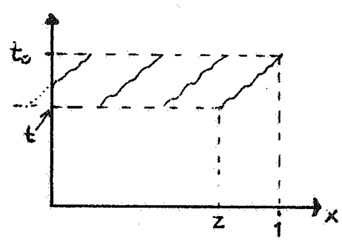

FIGURE 2. The integration-interval for $\overline{x}$ at time t runs from 0 to z ($z=x_0+\int_{t_0}^{t}\overline{W}(t',\overline{x}))\vee 0)dt$)

(Note that along $x=0$ boundary-conditions are given and that $(\overline{W}\vee 0)\geq 0\ \forall t,x$). Now we can deduce from

$$x_0 = \overline{x}(t)+\int_t^{t_0}(\overline{W}(t',\overline{x}(t'))\vee 0)dt'$$

that $\dfrac{\partial x_0}{\partial \overline{x}(t)}$ is positive and bounded. (This can be done by differentiating, using a Gronwall-inequality and the fact that $|\dfrac{\partial \overline{W}}{\partial x}|$ is bounded).

What we get then is

$$\|(X-\overline{X})_{t_0}\|^2 \leq C_3\int_0^{t_0}\|W_t-\overline{W}_t\|^2dt+C_4\int_0^{t_0}\|\frac{\partial W_t}{\partial x}-\frac{\partial \overline{W}_t}{\partial x}\|^2dt \quad \square$$

COROLLARY 3.1.

$$\|(X-\overline{X})_{t_0}\|^{2p} \leq C_5\int_0^{t_0}\|W_t-\overline{W}_t\|^{2p}+\|\frac{\partial W_t}{\partial x}-\frac{\partial \overline{W}_t}{\partial x}\|^{2p}dt \quad \square$$

3.5. Combining the two equations
In this section we will combine the velocity and the conservation equation and obtain a joint local solution.

THEOREM 3.3. *The system of equations (1) and (3) has a unique local solution* $(W^{(LN)}, X^{(LN)})$ *which belongs to the space* $(R^P_\alpha + W_{\text{in}}, R^P)$ *for each pair L and N. Furthermore,* $X^{(LN)}(t, x)$ *is differentiable w.r.t. t and x and* $\frac{\partial^2 W^{(LN)}}{\partial x^2}(t, x)$ *is Hölder-continuous w.r.t. t and x. The constants L and N are the truncating constants in* $\sigma^{(L)}$ *and* $G^{(N)}$.

PROOF. We will define, for fixed L and N a sequence of elements of R^P, which will be denoted by $\{X_n^{(LN)}, n \geq 1\}$. Of course $X_1^{(LN)}(t) = X_0$ for all t (i.e. the first element of the sequence is equal to the initial value X_0 for all t).

If we insert this $X_1^{(LN)}$ into the velocity-equation we obtain a solution $W_1^{(LN)}$ (the subscript referring to $X_1^{(LN)}$).

We take a continuous version of this $W_1^{(LN)}$ and restrict this version to the set

$$Q_1^M = \{\omega: \sup_{x,t} |\frac{\partial^2 W_1^{(LN)}}{\partial x^2}| \leq M\}.$$

Define $\overline{W}_1^{(LN)}$ to be equal to $W_1^{(LN)}$ on Q_1^M and equal to zero on $\Omega \setminus Q_1^M$.

Now, substituting this $\overline{W}_1^{(LN)}$ into the (modified!) conservation equation gives us a solution that will be denoted by $X_2^{(LN)}$, the second element of the sequence, that is equal to X_0 (all t) on $\Omega \setminus Q_1^M$.

$X_3^{(LM)}$ is obtained by going through this cycle again with the change that

$$Q_2^M = Q_1^M \cap \{\omega: \sup_{x,t} |\frac{\partial^2 W_2^{(LN)}}{\partial x^2}| \leq M\}.$$

The whole sequence is obtained by iterating the cycle. Every time we have

$$Q_{n+1}^M = Q_n^M \cap \{\omega: \sup_{x,t} |\frac{\partial^2 W_{n+1}^{(LN)}}{\partial x^2}| \leq M\}.$$

Note that, for each n,

$$\sup_{x,s,\omega} |X_n(s, x)| \leq C(L,N)$$

and

$$\sup_s E\|\frac{d}{dx} X_n(s)\|^{2p} < \infty.$$

By definition Q_n^M is a decreasing sequence Ω and has a limit which will be denoted by Q_∞^M.

All the sets Q_n^M and also Q_∞^M are measurable. We want to show that $\{X_n^{(LN)})$ is a Cauchy-sequence. To do this we first 'restrict' each $X_n^{(LN)}$ to the set Q_∞^M, and we calculate:

$$E(\|X_{n+1}^{(LN)}(t) - X_n^{(LN)}(t)\|^{2p} 1_{\{\omega \in Q_\infty^M\}}) \leq C_1 \int_0^t E(\|\frac{\partial W_n^{(LN)}}{\partial x}(s) - \frac{\partial W_{n+1}^{(LN)}}{\partial x}(s)\|^{2p} 1_{\{\omega \in Q_\infty^M\}}) ds$$

$$\leq C_2 \int_0^t \int_0^s (\sum_i e^{-\lambda_i(s-u)} E(\|X_n^{(LN)}(u) - X_{n-1}^{(LN)}(u)\|^{2p} 1_{\{\omega \in Q_\infty^M\}}) du\, ds$$

$$\equiv C_2 \int_0^t \int_0^s q(s-u) E(\|X_n^{(LN)}(u) - X_{n-1}^{(LN)}(u)\|^{2p} 1_{\{\omega \in Q_\infty^M\}}) du\, ds.$$

Next proceeding by iteration we obtain:

$$E(\|X_{n+1}^{(LN)}-X_n^{(LN)}(t)\|^{2p}1_{Q_\infty^M}) \quad \text{for} \quad n>m$$

$$\leqslant C_2\int_0^t\int_0^s q(s-u)C_2\int_0^u\int_0^v q(v-r)......dr\,dv\,du\,ds$$

$$\leqslant \sup_t E(\|X_{m+1}^{(LN)}-X_m^{(LN)}(t)\|^{2p}1_{Q_\infty^M})C_3^{n-m}\frac{1}{(n-m)!}t^{n-m}$$

$$\leqslant C_3^{n-m}\frac{1}{(n-m)!}t^{n-m}\sup_t E(\|X_2^{(LN)}-X_1^{(LN)}(t)\|^{2p}C_3^{m-1}\frac{1}{(m-1)!}T^{m-1}.$$

Now we will compare $X_l^{(LN)}$ and $X_m^{(LN)}$ for $l>m$ and m sufficiently large:

$$E(\|X_l^{(LN)}(t)-X_m^{(LN)}(t)\|^{2p} \leqslant E(\|X_l^{(LN)}(t)-X_m^{(LN)}(t)\|^{2p}1_{Q_\infty^M})+C_4P\{Q_\infty^M\setminus Q_m^M\}$$

$$\leqslant \sum_{n=m}^{\infty}C_3^{n-m}\frac{1}{(n-m)!}t^{n-m}C_5C_3^{m-1}\frac{1}{(m-1)!}T^{m-1}+C_4P\{Q_\infty^M\setminus Q_m^M\}$$

$$\leqslant C_6C_3^{m-1}T^{m-1}\frac{1}{(m-1)!}+C_4P\{Q_\infty^M\setminus Q_m^M\}$$

$\rightarrow 0$ as $m\rightarrow\infty$.

Now this convergence is uniform in t, i.e.

$$\sup_{0\leqslant t\leqslant T} E(\|X_l^{(LN)}(t)-X_m^{(LN)}(t)\|^{2p} \rightarrow 0$$

for $l>m$ and $m\rightarrow\infty$. So we can conclude that $\{X_n^{(LN)}$ is a Cauchy-sequence in R^P. We will denote the limit-point by $X^{(LN)}$.

At this point several question arise, e.g.:

- is this limit-point a 'partial' solution, i.e. a solution on Q_∞^M?
- is the limit-point 'non-trivial', i.e. is $X^{(LN)}$ not equal to $X_1^{(LN)}$? or, in other words, does Q_∞^M have positive measure?

The next two lemma's are devoted to these questions.

LEMMA 3.2. *Q_∞^M has positive measure.*

PROOF. We will show that

$$P(\sup_n\sup_{x,t}|\frac{\partial^2W_n^{(LN)}}{\partial x^2}(t,x)|>M)<\frac{C}{M^2}$$

where the constant C is only depending on N and L. For then

$$Q_\infty^M = \bigcap_n\{\sup_{x,t}|\frac{\partial^2W_n^{(LN)}}{\partial x^2}(t,x)|\leqslant M)\}=\Omega\setminus\{\sup_n\sup_{x,t}|\frac{\partial^2W_n^{(LN)}}{\partial x^2}|>M\}$$

has positive measure. Furthermore we get $Q_\infty^M\rightarrow\Omega$ as $M\rightarrow\infty$.

For $W_n^{(LN)}$ we can write (cf. Prop. 3.2)

$$\frac{\partial^2W_n^{(LN)}}{\partial x^2} = \frac{\partial^2}{\partial x^2}U_t(W_0-W_{\text{in}})+\int_0^t U_{t-s}\frac{d^2}{dx^2}\sigma^{(L)}(W_n^{(LN)})dB_s$$

$$+\frac{\lambda}{K}(V(X_n^{(LN)})-W_n^{(LN)})-\frac{\lambda}{K}U_t(V(X_0)-W_0)$$

$$+\frac{1}{K}\frac{d}{dx}G^{(N)}(W_n^{(LN)})-\frac{1}{K}U_t\frac{d}{dx}G^{(N)}(W_0).$$

We have the following upper bound

$$E(\sup_{n,t,x}|\frac{\partial^2 W_n^{(LN)}}{\partial x^2}(t,x)|^2) \leqslant C_1+C_2\{E(\sup_{n,t,x}|\int_0^t \frac{d^2}{dx^2}U_{t-s}\sigma^{(L)}(W_n^{(LN)})dB_s|^2)$$

$$+C_3E(\sup_{n,t,x}|W_n^{(LN)}|^2)+C_4E(\sup_{n,t,x}|\frac{d}{dx}G^{(N)}(W_n^{(LN)})|^2).$$

From

$$W_n^{(LN)} = U_t(W_0-W_{\text{in}})+W_{\text{in}}+\int_0^t U_{t-s}\sigma^{(L)}(W_n^{(LN)})dB_s$$

$$+\lambda\int_0^t U_{t-s}V(X_n^{(LN)})-W_n^{(LN)})ds+\int_0^t U_{t-s}\frac{d}{dx}G_{(N)}(W_n^{(LN)})ds$$

we deduce

$$\sup_{n,x,t}|W_n^{(LN)}(t,x)|^2 \leqslant C_1+C_2\sup_{n,x,t}|\int_0^t U_{t-s}\sigma^{(L)}(W_n^{(LN)})dB_s|^2$$

$$+C_3\int_0^T \sup_{n,x,t\leqslant s} W_n^{(LN)}(t,x)^2ds+C_4\int_0^T\sup_{n,x}(\frac{d}{dx}G^{(N)}(W_n^{(LN)}(s,x)))^2ds.$$

From this, using Gronwalls lemma, we obtain

$$\sup_{n,x,t}W_n^{(LN)}(t,x)^2 \leqslant C_5+C_6\sup_{n,x,t}|\int_0^t U_{t-s}\sigma^{(L)}(W_n^{(LN)})dB_s|^2+C_7\int_0^T\sup_{n,x}(\frac{d}{dx}G^{(N)}(W_n^{(LN)}(s,x)))^2ds.$$

This intermediate result and the boundedness of $\frac{d}{dx}G^{(N)}(W_n^{(LN)})$ now give

$$E(\sup_{n,t,x}|\frac{\partial^2 W_n^{(LN)}}{\partial x^2}(t,x)|^2) \leqslant C_1+C_2\{E(\sup_{n,t,x}\int_0^t\frac{d^2}{dx^2}U_{t-s}\sigma^{(L)}(W_n^{(LN)})dB_s|^2).$$

What remains is to estimate

$$E(\sup_{n,t,x}|\int_0^t\frac{d^2}{dx^2}U_{t-s}\sigma^{(L)}(W_n^{(LN)})dB_s|^2).$$

Define for this purpose

$$\tilde{W}^{(LN)}(s,x,\omega) \equiv \sum_n W_n^{(LN)}(s,x,\omega)1_{\{\sup_{x,t}|\int_0^t\frac{d^2}{dx^2}U_{t-s}\sigma^{(L)}(W_n^{(LN)})dB_s|\geqslant\sup_{m,x,t}|\int_0^t\frac{d^2}{dx^2}U_{t-s}\sigma^{(L)}(W_m^{(LN)})dB_s|\}};$$

it is easy to verify that $\tilde{W}^{(LN)}$ is measurable. Now we can estimate

$$E(\sup_{n,t,x}|\int_0^t\frac{d^2}{dx^2}U_{t-s}\sigma^{(L)}(W_n^{(LN)})dB_s|^2) \leqslant E(\sup_{t,x}|\int_0^t\frac{d^2}{dx^2}\sigma^{(L)}(\tilde{W}^{(LN)})dB_s|^2)$$

$$\leqslant C \qquad (C \text{ depending on } L).$$

The last inequality follows from the fact that

$$E(SI(t_1,x_1)-SI(t_2,x_2))^{2p} \leqslant C_1|t_1-t_2|^{\frac{1}{2}p}+C_2|x_1-x_2|^p$$

where SI means 'stochastic integral'. (c. Prop. 3.2). Applying Kolmogorovs continuous-version theorem (see [17] p. 273) yields the estimate (note that $E\|\sigma^{(L)}(\tilde{W}^{(LN)}\|_T^{2p}<\infty$). So, finally, we can conclude that

$$P(\sup_{n,t,x}|\frac{\partial^2 W_n^{(LN)}}{\partial x^2}(t,x)|>M) \leqslant \frac{1}{M^2}E(\sup_{n,t,x}|\frac{\partial^2 W_n^{(LN)}}{\partial x^2}(t,x)|^2)$$
$$\leqslant \frac{C}{M^2} \qquad (C \text{ depending on } L \text{ and } N). \ \square$$

LEMMA 3.3. *The limit-point $X^{(LN)}$ is differentiable w.r.t. x; both $X^{(LN)}$ and $\frac{\partial X^{(LN)}}{\partial x}$ are bounded on Q_∞^M and the corresponding $W^{(LN)}$ belongs to $R_\alpha^P + W_{\text{in}}$.*

PROOF. As $X_n^{(LN)} \to X^{(LN)}$ in R^P, simultaneously $W_n^{(LN)} \to W^{(LN)}$ in $R_{1/2}^P$. Solving the density-equation with $W = W^{(LN)}$ gives us a density, say $\tilde{X}^{(LN)}$. We calculate

$$\|X_{n+1}^{(LN)} - \tilde{X}^{(LN)})_t\|^2 \leqslant C_1 \int_0^t (W_n^{(LN)} - W^{(LN)})_s\|^2 ds + C_s \int_0^t \|\frac{\partial W_n^{(LN)}}{\partial x} - \frac{\partial W^{(LN)}}{\partial x}\|^2 ds$$

(see Prop. 3.5, note that $\frac{\partial W^{(LN)}}{\partial x}$ satisfies the Lipschitz-condition, even if $\frac{\partial W_n^{(LN)}}{\partial x}$ does not). So we conclude that $X_n^{(LN)} \to \tilde{X}^{(LN)}$ in R^P or, $\tilde{X}^{(LN)} = X^{(LN)}$ in R^P and as $\tilde{X}^{(LN)}$ is differentiable, we may choose $X^{(LN)}$ to be differentiable as well. Correspondingly we can choose $W^{(LN)}$ to be in $R_\alpha^P + W_{\text{in}}$. □

We will continue the proof of Theorem 3.3. Lemma 3.3 says that on Q_∞^M $X^{(LN)}$ (and $W^{(LN)}$) is a solution for the system of equations (1) and (3) (see section 3.2), provided we replace σ by $\sigma^{(L)}$, G by $G^{(N)}$ and use the modification of the density-equation that eliminates the effect of possibly negative velocities. As Lemma 3.3 is valid for all M, and $Q_\infty^M \to \Omega$ (as $M \to \infty$), we obtain a solution $(\tilde{W}^{(LN)}, X^{(LN)})$ on Ω by letting M go to infinity. Furthermore, as $M \to \infty$, the boundedness of $\tilde{V}$ guarantees that $W^{(LN)}$ stays in $(R_1^P + W_{\text{in}})$ (cf. Prop. 3.2 and Theorem 3.4 in section 3.6) and consequently $X^{(LN)}$ stays in R^P. □

Now, even if one of the truncating-constants (i.e. M) has been removed, L and N cannot be removed, at least not very easily. Because of the presence of these parameters we again (cf. section 3.4) call the solution we obtained local. If we define $\tau_N \equiv \inf\{t, \sup_x|\frac{\partial^2 W^{(LN)}}{\partial x^2}| \geqslant N\}$ (and for the moment $L = N$) then $(X^{(LN)}, W^{(LN)})$ is the solution we are looking for, for $0 \leqslant t \leqslant \tau_N$.

3.6. A global solution

As the last step in the construction of the model we want to obtain a global solution by 'removing' the parameters L and N.

For this purpose we make specific choices for $\tilde{\sigma}$ and $\tilde{G}$:

1) $\tilde{\sigma}(x) = a \cdot x$ ('a' is a positive const.)

2) $\tilde{G}'(\cdot)$ is bounded.

These choise are rather restrictive; $\tilde{\sigma}$ is more or less wholly determined in this way and $\tilde{G}(x)$ cannot be equal to $\mu x - ½x^2$ as we would prefer, but we have to assume instead that e.g. $\tilde{G}'(x) = (-H) \vee (\mu - x) \wedge H$ (H some large positive constant).

Now we can formulate a global result.

THEOREM 3.4. *The local solution obtained in the theorem of section 3.6 is a global one (i.e. for all t such that $0 \leqslant t \leqslant T$ it satisfies the equations (1) and (3)) if we choose $\tilde{\sigma}$ and $\tilde{G}$ as above.*

PROOF. The first step will be to 'remove' N. For this purpose define: $\tau_N \equiv \inf\{t: |\frac{\partial^2 W^{(LN)}}{\partial x^2}| \leqslant N\}$ and write for $\frac{\partial^2 W^{(LN)}}{\partial x^2}$ (cf. Prop. 3.2)

$$\frac{\partial^2 W^{(LN)}}{\partial x^2} = \frac{\partial^2}{\partial x^2} U_t(W_0 - W_{\text{in}}) + \int_0^t U_{t-s} \frac{d^2}{dx^2} \sigma^{(L)}(W^{(LN)}) dB_s$$
$$+ \frac{\lambda}{K}(V(X^{(LN)}) - W^{(LN)}) - \frac{\lambda}{K} U_t(V(X_0) - W_0)$$
$$+ \frac{1}{K}\frac{d}{dx} G^{(N)}(W^{(LN)}) - \frac{1}{K} U_t \frac{d}{dx} G^{(N)}(W_0).$$

Using this expression we calculate

$$E(\sup_{x,t \leqslant \tau_N \wedge T} |\frac{\partial^2 W^{(LN)}}{\partial x^2}|^{2p}) \leqslant C_1 + C_2 E(\sup_{x,t \leqslant \tau_N} |\int_0^t U_{t-s} \frac{d^2}{dx^2} \sigma^{(L)}(W^{(LN)}) dB_s|^{2p})$$
$$+ C_3(\sup_{x,t \leqslant \tau_N} |V(X^{(LN)}) - W^{(LN)}|^{2p}) + C_4(\sup_{x,t \leqslant \tau_N} |\frac{d}{dx} G^{(N)}(W^{(LN)})|^{2p}).$$

We will use the following:

$$E(\sup_{x,t \leqslant \tau_N} |W^{(LN)}|^{2p} \leqslant C_1 + C_2 E(\sup_{x,t \leqslant \tau_N} |\int_0^t U_{t-s} \sigma^{(L)}(W^{(LN)}) dB_s|^{2p})$$
$$C_3 E(\int_0^T \sup_{x,t \leqslant \tau_N \wedge s} |\frac{d}{dx} G^{(N)}(W^{(LN)})|^{2p} ds)$$

(cf. the proof of Lemma 3.2) and:

$$E(\sup_{x,t \leqslant \tau_N} |\frac{d}{dx}(W^{(LN)})|^{2p}) \leqslant C_1 + C_2 E(\sup_{x,t \leqslant \tau_N} |\int_0^t \frac{d}{dx} U_{t-s} \sigma^{(L)}(W^{(LN)}) dB_s)^{2p})$$
$$+ C_3 E(\sup_{x,t \leqslant \tau_N} |\int_1^x W^{(LN)} dy|^{2p}) + C_4 E(\sup_{x,t \leqslant \tau_N} |G^{(N)}(W^{(LN)})|^{2p})$$

so that

$$E(\sup_{x,t \leqslant \tau_N \wedge T} |\frac{d}{dx} G^{(N)}(W^{(LN)})|^{2p}) \leqslant C_1 E(\sup_{x,t \leqslant \tau_N \wedge T} |\frac{d}{dx}(W^{(LN)})|^{2p})$$
$$\leqslant C_2 + C_3 E(\sup_{x,t \leqslant \tau_N} |\int_0^t \frac{d}{dx} U_{t-s} \sigma^{(L)}(W^{(LN)}) dB_s||^{2p})$$
$$+ C_4 E(\sup_{x,t \leqslant \tau_N} |\int_0^t U_{t-s} \sigma^{(L)}(W^{(LN)}) dB_s||^{2p}) + C_5 E \int_0^T \sup_{x,t \leqslant \tau_N \wedge s} |\frac{d}{dx} W^{(LN)}|^{2p} ds$$

which implies that

$$E(\sup_{x,t \leqslant \tau_N \wedge T} |\frac{\partial}{\partial x} W^{(LN)}|^{2p}) \leqslant C_1 + C_2 E(\sup_{x,t \leqslant \tau_N \wedge T} |\int_0^t U_{t-s} \frac{d}{dx} \sigma^{(L)}(W^{(LN)}) dB_s|^{2p})$$

and the same (with other constants) is true for

$$E(\sup_{x,t \leqslant \tau_N \wedge T} |W^{(LN)}|^{2p}).$$

The last two estimates lead to the (intermediate) conclusion that

$$E(\sup_{x,t\leqslant\tau_N\wedge T}|\frac{\partial^2 W^{(LN)}}{\partial x^2}|^{2p}) \leqslant C_1+C_2E(\sup_{x,t\leqslant\tau_N\wedge T}|\int_0^t \frac{d^2}{dx^2}U_{t-s}\sigma^{(L)}(W^{(LN)})dB_s|^{2p}) \leqslant C_1+C$$

C depending on L (cf. Lemma 3.2). Then our final conclusion can be

$$P(\tau_N<T) = P(\sup_{x,t\leqslant\tau_N\wedge T}|\frac{\partial^2 W^{(LN)}}{\partial x^2}|\geqslant N) \leqslant \frac{1}{N^2}E(\sup_{x,t\leqslant\tau_N\wedge T}|\frac{\partial^2 W^{(LN)}}{\partial x^2}|) \leqslant \frac{(C_1+C)}{N^2}$$

or, in words, for N going to infinity (and L constant) τ_n will a.s. become larger than T, which means that

$$W_t^{(L)} \equiv W_t^{(LN)} \quad \text{for} \quad 0\leqslant t\leqslant\tau_N$$

is well-defined (cf. [19]).

Remark. In the calculations above $c_1, c_2, \cdots$ are different constants only within one line.

As a first step we have removed N (for fixed L). Let us now in the second step remove L. To do this we first notice that

$$E\|\frac{\partial^2 W^{(L)}}{\partial x^2}\| \leqslant J \quad J \text{ independent of } L$$

which is proven along the same lines as the previous result (note only that

$$\|\sigma^{(L)}(W^{(L)})\|_1 \leqslant a\|W^{(L)}\|_1$$

where the constant 'a' comes from: $\tilde{\sigma}(x)=a\cdot x$). Then we have

$$E(\sup_{x,t\leqslant T}|\frac{\partial^2 W^{(LN)}}{\partial x^2}|^{2p}) \leqslant C_1+C_2E(\sup_{x,t}\int_0^t \frac{d^2}{dx^2}U_{t-s}\sigma^{(L)}(W^{(LN)})dB_s)^2$$

$$\leqslant C_3 \quad (C_3 \text{ independent of } L)$$

where we used again the Hölder-continuity of the stochastic integral which results from Kolmogorovs continuous version theorem (cf. Lemma 3.2); note that the moduli of continuity are independent of L as J (see above) is independent of L. Finally, we conclude that

$$P(\tau_L<T) = P(\sup_{x,t\leqslant\tau_L}|\frac{\partial^2 W^{(L)}}{\partial x^2}|\geqslant L) \leqslant P(\sup_{x,t}|\frac{\partial^2 W^{(L)}}{\partial x^2}|\geqslant L) \leqslant \frac{1}{L^2}E(\sup_{x,t}|\frac{\partial^2 W^{(L)}}{\partial x^2}|^2)$$

$$\to 0 \quad \text{as} \quad L\to\infty$$

where, of course, we have defined

$$\tau_L \equiv \inf\{t, |\frac{\partial^2 W^{(L)}}{\partial x^2}|\geqslant L\}.$$

This means that the definition

$$W_t \equiv W_t^{(L)} \quad \text{for} \quad 0\leqslant t\leqslant\tau_L$$

is a proper one; so for the special choices of $\tilde{\sigma}$ and $\tilde{G}$ we have now obtained a global solution W (and of course also the corresponding X). □

3.7. Discussion

It turned out to be quite difficult to obtain a global solution. Rather restrictive choices for $\tilde{\sigma}$ and $\tilde{G}$ were necessity. At a certain point in the calculations no less than 3 truncation constants were being used.

One source of these difficulties is the fact that we were trying to mach two rather unrelated equations; one equation (the velocity-equation) essentially situated in L^2-type function-spaces; the other equation formulated in terms of characteristic curvers. Another source of difficulties was the necessity (partly for computational reasons) to solve the velocity-equation in a function space that contains very smooth functions. This caused problems in handling the 'parameters' $\tilde{\sigma}$ and $\tilde{G}$.

We hope to present a different approach in a subsequent paper. Section 3.1 contains some remarks concerning the possible direction of such an alternative approach.

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