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ON THE PROPAGATION SPEED IN RELATIVISTIC QUANTUM MECHANICS

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A new definition of propagation speed is proposed. Assuming that there is a maximal speed, we derive an operator condition that resembles the condition of local commutativity in quantum field theory. The relativistic energy-momentum relation implies that the maximal speed is not smaller than the speed of light. We discuss an example that shows a superluminal propagation speed.

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If one wants to investigate whether or not physical effects can propagate faster than light in relativistic quantum mechanics, one is faced with the problem that velocity cannot simply be defined as the time derivative of position, because the concept of position itself is disputed [1]. An alternative definition of velocity has been given by Ahmad and Wigner [2] (see also ref. [3]), but that only applies to a freely moving particle. We need a more general definition.

For inspiration, we first look at the propagation of disturbances in classical continuum physics. There the simplest way to determine experimentally the propagation speed in a medium is by creating a perturbation in a small region at a certain time and by measuring how long it takes before the resulting disturbance reaches a detector placed at a given distance. For simplicity, let us assume that the perturbation begins at time $t=0$. Let it act only within a region of radius b around the point with coordinate vector $\vec{x}_C$, as measured with respect to some inertial frame of reference. Assume that the detector is of such size that the smallest fictitious sphere in which it can be enclosed has radius a and let its position be such that the centre of that sphere has coordinate vector $\vec{x}_D$. If the propagation speed is not larger than v, then the detector will not register the disturbance at time t as long as

$$|\vec{x}_D - \vec{x}_C| > r + vt, \qquad t > 0, \tag{1}$$

where $r = a + b$.

If we restrict ourselves to translating the detector without rotating it, then instead of $\vec{x}_D$ we may also use the coordinate $\vec{x}$ of an arbitrary marking point on the detector to fix its position. For in that case the vector $\vec{\xi}_D := \vec{x}_D - \vec{x}$, which connects the marking point with the fictitious centre $\vec{x}_D$, is a constant vector and therefore the inequality (1) is equivalent to

$$|\vec{x} - \vec{x}_0| > r + vt, \quad t > 0, \tag{2}$$

where $\vec{x}_0 := \vec{x}_C - \vec{\xi}_D$ denotes the position of the marking point when the centre of the detector coincides with the centre of the perturbation. This inequality has the advantage over condition (1) that it does not depend on the knowledge of the exact position of the detector, but only on the displacement $\vec{x} - \vec{x}_0$ of the marking point. Therefore it can also be used in relativistic quantum mechanics, where translation is well defined, while position is not.

We now turn to that quantum mechanical situation. As unperturbed system we choose a conservative system with Hamiltonian H acting in a Hilbertspace $\mathcal{H}$. We assume that the system is translationally invariant so that H commutes with the three mutually commuting components of the total momentum operator $\vec{P} = (P_1, P_2, P_3)$. Later on we shall demand H and $\vec{P}$ to obey in addition the relativistic spectral condition (units are chosen such that $\hbar = c = 1$)

$$H \geqslant 0 \quad \text{and} \quad H^2 \geqslant \vec{P}.\vec{P}. \tag{3}$$

As above, we introduce at $t = 0$ a perturbation, which may be variable in time. The unitary evolution operator associated with the perturbed system will be denoted by $\hat{U}_t$, with the convention that $\hat{U}_t = U_t$ for $t \leqslant 0$, where $U_t := e^{-itH}$ is the evolution operator of the unperturbed system.

Let A be the bounded selfadjoint operator that represents the detector when it is in some arbitrary chosen, but henceforth fixed position with respect to the inertial frame of reference. Then the same detector, when translated by $\vec{x} \in \mathbb{R}^3$, is represented by

$$A_{\vec{x}} := e^{-i\vec{x}.\vec{P}} A \, e^{i\vec{x}.\vec{P}}. \tag{4}$$

It may be that in the perturbed situation a different representation must be assigned to the detector. That will be the case, for instance, when the detector measures an electric or magnetic quantity and the perturbation changes the electromagnetic field. We shall denote the representation in the perturbed situation by $\hat{A}_{\vec{x}}$, even when it is not different from the representation (4).

The effects of the perturbation are not measurable (in the quantum mechanical sense) by the detector at the position indicated by $\vec{x}$ at time t, as long as the expectation value

$$\hat{\mathcal{E}}(t,\vec{x}) := \langle \hat{U}_t \phi_0, \; \hat{A}_{\vec{x}} \hat{U}_t \phi_0 \rangle = \langle \phi_0, \; \hat{U}_t^\dagger \hat{A}_{\vec{x}} \hat{U}_t \phi_0 \rangle \tag{5}$$

in the perturbed situation is equal to the corresponding expectation value

$$\mathcal{E}(t,\vec{x}) := \langle U_t\phi_0,\ A_{\vec{x}}\, U_t\phi_0 \rangle = \langle \phi_0,\ U_t^{\dagger} A_{\vec{x}}\, U_t \phi_0 \rangle \tag{6}$$

in the unperturbed case. Here $\langle .,. \rangle$ denotes the inner product on $\mathcal{H}$ and $\dagger$ the adjoint operator. ϕ_0 is the unit vector in $\mathcal{H}$ that represents the state at $t = 0$, which by assumption is the same for both systems. (Strictly speaking, the disturbance is still detectable if the higher moments of the statistical distributions are not equal, too. But their equality follows automatically when we assume that $\hat{\mathcal{E}}(t,\vec{x}) = \mathcal{E}(t,\vec{x})$ for all unit vectors $\phi_0 \in \mathcal{H}$, as we shall presently do.)

If the region in IR^3 where $\hat{\mathcal{E}}(t,.)$ is different from $\mathcal{E}(t,.)$ grows with t, then we have clearly an expanding disturbance. In analogy with the classical situation and especially in reference to condition (2), we shall say in quantum mechanics that the detector and the perturbator have *finite extent* and that the disturbance *spreads* with velocity larger than v, if

$$\hat{\mathcal{E}}(t,\vec{x}) = \mathcal{E}(t,\vec{x}) \quad \text{for} \quad |\vec{x}-\vec{x}_0| > r+vt, \quad t>0, \tag{7}$$

where r and v are positive constants and $\vec{x}_0 \in \mathbb{R}^3$.

Notice that this condition furnishes not only a bound v for the spreading velocity, but also an estimate r for the sum of the sizes of perturbator and detector. It may be that we do not find the same values r and v for a different initial state ϕ_0. However, if the detector and the perturbator are really of finite size, and if there is a maximal propagation speed, just as in classical relativistic physics, then condition (7) with r and v replaced by some fixed r_m and v_m, should hold for all unit vectors $\phi_0 \in \mathcal{H}$. This is equivalent to the requirement that

$$\hat{U}_t^{\dagger} A_{\vec{x}} \hat{U}_t = U_t^{\dagger} A_{\vec{x}} U_t \quad \text{for} \quad |\vec{x}-\vec{x}_0| > r_m + v_m t, \quad t > 0. \tag{8}$$

The smallest value of v_m for which (8) still holds, will be called the maximal asymptotic spreading (MAS) velocity.

We shall investigate now under what conditions equation (8) may be valid. If the perturbation is represented by a time independent bounded selfadjoint operator B, then $\hat{U}_t = \exp[-it(H+B)]$ and $d\hat{U}/dt = -i(H+B)\hat{U}_t = -i\hat{U}_t(H+B)$ for $t > 0$. Hence, by differentiating (8) with respect to t we find in that case the condition

$$A_{t,\vec{x}} B = B A_{t,\vec{x}} \quad \text{for} \quad |\vec{x}-\vec{x}_0| > r_m + v_m t, \quad t>0, \tag{9}$$

where

$$A_{t,\vec{x}} := U_t^{\dagger} A_{\vec{x}} U_t = e^{itH - i\vec{x}.\vec{P}}\, A\, e^{-itH + i\vec{x}.\vec{P}}. \tag{10}$$

If B is time dependent, then $H+B$ does not commute with $\hat{U}_t$, in general, and therefore (9) is not valid. However, if we restrict ourselves to perturbations of finite duration, then we can derive a similar condition. For if the perturbation ends at $t = \tau > 0$, say, then $\hat{U}_t = U_{t-\tau}\hat{U}_\tau$ for $t \geqslant \tau$. Inserting this into (8), multiplying the result by $\hat{U}_\tau$ from the left and by $U_\tau^{\dagger}$ from the right and taking into account that $\hat{A}_{\vec{x}} = A_{\vec{x}}$ for $t > \tau$, we obtain

$$A_{t',\vec{x}} W = W A_{t',\vec{x}} \quad \text{for} \quad |\vec{x}-\vec{x}_0| > r_m' + v_m t', \quad t'>0, \tag{11}$$

where

$$W := \hat{U}_\tau U_\tau^{\dagger}, \quad t' := t-\tau \quad \text{and} \quad r_m' := r_m + v_m \tau. \tag{12}$$

W is a unitary operator that does not depend on the time t. From the relation $\hat{U}_\tau = W U_\tau$ it follows that W assigns to the state $U_\tau\phi_0$ of the unperturbed system at time τ the state $\hat{U}_\tau\phi_0$, which in the perturbed system at that time has evolved from the common initial state ϕ_0. As the perturbation stops at $t = \tau$ and the subsequent evolution is again governed by the original Hamiltonian H, the whole effect of the perturbation is for $t \geqslant \tau$ completely fixed by W. It is not surprising then, that the only quantity in (9) that bears on the type of perturbation is just this operator W.

Henceforth we shall discuss only condition (9). The theorems and examples that will be given, can be made to apply to (11) simply by taking $W = e^{iB}$, where B is selfadjoint.

We observe that, although we have been working in the context of ordinary quantum mechanics, condition (9) agrees with the concept of locality in quantum field theory (henceforth QFT) as introduced by Haag [4]. From the covariance of the field and from local commutativity it follows that local observables, which are associated with bounded space-time regions, satisfy (9) (with $v_m = 1$) and therefore have finite extent by our definition. Thus, if we take $\mathcal{H}$ to be a Hilbert space of QFT, it is simple to find operators B and A which produce and measure a finite disturbance that propagates with a velocity, that is not larger than the speed of light.

The postulate of local commutativity is based on the assumption that no physical influence can propagate faster than light (Einstein causality). In contrast with this, there is no a priori restriction to the velocity in our framework. It turns out, however, that the relativistic spectral condition sets a lower bound to the MAS velocity. For we can prove the following theorem.

Theorem 1. Let H and $\vec{P}$ obey the relativistic condition (3). If A and B are bounded operators on $\mathcal{H}$ with the property that (9) holds with $v_m < 1$, then

$$A_{t,\vec{x}}B = BA_{t,\vec{x}} \quad \text{for all } \vec{x} \in \mathbb{R}^3 \quad \text{and} \quad t > 0. \tag{13}$$

Thus, either we find a maximal speed that is at least equal to the speed of light, or we measure no disturbance at all. The latter situation will occur, for example, when the detector is not sensitive to the perturbation concerned.

Theorem 1 may be regarded as the counterpart of Araki's generalized version of Borchers' theorem in QFT [5]. Since the details of the proof are not essential for the understanding of the rest of this article, we omit it here.

Remark. If in theorem 1 we impose instead of (3) the nonrelativistic spectral condition $H \geq \vec{P}.\vec{P}\,/2M +$ constant, then the assumption that (9) holds for some finite value of v_m already implies (13).

We have not yet found examples of operators A and B in ordinary quantum systems that satisfy (9) with a finite MAS velocity. The method for constructing local observables in QFT depends essentially on the existence of a translationally invariant vacuum. Therefore it cannot be mimicked in ordinary quantum mechanics, where no such vacuum state exists. We can prove that if there are operators of finite extent in a relativistic one-particle system, then the corresponding MAS velocity is equal to 1. In systems of two or more particles, however, superluminal velocities may occur. Actually, we have found:

Theorem 2. Let $\mathcal{H}$ be the Hilbertspace of a system of two free particles of mass m. If A and B are bounded operators on $\mathcal{H}$ which satisfy (9) with MAS velocity $\gamma \geq 1$, then for each $\alpha > 1$ there exist also a pair $A^{(\alpha)}, B^{(\alpha)}$ which satisfies (9) with MAS velocity equal to $\alpha\gamma$.

Proof: The construction of that pair is most easily explained when we use a representation in which the total momentum $\vec{P} = \vec{p}_1 + \vec{p}_2$ and the relative momentum $\vec{Q} := \vec{p}_1 - \vec{p}_2$ of the two particles act according to

$$\vec{P}\psi(\vec{k},\vec{l}) = \vec{k}\psi(\vec{k},\vec{l}) \quad , \quad \vec{Q}\psi(\vec{k},\vec{l}) = \vec{l}\psi(\vec{k},\vec{l}) \tag{14}$$

for all those $\psi \in \mathcal{H}$ which are in the domain of $\vec{P}$ or $\vec{Q}$. The unperturbed Hamiltonian H acts on $\psi \in D(H)$ as

$$H\psi(\vec{k},\vec{l}) = \epsilon(\vec{k},\vec{l})\,\psi(\vec{k},\vec{l}), \quad \text{where} \quad \epsilon(\vec{k},\vec{l}) := \left[(\frac{\vec{k}+\vec{l}}{2})^2 + m^2\right]^{\frac{1}{2}} + \left[(\frac{\vec{k}-\vec{l}}{2})^2 + m^2\right]^{\frac{1}{2}}. \tag{15}$$

When $\vec{k}$ is kept fixed, the energy ranges over the halfline $\epsilon \geq (k^2 + 4m^2)^{\frac{1}{2}}$ in such a way that for each $\vec{l} \neq \vec{0}$ there is (for the given $\alpha > 1$) precisely one $\lambda_\alpha > 1$ such that $\epsilon(\vec{k}, \lambda_\alpha \vec{l}) = \alpha\epsilon(\vec{k},\vec{l})$. Hence there exists a partial isometry S_α on $\mathcal{H}$ with the property that $S_\alpha\psi(\vec{k},\vec{l}) = \psi(\vec{k},\lambda_\alpha\vec{l})\sqrt{J_\alpha}$ for each $\psi\in\mathcal{H}$, where J_α is the Jacobian of the transformation $(\vec{k},\vec{l})\mapsto(\vec{k},\lambda_\alpha\vec{l})$. The range of S_α is all of $\mathcal{H}$, the kernel of S_α is the set of functions ψ whose support contains only those points $(\vec{k},\vec{l})$ for which $\epsilon(\vec{k},\vec{l}) < \alpha(k^2+4m^2)^{\frac{1}{2}}$.

From the construction it follows that $S_\alpha H = \alpha H S_\alpha$ and $S_\alpha\vec{P} = \vec{P}S_\alpha$. This means that S_α enhances the energy by a factor α, while the total momentum is unaffected. Consequently, if we define $A^{(\alpha)}: = S_\alpha^\dagger B S_\alpha$, then

$$A_{t,\vec{x}}^{(\alpha)} : = S_\alpha^\dagger \, e^{-i\alpha t H - i\vec{x}.\vec{P}} \, A \, e^{-i\alpha t H + i\vec{x}.\vec{P}} \, S_\alpha = S_\alpha^\dagger \, A_{\alpha t,\vec{x}} \, S_\alpha. \tag{16}$$

Therefore

$$A_{t,\vec{x}}^{(\alpha)} B^{(\alpha)} = B^{(\alpha)} A_{t,\vec{x}}^{(\alpha)} \qquad \text{for } |\vec{x}-\vec{x}_0| > r_m + \alpha v_m t, \ t > 0 \, , \tag{17}$$

if A and B satisfy condition (9). On the other hand (17) implies (9). Thus, we have proved that if γ is the smallest value of v_m for which (9) holds, then (17) is satisfied with MAS velocity $\alpha\gamma$. □

A similar construction can be used for two particles with unequal masses and for systems of more than two particles. The trick does not work for a single particle, because in that case the absolute value of the momentum determines the energy ϵ uniquely, so that it is not possible to enhance the energy while keeping the momentum fixed.

We deduce from the above that in any system of two or more freely moving particles for which operators A and B of finite extent exist, we can create with the help of the enhanced perturbation $B^{(\alpha)}$ a disturbance, which appears to be travelling with superluminal velocity, when measured by the detector $A^{(\alpha)}$. However, before we jump to the conclusion that we have here a clear case of violation of Einstein causality, we had better first examine a concrete example in more detail. For that we turn again to QFT.

It appears that the construction of theorem 2 can also be applied to local field operators of the form $\Phi_f: = \int f(y_1,y_2)\Phi(y_1)\Phi(y_2)d^4y_1d^4y_2$, where $\Phi(y)$ is a free quantum field of mass m and f is a C^∞-function of compact support. The operators $A: = \Phi_f$ and $B: = \Phi_g$, where g is another C^∞-function of compact support, satisfy condition (9) with $v_m = 1$ and a finite value of r_m. By enhancing the energy by a factor $\alpha>1$ (this is most easily done by first changing to the momentum representation), we obtain operators $A^{(\alpha)} = \Phi_{f_\alpha}$ and $B^{(\alpha)} = \Phi_{g_\alpha}$, that satisfy (9) with $v_m = \alpha$. Hence, superluminal MAS velocities do occur in QFT, too. On closer inspection, however, it turns out that f_α and g_α are not of compact support. This is due to the fact that the mapping in momentum space, which enhances the energy, is not entire analytic. Because of that the Fourier transforms $\tilde{f}$ and $\tilde{g}$ of f and g, which are entire analytic functions, are changed under that mapping into non-entire functions $\tilde{f}_\alpha$ and $\tilde{g}_\alpha$. This implies that their inverse Fourier transforms f_α and g_α are not of compact support. (Actually, the situation is a bit more complicated, because the energy-enhancing map is only defined on the energy-shell. But that point is easily taken care of.)

Thus, we find that operators $A^{(\alpha)}$ and $B^{(\alpha)}$ can be constructed which satisfy condition (9) for a finite value of r_m and v_m, and therefore have finite extent according to our definition, but which are not of finite extension by the standards of QFT. This indicates that our definition is too broad. Probably, the fact that our definition refers to only one combination of perturbator and detector is the cause of that. In this connection we observe that if the perturbator B is truly of finite size and the propagation speed is bounded, then condition (9) must hold not merely for one particular detector A, but also for any other operator A' that represents a detector of finite size (with the value of r_m adapted to that size, of course). Hence, from all couples (A, B) which satisfy (9), only those should be accepted which belong to the largest subset $\mathfrak{L}$ with the property that with any two couples (A_i,B_i) and (A_j,B_j) from $\mathfrak{L}$ also the mixed pairs (A_i,B_j) and (A_j,B_i) belong to $\mathfrak{L}$. We shall call this the test for consistency. The field operators Φ_{f_α} and Φ_{g_α} constructed above do not stand this test, because the pairs (Φ_f,Φ_{g_α}) and (Φ_{f_α},Φ_g) do not satisfy (9) for arbitrary f and g of compact support. This shows that the consistency test is effective in exposing perturbators and detectors of pseudo-finite extent. We have not yet found out whether the operators $A^{(\alpha)}$ and $B^{(\alpha)}$ of theorem 2 also fail the consistency test.

Finally we mention that the localization operators of Wightman [6] and of Jauch and Piron [7] do not satisfy condition (9) for any finite value of v_m. (This agrees with a result obtained by Schlieder [8] in the context of QFT.) However, it appears that for such operators the effect of that part of the disturbance, which travels faster than light, drops off very rapidly with increasing distance. For example, if A and B are projection operators on the states of a relativistic particle of mass m which are localized in balls with centre in the origin and radii a and b, respectively, then we find the estimate

$$\|A_{t,\vec{x}}B - BA_{t,\vec{x}}\| < \text{constant} \times e^{-m(d^2-t^2)^{\frac{1}{2}}} \quad \text{for } d := |\vec{x}| - a - b > |t|. \tag{18}$$

This implies that superluminal effects are vanishing small at macroscopic distances. (A similar conclusion was reached in [9].)

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