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DYNAMICAL SYSTEMS AND NUMERICAL INTEGRATION

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A few schemes for the numerical integration of ordinary differential equations are considered as discrete dynamical systems. The properties of those systems centered around the Poincaré-Birkhoff theorem and the Kolmogorov-Arnold-Moser (KAM) theorem may give a deeper understanding in the global behaviour of integration schemes and may explain the occurrence of unwanted phenomena such as "chaotic" oscillations. This approach is of wider generality than the usual technique of Taylor expansions, a technique which is not always justified and may be even wrong.

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1. INTRODUCTION

When trying to integrate differential equations by means of numerical methods one often observes the appearance of unwanted spurious solutions or unrealistic oscillation. Highly efficient numerical schemes have been devised which do not have such complications, at least when the discretisation parameters are sufficiently small and if solutions are considered in a proper compact subspace without singularities. In order to be more specific we consider only ordinary differential equations which can be written in the form of an autonomous system

$$(1.1) \qquad \dot{x}_i = f_i(x_1,x_2,\ldots,x_N), \qquad i = 1,2,\ldots,N,$$

subjected to an initial condition.
Switching to vector notation we may approximate this by a one-step numerical process

$$(1.2) \qquad \vec{x}_{n+1} = \vec{x}_n + hf(\vec{x}_n)$$

or more generally

$$(1.3) \qquad \vec{x}_{n+1} = F(\vec{x}_n,\vec{x}_{n+1},\ldots,\vec{x}_{n+p},h)$$

where h is the discretisation of time t. Usually such a scheme is analysed by means of Taylor expansions of the kind

$$(1.4) \qquad \vec{x}_{n+1} = \vec{x}_n + h\dot{\vec{x}}_n + \tfrac{1}{2}h^2\ddot{\vec{x}}_n + \ldots .$$

However, it is not a priori clear that this is always a correct procedure. Simple examples show that it can be true almost always or almost never. The reason for this may be as follows. The solution space of (1.3) can be much larger than that of (1.1). To a smooth trajectory of (1.1) we may associate a discrete orbit of (1.3) for which (1.4) is valid. However, an arbitrary orbit of (1.3) does not necessarily converge to a trajectory of (1.1) as $h \to 0$, and $\vec{x}_{n+1}$ and $\vec{x}_n$ may not be close together for $h \to 0$. Only orbits of some invariant subspace of (1.3) may converge to trajectories of (1.1) and only for those orbits the Taylor procedure may be justified.

It seems imperative to investigate the behaviour of a discrete iteration map (1.3) by a method which does not use the Taylor expansion (1.4) as the only instrument. Fortunately discrete dynamical systems like (1.3) have been the object of intensive research and important and far-reaching results have been obtained in the past few decennia. The author believes that the theory of numerical integration schemes may profit greatly from what is known and still will be known about discrete dynamical systems. Our ideas will be made clear by considering a few special cases of the utmost simplicity. In fact we only try to solve the differential equations

$$\dot{x} = f(x) \text{ and } \ddot{x} = f(x)$$

by means of some standard numerical procedure. The overall picture is that of a perturbation of a regular dynamical system where h acts on the bifurcation parameter. In some cases the behaviour is that of a non-integrable Hamiltonian system which is a perturbation of a integrable system. Then all features predicted by the Poincaré-Birkhoff theorem and the KAM-theorem may appear. Closed orbits of the continuous system may remain closed as invariant tori or may break up into island chains and chaotic structures.

This research started when we looked for nice iterative two-dimensional maps where the techniques of discrete dynamical systems could be applied. Apparently a similar line of thought has been followed by Yamaguti and Ushiki [2], [3] who studied in particular the logistic equation $\dot{x} = 1-x^2$ using a central difference scheme. In section 3 we consider the same model is a more general setting. This is preceded by the even simple model $\dot{x} = 0$ in section 2. Although looking trivial the discussion of this most simple differential equation reveals almost dramatically the possible effects of a discretisation. In section 4 we consider the numerical integration of the equation of a non-linear pendulum $\ddot{x} + f(x) = 0$. Its discretisation can be considered as a perturbation of a Hamiltonian system. The corresponding map either dissipative or area-preserving shows a wealth of bifurcation phenomena. Closed trajectories may be turned into spirals with a possible convergence to a limit cycle or they may break up into island chains. There may be even chaos. Similar problems are also considered in a recent paper by PEITGEN [4].

2. THE SIMPLEST DIFFERENTIAL EQUATION

We consider the d.e.

(2.1) $$\dot{u} = 0 , \quad t > 0$$

to be solved numerically using the discretisation

(2.2) $$\dot{u} \approx \frac{(1+s)u_{n+1}-2su_{n}-(1-s)u_{n-1}}{2h}, \qquad s > -1.$$

Thus (2.1) is replaced by the interative process

(2.3) $$(1+s)u_{n+1} - 2su_{n} - (1-s)u_{n-1} = 0.$$

This is a linear difference equation the general solution of which is

(2.4) $$u_{n} = C_{1}\lambda_{1}^{n} + C_{2}\lambda_{2}^{n}$$

where λ_1, λ_2 are the roots of

(2.5) $$(1+s)\lambda^{2} - 2s\lambda - (1-s) = 0,$$

giving

(2.6) $$\lambda_{1} = 1 \qquad \lambda_{2} = \frac{s-1}{s+1}.$$

In order to obtain a unique solution of (2.3) we need two starting values u_0 and u_1. Then we find

(2.7) $$u_{n} = \tfrac{1}{2}(1-s)u_{0} + \tfrac{1}{2}(1+s)u_{1} + C\left(\frac{s-1}{s+1}\right)^{n}$$

where C is determined by taking n = 0. For s > 0 the process converges and

(2.8) $$u_{n} \to \tfrac{1}{2}(1-s)u_{0} + \tfrac{1}{2}(1+s)u_{1},$$

the weighted mean of the two starting values.

For $s = 0$ (2.2) becomes the central difference approximation

(2.9) $$\dot{u} \approx (u_{n+1}-u_{n-1})/(2h).$$

Since then $\lambda_2 = -1$ the process (2.3) becomes weakly stable with

(2.10) $$\begin{cases} u_n = u_0 & \text{for} \quad n \text{ even,} \\ u_n = u_1 & \text{for} \quad n \text{ odd.} \end{cases}$$

Finally for $s < 0$ the process diverges.

It is instructive to describe (2.3) as an iterative planar map by introducing a second variable v_n

(2.11) $$\begin{cases} u_{n+1} = v_n \\ v_{n+1} = (1-a)u_n + av_n \end{cases}$$

with

$$a = \frac{2s}{1+s} .$$

For $a = 1$ (2.11) reduces to the identity. For $0 < a < 1$ is an oblique reflection with respect to the axis $u = v$ and with the reduction factor $1-a$ as shown in fig.2.1

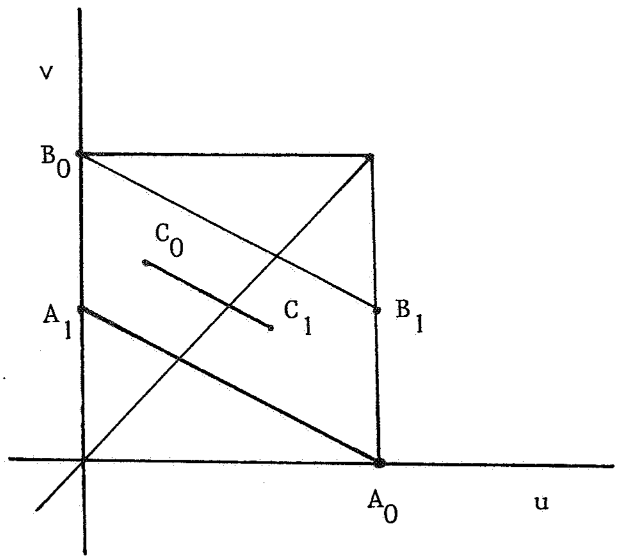

Fig. 2.1

Any sequence $P_0P_1P_2$.. converges at a point of the diagonal $u = v$.
For $a = 0$ (2.11) is the mirror-reflection with respect to $u = v$.
For $a > 1$ the map preserves orientation and is area-reducing with the constant factor $a-1$. It is illustrated in fig.2.2

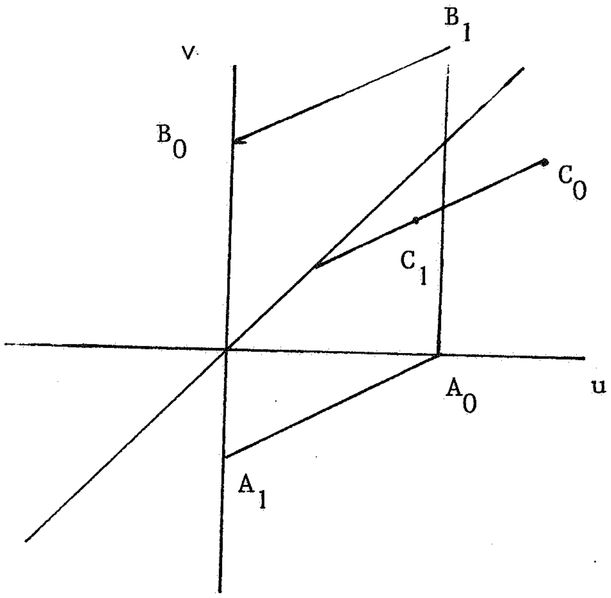

fig. 2.2.

3. A SIMPLE DIFFERENTIAL EQUATION

We consider the d.e.

(3.1) $\dot{u} = 1-u^2, \quad t > 0.$

Up to an arbitrary time shift this equation has the solution

(3.2) $u(t) = \dfrac{\exp(2t)-1}{\exp(2t)+1}$

which represents the well-known logistic curve. The simplest discretisation, (2.2) with $s = 1$, gives

(3.3) $u_{n+1} = u_n + h(1-u_n^2),$

or

$$(3.4) \qquad x_{n+1} = ax_n(1-x_n),$$

with

$$(3.5) \qquad u_n = -1 + \frac{1+2h}{h} x_n$$

and

$$(3.6) \qquad a = 1 + 2h.$$

Thus (3.3) is equivalent to the well-known and very popular logistic difference equation. We know that for any start in (3.1) x_n converges to the limit $1-1/a$ provided $1 < a \leq 3$ and that it converges for $3 < a < 1+\sqrt{6}$ to a periodic cycle of period 2 etc. For $a > 4$ the process always diverges For (3.3) this means that for any starting value u_0 satisfying

$$0 < u_0 < 1+1/h$$

the iterative process converges to $u = 1$ provided $h \leq 1$. For $0 < h < \frac{1}{2}$ convergence is monotonous but for $\frac{1}{2} < h < 1$ we have an oscillating behaviour due to a negative multiplier of the attracting fixed point. At $h = 1$ a multiplier becomes -1 as a prelude to flip bifurcation.
For $1 < h < \sqrt{3}/2 = 1.225...$ u_n converges to a period-two cycle. In table 3.1 a few values of u_n up to $h = 20$ are given for $h = 0.2, 0.4, 0.8, 1.1$ with the start $u_0 = -0.9$.

As the next special case we take the discretisation (2.2) with $s = 0$, the central difference approximation,

$$(3.7) \qquad u_{n+1} - u_{n-1} = 2h(1-u_n^2)$$

or

$$u_{n+1} = v_n,$$
(3.8)
$$v_{n+1} = u_n + 2h(1-v_n^2)$$

in the form of an iterative planar map.

0.2	0.4	0.8	1.1
-0.900	-0.900	-0.900	-0.900
-0.862	-0.824	-0.748	-0.691
-0.811	-0.696	-0.396	-0.116
-0.742	-0.489	0.279	0.969
-0.652	-0.185	1.017	1.036
-0.537	0.202	0.990	0.955
-0.395	0.585	1.006	1.052
-0.226	0.848	0.996	0.935
-0.036	0.960	1.002	1.073
0.163	0.991	0.999	0.906
0.358			1.103
0.532			0.865
0.676			1.142
0.784			0.807
0.861			1.191
0.913			0.731
0.946			1.243
0.967			0.644
0.980			1.288
0.988			0.563

Table 3.1.

The map (3.8) is area-preserving with the negative sign. Its bifurcation phenomena are present already for small values of h but in the part of the plane that interests us they are perhaps too small for causing trouble. The usual fixed points (-1,-1) and (1.1) are both saddles with multipliers

$2h \pm \sqrt{4h^2+1}$ and $-2h \pm \sqrt{4h^2+1}$. The points (1,-1) and (-1,1) form a period-two cycle. The corresponding multipliers are the root, of

$$\lambda^2 - 2(1-16h^2)\lambda + 1 = 0.$$

Thus they are saddles for $h^2 > 1/8$ and elliptic for $h^2 < 1/8$.
For $h \to 0$ the map (3.8) desintegrates to the mirror map of fig.2.2.
This shows that the usual technique of smoothing (3.8) by means of Taylor expansions must fail here since (u_{n+1}, v_{n+1}) and (u_n, v_n) are not close for small h. However, the squared map

$$\text{(3.9)} \qquad \begin{cases} u_{n+2} = u_n + 2h(1-v_n^2) \\ v_{n+2} = v_n + 2h\{1-(u_n+2h-2hv_n^2)\} \end{cases}$$

converges to the identity map for $h \to 0$ so that the Taylor technique can be used here. The approximations

$$\text{(3.10)} \qquad \begin{cases} u_{n+2} = u_n + 2h\dot{u}, \\ v_{n+2} = v_n + 2h\dot{v}, \end{cases}$$

show that (3.9) can be approximated by the continuous dynamical systems

$$\text{(3.11)} \qquad \begin{cases} \dot{u} = 1-v^2, \\ \dot{v} = 1-u^2. \end{cases}$$

Integration is elementary and we find trajectories determined by

$$\text{(3.12)} \qquad (v^3-u^3) - 3(v-u) = C.$$

The trajectories through the fixed points $(\pm 1, \mp 1)$ are composed of the line $u = v$ and the ellipse $u^2+uv+v^2 = 3$.
In fig.3.1 a few trajectories of (3.11) are given. In fig.3.2 we have given

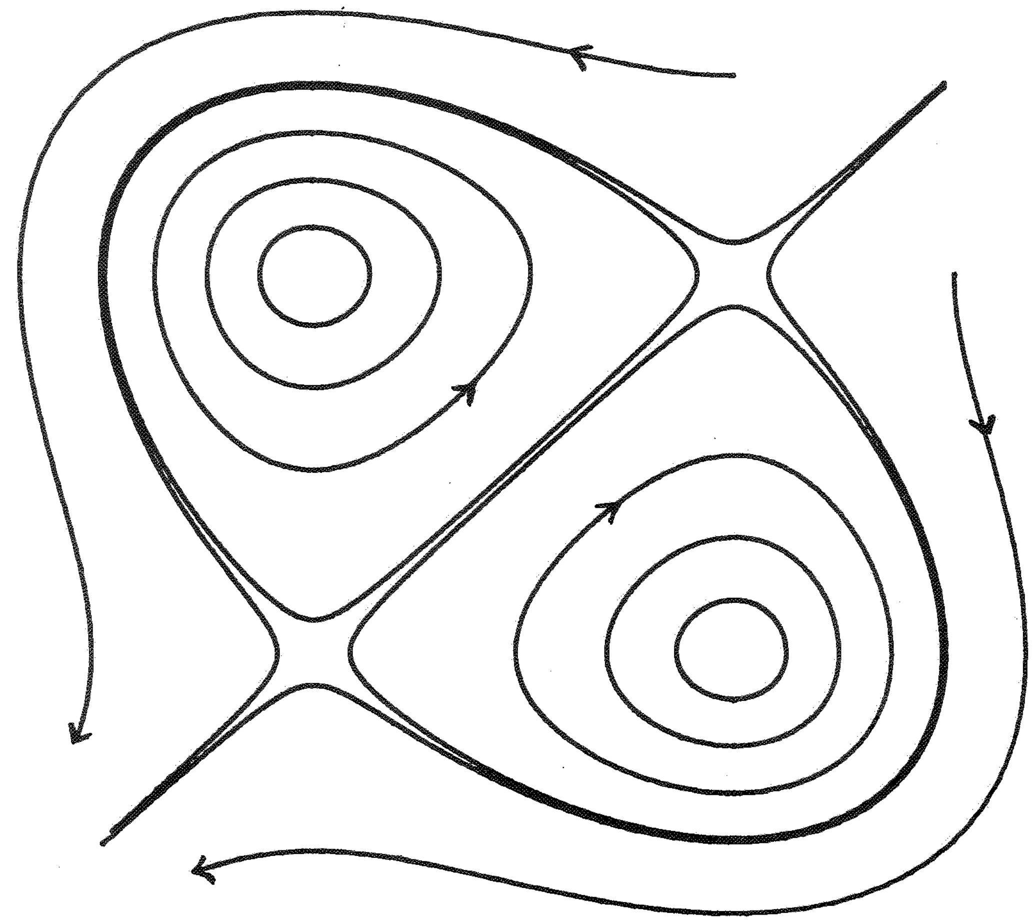

Fig. 3.1. $x' = 1 - y^2$, $y' = 1 - x^2$.

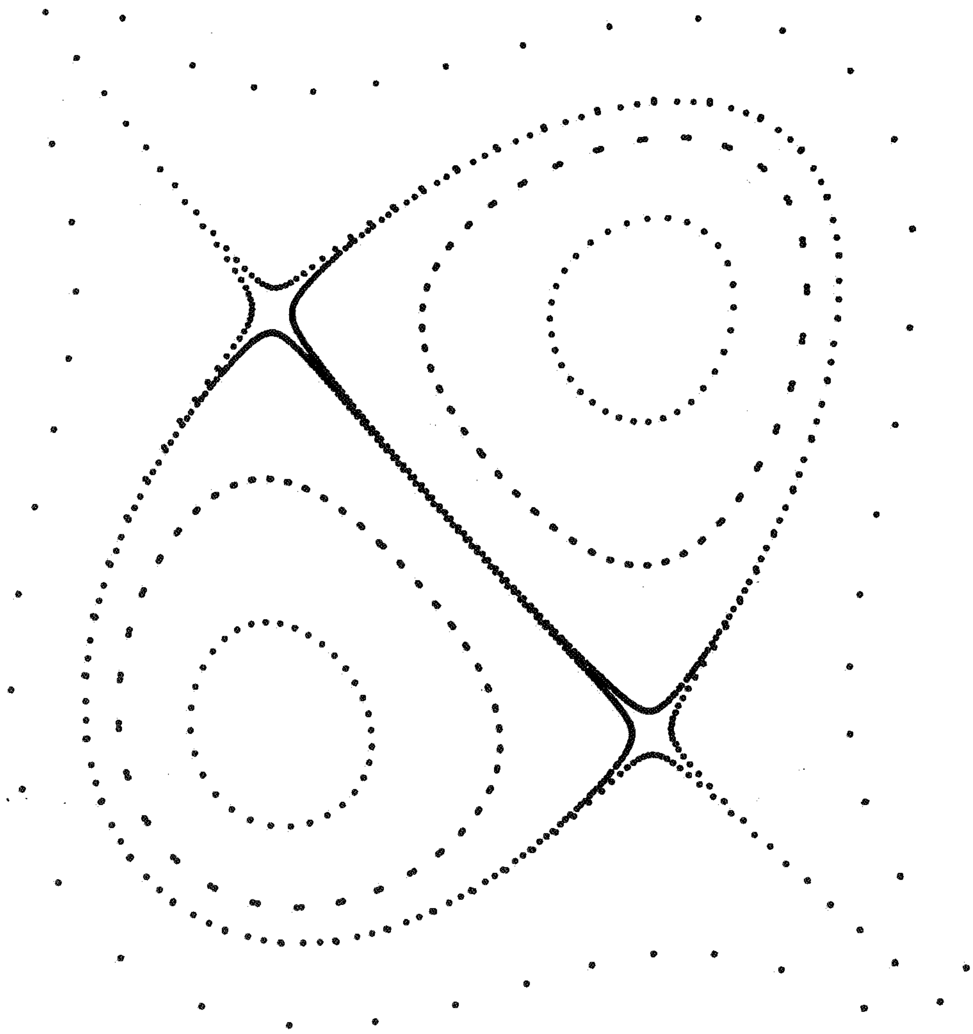

Fig. 3.2. $u' = v, \ v' = u + 2h(1-v^2)$
$h = 0.05.$

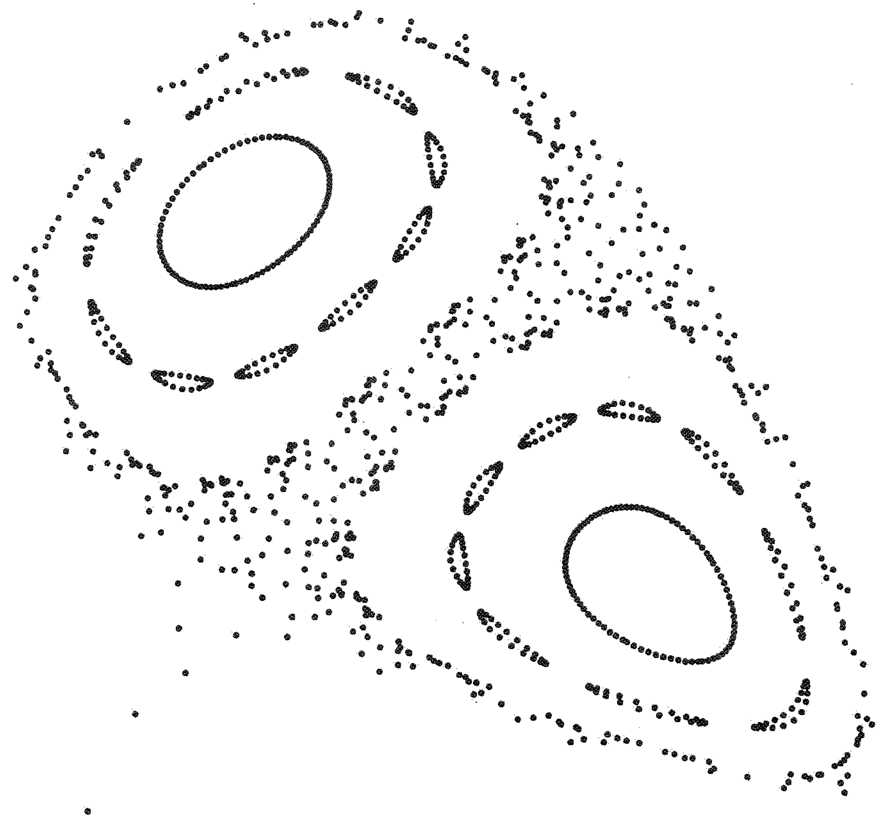

Fig. 3.3. $u' = v$, $v' = u + 2h\,(1-v^2)$

$h = 0.2$

a few orbits of the iterative map (3.8) with $h = 0.05$. Numerical integration of the original d.e. means that we take a starting point (u_0,v_0) close to the fixed point, say $(-1,-1)$. The successive points are close to the unstable manifold of $(-1,-1)$ which they follow in a zig-zag pattern. They trace out a perhaps imperceptibly perturbed closed orbit, a form of the KAM-theorem, and eventually they leave the neighbourhood of $(1\ 1)$ along the elliptical part of the separatrix.

Fig.3.3 shows what would happen for much larger values of h. For $h = 0.1$ we observe the familiar Birkhoff pattern of island formation.

As the final example in this section we try to solve (3.1) by a Runge-Kutta scheme. If u is the starting value of u(t) for $t = t_0$ of the d.e.

$$\dot{u} = f(u) \tag{3.13}$$

next values u' and u" are determined by

$$\begin{cases} u' = u + hf(u) \\ u'' = u + hf(u') \end{cases}$$

and the value of u for $t = t_0 + h$ is taken as $\frac{1}{2}(u'+u'')$. This gives the iterative process

$$u_{n+1} = u_n + \tfrac{1}{2}hf(u_n) + \tfrac{1}{2}hf(u_n+hf(u_n)). \tag{3.14}$$

In the special case (3.1) we have

$$u_{n+1} = u_n + h(1-u_n^2) - h^2u_n(1-u_n^2) - \tfrac{1}{2}h^3(1-u_n^3)^2. \tag{3.15}$$

Again we have fixed points at ± 1 of the usual kind. However, if h is increasing at $h = 1$ two new fixed points are appearing

$$x = (1\pm\sqrt{h^2-1})/h.$$

They are both stable in the interval $1 < h < \sqrt{2}$ but at $h = \sqrt{2}$ they are subjected to flip bifurcation as the beginning of a period-doubling sequence.

For $b \to 0$ the map (3.13) becomes the linear map (2.11). For the latter map successive points (u_n,v_n) and (u_{n+1},v_{n+1}) are not close unless we consider points of the line $u = v$ which are fixed points each. This means that for small h generally successive points of (3.13) are not close unless we consider points on the invariant manifold connecting the two fixed points $(-1,-1)$ and $(1,1)$.

Therefore the usual technique of analysing the iterative process (3.13) by means of a continuous system using Taylor expansions of the kind

$$u_{n+1} = u_n + h\dot{u}_n + \tfrac{1}{2}h^2\ddot{u}_n + \ldots$$

is doomed to fail. This procedure is only justified if we are on the invariant manifold between the two fixed points.

The reason for this is that the interative process has more solutions than the original differential equation (3.1) which it approximates. A solution of the differential equation requires a single initial condition $u = u_0$ for $t = 0$ whereas a solution of (3.13) requires two initial values u_0 and v_0. This suggests that for obtaining a good approximation to the solution of the d.e. it is important to follow the invariant curve, i.e. the unstable manifold of $(-1,-1)$ as close as possible.

4. A NONLINEAR PENDULUM

In this section we consider a few numerical difference schemes for the equation of the nonlinear pendulum

(4.1) $$\ddot{x} + f(x) = 0.$$

We consider in particular the two special cases

(4.2) $$f(x) = x^2-1,$$

and

(4.3) $$f(x) = \frac{x(x^2-1)}{x^2+1}.$$

The d.e. (4.1) can be written as the dynamical system

$$(4.4)\qquad \begin{cases} \dot{x} = y \\ \dot{y} = -f(x). \end{cases}$$

Integration gives

$$(4.5)\qquad y^2 + 2\int^x f(\xi)d\xi = C$$

as the equation of the trajectories.
Using the discretisation

$$(4.6)\qquad \ddot{x} \approx \frac{x_{n+1} - 2x_n + x_{n-1}}{h^2}$$

we may approximate (4.1) by the two-step iterative process

$$(4.7)\qquad x_{n+1} - 2x_n + x_{n-1} = -h^2 f(x_n),$$

written as the iterative planar map

$$(4.8)\qquad \begin{cases} x_{n+1} = x_n + hy_n \\ y_{n+1} = y_n - hf(x_{n+1}). \end{cases}$$

Both (4.4) and (4.8) are area-preserving Hamiltonian systems. The discrete system (4.8) can be interpreted as a nonlinear perturbation of the identity, a trivially integrable system. The fixed points of (4.8) coincide with those of (4.4). Their nature depends on the sign of $f'(x_0)$ where $f(x_0) = 0$. For $f'(x_0) > 0$ the fixed point $(x_0,0)$ is elliptic, for $f'(x_0) < 0$ it is hyperbolic.

In the special case (4.2) the interative map (4.8) is a quadratic Hamiltonian map

$$(4.9)\qquad \begin{cases} x_{n+1} = x_n + hy_n, \\ y_{n+1} = y_n - h((x_n+hy_n)^2-1). \end{cases}$$

The fixed point (1,0) is elliptic with the multipliers exp $\pm i\alpha$ with $h = 1 - \cos\alpha$. The fixed point (-1,0) is hyperbolic with the multipliers exp $\pm\beta$ with $h = \cosh\beta - 1$. The strong requirement of being are-preserving makes that all quadratic Hamilton maps can be reduced to the standard form

$$(4.10)\qquad z_{n+1} = e^{i\alpha}(z_n - \tfrac{1}{4}i(z_n+\bar{z}_n)^2)$$

by a single linear transformation of x,y to complex conjugate coordinates $z,\bar{z}$. In real form (4.10) is written as

$$(4.11)\qquad \begin{cases} x_{n+1} = x_n\cos\alpha - y_n\sin\alpha + x_n^2\sin\alpha, \\ y_{n+1} = x_n\sin\alpha + y_n\cos\alpha - x_n^2\cos\alpha. \end{cases}$$

The relation between the old variables of (4.9) and the new ones of (4.11) is of the form

$$\begin{cases} x \to 1 + a_{11}x + a_{12}y, \\ y \to \phantom{1 + {}} a_{21}x + a_{22}y. \end{cases}$$

The parameter α is the phase of the multiplier of the fixed point (1,0) of (4.9) which is located in the origin of (4.10) and (4.11). The map (4.11) has been studied numerically by HENON [1] in a pioneering paper. His many numerical experiments show all the complexities and beauties of a perturbed Hamiltonian map. Theory, the KAM-theorem etcetera, predicts that already for very small values of α, i.e. a small discretisation value h, strange effects may occur although they may escape attention in practial cases. However, if one is interested in the numerical determination of a trajectory close to a hyperbolic point difficulties are bound to appear. In fig.4.1 we reproduce one of Hénon's beautiful pictures, the case $\cos\alpha = 0.24$ for which $h = 0.87$ - no one would take such a discretisation in an actual numerical integration – . The elliptical fixed point is surrounded by a few KAM-curves. For the winding number with a 5-resonance the resonating orbit is broken up in a ring of 10 secondary fixed points in an alternating order of elliptic and hyperbolic in accordance with the Poincaré-Birkhoff theorem.

Next we consider a Runge-Kutta scheme (improved Euler method) for (4.4) as given by

$$\begin{cases} x' = x + hy, \\ y' = y - hf(x), \end{cases} \qquad \begin{cases} x'' = x + hy', \\ y'' = y - hf(x'), \end{cases}$$

$$\begin{cases} x \to \tfrac{1}{2}(x'+x''), \\ y \to \tfrac{1}{2}(y'+y''). \end{cases}$$

Written as an iterative map we have

$$\text{(4.12)} \qquad \begin{cases} x_{n+1} = x_n + hy_n - \tfrac{1}{2}h^2 f(x_n), \\ \\ y_{n+1} = y_n - \tfrac{1}{2}hf(x_n) - \tfrac{1}{2}hf(x_n+hy_n). \end{cases}$$

Again the map is a perturbation of the identity but it is no longer Hamiltonian. The Jacobian determinant is

$$\text{(4.13)} \qquad 1 + \tfrac{1}{4}h^2 f'(x)f'(x+hy)$$

showing the map is dissipative even for linear $f(x)$.
In the illustrative case (4.2) we have

$$\text{(4.14)} \qquad \begin{cases} x_{n+1} = x_n + hy_n + \tfrac{1}{2}h^2(1-x_n^2), \\ \\ y_{n+1} = y_n + h(1-x_n^2) - h^2 x_n y_n - \tfrac{1}{2}h^3 y_n^3. \end{cases}$$

The fixed point (1,0) has the multipliers $\lambda = 1-h^2 \pm ih\sqrt{2}$ so it is repelling with the winding number

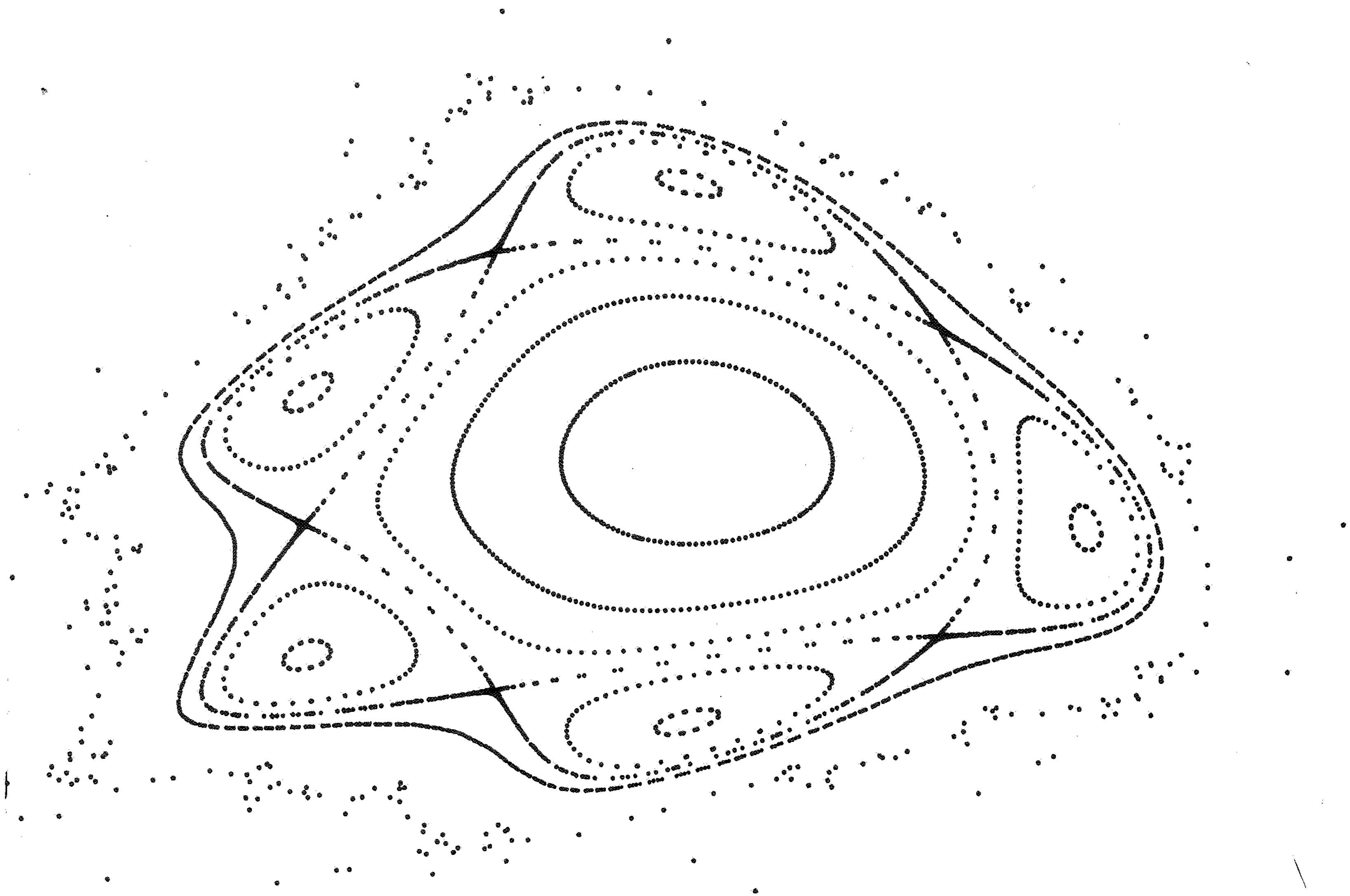

Fig. 4.1. Henon's quadratic map, $\cos\alpha = 0.24$.

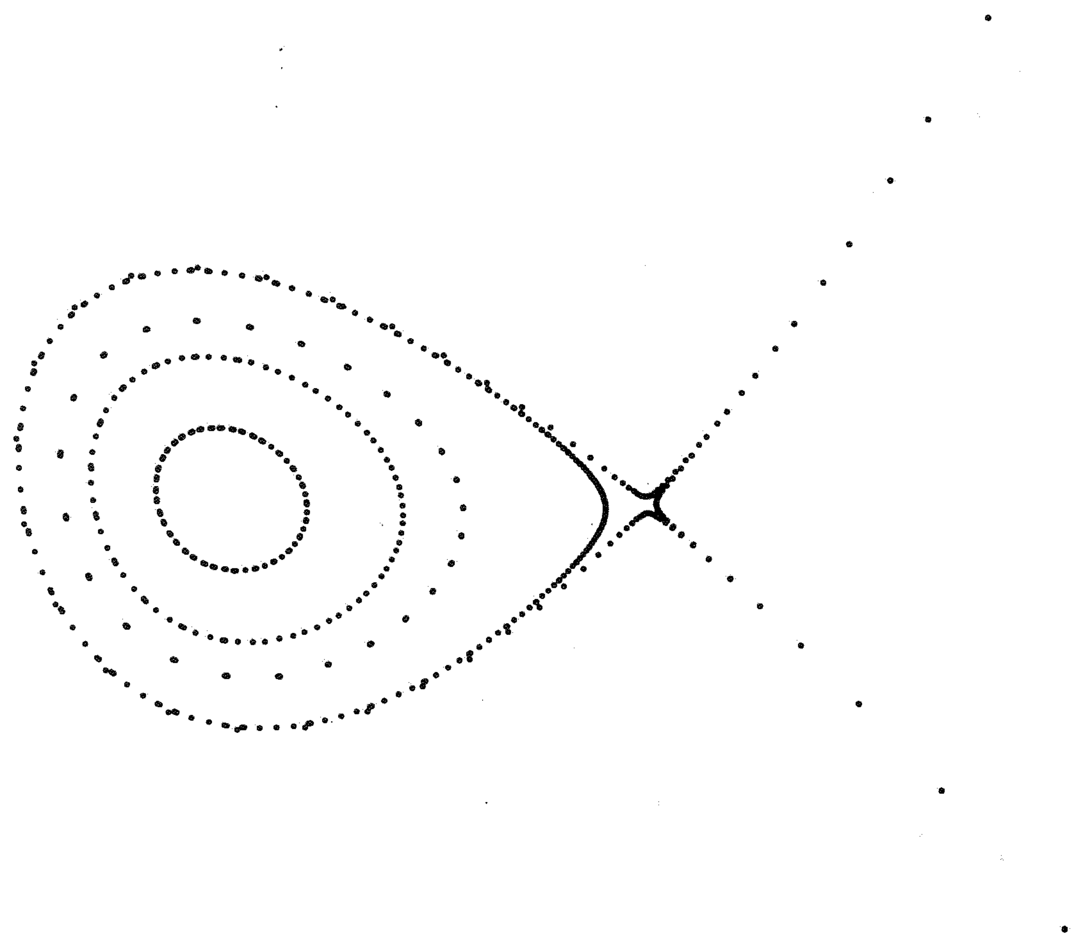

Fig. 4.2. $x' = x + hy$, $y' = y - h(x^2 - 1)$, $h = 0.02$.

$$\frac{1}{2\pi} \arctan \frac{h\sqrt{2}}{1-h^2} .$$

In the complex plane the multipliers are situated on the parabole

$$(\text{Im}\lambda)^2 = 2(1-\text{Re}\lambda)$$

osculating the unit circle at $\lambda = 1$.

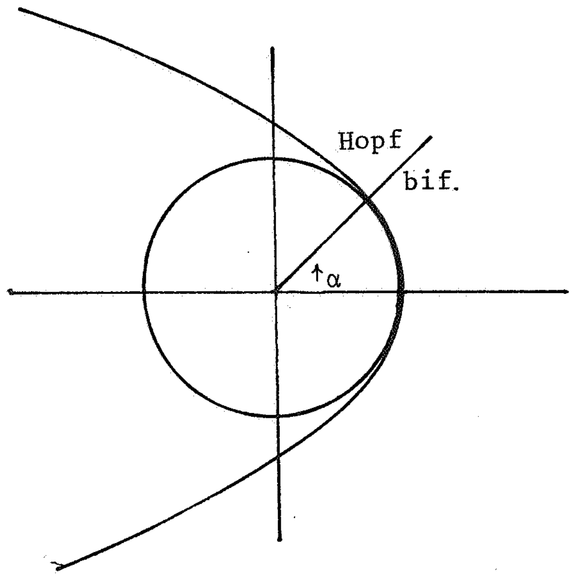

fig.4.3.

Since for moderate values of h λ is still close to the unit circle, $|\lambda| = \sqrt{1+h^4}$, we may expect Hopf bifurcation, the existence of a closed invariant curve. In fig. we have illustrated this for h = 0.2 where the winding number is 0.046 ≈ 1:22. A single orbit is shown of an 5000 odd points starting from x = 1, y = 0.1. The orbit is spiralling slowly outwards approaching an egg-shaped limit curve. We observe that as a rule by the Runge-Kutta process closed curves are turned into spirals with the exception of a single Hopf curve. Theory predicts that the linear size of this invariant curve is proportional to h^2.

As a final illustration we consider the iterative map (4.8) with (4.3). However, first we make a few general remarks with respect to (4.7) when f(x) is an odd function. Obviously each zero of f(x) generates a fixed point. If x = p is a non-zero solution of

(4.15) $$4x = -h^2 f(x)$$

we obtain a two-cycle

(4.16) $$p,\ -p,\ p, -p, \ldots$$

If $x = q$ is a non-zero solution of

(4.17) $$2x = -h^2 f(x)$$

we obtain a four-cycle

(4.18) $$q, 0, -q, 0, q, 0, \ldots$$

As an example of a cycle of higher order we may have an 8-cycle with the pattern

(4.19) $$p, p, q, -q, -p, -p, -q, q, p, p, \ldots\ .$$

In the special case (4.3) this means a two-cycle for $h > 2$ with

4.20) $$p = \sqrt{\frac{h^2+4}{h^2-4}}\ ,$$

a four-cycle for $h > \sqrt{2}$ with

(4.21) $$p = \sqrt{\frac{h^2+2}{h^2-2}}\ ,$$

and an 8-cycle of the kind (4.19) for $h > \sqrt{2-\sqrt{2}} = 0.765$. The bifurcation value $h^2 = 2-\sqrt{2}$ for this case is easily obtained from (4.7) by considering its far-away approximation

(4.22) $$x_{n+1} - 2x_n + x_{n-1} = -h^2 x_n.$$

This gives generally

(4.23) $$h = 2\sin \pi/m,$$

where m is the order of the given cycle. If h is increasing starting with a very small (postive) value we have at first cycles of high order but at each bifurcation value $h_m = 2 \sin \pi/m$ a pair of elliptic and hyperbolic cycles of order m is born at infinity and moves towards the origin as h further increases.

In figure 4.5 a few orbits are given of the autonomous system

(4.24) $$\dot{x} = y, \quad \dot{y} = -\frac{x(x^2-1)}{x^2+1}.$$

Explicitly they are given by

(4.25) $$x^2 + y^2 - 2\log(x^2+1) = C.$$

The singularities $(\pm 1,0)$ are elliptic whereas (0,0) is hyperbolic. The invariant manifold of the latter equilibrium is a lemniscate-like curve given by (4.25) for C = 0.

The corresponding iterative map is

(4.26) $$\begin{cases} x_{n+1} = x_n + hy_n \\ y_{n+1} = y_n - h\dfrac{x_{n+1}(x_{n+1}^2-1)}{x_{n+1}^2+1}. \end{cases}$$

In figure 4.6 we have illustrated the case h = 0.8. A few orbits are given illustrating the familiar Poincaré-Birkhoff behaviour. In this case we have

$$0.765 < 2\sin\frac{\pi}{8} < h < 2\sin\frac{\pi}{7} = 0.868$$

so that the lowest order cycle has period 8. The elliptic cycle has (6.17163,0) as one of its points.

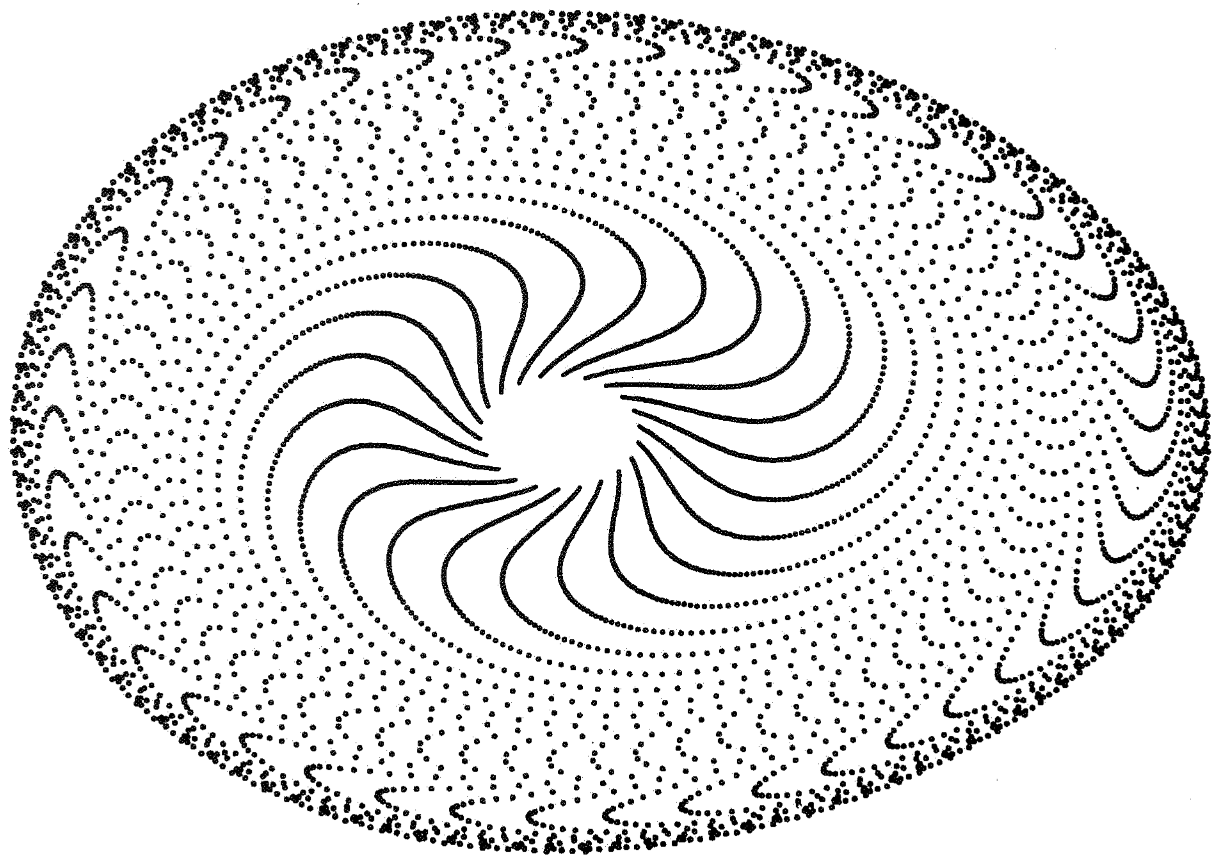

Fig. 4.4 Runge-Kutta, h = 0.2, start 1,01, 5000 points.

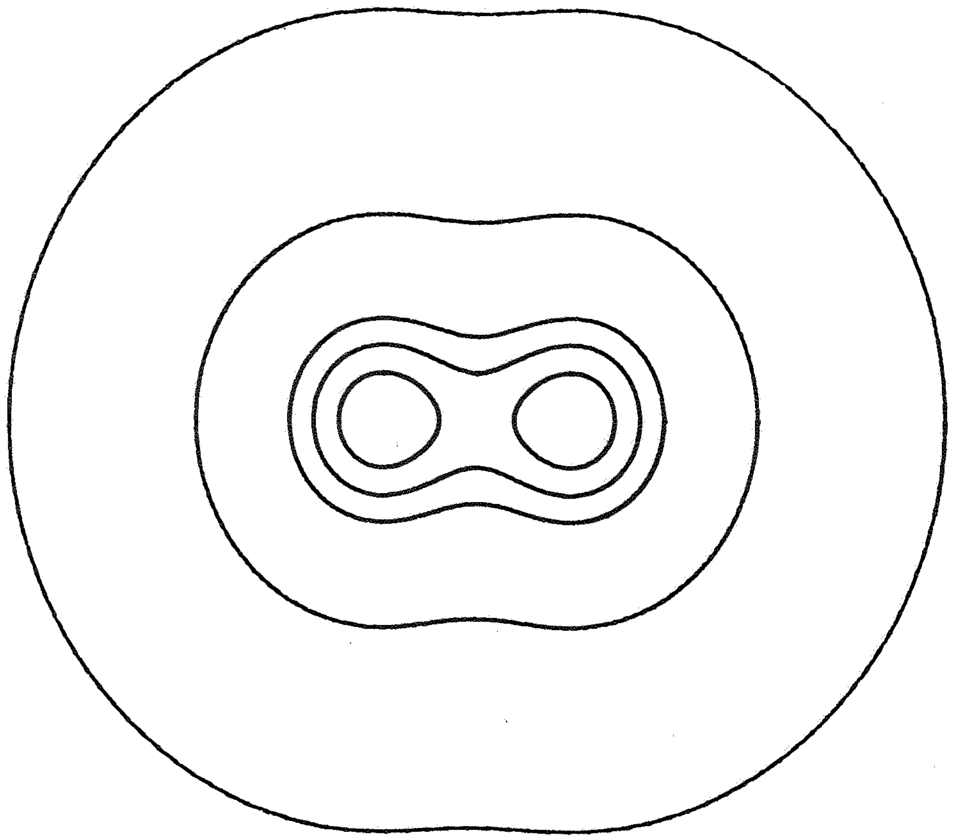

Fig. 4.5. $x' = y$
$y' = -\frac{-x(x^2-1)}{x^2+1}.$

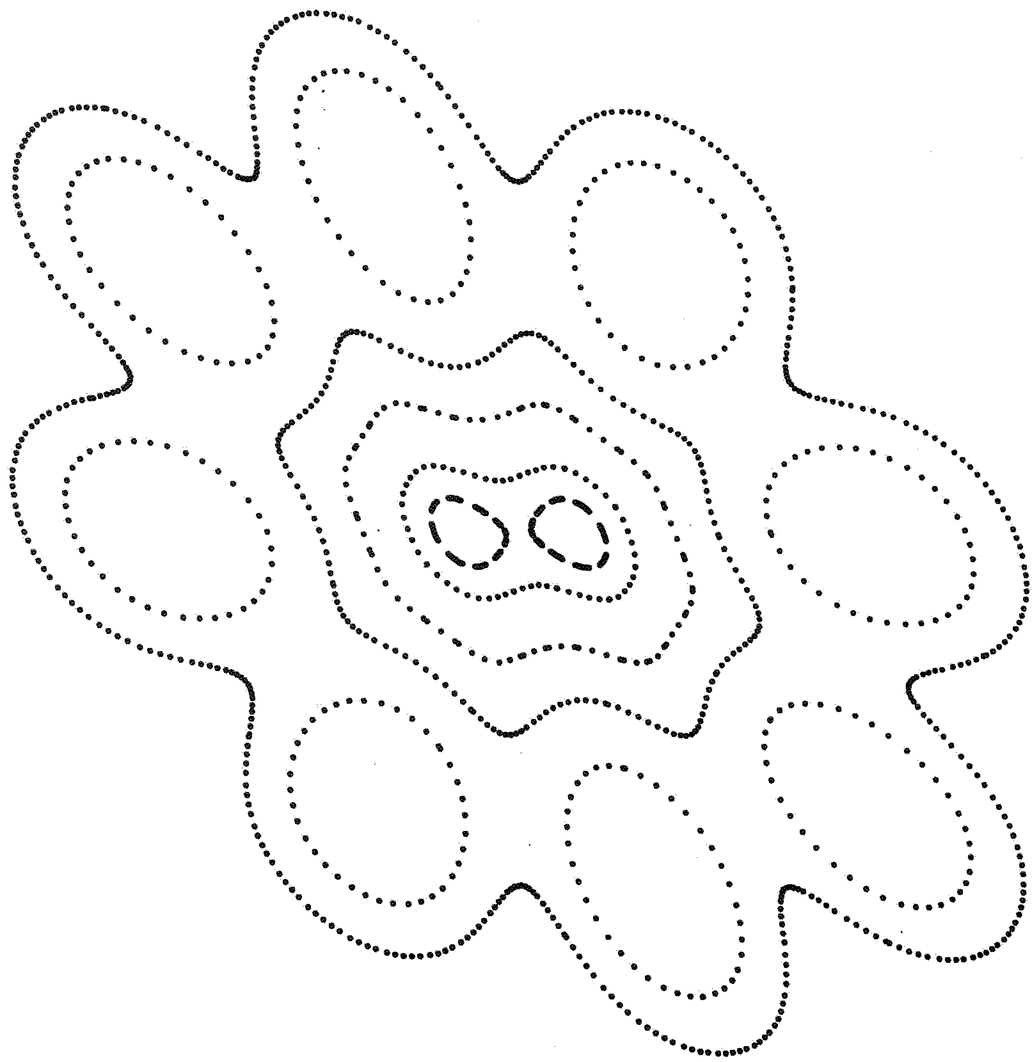

Fig. 4.6. $x' = x + hy$ $h = 0.8$
$y' = y - h.f(x')$

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