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On the Equivalence of BS-Stability and B-Consistency

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Among several stability and consistency concepts for Runge-Kutta methods applied to stiff initial value problems, BS-stability and B-consistency turn out to be equivalent for initial value problems with a one-sided Lipschitz constant $m \geq 0$. In addition to this result, it is shown that the same holds for their internal counterparts.

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1. Introduction

In the last decade several stability concepts have been introduced for assessing convergence of Runge-Kutta methods applied to stiff nonlinear differential equations $y'=f(t,y)$ with f satisfying a one-sided Lipschitz condition with constant m. Among these, B-stability and the equivalent criterium of algebraic stability are well-known (see [2], [3], [6]); they guarantee stability w.r.t. perturbations of the initial value for $m \leq 0$. Moreover, they enable establishing B-convergence (i.e. global error bounds independent of the stiffness, see [7]) for $m<0$ ([5], [11]); for $m \geq 0$, however, B-stability is not sufficient to have such convergence property (as was shown by means of a counterexample in [12]). For a nonnegative m B-convergence can be proved if, in addition, BS - and BSI - stability are assumed, ensuring stability per step w.r.t. perturbations of the internal stages of the Runge-Kutta method ([8], [9], [10]).

In this paper it will be shown that for any $m \geq 0$ BS-stability is equivalent to B-consistency (i.e. local error bounds independent of the stiffness, see [7]), revealing the necessity of BS-stability for stiffness independent error bounds; for BSI-stability a similar equivalence will be derived.

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2. PRELIMINARIES

Consider the numerical solution of a stiff initial value problem

$$y' = f(t,y),\ f: [0,T]\times\mathbb{R}^n \to \mathbb{R}^n,\ T>0, \tag{2.1a}$$

$$y(0) = y_0,\ y_0 \in \mathbb{R}^n \tag{2.1b}$$

by an s-stage Runge-Kutta method

$$Y_i^k = \eta_{k-1} + h\sum_{j=1}^{s} a_{ij} f(t_{k-1}+c_j h, Y_j^k),\ 1\leq i\leq s, \tag{2.2a}$$

$$\eta_k = \eta_{k-1} + h\sum_{j=1}^{s} b_j f(t_{k-1}+c_j h, Y_j^k) \tag{2.2b}$$

with $\eta_0 = y_0$ and $\eta_k \approx y(t_k)$ for $k = 1,2,...$ and $t_k = kh \leq T$. It will be assumed, for convenience, that the nodes are such that $0\leq c_1 \leq \cdots \leq c_s \leq 1$. Using the abbreviations

$$Y^k = \begin{bmatrix} Y_1^k \\ \vdots \\ Y_s^k \end{bmatrix},\quad F(t,U) = \begin{bmatrix} f(t+c_1 h, U_1) \\ \vdots \\ f(t+c_s h, U_s) \end{bmatrix}$$

for $U = (U_i)$, $U_i \in \mathbb{R}^n$, and $e = (1,...,1)^T \in \mathbb{R}^s$, system (2.2) will be written from now on as

$$Y^k = \mathbf{e}\eta_{k-1} + h\mathbf{A}F(t_{k-1}, Y^k) \tag{2.3a}$$

$$\eta_k = \eta_{k-1} + h\mathbf{b}^T F(t_{k-1}, Y^k). \tag{2.3b}$$

Here, boldface letters indicate Kronecker products with I_n to make $A = (a_{ij})$, $b = (b_i)$ and e of appropriate dimension. Further, $<v,w> = v^T w$ denotes the Euclidean inner product on $\mathbb{R}^l, l\in\mathbb{N}$, with $\|\cdot\|$ as the corresponding norm.

The righthand side f of (2.1a) is assumed to be differentiable w.r.t. y and to satisfy a one-sided Lipschitz condition (with a one-sided Lipschitz constant m)

$$<f(t,y_1)-f(t,y_2), y_1-y_2> \leq m\|y_1-y_2\|^2 \tag{2.4}$$

for all $t\in[0,T]$ and $y_1,y_2 \in \mathbb{R}^n$. The set of all functions f in (2.1a) satisfying (2.4) for a given $m\in\mathbb{R}$ will be denoted by $\mathcal{F}_m$.

Local stability and local errors of a Runge-Kutta method can be studied by considering the difference between an unperturbed step

$$Y = \mathbf{e}\eta_0 + h\mathbf{A}F(t_0, Y) \tag{2.5a}$$

$$\eta_1 = \eta_0 + h\mathbf{b}^T F(t_0, Y) \tag{2.5b}$$

and a perturbed one,

$$\tilde{Y} = \mathbf{e}\eta_0 + h\mathbf{A}F(t_0, \tilde{Y}) + \Delta \tag{2.6a}$$

$$\tilde{\eta}_1 = \eta_0 + h\mathbf{b}^T F(t_0, \tilde{Y}) + \delta, \tag{2.6b}$$

with perturbations $\Delta \in \mathbb{R}^{ns}$ and $\delta \in \mathbb{R}^n$. Introducing $V = \tilde{Y} - Y$, $v = \tilde{\eta}_1 - \eta_1$ and $W = hF(t_0, \tilde{Y}) - hF(t_0, Y)$, the difference between (2.6) and (2.5) can be written as

$$V = \mathbf{A}W + \Delta \tag{2.7a}$$

$$v = \mathbf{b}^T W + \delta. \tag{2.7b}$$

Now, the Runge-Kutta method is called *BSI-stable* on $\mathcal{F}_m$ if there are constants $\alpha_0, D_0 > 0$ such that

$$\|V\| \leqslant D_0 \|\Delta\| \tag{2.8}$$

for every $h > 0$ with $hm \leqslant \alpha_0$, and for all $\Delta \in \mathbb{R}^{ns}$ and $f \in \mathcal{F}_m$. The Runge-Kutta method is called *BS-stable* on $\mathcal{F}_m$ if there are constants α_1, $D_1 > 0$ such that

$$\|v\| \leqslant D_1(\|\Delta\| + \|\delta\|) \tag{2.9}$$

for every $h > 0$ with $hm \leqslant \alpha_1$, and for all $\Delta \in \mathbb{R}^{ns}$, $\delta \in \mathbb{R}^n$ and $f \in \mathcal{F}_m$.

In order to study local errors let $y(t)$ be a solution of (2.1a) with $f \in \mathcal{F}_m$; further, let $M_j = \max\{\|y^{(j)}(t)\| \mid 0 \leqslant t \leqslant T\}$. Then the method is called *BI-consistent* on $\mathcal{F}_m$ if there are constants $\beta_0 > 0$, $q_0 > 0$ (depending only on the method) and $C_0 > 0$ (depending on some of the M_j's but independent of the stiffness of the problem) such that

$$\|y(t_0 + c_i h) - Y_i\| \leqslant C_0 h^{q_0}, \quad 1 \leqslant i \leqslant s, \tag{2.10}$$

for every $h > 0$ with $hm \leqslant \beta_0$ and for every solution y of (2.1a) with $f \in \mathcal{F}_m$. Finally, the method is called *B-consistent* on $\mathcal{F}_m$ if there are constants $\beta_1 > 0$, $q_1 > 0$ and $C_1 > 0$ such that

$$\|y(t_1) - \eta_1\| \leqslant C_1 h^{q_1} \tag{2.11}$$

for every $h > 0$ with $hm \leqslant \beta_1$ and for every solution y of (2.1a) with $f \in \mathcal{F}_m$; again, for β_1, q_1 and C_1 the same holds as said above.

Both $B(I)$-consistency and $BS(I)$-stability are crucial for assessing B-convergence results for

Runge-Kutta methods on $\mathcal{F}_m, m \geq 0$ (see e.g. [9]).

There has been no specification yet about the dimension n. For convenience, n will be considered as a fixed but arbitrary integer, which means that all bounds D_i, C_i are allowed to depend on n. Results uniformly in n, similar to those that will be derived for fixed n, can be obtained in the same way by considering all the spaces $\mathbb{R}^n$ as subspaces of the Hilbert space l_2. Furthermore, also T is considered as a fixed number; its value is irrelevant since $BS(I)$-stability and $B(I)$-consistency are all local properties related to one step of the Runge-Kutta scheme.

It is pointed out, finally, that throughout this paper it is tacitly assumed that the system of algebraic equations defining the internal vectors Y_i has a unique solution, although the question of solvability is somewhat related to the concept of BSI-stability.

3. Equivalence of $BS(I)$-stability and $B(I)$-consistency

The main theorem of this paper is stated here without proof; it will be given in the next section.

Theorem 3.1. *Suppose $e_1^T A \neq 0$, $c_i \neq c_j$ whenever $i \neq j$, and let $m \geq 0$. Then the Runge-Kutta method is BS-stable on $\mathcal{F}_m$ if and only if it is B-consistent on $\mathcal{F}_m$. Likewise, the method is BSI-stable on $\mathcal{F}_m$ if and only if it is BI-consistent on $\mathcal{F}_m$.*

The condition $e_1^T A \neq 0$ cannot be omitted as the trapezoidal rule would then pose a counterexample to the theorem: for any $m \in \mathbb{R}$ it is B- and BI-consistent on $\mathcal{F}_m$ but neither BS- nor BSI-stable on $\mathcal{F}_m$. Note further that the theorem is not valid for $m<0$ since the Lobatto IIIC scheme with three stages is known to be B- and BI-consistent on $\mathcal{F}_m$ for $m<0$ (see [5], [11]), but neither BSI-stable (see [4]) nor BS-stable on $\mathcal{F}_m$ for any $m \in \mathbb{R}$, as can easily be shown.

4. Proofs

In a first step algebraic criteria for BSI- and BS-stability will be given for nonconfluent methods; nonconfluency means that $c_i \neq c_j$ whenever $i \neq j$. The logarithmic norm of a matrix Z will be denoted by $\mu(Z)$ and $\mathcal{D}_m$ will stand for the collection of matrices $\mathbf{Z} = \text{blockdiag}(Z_1, \ldots, Z_s) \in \mathbb{R}^{ns \times ns}$ with $Z_i \in \mathbb{R}^{n \times n}$ and $\mu(Z_i) \leq m$. The identity matrix of dimension ns will be denoted by $\mathbf{I}$.

Lemma 4.1. *A nonconfluent Runge-Kutta method is BSI-stable on $\mathcal{F}_m$ if and only if there are constants $\alpha_0, D_0 > 0$ such that*

$$\|(\mathbf{I} - h\mathbf{A}\mathbf{Z})^{-1}\| \leq D_0$$

for every $h>0$ with $hm \leqslant \alpha_0$ and $\mathbf{Z} \in \mathfrak{D}_m$. The method is BS-stable on $\mathfrak{F}_m$ if and only if there are constants $\alpha_1, D_1 > 0$ such that

$$\|h\mathbf{b}^T\mathbf{Z}(\mathbf{I}-h\mathbf{A}\mathbf{Z})^{-1}\| \leqslant D_1$$

for every $h>0$ with $hm \leqslant \alpha_1$ and $\mathbf{Z} \in \mathfrak{D}_m$.

Proof. The difference (2.7) between an unperturbed and a perturbed step can be written as

$$\begin{aligned} V &= h\mathbf{A}\mathbf{Z}V + \Delta \\ v &= h\mathbf{b}^T\mathbf{Z}V + \delta \end{aligned} \tag{4.1}$$

where $\mathbf{Z}$ is the blockdiagonal matrix with blocks $Z_i = \int_0^1 f_y(t_0+c_ih, Y_i+\theta(\tilde{Y}_i - Y_i))d\theta$, and - due to (2.4) - $\mathbf{Z} \in \mathfrak{D}_m$. So (4.1) takes the form

$$\begin{aligned} V &= (\mathbf{I}-h\mathbf{A}\mathbf{Z})^{-1}\Delta \\ v &= h\mathbf{b}^T\mathbf{Z}(\mathbf{I}-h\mathbf{A}\mathbf{Z})^{-1}\Delta + \delta \end{aligned} \tag{4.2}$$

and both *BSI*- and *BS*-stability directly follow from the upper bounds given in the lemma.

The necessity of these bounds can easily be seen by considering out of $\mathfrak{F}_m$ the class of problems

$$y' = \Lambda(t)y, \quad \Lambda(t) \in \mathbb{R}^{n \times n}, \quad \mu(\Lambda(t)) \leqslant m, \quad t \in [0,T],$$

and taking $\delta = 0$. Then $Z_i = \Lambda(t_0+c_ih)$ can be chosen arbitrary and independent of each other as $c_i \neq c_j$ whenever $i \neq j$ was assumed. □

In order to relate $BS(I)$-stability and $B(I)$-consistency, particular perturbations will be considered. For any differentiable function $g:[0,T] \to \mathbb{R}^n$ define $\Delta(g) \in \mathbb{R}^{ns}$ and $\delta(g) \in \mathbb{R}^n$ by

$$\Delta_i(g) = g(t_0+c_ih) - g(t_0) - h\sum_{j=1}^{s} a_{ij}g'(t_0+c_jh), \quad 1 \leqslant i \leqslant s, \tag{4.3a}$$

$$\delta(g) = g(t_1) - g(t_0) - h\sum_{i=1}^{s} b_i g'(t_0+c_ih). \tag{4.3b}$$

Lemma 4.2. *Suppose the Runge-Kutta method is nonconfluent and $e_1^TA \neq 0$. Then for any $\Delta \in \mathbb{R}^{ns}$ and $h>0$ there is a differentiable function g such that $\Delta(g) = \Delta$.*

Proof. Consider arbitrary $u_0, u_i, w_i \in \mathbb{R}^n, 1 \leqslant i \leqslant s$, with the restriction that $u_0 = u_1$ in the case $c_1 = 0$. Since the Runge-Kutta method was assumed to be nonconfluent the function g can be chosen as a polynomial on $[0,T]$ with coefficients in $\mathbb{R}^n$ such that $g(t_0) = u_0$, $g(t_0+c_ih) = u_i$, and $g'(t_0+c_ih) = w_i$ for $1 \leq i \leq s$. With $u = (u_i)$ and $w = (w_i) \in \mathbb{R}^{ns}$ the equation $\Delta(g) = \Delta$ then holds if and only if $u - eu_0 - hAw = \Delta$ holds. For any given Δ and h, though, such vectors u_0, u and w exist because of the assumption that $e_1^TA \neq 0$ for the case $c_1 = 0$. □

With the help of the above two lemmas Theorem 3.1 can now be proved.

Consider first (2.6) with $\Delta = \Delta(y)$ and $\delta = \delta(y)$ where y denotes the solution of (2.1); then $\tilde{Y}_i = y(t_0+c_ih)$, $\tilde{\eta}_1 = y(t_1)$, and thus $BS(I)$-stability on $\mathcal{F}_m$ implies $B(I)$-consistency on $\mathcal{F}_m$ for arbitrary $m \in \mathbb{R}$ (see also [9]).

Assume now a nonconfluent Runge-Kutta method with $e_1^TA \neq 0$ to be BI-consistent on $\mathcal{F}_m$, $m \geq 0$. Choose out of $\mathcal{F}_m$ the class of problems

$$\begin{aligned} y' &= \Lambda(t)(y - g(t)) + g'(t) \\ y(0) &= g(0) \end{aligned} \tag{4.4}$$

with $\Lambda(t) \in \mathbb{R}^{n \times n}$ such that $\mu(\Lambda(t)) \leq m$ for all $t \in [0,T]$ and with arbitrary $g: [0,T] \to \mathbb{R}^n$ as the solution $y(t)$. The BI-consistency inequality (2.10) together with (4.2) implies

$$\|(\mathbf{I} - h\mathbf{A}\mathbf{Z})^{-1}\Delta(g)\| \leq \sqrt{s}\, C_0 h^{q_0}$$

for every $h > 0$ with $hm \leq \beta_0$, where $\mathbf{Z} = \text{blockdiag}(Z_1, \ldots, Z_s)$ and $Z_i = \Lambda(t_0 + c_ih)$; the constant C_0 depends on some bounds M_j for $\|g^{(j)}(t)\|, t \in [0,T]$, but is independent of the stiffness of the problem. From the assumption that $c_i \neq c_j$ whenever $i \neq j$ it follows that $\mathbf{Z}$ can be any matrix in $\mathcal{D}_m$ by choosing $\Lambda(t)$ appropriately. Lemma 4.2 now implies the existence of a positive function $\phi(h, \Delta)$ such that

$$\|(\mathbf{I} - h\mathbf{A}\mathbf{Z})^{-1}\Delta\| \leq \phi(h, \Delta)$$

for any $\Delta \in \mathbb{R}^{ns}, \mathbf{Z} \in \mathcal{D}_m$ and $0 < h \leq H = \min\{T, \beta_0/m\}$ ($H = T$ if $m = 0$). Note that $\mathbf{Z} \in \mathcal{D}_m$ if and only if $h\mathbf{Z} \in \mathcal{D}_{hm}$ and that $\mathcal{D}_{hm} \subset \mathcal{D}_{Hm}$ as $m \geq 0$; consequently,

$$\sup_{\mathbf{Z} \in \mathcal{D}_m} \|(\mathbf{I} - h\mathbf{A}\mathbf{Z})^{-1}\Delta\| \leq \sup_{\mathbf{Z} \in \mathcal{D}_m} \|(\mathbf{I} - H\mathbf{A}\mathbf{Z})^{-1}\Delta\|.$$

Taking $\psi(\Delta) = \phi(H, \Delta)$ gives

$$\|(\mathbf{I} - h\mathbf{A}\mathbf{Z})^{-1}\Delta\| \leq \psi(\Delta)$$

for any $\Delta \in \mathbb{R}^{ns}$, $\mathbf{Z} \in \mathcal{D}_m$ and $0 < h \leq H$. From the principle of uniform boundedness (see e.g. [1]) it now follows that $(\mathbf{I} - h\mathbf{A}\mathbf{Z})^{-1}$ is uniformly bounded for $\mathbf{Z} \in \mathcal{D}_m$ and $0 < h \leq H$. By the characterization of

Lemma 4.1 the method is thus *BSI*-stable on $\mathcal{F}_m, m \geqslant 0$.

Finally, assuming *B*-consistency on $\mathcal{F}_m, m \geqslant 0$, and choosing again the class of problems (4.4), the *B*-consistency inequality (2.11) together with (4.2) implies

$$\|h\mathbf{b}^T\mathbf{Z}(\mathbf{I}-h\mathbf{A}\mathbf{Z})^{-1}\Delta(g)\| \leqslant C_1 h^{q_1} + \|\delta(g)\|$$

for every $h > 0$ with $hm \leqslant \beta_1$ and $\mathbf{Z} \in \mathcal{D}_m$. Expanding $\delta(g)$ into a Taylor series yields

$$\|\delta(g)\| \leqslant CM_q h^q$$

for some $C > 0, q \in \mathbb{N}$. Again, by using the principle of uniform boundedness and the characterization of Lemma 4.1 *BS*-stability can be established.

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