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Pressure Correction Splitting Methods for the Computation of Oscillatory Free Convection in Low Pr Fluids

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In this report we consider splitting methods for the time-integration of the incompressible Navier-Stokes equations in the Boussinesq approximation. These methods are combined with the pressure correction method in order to decouple the pressure computation from the velocity computation. The resulting pressure correction splitting methods are used to compute the (oscillatory) free convection of low Pr fluids in a long rectangular cavity.

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1. Introduction.

In this paper we consider the oscillatory free convection of low Pr fluids in a long, rectangular cavity. For this problem we use the primitive variable formulation (velocity, pressure and temperature) and the governing equations are the Navier-Stokes equations in Boussinesq approximation [3], and the transport equation for the temperature.

For the time-integration of these equations one can use an explicit method, an implicit method or a splitting method. Explicit methods are very cheap (per time step), but stability of these methods is subject to severe time step restrictions. Implicit methods are usually unconditionally stable, but are expensive to apply since they require the solution of a large set of algebraic equations at each time step. The purpose of splitting methods is to break down such a large algebraic system in a series of simple (small) systems in order to reduce the computational complexity, and at the same time maintain good stability properties [6]. In this report we restrict ourselves to splitting methods. The splitting methods we consider are the alternating direction implicit (ADI) method and the odd-even hopscotch (OEH) method.

In order to decouple the computation of the pressure from the computation of the velocity and the temperature, we combine the splitting methods with the pressure correction approach [2,7]. This approach leads to a predictor-corrector type method that decouples the pressure computation from that of the velocity and temperature. This approach requires per time step the solution of a Poisson equation for the computation of the pressure.

In Section 2 a short description is given of the pressure correction method, in combination with a splitting method for the time-integration. Section 3 is devoted to the pressure computation and computational results are presented in Section 4. Finally, some conclusions are formulated in Section 5.

2. Description of the method.

The primitive variable formulation of the incompressible Navier-Stokes equations, in Boussinesq approximation, can be written as

$$\mathbf{u}_t = \mathbf{f}(\mathbf{u}) - \nabla p, \text{ with } \mathbf{f}(\mathbf{u}) = -V_i \nabla \cdot (\mathbf{u}\mathbf{u}) + V_d \nabla^2 \mathbf{u} - V_b \theta \mathbf{i} \tag{2.1}$$

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$$\nabla \cdot \mathbf{u} = 0, \tag{2.2}$$

where $\mathbf{u}$, p and θ are respectively the (nondimensional) velocity, pressure and temperature. The parameters V_i, V_d and V_b are defined as: $V_i = Gr^{1/2}$, $V_d = 1$ and $V_b = -Gr^{1/2}$, where Gr is the Grashof number. In what follows, we assume that θ is a given function (case $Pr = 0$). Extension to the case where θ is not a priori known, in which case the transport equation for θ has to be included, is straightforward. A more detailed description of the free convection problem is given in Section 4. The partial differential equations (PDEs) (2.1)-(2.2) are defined on a connected space domain Ω with boundary Γ, on which conditions for the velocity $\mathbf{u}$ are specified. Notice that the boundary values for $\mathbf{u}$ must satisfy

$$\oint_{\Gamma} \mathbf{u} \cdot \mathbf{n}\, ds = \iint_{\Omega} \nabla \cdot \mathbf{u}\, dS = 0, \tag{2.3}$$

where $\mathbf{n}$ is the unit outward normal on Γ.

Following the method of lines approach, we assume that by an appropriate space discretization technique the PDE problem (2.1)-(2.2) is replaced by the following system of differential/algebraic equations (DAEs) [7,9,10]

$$\dot{\mathbf{U}} = \mathbf{F}(\mathbf{U}) - GP \tag{2.4}$$

$$D\mathbf{U} = B. \tag{2.5}$$

In (2.4) the variables $\mathbf{U}$ and P are grid functions defined on a space grid covering Ω and $\mathbf{F}(\mathbf{U})$ is the discrete approximation of $\mathbf{f}(\mathbf{u})$. The operators G and D are the discrete approximations of the gradient- and divergence operator, respectively, and B is a term containing boundary values for the velocity $\mathbf{u}$. For space discretization, we use standard central differences on a staggered grid; see e.g. [7-10].

First, consider (2.4) and suppose for the time being that GP is a known forcing term. Let $\mathbf{F}(\mathbf{U})$ be split into two terms, i.e.

$$\mathbf{F}(\mathbf{U}) = \mathbf{F}_1(\mathbf{U}) + \mathbf{F}_2(\mathbf{U}). \tag{2.6}$$

The precise form of $\mathbf{F}_1(\mathbf{U})$ and $\mathbf{F}_2(\mathbf{U})$ will be specified later. A two-stage, second order accurate formula for the integration of (2.4), which is based upon the splitting (2.6), is the formula of Peaceman and Rachford [6]

$$\tilde{\mathbf{U}} = \mathbf{U}^n + \frac{1}{2}\tau\mathbf{F}_1(\mathbf{U}^n) + \frac{1}{2}\tau\mathbf{F}_2(\tilde{\mathbf{U}}) - \frac{1}{2}\tau GP^n, \tag{2.7a}$$

$$\mathbf{U}^{n+1} = \tilde{\mathbf{U}} + \frac{1}{2}\tau\mathbf{F}_1(\mathbf{U}^{n+1}) + \frac{1}{2}\tau\mathbf{F}_2(\tilde{\mathbf{U}}) - \frac{1}{2}\tau GP^{n+1}, \tag{2.7b}$$

where τ is the time step. Note that in (2.7a) GP is set at time level $t_n = n\tau$ and in (2.7b) at time level $t_{n+1} = (n+1)\tau$, in order to maintain second order accuracy.

Consider (2.7a)-(2.7b) coupled with the (time discretized) set of algebraic equations

$$D\mathbf{U}^{n+1} = B^{n+1}. \tag{2.7c}$$

The computation of $\mathbf{U}^{n+1}$ and P^{n+1} requires the simultaneous solution of (2.7b)-(2.7c). In order to avoid this, we follow the well-known pressure correction approach [2,7] in which the computation of P^{n+1} is decoupled in a predictor-corrector fashion. Substitution of P^n for P^{n+1} in (2.7b) defines the predicted velocity $\tilde{\tilde{\mathbf{U}}}$:

$$\tilde{\tilde{\mathbf{U}}} = \tilde{\mathbf{U}} + \frac{1}{2}\tau\mathbf{F}_1(\tilde{\tilde{\mathbf{U}}}) + \frac{1}{2}\tau\mathbf{F}_2(\tilde{\mathbf{U}}) - \frac{1}{2}\tau GP^n. \tag{2.8}$$

The corrected velocity and pressure (which we hereafter also denote by $\mathbf{U}^{n+1}$ and P^{n+1}) are then defined by replacing $\mathbf{F}_1(\mathbf{U}^{n+1})$ in (2.7b) by $\mathbf{F}_1(\tilde{\tilde{\mathbf{U}}})$:

$$\mathbf{U}^{n+1} = \tilde{\mathbf{U}} + \frac{1}{2}\tau\mathbf{F}_1(\tilde{\tilde{\mathbf{U}}}) + \frac{1}{2}\tau\mathbf{F}_2(\tilde{\mathbf{U}}) - \frac{1}{2}\tau GP^{n+1}, \tag{2.9}$$

together with the discrete continuity equation (2.7c). From (2.8) and (2.9) we trivially obtain

$$\mathbf{U}^{n+1}-\tilde{\tilde{\mathbf{U}}} = -\tfrac{1}{2}\tau G Q^n, \ Q^n := P^{n+1}-P^n. \tag{2.10}$$

The idea of the pressure correction approach is now to multiply (2.10) by D and to write, using (2.7c),

$$LQ^n = \frac{2}{\tau}(D\tilde{\tilde{\mathbf{U}}}-B^{n+1}), \ L := DG. \tag{2.11}$$

Since $L=DG$ is a discretization of the Laplae operator $\nabla\cdot(\nabla)$, the computation of the pressure-increment Q^n requires the solution of a (discrete) Poisson equation. Once Q^n is known, the new velocity $\mathbf{U}^{n+1}$ can be directly obtained from (2.10).

To summarize, we get the following pressure correction scheme based upon the splitting (2.6):

$$\tilde{\mathbf{U}}=\mathbf{U}^n+\tfrac{1}{2}\tau\mathbf{F}_1(\mathbf{U}^n)+\tfrac{1}{2}\tau\mathbf{F}_2(\tilde{\mathbf{U}})-\tfrac{1}{2}\tau GP^n \tag{2.12a}$$

$$\tilde{\tilde{\mathbf{U}}}=\tilde{\mathbf{U}}+\tfrac{1}{2}\tau\mathbf{F}_1(\tilde{\tilde{\mathbf{U}}})+\tfrac{1}{2}\tau\mathbf{F}_2(\tilde{\mathbf{U}})-\tfrac{1}{2}\tau GP^n \tag{2.12b}$$

$$LQ^n=\frac{2}{\tau}(D\tilde{\tilde{\mathbf{U}}}-B^{n+1}), \ P^{n+1}=P^n+Q^n \tag{2.12c}$$

$$\mathbf{U}^{n+1}=\tilde{\tilde{\mathbf{U}}}-\tfrac{1}{2}\tau GQ^n. \tag{2.12d}$$

A pressure correction scheme based upon an ADI spitting is presented in [7]. Another scheme, which is based upon the OEH splitting [9,10], is discussed in some detail now.

Consider the chequer-board ordening of the grid. Let the grid be divided into two subsets, viz. the odd cells (corresponding to the white cells) and the even cells (corresponding to the black cells). In the OEH method, $\mathbf{F}_1(\mathbf{U}) := \mathbf{F}_O(\mathbf{U})$, i.e. the restriction of $\mathbf{F}(\mathbf{U})$ to the odd cells (likewise $\mathbf{F}_2(\mathbf{U}) := \mathbf{F}_E(\mathbf{U})$, the restriction of $\mathbf{F}(\mathbf{U})$ to the even cells). Using this definition of $\mathbf{F}_1(\mathbf{U})$ and $\mathbf{F}_2(\mathbf{U})$, one can easily obtain the following scheme (Cf. 2.12))

$$\tilde{\mathbf{U}}_O = \mathbf{U}_O^n + \tfrac{1}{2}\tau\mathbf{F}_O(\mathbf{U}^n)-\tfrac{1}{2}\tau(GP^n)_O \tag{2.13a}$$

$$\tilde{\mathbf{U}}_E = \mathbf{U}_E^n + \tfrac{1}{2}\tau\mathbf{F}_E(\tilde{\mathbf{U}})-\tfrac{1}{2}\tau(GP^n)_E \tag{2.13b}$$

$$\tilde{\tilde{\mathbf{U}}}_E = \tilde{\mathbf{U}}_E + \tfrac{1}{2}\tau\mathbf{F}_E(\tilde{\mathbf{U}})-\tfrac{1}{2}\tau(GP^n)_E = 2\tilde{\mathbf{U}}_E-\mathbf{U}_E^n \tag{2.13c}$$

$$\tilde{\tilde{\mathbf{U}}}_O = \tilde{\mathbf{U}}_O + \tfrac{1}{2}\tau\mathbf{F}_O(\tilde{\tilde{\mathbf{U}}})-\tfrac{1}{2}\tau(GP^n)_O \tag{2.13d}$$

$$LQ^n = \frac{2}{\tau}(D\tilde{\tilde{\mathbf{U}}}-B^{n+1}), \ P^{n+1} = P^n+Q^n \tag{2.13e}$$

$$\mathbf{U}^{n+1} = \tilde{\tilde{\mathbf{U}}}-\tfrac{1}{2}\tau GQ^n. \tag{2.13f}$$

The above scheme is referred to as the odd-even hopscotch pressure correction (OEH-PC) scheme. The essential feature of the scheme is the alternating use of the explicit and implicit Euler rule. One can easily see that, in combination with a central difference space discretization technique, the OEH-PC scheme is only diagonally implicit [9]. Hence the scheme is very fast per time step. An additional advantage of the scheme is that explicit evaluations can be saved using the so-called fast form (Cf. (2.13c)).

We conclude this section with two remarks concerning the OEH method. Consider to this purpose the linear convection-diffusion equation

$$f_t + (\mathbf{q}\cdot\nabla)f = \epsilon\nabla^2 f, \ \mathbf{x}\in\mathbb{R}^d, \ t>0 \tag{2.14}$$

where $\mathbf{q} := (q_1, \ldots, q_d)^T$ is the (constant) convective velocity and $\epsilon>0$ the viscosity parameter. Suppose that for space discretization we use standard central differences, with gridsize h in all space directions. First, von Neumann stability analysis applied to the OEH scheme for (2.14) gives the following necessary and sufficient time step restriction [9]

$$d(\frac{\tau}{h})^2 \sum_{k=1}^{d} q_k^2 \leq 4. \tag{2.15}$$

Thus, the OEH scheme for (2.14) is conditionally stable uniformly in ϵ, i.e. $\tau = O(h)$ independent of ϵ. The second remark concerns the so-called Du Fort-Frankel (DFF) deficiency [9]. By this we mean that for $\tau, h \to 0$ the solution of the OEH scheme for (2.14) converges to the solution of the problem

$$f_t + (\mathbf{q}\cdot\nabla)f = \epsilon\nabla^2 f - \epsilon d(\frac{\tau}{h})^2 f tt. \tag{2.16}$$

In general, for convergence it thus is necessary that $\tau = o(h)$. For many practical problems however, $\frac{\tau}{h}$ and the viscosity parameter ϵ are relatively small, so that the DFF deficiency has only a minor influence on the accuracy.

3. Computation of the pressure.

For the computation of the pressure (-increment) we have to solve the (discrete) Poisson equation

$$LQ^n = \frac{2}{\tau} r, \; r := D\tilde{\tilde{\mathrm{U}}} - B^{n+1}. \tag{3.1}$$

Considered as a matrix, L has a few attractive properties such as symmetry, non-positive definiteness and a pentadiagonal structure. However, L is singular with $Le = 0$, where $e = (1, \ldots, 1)^T$, and therefore the set of equations (3.1) has only a solution if $(e,r) = 0$. In [10] it is shown that this condition is the discrete equivalent of (2.3). Hence for our flow problem, the condition $(e,r) = 0$ is automatically satisfied.

There are many methods available for the solution of (3.1). In order to obtain a fast pressure correction method, it is important to employ a fast Poisson solver. In our computations we used a full multigrid method very similar to the multigrid method *MG*00 [4]. It is a*V*-cyclic method with red-black Gauss Seidel relaxation, half injection for the restriction operator and bilinear interpolation for the prolongation operator. The multigrid process is repeated until the l_2-norm of the residual is less that 10^{-4}.

4. Computational results.

The equations describing the oscillatory free convection of a low *Pr* fluid can be written in the following (non-dimensional) form:

$$u_t + V_i((u^2)_x + (uv)_y) = V_d(u_{xx} + u_{yy}) - p_x \tag{4.1a}$$

$$v_t + V_i((uv)_x + (v^2)_y) = V_d(v_{xx} + v_{yy}) - V_b\theta - p_y \tag{4.1b}$$

$$u_x + v_y = 0 \tag{4.2}$$

$$\theta_t + T_i((u\theta)_x + (v\theta)_y) = T_d(\theta_{xx} + \theta_{yy}), \tag{4.3}$$

where u and v are respectively the horizontal- and vertical velocity component, p is the pressure and θ the temperature. The parameters in (4.1)-(4.3) are defined as follows: $V_i = T_i = Gr^{1/2}$, $V_d = 1$, $T_d = Pr^{-1}$ and $V_b = -Gr^{1/2}$. The computational domain is $\Omega = [0,4] \times [0,1]$. The PDEs (4.1)-(4.3) are completed with the following set of boundary conditions:

$u = v = 0$ for $x = 0$, $x = 4$ and $y = 0$ (rigid walls),

$u = v = 0$ for $y = 1$ (rigid upper wall, case A) or

$u_y = v = 0$ for $y = 1$ (stear stress-free upper wall, case B),

$\theta(0,y)=0,\ \theta(4,y)=4$ (isothermal vertical walls),

$\theta(x,0)=\theta(x,1)=x$ (conducting horizontal walls).

One can easily see that for $Pr=0$ equation (4.3) reduces to the Laplace equation for θ. Taking into account the boundary conditions, this equation has the solution $\theta=x$. Thus, θ is in this case independent of the velocity-field $\mathbf{u}=(u,v)$ and the pressure p.

We have computed the solution of the free convection problem for $Pr=0$ (case A and B). For the time-integration of the Navier-Stokes equations we applied the OEH scheme and an ADI scheme. However the results presented in this section were all obtained with the OEH scheme. One reason for this is that the OEH scheme is much faster (per time step) than the ADI scheme; see e.g. [5] where both schemes, when applied to the Burgers' equations, are examined for use on a vectorcomputer. A second reason for using the OEH scheme is that, by our experience, the ADI scheme often requires small time steps for stability, at least in the present fluid flow computations. This is in contrast with the observation that the scheme is unconditionally stable in the sense of von Neumann for (2.14). The ADI time step can become close to the critical time step (for stability) of the OEH scheme. In such a situation we prefer to use the OEH scheme due to its low costs per time step and low storage demand.

We have used the transient code for all values of *Gr*, even if the solution tends to a steady state. All computations were performed on a uniform 128*32 grid. The initial solution for the smallest values of *Gr* is the asymptotic solution proposed by Ben Hadid et. al. [1]. As initial solution for larger *Gr*-values, we used a (possibly) steady solution obtained at the previous value of *Gr*. Details of the computations are given in Table 1. All computations have been carried out on a (2-pipe) cyber 205.

For the steady solutions, a few characteristic values are presented in Table 2. These are the following extrema

$$v_{max} := \max(v(x,0{\cdot}5)),\ v_{min} := \min(v(x,0{\cdot}5,))$$

$$u_{max} := \max(u(1{\cdot}0,y)),\ u_{min} := \min(u(1{\cdot}0,y))\quad (\text{for case } A)$$

$$u_{min} := \min(u(x,1{\cdot}0))\ (\text{for case } B),$$

and their locations. Streamlines and velocity profiles for the steady solutions are presented in Figure 1. The solutions for case A are centre-symmetric. The flow for $Gr=2*10^4$ contains one vortex in the centre of the cavity, and for $Gr=2{\cdot}5*10^4$ it contains one primary vortex and two secondary vortices. For case B ($Gr=10^4$), the flow is non-symmetric and contains only one vortex near the cold wall.

The unsteady solutions are characterized by the following maximum values (as a function of time)

$$v_{max} := \max|v(x,0{\cdot}5)|$$

$$u_{max} := \max|u(1{\cdot}0,y)|\quad (\text{for case } A)$$

$$u_{max} := \max|u(x,1{\cdot}0)|\quad (\text{for case } B).$$

The extrema (in time) of the characteristic quantities u_{max} and v_{max}, and the frequency f of the flows are presented in Table 3. The frequency f is computed by measuring the distance between two consecutive maxima of u_{max}. The time-history of u_{max} is given in Figure 2. Note that the solution for case A, $Gr=3*10^4$ needs a rather long adjustment time before it becomes truly periodic im time. This behaviour depends on the initial solution chosen. The solution for case A, $Gr=4*10^4$ tends much faster to a periodic behaviour. However, for $t\approx 0{\cdot}8$ disturbances start to develop, indicating that the solution contains a small component with frequency $f/2$. In the time interval $0\leqslant t\leqslant 1$, we did not find a steady solution for $Gr=4*10^4$. The solution for case B, $Gr=1{\cdot}5*10^4$ shows a damped oscillatory behaviour. For $t\rightarrow\infty$, the solution probably tends very slowly to steady state, therefore we did not include the extrema for u_{max} and v_{max} in this case. Finally, the solution for case B, $Gr=2*10^4$ shows a nice periodic behaviour in time.

To further demonstrate the periodicity of the flows, Figure 3 presents the streamline patterns for

Table 1: Computational details.

Case	Gr	τ	time-interval	type
A	$2*10^4$	$5*10^{-5}$	[0·0, 0·3]	steady
	$2{\cdot}5*10^4$	$5*10^{-5}$	[0·0, 0·3]	steady
	$3*10^4$	$5*10^{-5}$	[0·0, 2·0]	oscillatory
	$4*10^4$	$2{\cdot}5*10^{-5}$	[0·0, 1·0]	oscillatory
B	10^4	$5*10^{-5}$	[0·0, 0·3]	steady
	$1{\cdot}5*10^4$	$2{\cdot}5*10^{-5}$	[0·0, 1·0]	oscillatory/steady
	$2*10^4$	$2{\cdot}5*10^{-5}$	[0·0, 1·0]	oscillatory

Table 2: Requested steady solutions.

Case A				
Gr	v_{max}/x	v_{min}/x	u_{max}/y	u_{min}/y
$2*10^4$	0·473/2·453	−0·473/1·547	0·667/0·141	−0·433/0·641
$2{\cdot}5*10^4$	0·572/2·453	−0·572/1·547	0·676/0·141	−0·451/0·609
Case B				
Gr	v_{max}/x	v_{min}/x	u_{min}/x	
10^4	0·514/1·391	−1·051/0·203	−1·943/0·938	

Table 3: Requested unsteady solutions.

Case A			
Gr	v_{max}: max/min	u_{max}: max/min	f
$3*10^4$	0·73/0·56	0·81/0·59	17·30
$4*10^4$	1·04/0·52	1·04/0·40	20·73
Case B			
Gr	v_{max}: max/min	u_{max}: max/min	f
$1{\cdot}5*10^4$	-	-	12·43
$2*10^4$	1·71/1·08	2·52/1·84	15·27

case A, $Gr=4*10^4$ and case B, $Gr=2*10^4$ during two periods. Let T denote the period of the flow, then streamlines are presented at $t_i=t_o+iT/4$, $i=0(1)7$, for some arbitrary t_o. The flow for case A has one primary vortex in the centre of the cavity and two secondary vortices, which alternate in size. Notice that the 4th picture ($t=t_3$) and the 8th picture ($t=t_7$) differ slightly in the main vortex. This also indicates that the flow has a small component with frequency $f/2$. The streamline patterns for case B have one primary vortex near the cold wall and a secondary vortex in the hot region, alternating in size.

5. Conclusion

The OEH-PC scheme is a suitable technique to predict the time-dependent (oscillatory) behaviour of free convection of an incompressible fluid. The scheme has a few attractive properties. First, it is fast per time step. Second, the scheme is easy to implement and extension to arbitrary domains (even 3-dimensional) is straightforward. Finally, the storage requirements of the scheme are very modest. A drawback of the scheme is its conditional stability and the DFF deficiency. However, for many flow problems this deficiency is only of minor importance. For the present fluid flow problem, we have found the OEH scheme competitive to the ADI scheme, due to the disappointing stability behaviour of this ADI scheme.

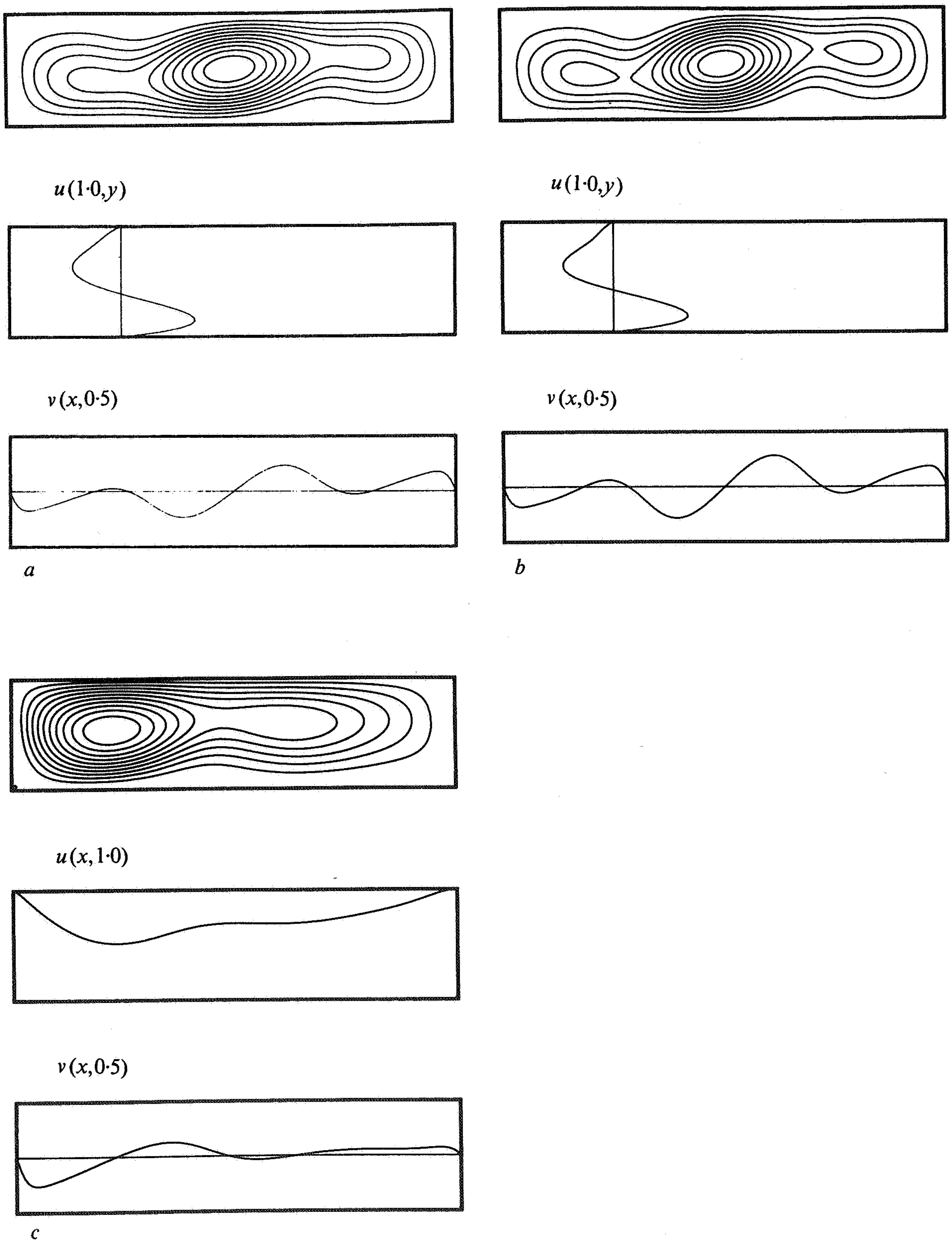


Fig. 1. Streamlines and velocity profiles.
Case A, $Gr = 2 * 10^4$ and $2{\cdot}5 * 10^4$ (a,b)
and Case B, $Gr = 10^4$ (c).

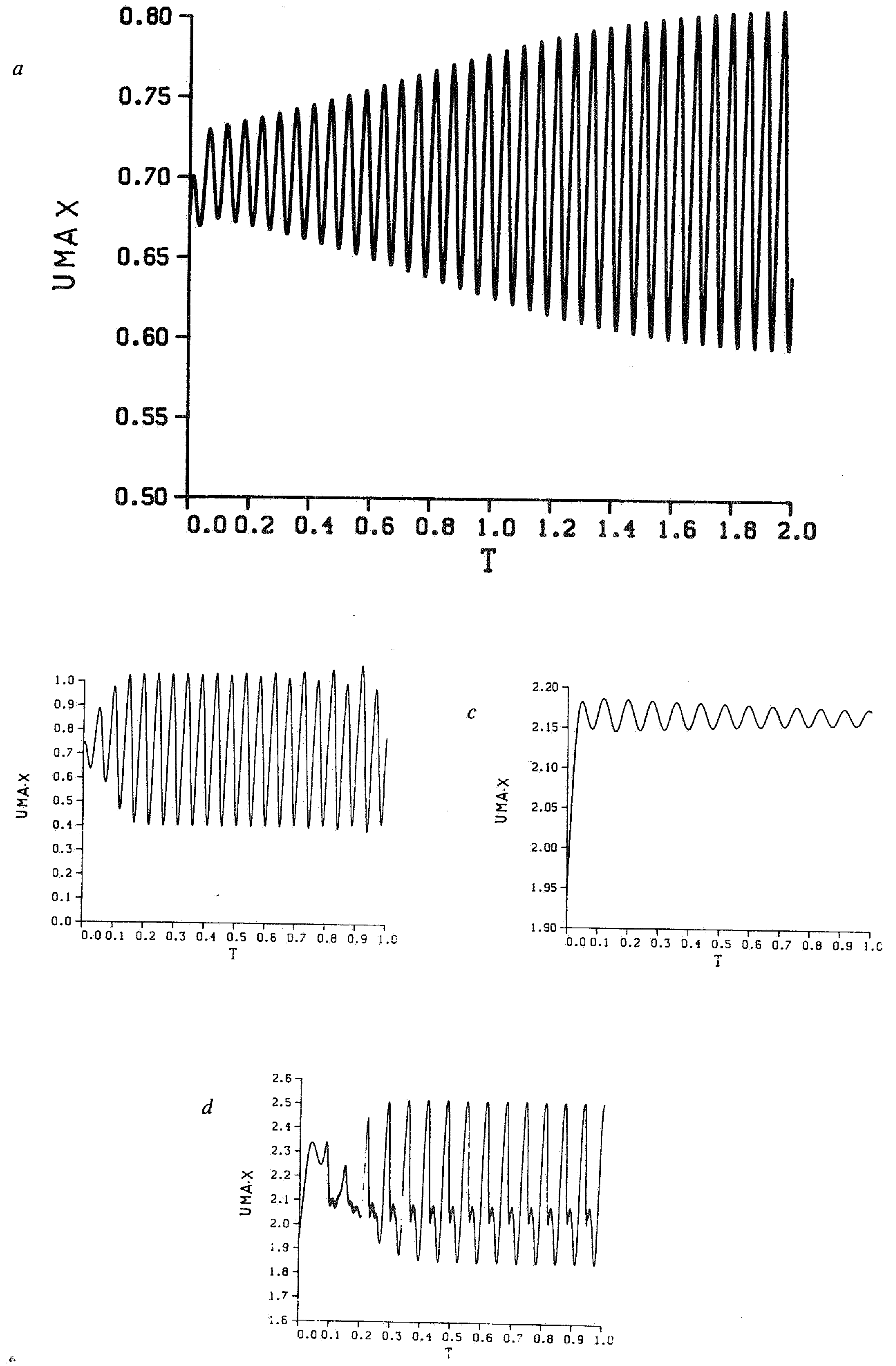
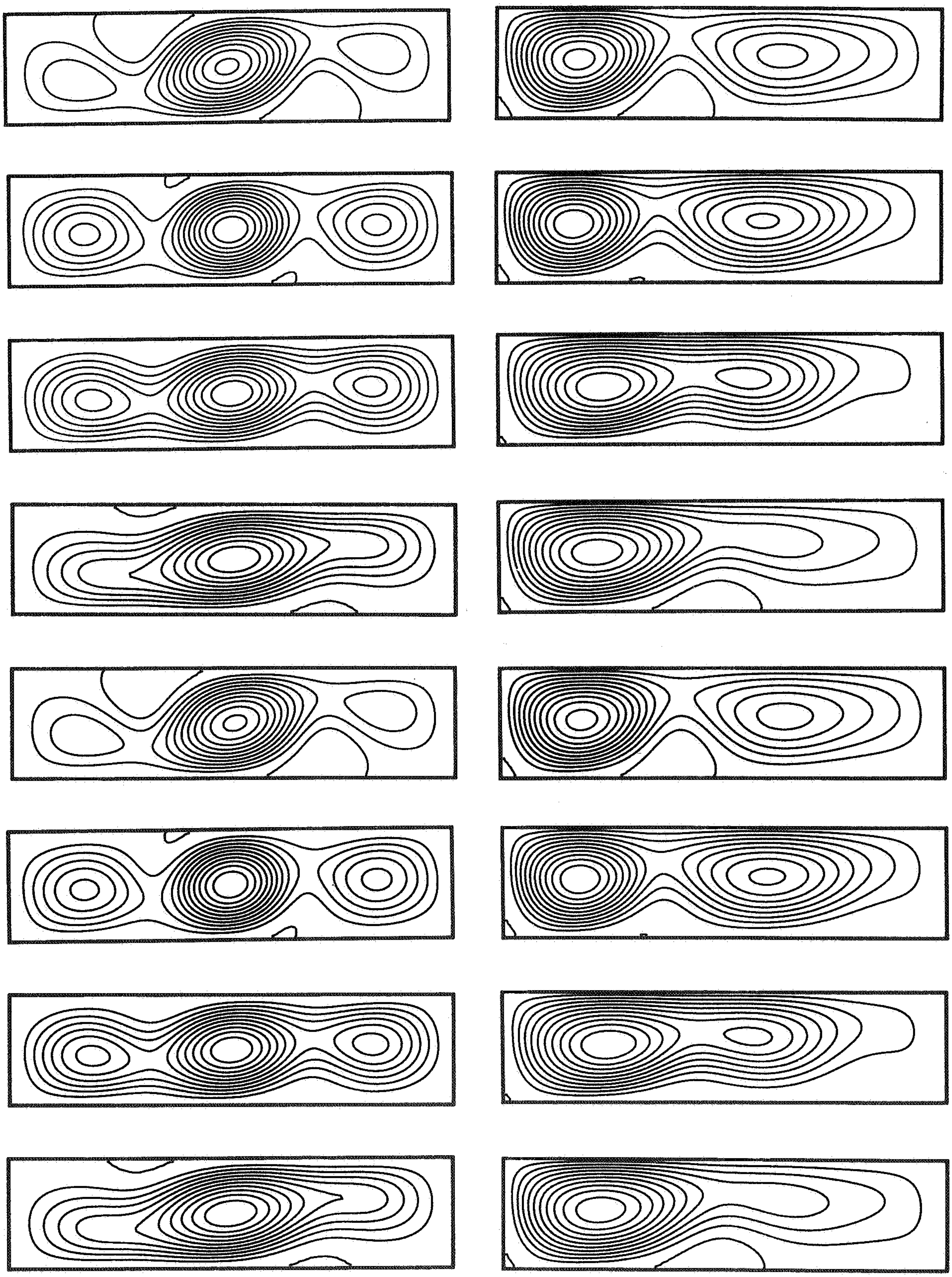


Fig. 2. Time-history of u_{max} for case A, $Gr = 3 * 10^4$ (a) and $Gr = 4 * 10^4$ (b) and for case B, $Gr = 1{\cdot}5 * 10^4 (c)$ and $Gr = 2 * 10^4 (d)$.



a *b*

Fig. 3. Streamlines during two periods of flow for case A, $Gr = 4 * 10^4 (a)$ and case B, $Gr = 2 * 10^4 (b)$.

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