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A Graph Theoretic Characterization for the Rank of the Transfer Matrix of a Structured System

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In this paper structured systems are considered and the generic rank of the transfer matrix of such systems is introduced. It is shown that this rank equals the maximum number of vertex disjoint paths from the input vertices to the output vertices in the graph that can be associated to the structured system. This maximum number of disjoint paths can be calculated using techniques from combinatorics. As an application a structural version is proposed of the well-known almost disturbance decoupling problem.

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Key Words and Phrases: Structured linear systems, generic rank of transfer matrix, graph theory.

1. Introduction

In this paper we consider the finite-dimensional linear time-invariant system given by

$$\dot{x}(t)=Ax(t)+Bu(t), \tag{1.a}$$

$$y(t)=Cx(t). \tag{1.b}$$

Here $x(t)\in\mathbb{R}^n$ denotes the state, $u(t)\in\mathbb{R}^m$ the input and $y(t)\in\mathbb{R}^p$ the output of the system. A,B and C are real matrices of dimensions $n\times n$, $n\times m$ and $p\times n$, respectively.
Linear systems of type (1) play an important role in system theory from a theoretical as well as from a practical point of view and may appear in many contexts. For instance, system (1) can represent the cascade interconnection of the following two (sub)systems that are both of the same type as system (1)

$$\dot{x}_1(t)=A_1x_1(t)+B_1u_1(t), \qquad \dot{x}_2(t)=A_2x_2(t)+B_2u_2(t),$$

$$y_1(t)=C_1x_1(t), \qquad y_2(t)=C_2x_2(t),$$

with $y_1(t)=u_2(t), u_1(t)=u(t)$ and $y_2(t)=y(t)$, and all vectors and matrices have appropriate dimensions. Then

$$A=\begin{bmatrix} A_1 & 0 \\ B_2C_1 & A_2 \end{bmatrix}, B=\begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \text{ and } C=\begin{bmatrix} 0 & C_2 \end{bmatrix}.$$

The zeroes in this representation of A,B and C are matrices with entries that are fixed zeroes. This means that the entries of these matrices always will be zero, no matter what the entries in the matrices of the two subsystems are. Such fixed zeroes in A,B and C are called structural zeroes. Entries in A,B and C that are not fixed zeroes are supposed to be unknown and may have any real value. These entries are called structural nonzeroes. The structural zeroes and structural nonzeroes determine the structure of system (1), or, what is the same, the structure of the matrices A,B and C.

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In this paper we assume that the structure of system (1) is known.
Given such a structured system (1), we let k be the number of structural nonzeroes in the structured matrices A,B and C. Then we can think of A,B and C as parametrized by a parameter $\lambda=(\lambda_1,\lambda_2,.......,\lambda_k)$ in the parameter space $\mathbb{R}^k$. Indeed, we can number the k structural nonzeroes in A,B and C from 1 upto k, and we can place λ_j at the j^{th} entry. Then it is clear that the structured matrices A,B and C depend on the parameter $\lambda\in\mathbb{R}^k$. To express this dependency on λ we frequently write A_λ,B_λ and C_λ instead of A,B and C.

EXAMPLE
1. $k=9, n=3, m=2, p=2.$

$$A_\lambda = \begin{bmatrix} 0 & \lambda_2 & 0 \\ \lambda_1 & 0 & 0 \\ 0 & \lambda_3 & \lambda_4 \end{bmatrix}, B_\lambda = \begin{bmatrix} \lambda_5 & 0 \\ 0 & \lambda_7 \\ \lambda_6 & 0 \end{bmatrix}, C_\lambda = \begin{bmatrix} 0 & \lambda_8 & 0 \\ 0 & 0 & \lambda_9 \end{bmatrix}.$$

$$C_\lambda(sI-A_\lambda)^{-1}B_\lambda = \frac{1}{s^3-s^2\lambda_4-s\lambda_1\lambda_2+\lambda_1\lambda_2\lambda_4}\begin{bmatrix} (s-\lambda_4)\lambda_1\lambda_5\lambda_8 & s(s-\lambda_4)\lambda_7\lambda_8 \\ (s^2-\lambda_1\lambda_2)\lambda_6\lambda_9+\lambda_1\lambda_3\lambda_5\lambda_9 & s\lambda_3\lambda_7\lambda_9 \end{bmatrix}.$$

2. $k=8, n=3, m=2, p=2.$

$$A_\lambda = \begin{bmatrix} 0 & \lambda_2 & 0 \\ \lambda_1 & 0 & \lambda_4 \\ 0 & \lambda_3 & 0 \end{bmatrix}, B_\lambda = \begin{bmatrix} \lambda_5 & 0 \\ 0 & \lambda_6 \\ 0 & 0 \end{bmatrix}, C_\lambda = \begin{bmatrix} 0 & \lambda_7 & 0 \\ 0 & 0 & \lambda_8 \end{bmatrix}.$$

$$C_\lambda(sI-A_\lambda)^{-1}B_\lambda = \frac{1}{s^3-s(\lambda_1\lambda_2+\lambda_3\lambda_4)}\begin{bmatrix} s\lambda_1\lambda_5\lambda_7 & s^2\lambda_6\lambda_7 \\ \lambda_1\lambda_3\lambda_5\lambda_8 & s\lambda_3\lambda_6\lambda_8 \end{bmatrix}.$$

2. GENERIC RANK OF TRANSFER MATRICES

Given a structured system (1) and a parameter $\lambda\in\mathbb{R}^k$ we say that rank $C_\lambda(sI-A_\lambda)^{-1}B_\lambda=q$ if there is a q^{th} order minor of $C_\lambda(sI-A_\lambda)^{-1}B_\lambda$ unequal to zero, while every $(q+1)^{th}$ order minor of $C_\lambda(sI-A_\lambda)^{-1}B_\lambda$, when defined, is equal to zero (as a rational function).
In the example 1 the 2^{nd} (and largest) order minor of $C_\lambda(sI-A_\lambda)^{-1}B_\lambda$ is equal to

$$\frac{-s\lambda_6\lambda_7\lambda_8\lambda_9}{s^3-s^2\lambda_4-s\lambda_1\lambda_2+\lambda_1\lambda_2\lambda_4},$$

and consequently rank$C_\lambda(sI-A_\lambda)^{-1}B_\lambda\leqslant 2$. In example 2 the 2^{nd} order minor of $C_\lambda(sI-A_\lambda)^{-1}B_\lambda$ is zero for every $\lambda\in\mathbb{R}^k$ which implies that rank $C_\lambda(sI-A_\lambda)^{-1}B_\lambda<2$.
Note that any minor in the above examples can considered to be a rational function in the indeteminate s with coefficients that are polynomials in the indeterminate $\lambda=(\lambda_1,\lambda_2,.......,\lambda_k)$. This fact is true for general structured systems of type (1) and will be proved and used in the proof of theorem 1 below.
Given a structured system (1) we define

$$r = \max_{\lambda\in\mathbb{R}^k}\left[\text{rank } C_\lambda(sI-A_\lambda)^{-1}B_\lambda\right] \tag{2}$$

and

$$W = \{\lambda \in \mathbb{R}^k \mid \text{rank}\ C_\lambda(sI - A_\lambda)^{-1} B_\lambda < r\} \tag{3}$$

A subset V of $\mathbb{R}^k$ is called a *variety* in $\mathbb{R}^k$ if V can be described as the locus of common zeroes of a finite number of polynomials $\psi_1, \psi_2, ..., \psi_t$ in the indeterminate $\tau = (\tau_1, \tau_2, ..., \tau_k)$, i.e. $V = \{(\tau_1, \tau_2, ..., \tau_k) \in \mathbb{R}^k \mid \psi_i(\tau_1, \tau_2, ..., \tau_k) = 0$ for all $i = 1, 2, ..., t\}$. A variety V in $\mathbb{R}^k$ is called *proper* if $V \neq \mathbb{R}^k$ (cf. Wonham [11]). Now we have the following.

THEOREM 1. *W is a proper variety in $\mathbb{R}^k$.*

PROOF. If $r = 0$ then $W = \phi$, where ϕ denotes the empty set, and it follows easily that W is a proper variety. If $r > 0$ then it follows from the above notion of rank that

$$W = \{\lambda \in \mathbb{R}^k \mid \text{every } r^{th} \text{ order minor of } C_\lambda(sI - A_\lambda)^{-1} B_\lambda \text{ is zero }\}.$$

By the definition of r and W it follows that there is a $\bar{\lambda} \in \mathbb{R}^k$ such that rank $C_{\bar{\lambda}}(sI - A_{\bar{\lambda}})^{-1} B_{\bar{\lambda}} = r$. Therefore, there is a an r^{th} order minor of $C_{\bar{\lambda}}(sI - A_{\bar{\lambda}})^{-1} B_{\bar{\lambda}}$ that is not equal to zero. Without loss of generality we may assume that this r^{th} order minor is $\det C'_{\bar{\lambda}}(sI - A_{\bar{\lambda}})^{-1} B'_{\bar{\lambda}}$. Here we have denoted B'_λ for the first r columns of B_λ, C'_λ for the first r rows of C_λ and substituted $\lambda = \bar{\lambda}$. Det stands for determinant. Note that $C'_\lambda, A_\lambda, B'_\lambda$ and powers of A_λ can be considered as matrices with entries in $\mathbb{R}[\lambda] = \mathbb{R}[\lambda_1, \lambda_2, ..., \lambda_k]$, the ring of polynomials with real coefficients in the indeterminate $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$. Also the trace of any power of A_λ can considered to be an element of $\mathbb{R}[\lambda]$. Because the Cayley-Hamilton theorem holds for square matrices with entries in $\mathbb{R}[\lambda]$ (cf. [4]), it follows by the Souriau-Frame-Faddeev algorithm for the computation of $(sI - A_\lambda)^{-1}$ (cf. [8]) that $C'_\lambda(sI - A_\lambda)^{-1} B'_\lambda = N(s,\lambda)/d(s,\lambda)$. Here $N(s,\lambda)$ is a matrix with enties that are polynomials in the indeterminate s and that have coefficients in $\mathbb{R}[\lambda]$, and $d(s,\lambda)$ is a monic polynomial in s also with coefficients in $\mathbb{R}[\lambda]$. By the Laplace expansion formula for the evaluation of the determinant (cf. [8]), it follows that $\det C'_\lambda(sI - A_\lambda)^{-1} B'_\lambda = \bar{n}(s,\lambda)/\bar{d}(s,\lambda)$, where $\bar{n}(s,\lambda)$ and $\bar{d}(s,\lambda)$ are polynomials in s with coefficients in $\mathbb{R}[\lambda]$, $\bar{d}(s,\lambda)$ is monic, and $\bar{n}(s,\lambda)$ and $\bar{d}(s,\lambda)$ have no factors in common. Then if we write $\bar{n}(s,\lambda) = \sum_{i=1}^{z} n_i(\lambda) s^{i-1}$ with $n_1, n_2, \ldots, n_z \in \mathbb{R}[\lambda]$, it follows that $\det C'_\lambda(sI - A_\lambda)^{-1} B'_\lambda = 0$ if and only if $n_i(\lambda) = 0$ for all $i = 1, 2, ..., z$. Hence, the set $\{\lambda \in \mathbb{R}^k \mid \det C'_\lambda(sI - A_\lambda)^{-1} B'_\lambda = 0\} = \{\lambda \in \mathbb{R}^k \mid n_i(\lambda) = 0$ for $i = 1, 2, ..., z\}$ is a variety in $\mathbb{R}^k$. Since $\det C'_{\bar{\lambda}}(sI - A_{\bar{\lambda}})^{-1} B'_{\bar{\lambda}}$ is a nonzero r^{th} order minor of $C_\lambda(sI - A_\lambda)^{-1} B_\lambda$, it follows that the set $\{\lambda \in \mathbb{R}^k \mid \det C'_\lambda(sI - A_\lambda)^{-1} B'_\lambda = 0\}$ is a proper variety in $\mathbb{R}^k$. By now we have proved that the set of parameters $\lambda \in \mathbb{R}^k$ for which a particular r^{th} order minor of $C_\lambda(sI - A_\lambda)^{-1} B_\lambda$, namely $\det C'_{\bar{\lambda}}(sI - A_{\bar{\lambda}})^{-1} B'_{\bar{\lambda}}$, is zero, is a variety in $\mathbb{R}^k$. Analogous to the above we can prove that this is the case for any r^{th} order minor of $C_\lambda(sI - A_\lambda)^{-1} B_\lambda$. Hence, the set of parameters $\lambda \in \mathbb{R}^k$ for which any r^{th} order minor of $C_\lambda(sI - A_\lambda)^{-1} B_\lambda$ is zero is a variety in $\mathbb{R}^k$. Furthermore, from the above it also follows that at least one of these sets is a proper variety in $\mathbb{R}^k$. Thus, we have now that W is the finite intersection of varieties in $\mathbb{R}^k$ of which at least one is proper. So, W is a proper variety in $\mathbb{R}^k$. □

The above theorem means that rank $C_\lambda(sI - A_\lambda)^{-1} B_\lambda = r$ for almost all $\lambda \in \mathbb{R}^k$. Here almost all is to be interpreted as everywhere except for a proper variety. We can think of r as the *generic rank* of $C(sI - A)^{-1} B$ (cf. Wonham [11]) and we therefore denote

$$r = \underset{(g)}{\text{rank}}\ C(sI - A)^{-1} B$$

3. Graphs

Given a triple structured matrices A,B and C with parameter space $\mathbb{R}^k$, we can construct a graph with $n+m+p$ vertices and k directed and labeled edges. This graph, denoted $G(\overline{V},\overline{E})$, is described by the set of vertices $\overline{V}$ and the set of edges (ordered pairs) $\overline{E}$, with

$$\overline{V} = \{u_1,u_2,....,u_m\} \cup \{x_1,x_2,....,x_n\} \cup \{y_1,y_2,....,y_p\}$$

and

$$\overline{E} = \{(u_j,x_i) \mid B_{i,j} \neq 0\} \cup \{(x_j,x_i) \mid A_{i,j} \neq 0\} \cup \{(x_j,y_i) \mid C_{i,j} \neq 0\}.$$

Here $\cup$ denotes the set theoretic union and $B_{i,j} \neq 0$ means that the entry of the matrix B in the i^{th} row and the j^{th} column is a structural nonzero. The edges of $G(\overline{V},\overline{E})$ can be numbered from 1 upto k and can be labeled by the parameters $\lambda_1,\lambda_2,....,\lambda_k$. The graphs associated to the examples 1 and 2 can be visualized as in figures 1 and 2, respectively.

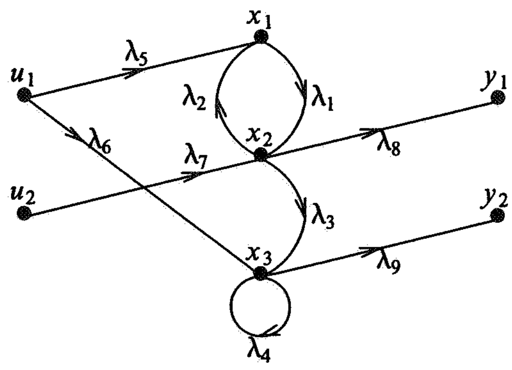

Figure 1.

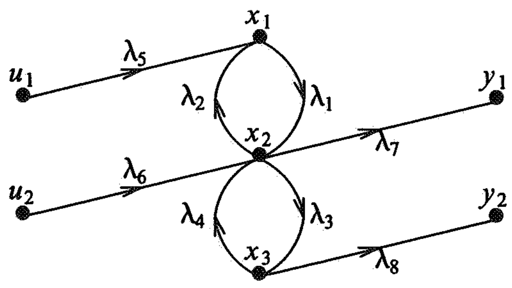

Figure 2.

Note there is an one-one relation between the triple structured matrices A,B and C with parameter space $\mathbb{R}^k$ and the associated directed and labeled graph $G(\overline{V},\overline{E})$.
To express the generic rank of $C(sI-A)^{-1}B$ in terms properties of graphs we need to consider an extension of the graph $G(\overline{V},\overline{E})$. This extension, denoted $G(V,E)$, is a graph with vertex set $V = \overline{V} \cup \{a\} \cup \{b\}$ and edge set $E = \overline{E} \cup \{(a,u_i) \mid i=1,2,...,m\} \cup \{(y_j,b) \mid j=1,2,...,p\}$. The vertices a and b are called the source and the sinc, respectively. The edges going out from a are labeled $\mu_1,\mu_2,....,\mu_m$, and the edges coming together in b are labeled $\rho_1,\rho_2,....,\rho_p$. The extended graph corresponding to example 1 can be visualized as in figure 3.

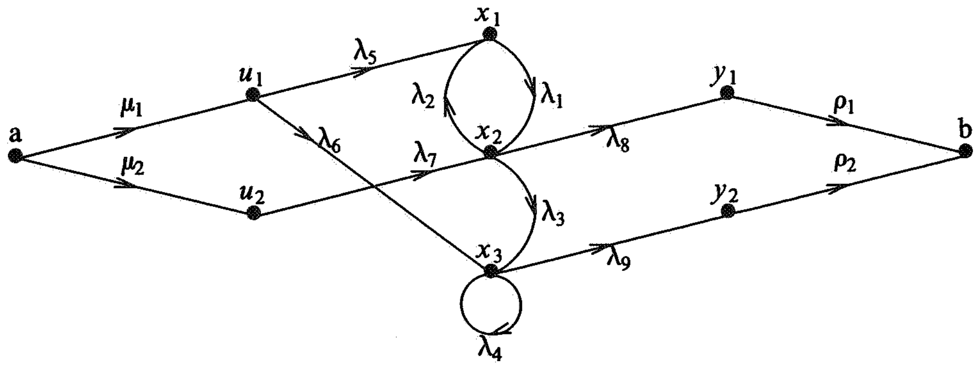

Figure 3.

Let v,w be two vertices in V. We say that the set of vertices $P=\{v_i \mid i=0,1,....,t\} \subseteq V$ forms a path of length t from v to w, if $v=v_0$, $w=v_t$ and $(v_i,v_{i+1}) \in E$ for all $i=0,1,....,t-1$. Two paths from v to w, not necesarily of the same length, are called *edge disjoint* if they have no edge in common and *vertex disjoint* if the only vertices that they have in common are v and w. Note that if two paths from v to w are vertex disjoint, they are also edge disjoint. We call an l-tuple of paths in $G(V,E)$ from v to w edge (vertex) disjoint if each pair of paths of the l-tuple is edge (vertex) disjoint.
If $e=(p,q) \in E$ with $p,q \in V$ is an edge of the graph $G(V,E)$ then we denote $\lambda(p,q)$ for the parameter that labels the edge (p,q). If $P=\{v_i \mid i=0,1,....,t\}$ is a path from v to w of length t then we denote $\Lambda(P)=\prod_{i=0}^{t-1} \lambda(v_i,v_{i+1})$. If $\{P_i \mid i=1,2,...,g\}$ is the set of *all* edge disjoint paths of length t from u_j to y_i, then $C_{\lambda,i} A_\lambda^{t-2} B_{\lambda,j} = \sum_{i=1}^{g} \Lambda(P_i)$ (cf. [5]), where $C_{\lambda,i}$ denotes the i^{th} row of C_λ and $B_{\lambda,j}$ denotes the j^{th} column of B_λ. From the description of the graph $G(V,E)$ it is clear that if there is no path from a to b, then there is no path from any input vertex u_j to any output vertex y_i. Hence, in that case $C_{\lambda,i} A_\lambda^{t-2} B_{\lambda,j}=0$ for all $i=1,2,...,p, j=1,2,...,m$, $t \geqslant 2$ and $\lambda \in \mathbb{R}^k$, which implies that $C_\lambda(sI-A_\lambda)^{-1} B_\lambda=0$ for all $\lambda \in \mathbb{R}^k$.
Let $e \in E$ be a directed edge of the graph $G(V,E)$. By the removal of e from $G(V,E)$ we mean the removal of e from the set E. In the structured system (1) given by the structured matrices A,B and C, the removal of an edge $e \in \overline{E}$ corresponds to the replacement of a structural nonzero by a structural zero. This replacement can be considered as the fixing to zero of the parameter that is associated to the edge e. Hence, the removal of an edge $e \in \overline{E}$ from $G(V,E)$ can be seen as the restricting of the parameter space $\mathbb{R}^k$ to a hyper plane (a linear subspace in $\mathbb{R}^k$ of dimension k-1) .
Let $v \in V$ be a vertex of the graph $G(V,E)$. By the removal of v from $G(V,E)$ we mean the removal of v from the set V and the removal of all the edges in E that have v as their beginpoint or endpoint. For the structured system (1) the removal of a vertex $v \in \overline{V}$ from $G(V,E)$ comes down to the deletion of a row, a column or both a row and a column in one or in all of the matrices A,B and C. To be more specific, if $v=u_j$ for some $j=1,2,...,m$, then the removal of v comes down to the deletion of the j^{th} column in the matrix B. Likewise, if $v=y_i$ for some $i=1,2,...,p$, then the removal of v comes down to the deletion of the i^{th} column in the matrix C. Finally, if $v=x_i$ for some $i=1,2,...,n$, then the removal of v means the deletion of the i^{th} row in B, the deletion of the i^{th} column in C, and the deletion of both the i^{th} row and the i^{th} column in A.
Every time when an edge or a vertex is removed from $G(V,E)$ we obtain a new graph corresponding to a new structured system from which again edges and vertices may be removed.
We now state an important and well-known result from graph theory which plays a crucial role in the present paper (cf. [3],[6]).

THEOREM 2. *The maximum number of vertex disjoint paths in $G(V,E)$ from a to b is equal to the minimum number of vertices in $\overline{V}$ whose removal from $G(V,E)$ results in a graph in which there is no path from a to b.*

The above theorem is due to Menger, and is closely related to the well-known "maximum flow theorem" of Ford and Fulkerson (cf. [2]). By a modification of the algorithm associated to this "maximum flow theorem", the maximum number of vertex disjoint paths from a to b in a given graph $G(V,E)$ can be calculated (cf. [1],[7]).

4. Main result

The Menger theorem helps us to prove the next theorem which is our main result.

THEOREM 3. *Consider the graph $G(V,E)$. Then the maximum number of vertex disjoint paths from a to b is equal to the generic rank of $C(sI-A)^{-1}B$.*

The proof of this theorem consists of the combination of the Menger theorem and the two lemmas stated below.

LEMMA 4. *Consider the graph $G(V,E)$. If there are l vertex disjoint paths in $G(V,E)$ from a to b, then $l \leq r$, where r is defined by (2).*

PROOF. Consider the l vertex disjoint paths in $G(V,E)$ from a to b, and remove from $G(V,E)$ all edges in E that do not occur in the l vertex disjoint paths from a to b. The result of this removal is a graph that consists of the l vertex disjoint paths, and, possibly, a number of isolated vertices. Next, in this graph remove the vertices a and b. The graph then obtained consists of l totally disjoint paths. Each path starts in an input vertex and ends in an output vertex. Number the l paths from 1 upto l and renumber the input and output vertices such that the j^{th} path starts in input vertex j and ends in output vertex j. The triple structured matrices, say $\hat{A},\hat{B}$ and $\hat{C}$, associated to the graph now obtained can be thought of as being obtained from the triple structured matrices A,B and C by a suitable permutation of inputs and outputs, and a restriction of the parameter space $\mathbb{R}^k$ to, say, L. It is easy to see that the system corresponding to the structured matrices $\hat{A},\hat{B}$ and $\hat{C}$ consists of l parallel, totally disconnected single input/single output systems, each having maximum rank 1. Therefore, since $L \subseteq \mathbb{R}^k$, it is clear that $l = \max_{\lambda \in L} [\text{rank } C_\lambda(sI-A_\lambda)^{-1}B_\lambda] \leq r$. □

LEMMA 5. *Consider the graph $G(V,E)$. If there exists a set of q vertices in $\overline{V}$ whose removal from $G(V,E)$ results in a graph in which there is no path from a to b, then $r \leq q$, where r is defined by (2).*

PROOF. Let H be a set of vertices in $\overline{V}$ whose removal from $G(V,E)$ results in a graph in which there is no path from a to b. Denote $U=\{u_1,u_2,....,u_m\}$, $X=\{x_1,x_2,....,x_n\}$ and $Y=\{y_1,y_2,....,y_p\}$. Let $U_1=H\cap U, X_1=H\cap X$ and $Y_1=H\cap Y$, where $\cap$ denotes the set theoretic intersection. Then $H=U_1\cup X_1\cup Y_1$. Furthermore, let U_2,X_2 and Y_2 be such that $U=U_1\cup U_2, X=X_1\cup X_2$ and $Y=Y_1\cup Y_2$, with $U_1\cap U_2=\phi, X_1\cap X_2=\phi$ and $Y_1\cap Y_2=\phi$. Referring to this decomposition, we can after permutation rewrite the structured system (1) as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix},$$
$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

A_{ij},B_{ij} and C_{ij} are structured matrices of sizes $n_i \times n_j, n_i \times m_j$ and $p_i \times n_j$, respectively, where n_i is the number of elements in X_i, m_i is the number of elements U_i, p_i is the number of elements Y_i and $i,j = 1,2$. Like A,B and C as in (1), the matrices A_{ij},B_{ij} and C_{ij} depend on the parameter λ. The removal of the set of vertices H from $G(V,E)$ results in a graph that corresponds to the structured system

$$\dot{x}_2(t)=A_{22}x_2(t)+B_{22}u_2(t),$$
$$y_2(t)=C_{22}x_2(t).$$

Since in the graph obtained by the removal of H there is no path from a to b, it follows that $C_{22}(sI-A_{22})^{-1}B_{22}=0$ for all $\lambda\in\mathbb{R}^k$. Now the proof of lemma 5 is completed by lemma 6 stated below. Indeed, from lemma 6 it follows that rank $C_\lambda(sI-A_\lambda)^{-1}B_\lambda \leqslant n_1+m_1+p_1=q$ for all $\lambda\in\mathbb{R}^k$. Hence, $r\leqslant q$. □

Before stating lemma 6 we give a proof of our main result.

Proof of theorem 3. Consider the graph $G(V,E)$. Let $l_{\max}$ denote the maximum number of vertex disjoint paths in $G(V,E)$ from a to b and let $q_{\min}$ denote the minimum number of vertices in $\overline{V}$ whose removal from $G(V,E)$ results in a graph in which there is no path from a to b. Then by theorem 2, and lemma 4 and 5 it follows that $l_{\max}=r=q_{\min}$. Since r is equal to the generic rank of $C(sI-A)^{-1}B$ the proof of theorem 3 is now completed. □

To complete the results of this section, it remains to prove the following lemma.

Lemma 6. *If* $M_{22}(sI-K_{22})^{-1}L_{22}=0$ *then*

$$\operatorname{rank}\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}(sI-\begin{bmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{bmatrix})^{-1}\begin{bmatrix}L_{11} & L_{12}\\ L_{21} & L_{22}\end{bmatrix}\leqslant n_1+m_1+p_1,$$

where for $i,j,=1,2$ K_{ij},L_{ij} *and* M_{ij} *are real matrices of sizes* $n_i\times n_j, n_i\times m_j$ *and* $p_i\times n_j$*, respectively.*

Proof. Denote

$$T_{ij}(s)=\begin{bmatrix}M_{i1} & M_{i2}\end{bmatrix}(sI-\begin{bmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{bmatrix})^{-1}\begin{bmatrix}L_{1j}\\ L_{2j}\end{bmatrix},\quad i,j=1,2.$$

Then it is clear that

$$\operatorname{rank}\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}(sI-\begin{bmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{bmatrix})^{-1}\begin{bmatrix}L_{11} & L_{12}\\ L_{21} & L_{22}\end{bmatrix}=\operatorname{rank}\begin{bmatrix}T_{11}(s) & T_{12}(s)\\ T_{21}(s) & T_{22}(s)\end{bmatrix}\leqslant$$
$$\operatorname{rank}[T_{11}(s)\ \ T_{12}(s)]+\operatorname{rank}[T_{21}(s)\ \ T_{22}(s)]\leqslant p_1+m_1+\operatorname{rank}T_{22}(s).$$

Since $M_{22}(sI-K_{22})^{-1}L_{22}=0$ there is a linear subspace $S_2\subseteq\mathbb{R}^{n_2}$ such that $K_{22}S_2\subseteq S_2$ and $im\,L_{22}\subseteq S_2\subseteq ker\,M_{22}$. For instance, $S_2=\sum_{i=0}^{n_2-1}K_{22}^i im L_{22}$ (cf. [11], Chapter 4). Here *im* denotes the image of a matrix and *ker* the kernel.
Let $Q_2=[Q_{21}\ Q_{22}]$ be an invertible matrix such that $S_2=im\,Q_{21}$. Then it follows easily that

$$Q_2^{-1}K_{22}Q_2=\begin{bmatrix}K'_{22} & \hat{K}_{22}\\ 0 & K''_{22}\end{bmatrix},\ Q_2^{-1}L_{22}=\begin{bmatrix}L'_{22}\\ 0\end{bmatrix}\text{ and }M_{22}Q_2=\begin{bmatrix}0 & M''_{22}\end{bmatrix}.$$

In addition denote

$$Q_2^{-1}K_{21}=\begin{bmatrix}K'_{21}\\ K''_{21}\end{bmatrix}\text{ and }K_{12}Q_2=\begin{bmatrix}K'_{12} & K''_{12}\end{bmatrix}.$$

Then $T_{22}(s)=M_{21}X_1(s)+M''_{22}X_3(s)$ and $-K''_{21}X_1(s)+(sI-K''_{22})X_3(s)=0$, where we have denoted

$$\begin{bmatrix}X_1(s)\\ X_2(s)\\ X_3(s)\end{bmatrix}=(sI-\begin{bmatrix}K_{11} & K'_{12} & K''_{12}\\ K'_{21} & K'_{22} & \hat{K}_{22}\\ K''_{21} & 0 & K''_{22}\end{bmatrix})^{-1}\begin{bmatrix}L_{12}\\ L'_{22}\\ 0\end{bmatrix}.$$

Hence,

$$T_{22}(s) = (M_{21} + M''_{22}(sI - K''_{22})^{-1}K''_{21})X_1(s),$$

from which it is follows that $\operatorname{rank} T_{22}(s) \leqslant \operatorname{rank} X_1(s) \leqslant \min(n_1, m_2) \leqslant n_1$. This completes the proof of the lemma 6. □

5. Application

In this section we present an application of our main result. To this end we consider the following extension of system (1)

$$\dot{x}(t) = Ax(t) + Bu(t) + Gd(t), \tag{4.a}$$

$$y(t) = Cx(t). \tag{4.b}$$

Here $x(t), u(t), y(t), A, B$ and C are as in the description of system (1), $d(t) \in \mathbb{R}^q$ denotes the disturbance input and G is a real $n \times q$ matrix. Like A, B and C, we assume that G is a structured matrix. We denote the total number of parameters in A, B, C and G by k', and collecting all parameters in $\lambda' \in \mathbb{R}^{k'}$, we express the dependency of A, B, C and G on λ' by $A_{\lambda'}, B_{\lambda'}, C_{\lambda'}$ and $G_{\lambda'}$. The matrices $A_{\lambda'}, B_{\lambda'}, C_{\lambda'}$ and $G_{\lambda'}$ can be considered as the nominal values of A, B, C and G for a given $\lambda' \in \mathbb{R}^{k'}$. Note that the compound matrix $[B, G]$ can be seen as an input matrix for system (4) in the same way as the matrix B can be seen as an input matrix for system (1). Similarly as to system (1) we can associate to system (4) a graph $G(\overline{V}', \overline{E}')$ and an extended graph $G(V', E')$. The graph $G(\overline{V}', \overline{E}')$ is consists of a vertex set $\overline{V}' = \overline{V} \cup \{d_1, d_2, \ldots, d_q\}$ and an edge set $\overline{E}' = \overline{E} \cup \{(d_j, x_i) \mid G_{i,j} \neq 0\}$ The graph $G(V', E')$ consists of a vertex set $V' = V \cup \{d_1, d_2, \ldots, d_q\}$ and an edge set $E' = E \cup \{(d_j, x_i) \mid G_{i,j} \neq 0\} \cup \{(a, d_i) \mid i = 1, 2, \ldots, q\}$.

Following Willems [9] we say that for a given $\lambda' \in \mathbb{R}^{k'}$ the *almost disturbance decoupling problem* for system (4) is solvable if for all $\epsilon > 0$ there is a real $m \times n$ matrix F_ϵ such that the H_∞-norm of $C_{\lambda'}(sI - (A_{\lambda'} + B_{\lambda'}F_\epsilon))^{-1}B_{\lambda'}$ is less then or equal to ϵ. We denote this problem $(ADDP)_{\lambda'}$.
Using the results of Willems [9] it can be shown that $(ADDP)_{\lambda'}$ is solvable if and only if

$$\operatorname{rank} C_{\lambda'}(sI - A_{\lambda'})^{-1}B_{\lambda'} = \operatorname{rank} C_{\lambda'}(sI - A_{\lambda'})^{-1}[B_{\lambda'}, G_{\lambda'}].$$

In the spirit of the present paper we say that the almost disturbance decoupling problem for the structured system (4) is *generically solvable* if the set

$$\{\lambda' \in \mathbb{R}^{k'} \mid \operatorname{rank} C_{\lambda'}(sI - A_{\lambda'})^{-1}B_{\lambda'} \neq \operatorname{rank} C_{\lambda'}(sI - A_{\lambda'})^{-1}[B_{\lambda'}, G_{\lambda'}]\}$$

is contained in a proper variety in $\mathbb{R}^{k'}$. This implies that if the almost disturbance decoupling problem for the structured system (4) is generically solvable then $(ADDP)_{\lambda'}$ is solvable for almost all $\lambda' \in \mathbb{R}^{k'}$. The following theorem is now an immediate consequence of theorems 1 and 3.

Theorem 7. *The almost disturbance decoupling problem for the structured system (4) is generically solvable if and only if the maximum number of vertex disjoint paths from a to b in $G(V,E)$ is equal to the maximum number of vertex disjoint paths from a to b in $G(V',E')$.*

6. Remarks and conclusions

In section 2 we showed that the generic rank of $C(sI - A)^{-1}B$ is equal to

$$\max_{\lambda \in \mathbb{R}^k} \left[\operatorname{rank} C_\lambda(sI - A_\lambda)^{-1}B_\lambda \right]$$

In this characterization it is assumed that each component of λ may have any real value. In practice however, the values that each component of λ can have, may be each restricted to a subset of $\mathbb{R}$. Hence, in practice the parameter λ may only take its values in some subset Ω in $\mathbb{R}^k$. Now, if Ω is an open subset in $\mathbb{R}^k$, there are parameters λ in Ω that are not in W, due to the fact that W is a closed subset in $\mathbb{R}^k$.

By the definition of r and W, see (2) and (3), it follows that

$$r = \max_{\lambda \in \Omega} \left[\text{rank } C_\lambda (sI - A_\lambda)^{-1} B_\lambda \right]$$

Thus, our main result remains valid also in the case that the parameter λ takes its values in some open subset Ω in $\mathbb{R}^k$.

In the present paper we considered finite-dimensional linear time-invariant systems that are structured and we introduced the notion of the generic rank of the transfer matrix for such systems. We showed that this generic rank can be determined by calculating the maximum number of vertex disjoint paths between two points in a graph that is easily related to the structured system. For simple systems the calculation of the maximum number of vertex disjoint paths can be done by hand, see the two examples, for complicated systems this can be done by the algorithm of Even and Tarjan (cf. [1]). The complexity of this algorithm is of order $O((k+m+p)\sqrt{(n+m+p+2)})$. Here $n+m+p+2$ is the total number of vertices in $G(V,E)$, and $k+m+p$ is the total number of edges.

As an application of our result we proposed a structural version of the well-known almost disturbance decoupling problem for the structured system (4) (cf. [9]), and we derived necessary and sufficient conditions for the solvability of the problem in terms of properties of the associated graph. Results concerning the solvability of structural versions of *the almost disturbance decoupled estimation problem* and *the almost disturbance decoupling problem by measurement feedback* (cf. [10]) can be derived in a similar way.

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