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# HARMONIC ANALYSIS AND RADON TRANSFORMS ON PENCILS OF GEODESICS

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We introduce pencils of geodesics as generalized points in classical geometries $X$. We generalize the spherical harmonic analysis on $X$ to pencils (in particular we consider spherical functions and spherical Fourier transforms on pencils). By means of "Radon transforms" (i.e. by changing the invariance type of a function through integration) we can relate the theories on different pencils. By evaluating the Radon transforms of spherical functions we get various product formules.

In the interesting case that $X = \mathbb{H}^n$ (hyperbolic $n$-space) we express anything explicitly. The spherical functions in particular can be expressed by Jacobi functions, the Radon transforms can be reduced to fractional integral transforms.

By a Radon transform we also transfer the convolution structure from $K$-invariant functions on $\mathbb{H}^n$ to $H$-invariant functions on $\mathbb{H}^n$ $(K = SO(n),\ H = SO_0(1,n-1))$.

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## 1. Introduction

In the Kleinian model, hyperbolic $n$-space $\mathbb{H}^n$ consists of the points of the open unit ball $B^n \subseteq \mathbb{R}^n \subseteq \mathbb{R}P^n$. Here we consider the elements of $\mathbb{R}P^n$ as "generalized points" of $\mathbb{H}^n$ and associate to them a "generalized spherical harmonic analysis". It is a new aspect of this generalization that we can relate the different theories obtained thus by means of "Radon transforms".

In chapter 2 we lay the geometrical basis. We introduce "pencils of geodesics" as generalized points in "classical geometries $X^n$". The set $\hat{X}^n$ of pencils may be identified with $\mathbb{R}P^n$ and the action of $G = I_0(X^n)$ extends to $\hat{X}^n$. A decomposition of $G$ is associated to each $\mathcal{P} \in \hat{X}^n$. We separate the pencils into "elliptic", "parabolic" and "hyperbolic" ones and choose a standard pencil of each type in $\hat{\mathbb{H}}^n$.

The geometry of "parabolic pencils" in the general context of symmetric spaces has been treated in detail in [14].

In chapter 3 we treat the harmonic analysis on a given pencil $\mathcal{P}$. We define radial functions on $\mathcal{P}$ and take the radial eigenfunctions of the Laplace -Beltrami operator $\Delta$ on $X^n$ as "spherical functions on $\mathcal{P}$". The "Fourier transform on $\mathcal{P}$" may then be treated as spectral decomposition of the radial part of $\Delta$. ([1])

By a simple general method we obtain the explicit expressions for $X^n = \mathbb{H}^n$. The Fouriertransform on $\mathcal{P}$ can be reduced to the ordinary Fourier transform ($\mathcal{P}$ parabolic) or to a "Jacobi transform" ([17]). For hyperbolic $\mathcal{P}$ a discrete spectral part occurs ($n \geqslant 4$) and the continuous part has multiplicity two (so "odd Jacobi functions" have to be introduced).

The harmonic analysis on elliptic ("classical spherical harmonic analysis") and parabolic pencils ([14]) is well-known for general symmetric spaces. The "even spherical functions" in the hyperbolic case occur also as trivial $K$-types in the harmonic analysis on $SO_0(1,n) / O(1,n-1)$ ([5]).

In chapter 4 and 5 we study "Radon transforms" between pairs of pencils in $\mathbb{H}^n$ (i.e. the process of changing the invariance type of a function by means of an integration)

In chapter 4 we obtain various "product formulas" by evaluating explicitly the Radon transforms of spherical functions and "spherical functions of second kind". "Universal factors" (expressible by the $c$-functions of chapter 3) appear in these formulas.

In particular we get "integral representations" by applying the Radon transforms to spherical functions on a parabolic pencil.

Our formulas generalize simultaneously the product formula for the common spherical functions on $\mathbb{H}^n$, their integral representation and the integral representation for the common $c$-function on $\mathbb{H}^n$ (see [17]).

In chapter 5 we study the Radon transforms themselves more precisely. The simple method of chapter 3 applies also here to give the explicit analytic expressions.

As an application, we transfer the convolution structure from the $K$-biinvariant functions on $G = SO_0(1,n)$ to functions which are left $O(1,n-1)$- and right $K$-invariant.

The Radon transforms generalize the "translation of $K$-biinvariant functions" and the "Abel transform" on $\mathbb{H}^n$ ([17]). Convolution products associated to the Jacobi transforms have been studied in [9], but only for parameters where no discrete spectral part occurs. (From our approach a certain obstruction rôle of the discrete part is appearent).

This paper is partially based on work done already in [1]. Also the list of properties enumerated in [7] has partially served as a program.

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## 2. Pencils of geodesics in classical geometries

Let $X$ be a Riemannian manifold, $I(X)$ its isometry group. Closed totally geodesic submanifolds of $X$ we call *planes.* Planes of dimension $m$ we call *m-planes.* By the *totally geodesic span* $<A>$ of a set $A \subseteq X$ we mean the intersection of all planes which include $A$. For a plane $Y$ in $X$ there is at most one involution in $I(X)$ for which $Y$ is a component of the fixed point set ([12], I.I.11.2.); we call it *reflection* $s_Y$ *in* $Y$ if it exists.

We call $X$ a *classical geometry* if for each $x \in X$ the linear isotropy group of $X$ at $x$ is the full orthogonal group of $X_x$. In the following $X$ is a classical geometry and $X^n$ a classical geometry of dimension $n$. It is well known that $X^n$ is either a sphere $S^n$, an elliptic space $\mathbb{R}P^n$, a euclidean space $\mathbb{R}^n$ or a hyperbolic space $\mathbb{H}^n$ (see e.g. [15], th. II. 3.1).

We call two distinct points of $X$ *antipodal* if their isotropy group is the same.

**Lemma 1.** Each subspace $Y_x$ of each tangent space $X_x$ is tangent to a plane of $X$. Each plane $Y$ in $X$ is itself a classical geometry with the induced Riemannian structure (and $Y$ is of the same type as $X$).

**Proof.** To the reflection of $X_x$ in $Y_x$ there corresponds an involution $s \in I(X)$. The $x$-component of the fixed point set of $s$ is a plane ([15], th.II. 5.1) with tangent space $Y_x$ at $x$.
Let $y \in Y$. Any orthogonal transformation of $Y_y$ is the restriction of some orthogonal transformation of $X_y$. So $Y$ is a classical geometry. (Note that the types are easily distinguished by curvature and antipodal points say)

**Lemma 2.** To any $x \in X$ there exists at most one antipodal point $y \in X$. If $x,y$ are antipodal then each geodesic through $x$ goes through $y$ too. If conversely two district points $x_1,x_2 \in X$ can be joined by two distinct geodesics $\gamma_1,\gamma_2$, then they are antipodal.

**Proof.** Suppose that $x$ and $y$ are antipodal. $x$ can be joined to $y$ by a geodesic $\gamma$. Since $X$ is two-point homogeneous, any geodesic through $x$ goes through $y$ too. From $s_x(y)=y$ we see that $y$ is unique.
Conversely let first $n \geqslant 3$. The pointwise stabilizers of $\gamma_1,\gamma_2$ generate together the full isotropy group of $X$ at $x$ and $y$ (note that so$(n-1)$ is maximal in $so(n)$ for $n \geqslant 3$). Now let $n=2$. Embed $X$ as a 2-plane in a space $X^3$. Even each geodesic through $x$ in $X^3$ goes through $y$ too.

**Lemma 3.** Let $A \subseteq X$. Then $<A>$ is a plane or consists of two antipodal points

**Proof.** Suppose that $x,y \in A$ are not antipodal. Then by (2) the geodesic $\gamma$ joining $x$ and $y$ is unique. So $\gamma \subseteq <A>$. Let $z \in A$. Then there is a $z_1 \in \gamma$, $z_1$ not antipodal to $z$. The joining geodesic lies in $<A>$. So $<A>$ is connected and thus a plane.

**Definition.** Embed $X=X^n$ as an $n$-plane in $X^{n+1}$. For a geodesic $\delta$ in $X^{n+1}$, $\delta \not\subseteq X$, define the *pencil* $\mathcal{P}(\delta)$ *of geodesics in* $X$ to be the set of nonempty intersections of 2-planes through $\delta$ with $X$. (These are geodesics by (3)). Set $\hat{X}=\{\mathcal{P}(\delta):\delta\subseteq X^{n+1},\delta\not\subseteq X\}$ and $\hat{X}(p) = \{\mathcal{P}(\delta) : p\in\delta\subseteq X^{n+1}\}$ for $p\in X^{n+1}\setminus X$.

**Proposition 4.**

(a) Let $x\in X$. If $x\notin\delta$, then there is a unique geodesic through $x$ in $\mathcal{P}(\delta)$. If $x\in\delta$, then $\mathcal{P}(\delta)$ is the set $\mathcal{P}(x)$ of all geodesics through $x$ in $X$.

(b) Any two geodesics $\gamma_1,\gamma_2\in\mathcal{P}(\delta)$ lie in a 2-plane.

(c) Let $\gamma_1,\gamma_2$ be geodesics in $X$, dim $<\gamma_1,\gamma_2>=2$. To each $p\in X^{n+1}\setminus X$ there is a unique $\delta$ through $p$ with $\gamma_1,\gamma_2\in\mathcal{P}(\delta)$ and $\mathcal{P}(\delta)$ does not depend of $p$ (i.e. $\hat{X}=\hat{X}(p)$).

**Proofs.** (a) If $x\notin\delta$, then $<\delta,x>$ is the unique 2-plane through $\delta$ and $x$ in $X^{n+1}$ and thus $<\delta,x>\cap X$ the unique geodesic through $x$ in $\mathcal{P}(\delta)$.
Now let $x\in\delta$, $y\in X\setminus\delta$. Then the geodesic through $y$ in $P(\delta)$ is $<\delta,y>\cap X=<x,y>$ (use (2)), i.e. $\mathcal{P}(\delta)=\mathcal{P}(x)$.
(b) Let $x\in\gamma_2\setminus\gamma_1$. Then $\gamma_2=<\delta,x>\cap X=<\delta,x>\cap<\gamma_1,x>$since dim $<\delta,\gamma_1,x>=3$.
(c) Let $\gamma_1,\gamma_2\in\mathcal{P}(\delta)$ and $p\in\delta$; then $\delta=<\gamma_1,p>\cap<\gamma_2,p>$. Conversely, if dim$<\gamma_1,\gamma_2>=2$ then $\delta=<\gamma_1,p>\cap<\gamma_2,p>$ is indeed a geodesic (as in (b)) and $\gamma_1,\gamma_2\in\mathcal{P}(\delta)$.
Now assume $\gamma_1,\gamma_2\in\mathcal{P}\in\hat{X}^n$ and $n\geqslant 3$. Then for $x\in\hat{X}^n\setminus<\gamma_1,\gamma_2>$ we have $\gamma=<\gamma_1,x>\cap<\gamma_2,x>\in\mathcal{P}$ by (b), (a) and then for $y\in<\gamma_1,\gamma_2>\setminus\gamma_1$ by the same argument $<\gamma,y>\cap<\gamma_1,y>\in\mathcal{P}$. Hence $\mathcal{P}\in\hat{X}^n$ is uniquely determined by $\gamma_1,\gamma_2$.
Now let $n=2$. Embed $X^3$ in $X^4$, choose an $y\in X^4\setminus X^3$, set $Y=<X^2,y>$. By (b) any pencil $\mathcal{P}\in\hat{X}^2(p)$ is the restriction $\{\gamma\in\tilde{\mathcal{P}}:\gamma\subset X^2\}$ of a pencil $\tilde{\mathcal{P}}\in\hat{Y}(p)$. So the uniqueness follows also for $n=2$.

As a consequence we may identify $\hat{X}^n$ with projective space $\mathbb{R}P^n$ (namely the set of geodesics through $p\in X^{n+1}\setminus X^n$). $X^n$ is mapped into $\hat{X}^n$ in a natural way ($x\rightarrow\mathcal{P}(x)$)

Also the action of $I(X^n)$ extends to $\hat{X}^n$ : To $g\in I(X^n)$ choose an extension $\tilde{g}\in I(X^{n+1})$ (such extensions exists if $g$ is in an isotropy group, thus for any $g\in I(X^n)$). Then $g(\mathcal{P}(\delta))$ is the pencil $\mathcal{P}(\tilde{g}(\delta))$.

**Lemma 5.** Let $\mathcal{P}\in\hat{X}^n$. If $g\in I(X^n)$ stabilizes some $\gamma\in\mathcal{P}$ pointwise, then $g\mathcal{P}=\mathcal{P}$.

**Proof.** Choose $x\in\gamma$ and denote the geodesic through $x$ in $X^{n+1}$ orthogonal to $X^n$ by $\beta$. Let $\tilde{g}\in I(X^{n+1})$ be the extension of $g$ which stabilizes also $\beta$ ( (1), proof). Choose a $\delta\subseteq<\gamma,\beta>$ with $\mathcal{P}=\mathcal{P}(\delta)$ ((4c), proof). Then $g\mathcal{P}(\delta) = \mathcal{P}(\tilde{g}\delta) = \mathcal{P}(\delta)$.

Let $G=I_0(X)$. Denote by $Z(S)=\{g\in G : gs = s\ \forall s\in S\}$ the "stabilizer in $G$", by $N(S)=\{g\in G : gS = S\}$ the "normalizer in $G$" of some set $S$ of geometrical objects in $X$.

**Definition.** Let $\mathcal{P}\in\hat{X}$. Denote the intersection with $G$ of the group generated by the stabilizers in $I(X)$ $Z_{I(X)}(\gamma)$ ($\gamma\in\mathcal{P}$) by $L = L(\mathcal{P})$ ("Isotropy group of $\mathcal{P}$"). Set $L' = L'(\mathcal{P})=N(\mathcal{P})$. (From (5) it follows that $L$ is a normal subgroup of $L'$).
The orbits of $L(\mathcal{P})$ in $X^n$ we call *spheres* of $\mathcal{P}$.

**Proposition 6.** Let $\gamma_0 \in \mathcal{P} = \mathcal{P}(\delta) \in \hat{X}^n$, $x \in \gamma_0 \setminus \delta$. (Note : $L \cdot x = x$ if $x \in \delta$). The sphere $L \cdot x$ of $\mathcal{P}$ is a closed submanifold of $X^n$ of dimension $n-1$. With the Riemannian structure induced from $X^n$ it is a classical geometry. It intersects each $\gamma \in \mathcal{P}$ once or each $\gamma \in \mathcal{P}$ twice orthogonally.

**Proof.** Let first $n \geqslant 3$. Denote the identity component of $Z(\gamma)$ by $Z_0(\gamma)$. Let $L_0$ be the group generated by the $Z_0(\gamma)$ $(\gamma \in \mathcal{P})$. $L_0$ is a connected Lie group, $L_0 \cdot x$ a submanifold of $X^n$.

Now let $\gamma \in \mathcal{P}$. $Z_0(\gamma) \cdot x$ intersects $<\gamma_1, \gamma_0>$ at most twice ([12], I.I.11.2.) and is invariant with respect to the reflection in $<\gamma_1, \gamma_0>$ since $s_{<\gamma_1,\gamma_0>} Z_0(\gamma) s_{<\gamma_1,\gamma_0>} \cdot x = Z_0(\gamma) \cdot x$. Hence $z_0(\gamma) \cdot x$ is orthogonal to $<\gamma_1, \gamma_0>$ and in particular to $\gamma_0$, thus $L_0 \cdot x$ is orthogonal to $\gamma_0$ at $x$ and dim $(L_0 \cdot x) \leqslant n-1$.

Conversely, for $\gamma \in \mathcal{P} \setminus \gamma_0$ we have $\dim(Z_0(\gamma) \cdot x) \geqslant 1$ and $L_0 \cdot x$ is $Z_0(\gamma_0)-$ invariant, hence $\dim(L_0 \cdot x) \geqslant n-1$. From this follows also that $L_0 \cdot (\gamma_0 \setminus \delta)$ is open in $X^n \setminus \delta$; this is true for any $\gamma_0 \in \mathcal{P}(\delta)$ and $X^n \setminus \delta$ is connected, thus $L_0 \cdot (\gamma_0 \setminus \delta) = X^n \setminus \delta$. Hence $L_0 \cdot \gamma = \mathcal{P}$ ((4a)) and $L_0 \cdot x$ intersects each $\gamma \in \mathcal{P}$.

Next, for $\gamma \in \mathcal{P}$ and $y \in L_0 \cdot x \cap \gamma$ we have $Z_{I(X_n)}(\gamma) \cdot y = y$, $Z_{I(X_n)}(\gamma) \cdot L_0 \cdot y = L_0 \cdot Z_{I(X_n)}(\gamma) \cdot y = L_0 \cdot y = L_0 \cdot x$. This shows that $L_0 \cdot x$ is a classical geometry and that $L_0 \cdot x = L \cdot x$. Moreover, if also $z \in L_0 \cdot x \cap \gamma$, $z \neq y$ then $z$ is antipodal to $y$ in $L_0 \cdot x$ and thus unique by (2). Using (3a) this shows also that $L_0 \cdot x$ is an embedded submanifold and thus closed.

Finally let $n=2$. Embed $X^2$ as a 2-plane in $X^3$. Let $\tilde{\mathcal{P}} \in X^3$ be the unique extension of $\mathcal{P}$ ((4c), proof). Obviously $L(\mathcal{P}) \cdot x \subseteq L(\tilde{\mathcal{P}}) \cdot x \cap X^2$. The converse follows from the fact that to any $y \in L(\mathcal{P}) \cdot x$ there is a $\gamma \in \tilde{\mathcal{P}}$ with $y = s_\gamma s_{\gamma_1} \cdot x$ (see (7), proof).

Also $\mathcal{P}$ itself (covered twice at most by $L \cdot x$) can be provided with an $L$-invariant Riemannian structure. For $n \geqslant 3$, $\mathcal{P}$ is then a nonspherical classical geometry (note that for $n \geqslant 3$, $\gamma_0$ is the unique geodesic normalized by $Z(\gamma_0)$).

$\mathcal{P}$ is determined by any of its spheres ((3a), (6)). So $L'$ is also the normalizer of the set of spheres of $\mathcal{P}$.

Call $x \in \gamma \in \mathcal{P}$ "regular point of $\mathcal{P}$" if $Z(x) \cap H = Z(\gamma)$; else call $x$ "singular". So $x$ is singular iff $s_x|_\gamma = h|_\gamma$ for some $h \in H$. There are at most two singular points on $\gamma$ ((6)). So $L$ is also the stabilizer of the set of spheres of $\mathcal{P}$ (and in particular closed).

We need some more group terminology. Fix $x_0 \in \gamma_0 \in \mathcal{P}$. As usual ([12]) define $K = Z(\{x_0\})$, $M = Z(\gamma_0)$, $M' = N(\gamma_0) \cap K$. $\gamma_0 = Ax_0$ (where $A = \exp\mathfrak{a}, \mathfrak{a} \subseteq \mathfrak{p}, \mathfrak{p} = \mathfrak{k}^\perp$ in $\mathfrak{g}$). Then $N(\gamma_0) = M'A$. Choose also a positive Weyl chamber $\mathfrak{a}^+ \subseteq \mathfrak{a}$.

The groups $N(\gamma_0) \cap L / M$, $N(\gamma_0) \cap L' / M$ act on $\gamma_0$. We denote their preimages in $I(\mathfrak{a})$ by $W = W(\mathcal{P})$ ("Weylgroup of $\mathcal{P}$") and $W' = W'(\mathcal{P})$ respectively. The significance of $W$ and $W'$ as symmetry groups in the harmonic analysis on $\mathcal{P}$ will become clear in chapter 3.

Let $\mathfrak{a}' = \{H_0 \in \mathfrak{a} : \exp H_0 \cdot x_0 \text{ regular}\}$ be the set of "regular elements of $\mathfrak{a}$" ($W$ can be shown to be generated by the reflections in the singular elements)

Define $R = R(\mathcal{P}, \gamma_0) = \{s_\gamma s_{\gamma_0} : \gamma \in \mathcal{P}\}$

**Corollary 7.** Let $\mathfrak{a}_0$ be an interval representing $W \setminus \mathfrak{a}'$.

We have the decompositions $H = RM$ and $G = R\cdot\exp\bar{\mathfrak{a}}_0\cdot K$ (unique in $H_0\in\bar{\mathfrak{a}}_0$). (The first decomposition is unique if $h\notin N(\gamma_0)\setminus M$. The second decomposition is unique if $gx_0\notin\gamma_0\setminus\exp\bar{\mathfrak{a}}_0\cdot x_0$ and if $gx_0$ is not singular)

**Proof.** Let $h\in H$, $g\in G$. Choose $x\in\exp\mathfrak{a}_0\cdot x_0$ and assume $gx_0\notin\delta$ ($\mathcal{P}=\mathcal{P}(\delta)$). Then there are "geodesic symmetries" $s_{\gamma 1},s_{\gamma 2}$ in the classical geometries $H\cdot x$, $H\cdot gx_0$ resp. with $s_{\gamma 1}hx = x$ and $s_{\gamma 2}gx_0\in\exp\bar{\mathfrak{a}}_0\cdot x_0$ resp. (unique if the additional conditions are satisfied); hence $s_{\gamma 0}s_{\gamma}h\in M$ and there is a unique $H_0\in\bar{\mathfrak{a}}_0$ with $s_{\gamma 0}s_{\gamma}gx = \exp H_0\cdot x_0$.

We conclude the chapter by choosing a standard pencil in each $G$-orbit of $\hat{\mathbb{H}}^n$.
First we separate the pencils $\mathcal{P}(\delta)$ in $\hat{X}$ into the three types "elliptic", "parabolic", "hyperbolic" according to "$\delta$ intersects $\gamma_0$", "$\delta$ is parallel to $\gamma_0$", "$\delta$ is ultraparallel to $\gamma_0$" in the 2-plane $<\gamma_0,\delta>$. (This amounts to $\mathcal{P}(\delta)$ beeing an elliptic, euclidean or hyperbolic space resp. for $n\geqslant 3$).

(8) Now let $X = \mathbb{H}^n$, $G = SO_0(1,n)$.

(K) In the elliptic case, $\mathcal{P}(\delta) = \mathcal{P}(\gamma_0\cap\delta)$. The spheres of $\mathcal{P}(\delta)$ are ordinary spheres. Our standard example is $\mathcal{P}(x_0)$. Then $L = K = \exp(\mathfrak{m}^{\perp}\cap\mathfrak{k})\cdot M \doteq SO(n)$. $W=M'/M$, $W'/W = \{e\}$

($\bar{N}$) In the parabolic case $\mathcal{P}(\delta)$ consists of all geodesics parallel to $\exp(\mathfrak{a}^+)\cdot x_0$ or of all geodesics parallel to $\exp(-\mathfrak{a}^+)x_0$. The spheres of $\mathcal{P}(\delta)$ are horospheres ([14]). We take $-\mathfrak{a}^+$ to get a standard example; let $G = \bar{N}AK$ be the associated Iwasawa decomposition. Then $L = \bar{N}M \doteq I_0(\mathbb{R}^n)$. $L' = \bar{N}AM$. $W = \{e\}$, $W' = \mathbb{R}$

(H) In the Hyperbolic case, $\gamma_0$ and $\delta$ have a common perpendicular ([2]). $\mathcal{P}(\delta)$ consists of the geodesics orthogonal to some hyperplane. The spheres of $\mathcal{P}(\delta)$ are equidistant surfaces ([2]). We choose the hyperplane orthogonal to $\gamma_0$ in $x_0$ to get a standard example. Then $L=H=\exp(\mathfrak{a}^{\perp}\cap\mathfrak{p})\cdot M\doteq SO\cdot(1,n-1)$ : $L' = HM' = H'$. $W=\{e\},W'=M'/M$.

So the partition of $\hat{\mathbb{H}}^n$ into types concides with its partition into $G$-orbits.

## 3. Spherical functions and Fourier transform on pencils

As in chapter 2 we fix $x_0,\gamma_0,\mathcal{P}$ with $x_0\in\gamma_0\in\mathcal{P}\in\hat{X}$. Moreover we choose the fixed *geodesic parametrization* $\gamma_0(t)=\exp(tH_0)\cdot x_0$ ($H_0\in\mathfrak{a}^+,\|H_0\| = 1$).

We call a function on $X$ *radial on* $\mathcal{P}$ if it is constant on the spheres of $\mathcal{P}$; or equivalently , if it is $L(\mathcal{P})$ - invariant. By restriction to $\gamma_0$ we may identify radial functions on $\mathcal{P}$ with $W$- invariant functions on $\mathbb{R}$ (2.(7)), by extension to $G$ with left $L$- right $K$-invariant functions.

We shall denote the *invariant extension* of a function $f$ on $W\setminus\mathbb{R}$ to $X$ (or to $G$) by $f^L$, i.e. $f^L(l\cdot\gamma_0(t))=f(t)$ $(\forall l\in L)$. But we shall also write shortly $f^L(t)$ instead of $f^L(\gamma_0(t))$ for the restriction of a radial function.

Now fix a Haar measure $dl$ on $L$ and normalize dm on $M$ such that $\int_M dm=1$. The measure on the sphere $L\cdot\gamma_0(t)$ induced from the Riemannian structure on $X^n$ can then be written as $\omega^L(t)\cdot d(lM)$. By denoting the Riemannian measure on $X$ by $dx$ we may restate this as integral formula for the decomposition 2.(7).

**Proposition 1.** $\int_X f(x)dx = \int_{W\setminus\mathbb{R}} \int_L f(l\cdot\gamma_0(t))dl\,\omega^L(t)dt \qquad (f\in L^1(X))$

Later we shall normalize $dx$ and $\omega^L(t)$ differently; but formula (1) will remain valid (see (8))

It is thus natural to define the space $L^p(L\setminus X)$ of radial $L^p$- functions on $X$ by identification with the space $L^p(W\setminus\mathbb{R},\omega^L(t)dt)$. Also "radial functions with compact support" we define in this way.

The Laplace - Beltrami operator $\Delta$ maps $C^\infty(L\setminus X)$ into itself ([11], 1.3.1). We denote the corresponding *radial part of* $\Delta$ on $\exp\mathfrak{a}'\cdot x_0$ by $\Delta^L$. Expressed in the geodesic coordinate, it is a second order differential operator on $\mathbb{R}$ (maybe singular for singular $t$) which can be written down explicitly in terms of the weight function $\omega^L(t)$:

**Proposition 2.** $\Delta^L = 1/\omega^L(t))d/dt(\omega^L(t)d/dt) = d^2/dt^2 + (\dot\omega^L(t)/\omega^L(t))d/dt$

**Proof.** This is just a "codimension 1 simplification " of Helgasons formula [11], th. 3.2. There is a simple proof for it too: let $[t_1,t_2]$ be any interval of $\mathbb{R}$ with $[t_1,t_2]\cdot H_0\subseteq\mathfrak{a}'$. Choose a ball $B$ in $L$ of measure one. Set $\Gamma=B\cdot\gamma_0[t_1,t_2]$. By Greens formula and (1) we obtain for $f\in C^\infty(W\setminus\mathbb{R})$, proving the proposition

$$\int_{t_1}^{t_2}(\Delta^L f)(t)\omega^L(t)dt = \int_\Gamma(\Delta f^L)(x)dx = \int_{\partial\Gamma}\partial f^L\setminus\partial n\cdot d\delta =$$

$$\omega^L(t)df/dt\,|_{t_1}^{t_2} = \int_{t_1}^{t_2}(1/\omega^L(t))d/dt(\omega^L(t)df/dt)\omega^L(t)dt$$

As the *spherical functions on* $\mathcal{P}$ we take the radial eigenfunctions of $\Delta$ on $X$.
Note that the individual radial eigenfunctions are $W$-invariant on $\mathbb{R}$, while the radial eigenspaces of $\Delta$ considered on $\mathbb{R}$ are even $W'$-invariant.

**Lemma 3.** If $W'/W$ contains a reflection, then the eigenspaces of $\Delta^L$ may be decomposed into even and odd functions. If $W'$ contains all translations of $\mathbb{R}$ (this is fullfilled for parabolic $\mathcal{P}$), then the eigenspaces of $\Delta^L$ are exponential functions (maybe times $t$)

**Proof.** If $W'$ contains all translations of $\mathbb{R}$, then $\Delta^L$ commutes with $d/dt$. Thus it has constant coefficients. (We have $A\subseteq L' = N(\mathcal{P})$ for parabolic pencils).

$\Delta^L$ is a symmetric second order differential operator in $L^2(W\setminus\mathbb{R},\ \omega^L(t)dt)$ by (2). So we can treat the *spherical Fourier transform on* $\mathcal{P}$ as the spectral decomposition of $\Delta^L$ in $L^2(W\setminus\mathbb{R},\omega^L(t)dt)$ (see [1], [4]).

We turn to the explicit calculations for our standard pencils 2.(8) in the case $X = \mathbb{H}^n$. We shall treat the well-known case $(K)$ and the trivial case $(\bar{N})$ too.

In computing first $\Delta^L$ and $\omega^L$ we can avoid hyperbolic trigonometry by using a more general group theoretic method. It is based on the consideration of simple invariant functions on the group $SL(2,\mathbb{R})$. These will be useful again in chapters 4,5.

We work with invariant functions on $G$. In this picture, the Laplacian $\Delta$ may be written (see [13])

$$\Delta = \tilde{H}_0^2 + \sum_{i=1}^{n-1} \tilde{Y}_i^2$$

(where $Y_i$ $(i=1,\ldots,n-1)$ is an orthonormal basis in $\mathfrak{a}^{\perp}\cap\mathfrak{p}$)

First we have for $f \in C^\infty(W\setminus\mathbb{R})$

$$(\tilde{H}_0^2 f)(t) = d^2/ds^2|_0 \ \ f^L(\exp(tH_0)\exp(sH_0)) = d^2/ds^2|_0 \ \ f(t+s) = d^2f/dt^2$$

Defining then $g(s),r(s)$ by $g(s)=\exp(tH_0)\cdot\exp(sY_i) \subseteq L\cdot\exp(r(s)\cdot H_0)\cdot K$ (in particular $r(0)=t$) we obtain for $f$

$$d/ds|_0 \ \ f^L(g(s)) = \dot{r}(0)df/dt(t) \tag{4a}$$

and, if we suppose $\dot{r}(0) = 0$

$$(\tilde{Y}_i^2 f) = d^2/ds^2|_0 \ \ f^L(g(s)) = \ddot{r}(0)df/dt(t) \tag{4b}$$

So $r(s)$ needs not be known: Any radial function is suitable to calculate the operator $\tilde{Y}_i^2$. Moreover $\tilde{Y}_i^2 = \tilde{Y}_1^2$ (by the $M$-invariance 2.(5) of $\mathcal{P}$) and $Y_1,H_0$ generate a Lie algebra $\cong sl(2,\mathbb{R})$. So we may restrict ourselves to the case $n=2$ and work with the group $G=SL(2,\mathbb{R})$. Assuming sectional curvature $-1$ in $\mathbb{H}^n$ we may set ([10], p.405)

$$H_0 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \quad Y_1 = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \quad Z_1 = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \tag{5}$$

(where $Z_1 = [H_0,Y_1]$). Our standard isotropy groups are now $K = \exp(\mathbb{R}Z_1)$, $\bar{N} = \exp(\mathbb{R}(Z_1 - Y_1))$, $H = \exp(\mathbb{R}Y_1)$. So we may apply (4) to the following simple radial functions

$$\begin{aligned} &f^K(g) = \tfrac{1}{2}tr(g.g^{tr}). && f^K(g(s)) = chs\cdot cht && \Rightarrow \dot{r}(0) = 0, \ \ \ddot{r}(0) = cotht \\ &f^{\bar{N}}(g) = (g\cdot g^{tr})_{11}. && f^{\bar{N}}(g(s)) = chs\cdot e^t && \Rightarrow \dot{r}(0) = 0, \ \ \ddot{r}(0) = +1 \\ &f^H(g) = tr(H_0\cdot g\cdot g^{tr}). && f^H(g(s)) = chs\cdot sht && \Rightarrow \dot{r}(0) = 0, \ \ \ddot{r}(0) = tanht \end{aligned} \tag{6}$$

Where we have used

$$g(s)\cdot g(s)^{tr} = \exp(tH_0)\cdot\exp(2sY_1)\cdot\exp(tH_0) = \begin{bmatrix} chse^t & * \\ * & chse^{-t} \end{bmatrix}.$$

Now $\Delta^L$ can be read off from (4), (6). By (2), $\omega^L$ is then determined up to a constant.

**Proposition 7.** Define $\rho = (n-1)/2$. Then we have for $\Delta^L$ and (up to a constant) for $\omega^L$

$$\Delta^K = d^2/dt^2 + 2\rho \text{ coth}t\, d/dt \qquad \omega^K(t) = (2sht)^{2\rho}$$
$$\Delta^{\overline{N}} = d^2/dt^2 + 2\rho d/dt \qquad \omega^{\overline{N}}(t) = e^{2\rho t}$$
$$\Delta^H = d^2/dt^2 + 2\rho \text{ tanh}t\, d/dt \qquad \omega^H(t) = (2cht)^{2\rho}$$

(8) *Normalization of measures.* We choose $dk$ on $K$ such that $\int_K dk = 1$. Next we take $dx$ on $X$ and then $d\overline{n}$ on $\overline{N}$, $dh$ on $H$ such that formula (1) holds for $\omega^K$, $\omega^{\overline{N}}$, $\omega^H$ as defined in (7).
(Note that we have to multiply $dx$ and $d\overline{n}$ by $2^{-2\rho}k_{n-1}$, $dh$ by $k_{n-1}$ in order to get the measures on $X$, $H\cdot x_0$, $\overline{N}\cdot x_0$ resp. induced from the Riemannian structure on $X$. Here $k_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ denotes the volume of the euclidean unit sphere in $\mathbb{R}^n$)

Now we will determine the spherical functions on our standard pencils. For $\lambda\in\mathbb{C}$ consider on $\mathbb{R}$ the eigenequation

$$(\Delta^L + \rho^2 + \lambda^2) f = 0 \tag{9}$$

For $L=K,H$ there is a unique *normed even solution* $\phi_\lambda^L$ of (9) (i.e. $\phi_\lambda^L$ even, $\phi_\lambda^L(0) = 1$). For $L=H$ there is also a unique *normed odd solution* $\psi_\lambda^H$ of (9) (i.e. $\psi_\lambda^H$ odd, $\dot\psi_\lambda^H(0) = 1$). (Remember also lemma 3).

In all cases $L = K,\overline{N}$ or $H$ there is (at least for $\lambda\notin -i\mathbb{N}$) also a unique *solution* of *second kind* $\Phi_\lambda^L$ of (9). $\Phi_\lambda^L>0$ is recessive at $+\infty$ for $\text{Im}\lambda>0$, it is defined for $\text{Im}\lambda\geq 0$ by $\lim_{t\to\infty} \Phi_\lambda^L(t)e^{(-i\lambda+\rho)t} = 1$ ([4], [6]) and is defined by analytic continuation for $\text{Im}\lambda<0$ ([6], [8]).

Note that $\Phi_\lambda^K$ is singular at $t=0$, but that $\lim_{t\to 0} t^{n-2}\Phi_\lambda^K(t)$ $(n>2)$, $\lim_{t\to 0} \Phi_\lambda^K(t)/logt$ $(n=2)$ resp. exist ([6]).

Now we recall (see [17]) that the *Jacobi functions* $\phi_\lambda^{(\alpha,\beta)}$ are defined for $\alpha,\beta,\lambda\in\mathbb{C}$, $\alpha\notin -\mathbb{N}$, as the normed even $C^\infty$ - solutions on $\mathbb{R}$ of the equation.

$$(d^2/dt^2 + ((2\alpha+1) \text{ coth}t + (2\beta+1) \text{ tanh}t)d/dt + \rho^2+\lambda^2)f = 0 \tag{10}$$

$$(\text{ here } \rho=\alpha+\beta+1)$$

The harmonic analysis with respect to the functions $\phi_\lambda^{(\alpha,\beta)}$ has been studied very much (see the nice survey article [17] which can also be taken as general reference in the following). Comparing (7), (9), (10) we see directly that the $\phi_\lambda^K$, $\phi_\lambda^H$ are Jacobi functions; but also the odd solutions $\psi_\lambda^H$ may be expressed explicitly by using Jacobi functions (the expression given below is checked readily)

**Corollary 11.** The $C^\infty$ - solutions of (9) on $W\setminus\mathbb{R}$ (i.e. the spherical functions on $\mathcal{P}$) are linear combinations of the following functions:

$(K)$ $\phi_\lambda^K(t) = \phi_\lambda^{(\rho-½,\, -½)}(t)$

$(\overline{N})$ $\phi_{\pm\lambda}^{\overline{N}}(t) = e^{(\pm i\lambda-\rho)t}$ $\quad(\lambda\neq 0;\ \ te^{-\rho t}$ is a second solution for $\lambda=0)$

$(H)$ $\phi_\lambda^H(t) = \phi_\lambda^{(-½,\, \rho-½)}(t)$, $\psi_\lambda^H(t) = sht\,\phi_\lambda^{(½,\, \rho-½)}(t) = \psi_\lambda^{(-½,\, \rho-½)}(t)$

(**Remark**. The functions $\phi_\lambda^H$ occur also as trivial $K$-types in the harmonic analysis on the pseudoriemannian symmetric space $H'\setminus G$ which is a "generalized Gelfand pair" ([2], [5]). The occurrence of the $\psi_\lambda^H$

suggests, that $L^2(H \setminus G)$ is not multiplicity free. This will be confirmed in chapter 5)

By (11) we can reduce the spherical Fourier transform on the pencils in $\mathbb{H}^n$ either to the ordinary Fourier transform ($\mathcal{P}$ parabolic) or else to the Jacobi transform. In order to formulate the results in the cases $L=K,H$ we have yet to introduce the *c-functions* $c(\lambda)$, $d(\lambda)$ by

$$\phi_\lambda^L = c^L(\lambda)\Phi_\lambda^L + c^L(-\lambda)\Phi_{-\lambda}^L \ , \psi_\lambda^H = d^H(\lambda)\Phi_\lambda^H + d^H(-\lambda)\Phi_{-\lambda}^H \quad (\lambda \notin i\mathbb{Z}) \tag{12}$$

These can be expressed explicitly in terms of the $\Gamma$-function ([17], 2. 18.; see also [1], [6] )

$$\begin{aligned} c^K(\lambda) &= \frac{2^{2\rho-1}\Gamma(\rho+\frac{1}{2})\Gamma(i\lambda)}{\sqrt{\pi}\Gamma(i\lambda+\rho)} \\ c^H(\lambda) &= \frac{\sqrt{\pi}2^{\rho-i\lambda}\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(i\lambda+\rho))\Gamma(\frac{1}{2}(i\lambda-\rho+1))} \\ d^H(\lambda) &= \frac{\sqrt{\pi}2^{\rho-i\lambda-1}\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(i\lambda+\rho+1))\Gamma(\frac{1}{2}(i\lambda-\rho+2))} \end{aligned} \tag{13}$$

Furthermore we have to take into consideration (for $L=H$) a discrete contribution to the Plancherel formula at the zeroes of $c^H(-\lambda)$, $d^H(-\lambda)$ in the upper half plane ([8]); these lie in $C = \{\mu = i(\rho-2k-1) : \rho>2k+1, \ k\in\mathbb{N}_0\}$, $D = \{\nu=i(\rho-2k-2) : \rho>2k+2, \ k\in\mathbb{N}_0\}$ respectively and the contribution there may be evaluated explicitly by taking residues (see [1], correcting incidently formula [8], A.11 by a factor $\frac{1}{2}$ in the case $\rho\in\mathbb{N}$):

$$\begin{aligned} c_\mu &= \frac{-2i\mu}{2^{2\rho}\pi} \ \frac{\Gamma(k+\frac{1}{2})\Gamma(\rho-k-\frac{1}{2})}{k!\Gamma(\rho-k)} \\ d_\nu &= \frac{-8i\nu}{2^{2\rho}\pi} \ \frac{\Gamma(k+\frac{3}{2})\Gamma(\rho-k-\frac{1}{2})}{k!\Gamma(\rho-k-1)} \end{aligned}$$

Now we can take over the general results of the Jacobi transform theory ([17], §2)

**Corollary 15.** The following transformations ("spherical Fourier transforms on pencils") $f\to\hat{f}$ are isometries of Hilbert spaces

$$(K) \quad \hat{f}(\lambda)=\int_0^\infty f(t)\phi_\lambda^K(t)\omega^K(t)dt \qquad \left( L^2(K\setminus X)\to L^2([0,\infty[,\frac{d\lambda}{2\pi|c^K(\lambda)|^2})\right)$$

$$f^K(x) = \frac{1}{2\pi}\int_0^\infty \hat{f}(\lambda)\phi_\lambda^K(x)\frac{d\lambda}{|c^K(\lambda)|^2}$$

$$(\overline{N}) \quad \hat{f}(\lambda)=\int_0^\infty f(t)\Phi_{-\lambda}^{\overline{N}}(t)\omega^{\overline{N}}(t)dt \qquad \left( L^2(\overline{N}\setminus X)\to L^2(\mathbb{R},\frac{d\lambda}{2\pi})\right)$$

$$f^{\overline{N}}(x)=\frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\lambda)\Phi_\lambda^{\overline{N}}(x)d\lambda$$

$$(H) \quad \hat{f}(\lambda)=\int_0^\infty f(t)\phi_\lambda^H(t)\omega^H(t)dt \qquad \left( L^2_{\text{even}}(H\setminus X)\to L^2([0,\infty[,\frac{d\lambda}{2\pi|c^H(\lambda)|^2})\oplus(\bigoplus_{\mu\in C}\mathbb{C}\phi_\mu^H)\right)$$

$$f^H(x) = \frac{1}{2\pi}\int_0^\infty \hat{f}(\lambda)\phi_\lambda^H(x)\frac{d\lambda}{|c^H(\lambda)|^2} + \sum_{\mu\in C} c_\mu \hat{f}(\mu)\phi_\mu^H(x)$$

The formulas for $L^2_{\text{odd}}(H \setminus X)$ are obtained by replacing everywhere $\phi$ by $\psi$, $c$ by $d$, $C$ by $D$ and $c_\mu$ by $d_\nu$.

**Remarks.** In all cases also a Paley - Wiener theorem holds ([16], th. 3.4). The even $L^1$-functions ($L=K,\overline{N},H$) and the odd $L^1$-functions ($L=\overline{N},H$) are mapped onto functions which are holomorphic in the strip $\{\lambda\in\mathbb{C}:|Im\lambda|<\rho\}$ and continuous and bounded on its closure.

For $L=H$ there is a discrete spectral part for $n\geqslant 4$ (even functions) and $n\geqslant 6$ (odd functions). We characterize now the "continuous part" in these cases.

**Lemma 16.** The equation $(\Delta+\rho^2+\lambda^2)g = f$ $(\lambda\in\mathbb{C})$ has a solution
(a) $g\in\mathcal{D}(H'\setminus X)$ for $f\in\mathcal{D}(H'\setminus X)$ iff $\hat{f}(\lambda)=0$ (and $g$ is unique)
(b) $g\in L^2(H'\setminus X)$ for $f\in L^2(H'\setminus X)$ and $\lambda\in C$ iff $\hat{f}(\lambda)=0$ (and there is a unique such $g$ with $\hat{g}(\lambda)=0$).
Analoguous statements hold also for odd functions. Also the $\hat{f}$ with several zeroes and multiple zeroes (case (a)) may be similarly characterized.

**Proof.** By the Paley-Wiener and Plancherel theorem for the Fourier transform on $H'\setminus X$ we have to find a function $\hat{g}(\lambda)$ with $(\mu^2-\lambda^2)\hat{g}(\lambda) = \hat{f}(\lambda)$ to solve the equation; $\hat{g}$ has to be even, entire, rapidly decreasing in the case (a); even, in $L^2([0,\infty[, d\lambda/|c^H(\lambda)|^2)$ and to be defined on $C$ in the case (b). In both cases this amounts to the stated conditions.

## 4. Product formulas and integral representations

In the next two chapters we study the process of changing the radiality of functions through integration. For a pencil $\mathcal{P}$ on $X$, $L=L(\mathcal{P})$ and suitable functions $f$ we define

$$(R^L f)(x)=\int_L f(lx)dl \tag{1}$$

$R^L f$ is radial on $\mathcal{P}$. We call the restriction of $R^{L_2}$ to functions which are already radial on $\mathcal{P}_1$ the *Radon transform from* $\mathcal{P}_1$ *to* $\mathcal{P}_2$ and denote it by $R^{L_2}_{L_1}$ $(L_i=L(\mathcal{P}_i))$.

Recall that by $f^L$ we denote invariant extensions of functions and also the "restrictions" of invariant functions to $\mathbb{R}$.

In this chapter we determine explicitly the Radon transforms of the spherical functions and the (extended) spherical functions of second kind in the cases of absolute convergence of the integral in (1). In this way we get various product formulas and integral representations (generalizing the common product formula of $\phi_\lambda^K$ and the common integral representations of $\phi_\lambda^K$ and $c^K(\lambda)$).

We restrict ourselves to the case $X=\mathbb{H}^n$. We may choose $\mathcal{P}_2$ standard and $\mathcal{P}_1$ conjugate to a standard pencil 2.(8) in order to treat all possible pairs of pencils up to the action of the group.

First, for $L_2=K$, we may write down $R^K f_\lambda$ explicitly for an arbitrary eigenfunction $f_\lambda$ of $\Delta$ on $X$

$$(\Delta + \rho^2 + \lambda^2)f_\lambda = 0 \tag{2}$$

**Proposition 3.** For any solution $f_\lambda$ of (2), $\lambda\in\mathbb{C}$ we have

$$\int_K f_\lambda(gkx)\,dk = f_\lambda(gx_0)\,\phi_\lambda^K(x)$$

Conversely, let $\tau\neq 0$ be a distribution on $X$ with $\int_K \tau(gkx)\,dk = r(x)\otimes\tau(gx_0)$ for all $g\in G$, all $x$ in some neighborhood of $x_0$ (here the left hand side has to be considered as a distribution on $X\times X$ in the obvious way). Then $\tau$ is a solution of (2) on $X$ for some $\lambda\in\mathbb{C}$ and $r(x) = \phi_\lambda^K(x)$.

**Remarks.** We get back the common product formula of $\phi_\lambda^K$ setting $f_\lambda = \phi_\lambda^K$ in (3). We get back the common integral representation of $\phi_\lambda^K$ by setting $g=e$, $f_\lambda = \Phi_\lambda^N$. For $f_\lambda = \Phi_\lambda^K$ the result has to be modified only in the region $d(x,x_0)>d(gx_0,x_0)$:

$$\int_K \Phi_\lambda^K(gkx)dk = \begin{cases} \Phi_\lambda^K(gx_0)\phi_\lambda^K(x) & d(x,x_0)\leq d(gx_0,x_0) \\ \phi_\lambda^K(gx_0)\Phi_\lambda^K(x) & d(x,x_0)\geq d(gx_0,x_0) \end{cases} \tag{4}$$

It may be worthwile to note, that the right hand side of (4) may be viewed (for $\mathrm{Im}\lambda>0$) as a Green's function multiplied by $W(\phi_\lambda^K,\Phi_\lambda^K) = -2i\lambda c^K(-\lambda)$ (see [3]; [7], A.6) for $\Delta^K$ on $L^2([0,\infty[,\omega^K(t)dt)$. $-\Phi_\lambda^K/2i\lambda c^K(-\lambda)$ is thus a fundamental solution for the operator $\Delta+\rho^2+\lambda^2$ on $X$ (use 3.(1)). Compare (4) also with the remark in [8], p.255.

**Corollary 5.** We have the following "mixed product formulas" for $\phi_\lambda^H$, $\psi_\lambda^H$.

$$\int_K \phi_\lambda^H(gkx)dk = \phi_\lambda^H(gx_0)\phi_\lambda^K(x), \quad \int_K \psi_\lambda^H(gkx)dk = \psi_\lambda^H(gx_0)\phi_\lambda^K(x)$$

Conversely, let $\tau$ be an $H$-invariant distribution on $X$ with $\int_K \tau(gkx)dk = r(x)\otimes\tau(gx_0)$ for all $g\in G$ and all $x$ in some neighborhood of $x_0$. Then $\tau$ is a linear combination of $\phi_\lambda^H$, $\psi_\lambda^H$ for some $\lambda\in\mathbb{C}$ and $r(x) = \phi_\lambda^K(x)$.

**Proofs.** As a function of $x\in X$, $\int_K f_\lambda(gkx)dk$ is obviously a $K$-invariant solution of (2), thus a multiple of $\phi_\lambda^K$. The factor $f_\lambda(gx_0)$ can be read off by setting $x=x_0$. For the converse, the proof given in [10], th. X. 7.2. is easily seen to apply to distributions too.

The proof shows also that (4) holds in the region $d(x,x_0)<d(gx_0,x_0)$; it applies also to the region $d(x,x_0)>d(gx_0,x_0)$ if the integral in (4) is treated there as a function of $y = gx_0$. Note also that the integral (4) converges still absolutely for $d(x,x_0) = d(gx_0,x_0)$ if $gx_0\neq x_0$. (By the asymptotics of $\Phi_\lambda^K(t),t\to 0$, see chapter 3).

In the following we will consider the Radon transforms of spherical functions which involve integrations over noncompact groups $L_2$. Only when working with spherical functions of the second kind and for $\mathrm{Im}\lambda$ big enough we may hope to obtain absolutely convergent integrals.

**Lemma 6.** The radial part of the operator $\tilde{H}_0 + \Sigma_{i=1}^{n-1}(\tilde{Y}_i^2 - \tilde{Z}_i^2)$ (see 3.(5)) on $\exp\mathbb{R}^+\cdot H_0$ for $H$-

biinvariant functions on $G$ is ${}^H\Delta^H = d^2/dt^2 + 2\rho$ coth$t\, d/dt$.

**Proof.** We apply the method of chapter 3, (4) - (6). As $H$-biinvariant function on $SL(2,\mathbb{R})$ we choose $f(g) = \frac{1}{2}tr(g\cdot\tau(g))$ where $\tau$ is the anti automorphism

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \to \begin{bmatrix} x_1 & -x_3 \\ -x_2 & x_4 \end{bmatrix}$$

For $\tilde{Z}_1^2$ we find $f(g(s)) =$ cht$\cdot$coss $\Rightarrow d/ds|_0\ f(g(s)) = 0,\ d^2/ds^2|_0\ f(g(s)) = -cht$ thus $-(\tilde{Z}_1^2 f)(t) = +$ coth$t\ d/dt$.

**Theorem 7.** Define $m(\lambda) = -2^{2\rho}c^K(-\lambda)/4i\lambda c^H(-\lambda)d^H(-\lambda)$. Then we have for Im$\lambda > \rho - 1$, with absolute convergence of the integrals (and $dh$ normalized by 3.(8))

$$\int_H \Phi_\lambda^K(ghx)dh = m(\lambda)\begin{cases} \Phi_\lambda^H(\sigma(g^{-1}x_0))\Phi_\lambda^H(x) & Hg^{-1}x_0 \leqslant Hx \\ \Phi_\lambda^H(g_\lambda^{-1}x_0)\Phi_\lambda^H(\sigma x) & Hg^{-1}x_0 \geqslant Hx \end{cases} \tag{a}$$

$$\int_H \Phi_\lambda^{\bar{N}}(ghx)dh = m(\lambda)\cdot{}^{\bar{N}}\Phi_\lambda^H(g)\Phi_\lambda^H(x) \qquad g \in \bar{N}AH \tag{b}$$

$$\int_H \Phi_\lambda^H(ghx)dh = m(\lambda)\cdot{}^{H}\Phi_\lambda^H(g)\Phi_\lambda^H(x) \qquad g \in HA^+H \tag{c}$$

Here $\sigma$ denotes the reflection in the plane $H\cdot x_0$ and the ordering on the spheres $H\cdot x$ is the one induced from $\gamma_0$: ${}^{\bar{N}}\Phi_\lambda^H(\bar{n}\cdot exptH_0\cdot h) = \Phi_\lambda^{\bar{N}}(t)$ $(\forall \bar{n} \in \bar{N}, h \in H, t \in \mathbb{R})$. ${}^H\Phi_\lambda^H(h_1 exptH_0 h_2) = \Phi_\lambda^K(t)$ $(\forall h_1, h_2 \in H, t \in \mathbb{R}^+)$.

**Proof.** We may assume $g = a = \exp(sH_0) \in A$, $x = \exp(tH_0)\cdot x_0$. For $h = \exp(u\cdot Y_1)$ (3.(5)) the radial measure on $H\cdot x_0$ and thus on $H$ is up to a constant $(shu)^{2\rho-1}du$ (3.(7)).

(i) First we prove the absolute convergence of the integrals. Let $L = K, \bar{N}$ or $H$. Define $\mathrm{v} \in \mathbb{R}$ by $\exp(sH_0)\exp(uY_1)\exp(tH_0) \in L\cdot\exp(\mathrm{v}H_0)\cdot K$. For $F^L = 2f^K, f^{\bar{N}}, 2f^H$ (see 3.(6)) and $\epsilon^L = 1,0,-1$ resp for $L = K,\bar{N},H$ resp. we find

$$F^L(\mathrm{v}) = 2sh^2(u/2)cht(e^s + \epsilon^L\cdot e^{-s}) + (e^{(s+t)} + \epsilon^L\cdot e^{-(s+t)}),$$

So $\mathrm{v} \to +\infty$ for $u \to \infty$ (remember that we have assumed $s>0$ for $L = H$). But $\Phi_\lambda^L(\mathrm{v}) = (F^L(\mathrm{v}))^{i\lambda-\rho}(1+o(1))$ for $\mathrm{v} \to +\infty$. Hence we may replace $\Phi_\lambda^L(\mathrm{v})$ by $(F^L(\mathrm{v}))^{i\lambda-\rho}$ in order to prove that $\int_0^\infty \Phi_\lambda^L(\mathrm{v})(shu)^{2\rho-1}du$ converges absolutely for Im$\lambda > \rho - 1$. (As in (4) no convergence problems arise at the singularity of $\Phi_\lambda^K$).

(ii) Next we show that $\lim_{s\to+\infty} \int_H \Phi_\lambda^L(ahx)/\Phi_\lambda^N(s)dh = \int_H \Phi_\lambda^{\bar{N}}(hx)dh$.
As in (i) the integrands on the left hand side may be replaced by $(F^L(ahx)/e^s)^{(i\lambda-\rho)}$. But the convergence of

$$F^L(agx_o)/e^s = (g\cdot g^{tr})_{11} + \epsilon^L\cdot e^{-2s}(g\cdot g^{tr})_{22}$$

to $F^{\bar{N}}(g) = (g.g^{tr})_{11}$ is obvious and moreover monotone since $(g\cdot g^{tr})_{22} > 0$. So the integrals converge too (by the dominated convergence theorem say).

(iii) As functions of $g$, the integrals in (7) are $L$-left $H$-right invariant eigenfunctions of $\tilde{H}_o^2 + \Sigma_{i=1}^{n-1}(\tilde{Y}_i^2 - \tilde{Z}_i^2)$. The corresponding radial parts we have already calculated (in 3.(7), 4.(6) resp., for $L=\bar{N}$ the dependence of $g$ can be read off directly). In (ii) we have shown that indeed we get the "eigenfunctions of second kind", the universal factor $\int_H \Phi_\lambda^{\bar{N}}(hx)dh$ giving the dependence of $x$ and $\lambda$.

(iv) To evaluate this factor explicitly, we consider $R_K^H\Phi_\lambda^K(x) = \int_H \Phi_\lambda^K(hx)dx$. We assume $\mathrm{Im}\lambda>\rho$. Then $\Phi_\lambda^K \in L^1(X)$, so $R_K^H\Phi_\lambda^K \in L^1(H' \setminus X)$ (see 5.(2)), so $R_K^H\Phi_\lambda^K$ is indeed a multiple of $\Phi_\lambda^H$ on $\gamma_0(\mathbb{R}^+)$. By using Wronskians ([4]; [8], A6) we can compute

$$-(\lambda^2+\rho^2)\int_0^\infty \Phi_\lambda^K(t)\omega^K(t)dt = -\dot{\Phi}_\lambda^K(t)\omega^K(t)|_{t=0} = -W(\Phi_\lambda^K,\phi_\lambda^K) = -2i\lambda c^K(-\lambda)$$

$$-(\lambda^2+\rho^2)\int_0^\infty \Phi_\lambda^H(t)\omega^H(t)dt = \cdots = -2i\lambda c^H(-\lambda)$$

Now by 3.(1) (or 5.(2))

$$\int_0^\infty \Phi_\lambda^K(t)\omega^K(t)dt = \int_X \Phi_\lambda^K(x)dx = 2\int_0^\infty (R_K^H\Phi_\lambda^K)(t)\omega^H(t)dt,$$

hence

$$R_K^H\Phi_\lambda^K = c^K(-\lambda)/2c^H(-\lambda)\Phi_\lambda^H \tag{8}$$

(valid for $\mathrm{Im}\lambda>\rho-1$ by analytic continuation)
Setting $g=e$ in (7a), $m(\lambda)$ can now be read off from (8) by using $2^{2\rho}\Phi_\lambda^H(0) = \cdot -2i\lambda d^H(-\lambda)$. (Obtainable from 3.(12), see [1]).

**Remarks (i)** The right hand side of (7a) has to be $-2i\lambda c^K(-\lambda)$ times the Green's function for $\Delta^H$ on $L^2(\mathbb{R},\omega^H(t)dt)$ for $\mathrm{Im}\lambda>0,\rho-1$. (7a) might also be proved directly in this way.
(ii) We get various integral representations setting $g=e$ in formula (7b). First this gives an integral representation for $\Phi_\lambda^H(Im\lambda>\rho-1)$. By taking the even and the odd part we also get an integral representation for $\phi_\lambda^H$, $\psi_\lambda^H$ $(Im\lambda>\rho-1)$. Setting also $x=x_0$ we get the integral representation

$$c^K(-\lambda)/2c^H(-\lambda) = \int_H \Phi_\lambda^{\bar{N}}(hx_0)dh \tag{9}$$

for the quotient of the $c$-functions. For $x=\exp(tH_0)\cdot x_0$, dividing by $e^{(i\lambda-\rho)t}$ we also get an integral representation for $m(\lambda)$ (see 5.(23), (24))
(iii) For $n=2$, $|\mathrm{Im}\lambda|<\frac{1}{2}$ absolutely convergent integrals are also obtained by integrating the functions $\phi_\lambda^K$; $\phi_\lambda^H$, $\psi_\lambda^H$ resp. The results can be obtained from (7) and 3.(12) (e.g. $\int_H \phi_\lambda^K(hx)dh = \dfrac{c^K(\lambda)c^K(-\lambda)}{2c^H(\lambda)c^H(-\lambda)}\phi_\lambda^H(x)$ for $G = SL(2,\mathbb{R})$. In the sense that we take the analytic continuation of the integrals in (8) and that we use 3.(12) such formulas hold for general $n$).

**Proposition 10.** We have for $\mathrm{Im}\lambda>0$, with absolute convergence of the integrals

$$\int_N \Phi_\lambda^K(gnx)dn = c^K(-\lambda)\begin{cases}\Phi_\lambda^N(g^{-1}x_0)\Phi_\lambda^N(x) & Ng^{-1}x_0\leq Nx\\ \Phi_\lambda^N(g^{-1}x_0)\Phi_{-\lambda}^N(x) & Ng^{-1}x_0\geq Nx\end{cases} \tag{a}$$

$$\int_N \Phi_\lambda^N(gnx)dn = c^K(-\lambda)\cdot {}^{\bar N}\Phi_\lambda^N(g)\Phi_\lambda^N(x) \qquad g\in \bar N MAN \tag{b}$$

$$\int_N \Phi_\lambda^N(gnx)dn = c^K(-\lambda)\cdot {}^{H}\Phi_\lambda^N(g)\Phi_\lambda^N(x) \qquad g\in HAN \tag{c}$$

Here $\Phi_\lambda^N(n\exp tH_0\cdot x_0) = e^{(i\lambda+\rho)t}$ $(\forall n\in N)$ and the ordering on the horospheres $N\cdot x$ is induced from $\gamma_0$. ${}^{\bar N}\Phi_\lambda^N(\bar n m\exp(tH_0)n) = \Phi_\lambda^{\bar N}(t) = {}^{H}\Phi_\lambda^N(h\exp tH_0 n)$ $(\forall h\in H, n\in N, \bar n\in\bar N)$

**Remarks.** We don't give the proof which proceeds step for step in the same way as the prove of (8). Moreover, both formulas (10a) and (10b) are well known in the classical spherical harmonic analysis on $X$: They are the "Abel transform of the function $\Phi_\lambda^K$" ((10a), for $g=e$) and the "integral representation of the function $c^K$" ((10b), $g=e, x=x_0$) resp. (see [17]).

We shall give explicit analytic expressions for the Radon transforms -and thus for the product formulas of this chapter -at the end of chapter 5.

## 5. Radon transforms on pencils

In chapter 4 we have evaluated the Radon transforms of certain special functions. Here we will study the transformation $R_{L_2}^{L_1}$ themselves more precisely. We shall also use the transformation $R_K^H$ to transfer the convolution structure from $K$- to $H$- invariant functions on $X=\mathbb{H}^n$.

The Radon transforms generalize both, the "translation $T_g$ of $K$-invariant functions on $X$" and the "Abel transform $\mathcal{A}$ on $X$" (see [17]):

$$(T_g f^K)(x) = \int f^K(gkx)dx = (R_{g^{-1}Kg}^{K}\lambda_{g^{-1}}f^K)(x) \qquad (\lambda_g f(x) = f(g^{-1}x))$$

$$(\mathcal{A}f^K)(x) = \Phi_{2i\rho}^N(x)\int_N f^K(nx)dn = \Phi_{2i\rho}^N(x)(R_K^N f^K)(x) \tag{1}$$

We begin with two general observations. $R^L$ is a positive operator ($f>0 \Rightarrow R^L f>0$) and it maps $L^1(X)$ onto $L^1(L\setminus X)$:

$$\int_X f(x)dx = \int_{W\setminus\mathbb{R}} (R^L f)(t)\omega^L(t)dt \tag{2}$$

by 3.(1). Together with the positivity this implies also $\|R^L f\|_1\leq\|f\|_1 \cdot \mathfrak{D}(X)$ is mapped onto $\mathfrak{D}(L\setminus X)$.

The two operators $R_{L_1}^{L_2}$, $R_{L_2}^{L_1}$ are real-adjoint. Assume namely that $f$ is $L_1$ and $g$ is $L_2$-invariant such that $f(x)\cdot g(x)\in L^1(X)$. Then we get by applying (2) in two ways to $\int_X f(x)g(x)dx$

**Lemma 3.** $\int_{W_1\setminus\mathbb{R}} f(t)(R_{L_1}^{L_2}g)(t)\omega^{L_1}(t)dt = \int_{W_2\setminus\mathbb{R}} (R_{L_1}^{L_2}f)(t)g(t)\omega^{L_1}(t)dt.$

Next we will study the transformation pair $R_K^H$, $R_H^K$. First we do this on the Fourier transformed side by using the results of chapters 3,4. In this way the geometrical relevance of the $c$-functions and the discrete part occurring in $L^2_{\text{even}}(H \setminus X) = L^2(H' \setminus X)$ will be appearant.

**Proposition 4.** Let $f \in L^1(K \setminus X)$, $|\mathrm{Im}\lambda| \leq \rho$ or $f \in \mathfrak{D}(K \setminus X)$, $\lambda \in \mathbb{C}$ resp.. Then ($\hat{}$ denoting the Fourier transform 3.(15) on $K \setminus X$, $H' \setminus X$ resp.)

$$\hat{f}(\lambda) = 2(R_K^H f)(\lambda).$$

**Proof.**

$$\hat{f}(\lambda) = \int_0^\infty f(t)\phi_\lambda^K(t)\omega^K(t)dt \overset{4(5)}{=} \int_0^\infty f(t)(R_H^K \phi_\lambda^H)(t)\omega^K(t)dt \overset{(3)}{=} \int_{\mathbb{R}} (R_K^H f)(t)\phi_\lambda^H(t)\omega^H(t)dt = 2(R_K^H f)(\lambda).$$

**Corollary 5.** $R_K^H$ is a bijection $\mathfrak{D}(K \setminus X) \to \mathfrak{D}(H' \setminus X)$. $R_K^H$ is an injection $L^1(K \setminus X) \to L^1(H' \setminus X)$.

**Proof.** The first statement follows from the Paley-Wiener theorems for the Jacobi transforms ([16], th.3.4), the second one from the injectivity of the Jacobi transforms on $L^1$- spaces ([9], th.3.2. The proof given there applies also to $L^1(H' \setminus X)$).

We will also study the operators $R_K^H$, $R_H^K$ as unbounded operators between the $L^2$- spaces. Here the quotient $c^H(\lambda) / c^K(\lambda)$ plays a significant rôle;
We write it down explicitly (3.13)

$$\frac{c^H(\lambda)}{c^K(\lambda)} = \frac{\Gamma(\frac{1}{2})}{\Gamma(\rho+\frac{1}{2})} \frac{\Gamma(\frac{1}{2}(i\lambda+\rho+1))}{\Gamma(\frac{1}{2}(i\lambda-\rho+1))} = p_\rho(\lambda) \tag{6}$$

Observe that $p_\rho(\lambda)$ is a polynomial for $\rho \in \mathbb{N}_0$, namely

$$p_\rho(\lambda) = q_\rho(-\lambda^2) \quad (\rho \in 2\mathbb{N}_0), \quad p_\rho(\lambda) = i\lambda q_\rho(-\lambda^2) \quad (\rho \in 2\mathbb{N}_0+1)$$

where

$$q_\rho(-\lambda^2) = \frac{\Gamma(\frac{1}{2})}{2^\rho / \Gamma(\rho+\frac{1}{2})} \prod_{k=1}^{[\rho/2]} (-\lambda^2-(\rho-(2k-1))^2)$$

**Corollary 7.** Consider $R_K^H : \mathfrak{D}(K \setminus X) \to \mathfrak{D}(H' \setminus X)$ and $R_H^K : L^2(H' \setminus X) \to \mathfrak{D}'(K \setminus X)$ as unbounded operators between the spaces $L^2(K \setminus X)$ and $L^2(H' \setminus X)$. Denote by $P_c$, $P_d$ resp. the projections of $L^2(H' \setminus X)$ onto its continuous part V and its discrete part resp.

(a) The operators $P_c R_K^H : L^2(K \setminus X) \to V$ and $R_H^K : V \to L^2(K \setminus X)$ are bounded operators for $\rho \notin 2\mathbb{N}_0+1$. For $\rho \in 2\mathbb{N}_0+1$ they are closed operators with domains $V_1 = \{f \in L^2(K \setminus X) : \hat{f}(\lambda)/\lambda \in L^2([0,\infty[, \frac{d\lambda}{|c^K(\lambda)|^2})\}$, $V_2 = \{f \in V : \hat{f}(\lambda)/\lambda \in L^2([0,\infty[, \frac{d\lambda}{|c^H(\lambda)|^2})\}$ resp.. They map into the Sobolev spaces $\{f \in V : \Delta^{[\rho/2]} f \in V\}$ and $\{f \in L^2(K \setminus X) : \Delta^{[\rho/2]} f \in L^2(K \setminus X)\}$ resp.

(b) $P_c R_K^H$ is an unbounded not closable operator (for $\rho > 1$). If $f \in L^2(H' \setminus X)$ is mapped by $R_H^K$ into

$L^2(K\setminus X)$, then $f\in V$.

**Proof.** (a) By using the isometries $f\to\hat f(\lambda)/c^K(\lambda)$, $\hat f(\lambda)/c^H(\lambda)$ resp. (From $L^2(K\setminus X)$ , $L^2(H'\setminus X)$ resp. onto $L^2([0,\infty[,d\lambda)$ ) (see 3.(15)), we can transfer the operators $P_c R_K^H$, $R_H^K$, $\Delta$ resp. to get multiplication by $c^K(\lambda)/2c^H(\lambda)$, $c^K(-\lambda)/c^H(-\lambda)$, $-(\lambda^2+\rho^2)$ resp. ((4), (3)).
For $\rho\notin 2\mathbb{N}_0+1$ the function $c^K(\lambda)/c^H(\lambda)$ and $(\lambda^2+\rho^2)^{[\rho/2]}c^K(\lambda)/c^H(\lambda)$ are bounded on $[0,\infty[$ (see (6) and note that $\Gamma(z+a)/\Gamma(z)\sim z^a$ for $z\to\infty$, $|arg z|<\pi-\epsilon$)
For $\rho\in 2\mathbb{N}_0+1$ they get bounded when they are multiplied by $\lambda$. We see also from the proof that for $f\in V$ ($V_2$ resp. for $\rho\in 2\mathbb{N}_0+1$), $\lambda\in[0,\infty[$

$$(R_H^K f)\hat{}(\lambda) = \hat f(\lambda)\frac{|c^K(\lambda)|^2}{|c^H(\lambda)|^2} \tag{8}$$

(b) Since $R_H^K$ is (real-) adjoint to $R_K^H$ (see (3)) it is sufficient to prove that $R_H^K$ is densely definable only on $V$ ([4]). For $\rho\notin 2\mathbb{N}_0+1$ this follows from (a) and from $R_H^K\phi_\mu^H = \phi_\mu^K$ (4.(5)) and $\phi_\mu^K\notin L^2(K\setminus X)$. For $\rho\in 2\mathbb{N}_0+1$ and $R_H^K f\in L^2(K\setminus X)$ also $R_H^K(\Delta+\rho^2)g\in L^2(K\setminus X)$ where $g\in L^2(H'\setminus X)$ is the unique solution of $(\Delta-\rho^2)g = f$ say. But for $(\Delta+\rho^2)g$ the previous argumentation shows that $((\Delta+\rho^2)g)\hat{}(\mu)=0$ ($\mu\in\mathbb{C}$). Thus also $\hat f(\mu) = 0$ ($\mu\in\mathbb{C}$).

So we may say that $R_K^H$, $R_H^K$ are regularizing (with the degree of regularization described by the behaviour of $c^K(\lambda)/c^H(\lambda)$, $\lambda\to\infty$). $R_K^H$ is not closable ($R_H^K$ not densely definable ) with the obstruction lying in the discrete part of $L^2(H'\setminus X)$

For $\rho\in\mathbb{N}_0$ also the mapping properties of $R_H^K$ on $\mathcal{D}(H'\setminus X)$ are quite nice (for $\rho\notin\mathbb{N}_0$ it can be shown that $R_H^K f$ never has compact support for $f\in\mathcal{D}(H'\setminus X)$, $f\neq 0$).

**Corollary 9.** Let $f\in\mathcal{D}(H'\setminus X)$, $\rho\in\mathbb{N}_0$.
$R_H^K f\in\mathbb{D}(K\setminus X)$ iff $\hat f$ has a zero of order 2 at any $\mu\in C$ (and also at $\mu=0$ for $\rho\in 2\mathbb{N}_0+1$).
In general, $R_H^K f$ is a linear combination of the functions $\Phi_\mu^K$, $\partial/\partial_\mu\Phi_\mu^k$ with $\mu\in C$ (and also of $\Phi_0$ if $\rho\in 2\mathbb{N}_0+1$) on each component $U$ of $\mathbb{R}\setminus$ suppf. (All functions are considered as functions on $\mathbb{R}$ by restriction to $\gamma_0$).

**Proof.** If $R_H^K f\in\mathbb{D}(K\setminus X)$, then $(R_H^K f)\hat{}(\lambda) = \hat f(\lambda)\dfrac{c^K(\lambda)\, c^K(-\lambda)}{c^H(\lambda)\, c^H(-\lambda)}$ ($\lambda\in\mathbb{C}$) by (8), but this is indeed an entire rapidly decreasing function of exponential type iff the poles of $1/p_\rho(\lambda)p_\rho(-\lambda)$ (see (6)) are compensated by zeros of $\hat f$.

$R_H^K f$ vanishes in the 0-compnent of $\mathbb{R}\setminus$suppf. So we may assume that $U$ is the component of $+\infty$. But $(\Delta+\rho^2)(g_\rho(\Delta+\rho^2))^2 f=0$ on $U$ (see (6); use [17], th. 2.1, refined version). So $f$ is on $U$ indeed a linear combination of solutions of $(\Delta+\rho^2+\mu^2)^2\phi=0$ (and of $(\Delta+\rho^2)\phi = 0$ for $\rho\in 2\mathbb{N}_0+1$). That only eigenfunctions of second kind are involved for $\mu\in C$ can be seen from $(\Delta+\rho^2)g_\rho(\Delta+\rho^2)(R_H^K f)$ $\in L^2(K\setminus X)$ ((7)). (Treat the case $\mu=0$ by considering $R_H^K(\Delta+\rho^2)(\Delta+\rho^2-\epsilon^2)^{-1} f\in\mathcal{D}'(K\setminus X)$ and let $\epsilon\to 0$).

**Remarks.** In the same way we might study any transformation pair $R_K^L$, $R_L^K$ by using 4.(3), since the behaviour of the functions $\phi_\lambda$, $\psi_\lambda$ as functions of $\lambda$ is known quite well ([6], [17] ).

(In particular assume for a fixed $g$ that $\psi_\lambda(g)$ does not vanish as a function of $\lambda$ on the closed interval $I \subseteq \mathbb{R}^+$. Let $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}$ resp. be the subspaces of $L^2(H' \setminus X)$, $L^2_{odd}(H \setminus X)$, $L^2(K \setminus X)$ resp. consisting of the $f$ with supp $\hat{f} \subseteq I$. Then $R^K$, $R^K\lambda_{g^{-1}} : L^2(H \setminus G) \to \mathcal{D}'(K \setminus G)$ are invariant with respect to the right action of $G$. $R^K|_{\mathcal{H}_1}$, $R^K\lambda_{g^{-1}}|_{\mathcal{H}_2}$ map in $\mathcal{H}$ and are bounded (as in (7)); they can even be composed with bounded invertible $G$-invariant operators $\tilde{\mathcal{H}} \to \tilde{\mathcal{H}}$ (where $\tilde{\mathcal{H}} \subseteq L^2(K \setminus G)$ is the closed $G$-span of $\mathcal{H}$) to get isometries on $\mathcal{H}_1$, $\mathcal{H}_2$ resp. But these extend to $G$-invariant isometries from $\tilde{\mathcal{H}}_i$ onto $\tilde{\mathcal{H}}$ (where $\tilde{\mathcal{U}}_i$ is the closed $G$-span of $\mathcal{H}_i$ in $L^2(H \setminus G)$). This shows that $L^2(H \setminus G)$ is not multiplicity free (since $\tilde{\mathcal{H}}_1 \cap \tilde{\mathcal{H}}_2 = \{0\}$) and a fortiori $(G,H)$ is not a "generalized Gelfand pair" ([3]).)

As another application, we will transfer now the convolution product from $\mathcal{D}(K \setminus X)$ to $\mathcal{D}(H' \setminus X)$.

**Proposition 10.** Define a "convolution product" $\otimes$ on $R_K^H L^1(K \setminus X)$ as follows:

$$R_K^H f_1 \otimes R_K^H f_2 = R_K^H(f_1 * f_2)$$

$R_K^H L^1(K \setminus X)$ gets a commutative Banach algebra, $\mathcal{D}(H' \setminus X)$ is a subalgebra.
Their characters are given by

$$\chi_\lambda : f \to \int_{\mathbb{R}} f(t)\phi_\lambda^H(t)\omega^H(t)dt$$

with $|\mathrm{Im}\lambda| \leq \rho$, $\lambda \in \mathbb{C}$ resp. and $\chi_\lambda = \chi_{-\lambda}$.

**Proof.** Anything may be read off from the fact, that $f^K \to 2(R_K^H f)^{\wedge}(\lambda)$ defines a character on the common convolution algebras $L^1(K \setminus X)$ $(|\mathrm{Im}\lambda| \leq \rho)$ and $\mathcal{D}(K \setminus X)$ $(\lambda \in \mathbb{C})$ by (4) and from $R_K^H \mathcal{D}(K \setminus X) = \mathcal{D}(H' \setminus X)$ by (5).

It is interesting to compare the convolution product $\otimes$ on $\mathbb{D}(H' \setminus X)$ with the ordinary convolution $*$ on $\mathcal{D}(H' \setminus X)$ (defined only formally). For the translations $S_a$, $T_a$ associated to these convolutions ([17]) we get, $a(t) = \exp tH_0$.

$$(R_K^H(f_1 * f))(x) = ((R_K^H f_1) * f)(x) = \int_G (R_K^H f_1)(g) f(g^{-1}x)dg$$

$$= \int_{H\mathbb{R}} (R_K^H f_1)(t) f(a(-t)h^{-1}x)dh\,\omega^H(t)dt$$

$$= \int_{\mathbb{R}} (R_K^H f_1)(t)(R^H\lambda_a(R_K^H)^{-1}R_K^H f)(t)\omega^H(t)dt$$

$$f_1 * f(x) = \int_G f_1(g) f(g^{-1}x)dg = \int_{H\mathbb{R}K} f_1(t) f(k^{-1}a(-t)\,h^{-1}x)dk\,\omega^H(t)dt\ dh$$

$$= \int_{\mathbb{R}} f_1(t)(R^H\lambda_a R_H^K f)(t)\omega^H(t)dt$$

$$(S_a f)(x) = (R^H\lambda_a(R_K^H)^{-1}f)(x) \tag{11}$$

$$(T_a f)(x) = (R^H\lambda_a R_H^K f)(x) \tag{12}$$

By (7), (8) we can compare $(R_K^H)^{-1}$ and $R_H^K$ on the Fourier transformed side. For $\rho \in \mathbb{N}_0$ in particular we have the inversion formulas (see also (6))

$$(R_K^H)^{-1} = Q_\rho(\Delta+\rho^2)\, R_H^K \tag{13}$$

where

$$Q_\rho(-\lambda^2) = \begin{cases} q_\rho^2(-\lambda^2) & \rho\in 2\mathbb{N}_0 \\ +\lambda^2 9_\rho^2(-\lambda^2 & \rho\in 2\mathbb{N}_0+1 \end{cases}$$

So that we may write then

$$S_a f = Q_\rho(\Delta+\rho^2) T_a f \tag{14}$$

So we may say that the translation $T_a$ has a regularizing effect, but that it can diverge at the discrete part. This regularizing effect is compensated and the divergent part eliminated when $S_a$ is taken instead.

Note that for $\rho\in\mathbb{N}_0$, $T_a f$ is a well defined function in $\mathcal{D}(H^{\text{c}}\setminus X)$ if $\hat{f}(\mu)$ has a zero of multiplicity 2 at any $\mu\in C$ and also at $\mu=0$ for $\rho\in 2\mathbb{N}_0+1$. Denote the space of such functions $f$ by $\mathcal{D}_0(H^{\text{c}}\setminus X)$.

**Corollary 15.** $(\mathcal{D}_0(H^{\text{c}}\setminus X),\star)$ is a commutative and associative convolution algebra for $\rho\in\mathbb{N}$. The mapping $f\to Q_\rho(\Delta+\rho^2)f$ is a bijective isomorphism $(\mathcal{D}(H^{\text{c}}\setminus X),\otimes)\to(\mathcal{D}_0(H^{\text{c}}\setminus X),\star)$. The characters of $(\mathcal{D}_0(H^{\text{c}}\setminus X),\star)$ are given by

$$f\to\int_{\mathbb{R}} f(t)\,\frac{\phi_\lambda^H(t)}{Q_\rho(-\lambda^2)}\,\omega^H(t)dt \quad (\lambda\notin C \text{ and also } \lambda\neq 0 \text{ for } \rho\in 2\mathbb{N}_0+1)$$

$$f\to\int_{\mathbb{R}} f(t)\,\frac{(\partial/\partial\mu)^2\phi_\mu^H(t)}{(d/d_\mu)^2 Q_\rho(-\mu^2)}\,\omega^H(t)dt \quad (\mu\in C \text{ and also } \mu=0 \text{ for } \rho\in 2\mathbb{N}_0+1)$$

**Proof.** From (14) we get $Q_\rho(\Delta+\rho^2)(f\otimes g) = (Q_\rho(\Delta+\rho^2)f)\otimes g = Q_\rho(\Delta+\rho^2)f\star Q_\rho(\Delta+\rho^2)g$, thus $f\mapsto Q_\rho(\Delta+\rho^2)f$ is indeed a homomorphism; moreover it is bijective by 3.(16). The rest can now be taken over from (10).

Finally we will get explicit analytic expressions in geodesic coordinates for the transforms $R_{L_1}^{L_2}$. Again we can use the invariant functions 3.(6). We treat here only the most interesting cases $R_{a^{-1}Ka}^{H}$, $R_{a^{-1}Ha}^{K}$, $R_N^H$.

In the following let $a = \exp(sH_0)$, $x = \exp(tH_0)\cdot x_0$ be fixed. For $L_2=K$ let $X = Z_1$, for $L_2 = H$ let $X = Y_1$ (see 3(5)). Denote the radial measure on $M\setminus L_2/M$ at $\ell = \exp(rX)$ by $\alpha^{L_2}(r)dr$ i.e. (3.(8))

$$\alpha^K(r) = \frac{k_{n-2}}{k_{n-1}}(\sin r)^{2\rho-1}dr \quad \alpha^H(r) = \frac{k_{n-2}}{k_{n-1}}(sh\, r)^{2\rho-1}dr \tag{16}$$

Denote the "$L_1$-radial projection" of $a\, exp(rX)\cdot x$ on $\gamma_0$ by $u(r)$, i.e. for any $L_1$-invariant function $f^{L_1}$ on $X$ we have

$$f^{L_1}(a\cdot\exp(rX)\cdot x) = f^{L_1}(u(r)). \tag{17}$$

So $\int_{L_2} f^{L_1}(alx)dl$ gets

$$\int_{L_2} f^{L_1}(alx)dl = \int_0^{r_0} f^{L_1}(a\ \exp(rX)\cdot x)\ \alpha^{L_2}(r)dr = \int_0^{r_0} f^{L_1}(u(r))\ \alpha^{L_2}(r)dr \tag{18}$$

(where $r_0=\pi$ for $L_2=K$, $r_0=\infty$ for $L_2=H$). Rewritten in "kernel form"

$$\int_{L_2} f^{L_1}(alx)dl = \int_{u(0)}^{u(r_0)} f(u)\ \alpha^{L_2}(r(u)dr/du\cdot du\cdot \tag{19}$$

For the explicit calculations we simply have to take one of the standard invariant functions 3.(6) for $f^{L_1}$. For $L_2=K$, $L_1=H$; $L_2=H$, $L_1=K$; and finally $L_2=H$, $L_1=\overline{N}$, $a=e$ resp. we get then explicitly for (17)

$$shu = sh(s+t)\cos^2(r/2) + sh(s-t)\quad \sin^2(r/2) = chs\ cht\ cosr + shs\ cht \tag{17a}$$

thus $chu\ du = -chs\ sht\ sinrdr$, $\sin^2 r = (sh(s+t)-shu)(shu-sh(s-t))/ch^2ssh^2t$

$$chu = ch(s+t)ch^2(r/2) + ch(s-t)sh^2(r/2) = chschtchr + shssht \tag{17b}$$

thus $shu\ du = chs\ cht\ shr\ dr$, $sh^2 r = (chu-ch(s+t))(chu+ch(s-t))/ch^2sch^2t$

$$e^u = e^t ch^2(r/2) + e^{-t}sh^2(r/2) = chtchr + sht \tag{17c}$$

thus $e^u\ du = cht\ sh\ r\ d\ r$, $sh^2 r = (e^u - e^t)(e^u + e^{-t})/ch^2t$

Substituting (17a), (17b), (17c) in (19) we get

**Theorem 20.** Let $a=exp(sH_0)$, $x=exp(tH_0)\cdot x_0$. Then we have explicitly in geodesic coordinates ($f$ beeing $H$-invariant in (a), $K$-invariant in (b) ):

$$\int_K f(akx)dk = (k_{n-2}/k_{n-1}(chs)^{2\rho-1}(sht)^{2\rho-1}) \int_{s-t}^{s+t} f(u)((sh(s+t)-shu)(shu-sh(s-t))^{\rho-1}chudu \tag{a}$$

$$\int_H f(ahx)dh = (k_{n-2}/k_{n-1}(chs)^{2\rho-1}(cht)^{2\rho-1}) \int_{s+t}^{\infty} f(u)((chu-ch(s+t))(chu+ch(s-t))^{\rho-1}shudu \tag{b}$$

**Corollary 21.**

$$(sht)^{2\rho-1}(R_H^K f)(t) = k_{n-2}/2^{\rho-2}k_{n-1}\cdot)\int_0^t f(u)(ch(2t)-ch(2u))^{\rho-1}chudu\quad (f\ \text{even}) \tag{a}$$

$$(cht)^{2\rho-1}(R_K^H f)(t) = (k_{n-2}/2^{\rho-1}k_{n-1})\int_t^{\infty} f(u)(ch(2u)-ch(2t))^{\rho-1}shudu \tag{b}$$

$$(cht)^{2\rho-1}(R_N^H f)(t) = (k_{n-2}/k_{n-1})\int_t^{\infty} f(u)((e^u-e^t)(e^u+e^{-t}))^{\rho-1}e^u du \tag{c}$$

All these transforms can easily be reduced to the Riemann-Liouville or Weyl fractional integral transform. Mapping properties and the inversion formulas for both are well known ([17], §5.2., 5.3.). (For the inversion of (21a), (21b) see also [16], p153.)

Explicit analytic expressions for the product formulas in chapter 4 can now easily be obtained. Also a refined study of the mapping properties of $R_K^H$, $R_H^K$ is possible. So an explicit inversion formula for $R_K^H$, $R_H^K$ is now also available for the even dimensions.

As an example, we write down explicitly the integral representation 4.(9) for the functions $c^K(-\lambda)/2c^H(-\lambda)$ and $m(\lambda)$. By substituting (17c) in (18) we obtain for 4.(8b)

$$m(\lambda)\Phi_\lambda^H(t) = (k_{n-2}/k_{n-1})\int_0^\infty (chtchr+sht)^{i\lambda-\rho}(shr)^{2\rho-1}dr \tag{22}$$

thus by setting $t=0$ (see 4.(9))

$$c^K(-\lambda)/2c^H(-\lambda) = (k_{n-2}/k_{n-1})\int_0^\infty (chr)^{i\lambda-\rho}(shr)^{2\rho-1}dr \tag{23}$$

and by using the other expression for $e^u$ in (17c), by dividing through $e^{(i\lambda-\rho)t}$ and by letting $t\to\infty$ (see 4.(9)).

$$m(\lambda) = 2^{2\rho}(k_{n-2}/k_{n-1})\int_0^\infty (chr)^{2i\lambda-1}(shr)^{2\rho-1}dr \tag{24}$$

## Bibliography


[1] BADERTSCHER E.: *Sphärische harmonische Analysis auf Geradenbüscheln*, Dissertation, Bern 1982.

[2] DIJK, G. van: *On generalized Gelfand pairs, a survey of results.* Proc. Jap. Acad. **60**, Serie A no.1 (1984)

[3] COXETER H.S.M.: *Non-Euclidian Geometry.* Toronto 1947.

[4] FARAUT J.: *Théorie spéctrale des opérateurs diff.* du second ordre (Vorlesungsskript, Tunis 1974/75).

[5] FARAUT J.: *Distributions sphériques sur les espaces hyperboliques.* J. Math. Pures Appl. (9) **58** (1979).

[6] FLENSTED-JENSEN M.: *Paley-Wiener type theorems for a differential operator connected with symmetric spaces.* Arkiv fòr Matematik **10** (1972).

[7] FLENSTED-JENSEN M.: *Spherical functions on rank one symmetric spaces and generalizations.* Proc. Symposia Pure Math. Vol. **26**, AMS Providence R.I., 1973.

[8] FLENSTED-JENSEN M.: *Spherical functions on a simply connected semisimple Lie group II.* Math. Ann. **228** (1977).

[9] FLENSTED-JENSEN M. and KOORNWINDER T.H.: *The convolution structure for Jacobi function expansions.* Arkiv fòr Matematik **11** (1973).

[10] HELGASON S.: *Differential geometry and symmetric spaces.* Academic Press, 1962.

[11] HELGASON S.: *A formula for the radial part of the Laplace - Beltrami operator.* Journal of Diff. Geom. **6** (1972)

[12] HELGASON S.: *Differential geometry, Lie groups and symmetric spaces.* Ac. press, New York 1978.

[13] HELGASON S.: *Invariant differential operators and eigenspace representations.* London Math. Soc. Lecture Note Series **34** (1980).

[14] KARPELEVIC F.J.: *The geometry of geodesics and the eigenfunctions of the Beltrami-Laplace operator on symmetric spaces.* Trans. Moscow Math. Soc. **14** (1965).

[15] KOBAYASHI S.: *Transformation groups in differential geometry.* Erg. Math. u.G. **70**, Springer (1972).

[16] KOORNWINDER T.H.: *A new proof of a Paley-Wiener type theorem for the Jacobi transform.* Arkiv fòr Matematik **12** (1974).

[17] KOORNWINDER T.H.: *Jacobi functions and analysis on non compact semisimple Lie groups.* In "Special functions: group theoretical aspects and applications" (R.A.S. Askey, T.H. Koornwinder and W. Schempp, eds.), Reidel, 1984, pp. 1-85.