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A CONVEXITY THEOREM FOR SEMISIMPLE SYMMETRIC SPACES

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We study an Iwasawa type projection related to a semisimple symmetric space and prove a generalization of Kostant's convexity theorem for it.

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CONTENTS

0. Introduction

In this paper we prove a generalization of a convexity theorem of Kostant (cf. [15]), related to a semisimple symmetric space G/H. Here G is a connected real semisimple Lie group with finite centre, τ an involution of G and H an open subgroup of $G^{\tau} = \{x \in G; \ \tau(x) = x\}$.

Let K be a τ-stable maximal compact subgroup of G (for its existence, cf. [6]) and let θ be the associated Cartan involution. We denote the infinitesimal involutions determined by θ and τ by the same symbols and write $\mathfrak{p}$, $\mathfrak{q}$ for their respective -1 eigenspaces in $\mathfrak{g}$, the Lie algebra of G. The +1 eigenspaces of θ and τ in $\mathfrak{g}$ are the respective Lie algebras $\mathfrak{k}$ and $\mathfrak{h}$ of K and H. Since θ and τ commute we have the simultaneous eigenspace decomposition

$$\mathfrak{g} = \mathfrak{k}\cap\mathfrak{q} \oplus \mathfrak{k}\cap\mathfrak{h} \oplus \mathfrak{p}\cap\mathfrak{q} \oplus \mathfrak{p}\cap\mathfrak{h} . \qquad (0.1)$$

Fix a maximal abelian subspace $\mathfrak{a}_{pq}$ of $\mathfrak{p}\cap\mathfrak{q}$ and let $\mathfrak{a}_p$ be a τ-stable maximal abelian subspace of $\mathfrak{p}$, containing $\mathfrak{a}_{pq}$. Then

$$\mathfrak{a}_p = \mathfrak{a}_{ph} \oplus \mathfrak{a}_{pq} ,$$

where $\mathfrak{a}_{ph} = \mathfrak{a}_p \cap \mathfrak{h}$. Let $E_{pq}: \mathfrak{a}_p \longrightarrow \mathfrak{a}_{pq}$ denote the corresponding projection.

The set $\Delta = \Delta(\mathfrak{g}, \mathfrak{a}_{pq})$ of restricted roots of $\mathfrak{a}_{pq}$ in $\mathfrak{g}$ is a (possibly non-reduced) root system (cf. [18]). Let Δ^+ be a choice of positive roots for Δ and Δ_p^+ a compatible choice of positive roots for $\Delta_p = \Delta(\mathfrak{g}, \mathfrak{a}_p)$. To the latter choice corresponds an Iwasawa decomposition

$$G = K A_p N , \tag{0.2}$$

where $A_p = \exp \mathfrak{a}_p$. The real analytic map $\mathcal{H} : G \longrightarrow \mathfrak{a}_p$ determined by

$$x \in K \exp \mathcal{H}(x) N \qquad (x \in G)$$

is called the corresponding Iwasawa projection.

The main result of this paper is, for any fixed $a \in A_{pq}$, a description of the image of the map $F_a : H \longrightarrow \mathfrak{a}_{pq}$ defined by

$$F_a(h) = E_{pq} \circ \mathcal{H}(ah) \tag{0.3}$$

(see Theorem 1.1). Here H is required to be connected (or to satisfy the slightly weaker condition (1.2)). If τ is a Cartan involution, then $\tau = \theta$, $H = K$, $\mathfrak{a}_p = \mathfrak{a}_{pq}$ and the result is precisely the Kostant convexity theorem.

In the present case the image of F_a is a vector sum

$$\operatorname{im}(F_a) = \operatorname{conv}(W_{K\cap H} \cdot \log a) + \Gamma(\Delta_-^+). \tag{0.4}$$

Here $W_{K\cap H}$ is a certain Weyl group, $\Gamma(\Delta_-^+)$ a closed convex polyhedral cone and we have used the notations "conv" for convex hull and "log" for the inverse of exp: $\mathfrak{a}_p \longrightarrow A_p$. The cone $\Gamma(\Delta_-^+)$ can be entirely described in terms of a set of roots Δ_-^+. In particular it is independent of a and equals $\operatorname{im}(F_e) = E_{pq} \circ \mathcal{H}(H)$.

We prove the characterization (0.4) by induction over centralizers in G, using ideas of Heckman [13] . However, since there exists no obvious infinitesimal version of (0.4), we cannot use his homotopy argument to reduce to an infinitesimal case. Consequently, we need to compute critical points and Hessians of F_a on the group. This is done in Sections 4 and 5, using ideas of [8].

Another complication is caused by the non-compactness of H. It is overcome by showing that the map F_a, apart from a right invariance, is proper (Lemma 3.3), and that its image does not fill up all of $\mathfrak{a}_{pq}$ (Lemma 3.9). These non-trivial facts are established in Section 3, by comparing F_a with another map P_a (Lemma 3.6). For a restricted class of symmetric spaces, the map P_a has been studied by Oshima and Sekiguchi [17], who pointed out its importance for the harmonic analysis on G/H. Lemma 3.3 follows from the properness of P_a, which in turn is based on a generalization of a result of [2] on the global holomorphic continuation of the Iwasawa projection. The full proof of this generalization will appear in another paper [5].

In the recent literature, Kostant's theorem for complex groups has been generalized to a Hamiltonian framework by Atiyah [1], and by Guillemin and Sternberg [11]. Duistermaat [7] obtained such a generalization for the real case. At present I do not know whether the result of this paper fits into such a framework or not.

It is a pleasure for me to thank Hans Duistermaat and Gert Heckman for some stimulating discussions on the subject of this paper.

1. A precise formulation of the result

The group N in the Iwasawa decomposition (0.2) is given by $N = \exp(\mathfrak{n})$, where

$$\mathfrak{n} = \sum_{\alpha \in \Delta_p^+} \mathfrak{g}^\alpha$$

is the Lie algebra of N.

Recall that $\Delta = \Delta(\mathfrak{g}, \mathfrak{a}_{pq})$ is a (possibly non-reduced) root system. If $\alpha \in \Delta$, we let H_α denote the element of $\mathfrak{a}_{pq}$ given by

$$H_\alpha \perp \ker \alpha \ , \qquad \alpha(H_\alpha) = 1.$$

Here $\perp$ denotes orthogonality with respect to the Killing form $\langle \cdot , \cdot \rangle$ of $\mathfrak{g}$. Moreover, if T is a subset of Δ^+, we put

$$\Gamma(T) = \sum_{\alpha \in T} \mathbb{R}_+ \cdot H_\alpha \ ,$$

where $\mathbb{R}_+ = [0, \infty)$.

Since θ and τ commute, $\theta \cdot \tau$ is an involution. The +1 and -1 eigenspaces of $\theta \cdot \tau$ are $\mathfrak{g}_+ = \mathfrak{h} \cap \mathfrak{k} \oplus \mathfrak{p} \cap \mathfrak{q}$ and $\mathfrak{g}_- = \mathfrak{k} \cap \mathfrak{q} \oplus \mathfrak{p} \cap \mathfrak{h}$ respectively. Now $\theta \cdot \tau$ acts as the identity on $\mathfrak{a}_{pq}$. Therefore, it leaves the root spaces $\mathfrak{g}^\alpha$ $(\alpha \in \Delta)$ invariant. Consequently, writing $\mathfrak{g}_+^\alpha = \mathfrak{g}^\alpha \cap \mathfrak{g}_+$ and $\mathfrak{g}_-^\alpha = \mathfrak{g}^\alpha \cap \mathfrak{g}_-$, we have

$$\mathfrak{g}^\alpha = \mathfrak{g}_+^\alpha \oplus \mathfrak{g}_-^\alpha \qquad (\alpha \in \Delta). \tag{1.1}$$

We define

$$\Delta_- = \{\alpha \in \Delta \; ; \quad \mathfrak{g}_-^{\alpha} \neq 0\} ,$$

and put $\Delta_-^+ = \Delta_- \cap \Delta^+$.

The notation $\Gamma(\Delta_-^+)$ in (0.4) has now been explained. In addition, the Weyl group $W_{K\cap H}$ is defined as

$$W_{K\cap H} = N_{K\cap H}(\mathfrak{a}_{pq}) \,/\, Z_{K\cap H}(\mathfrak{a}_{pq}),$$

the normalizer modulo the centralizer of $\mathfrak{a}_{pq}$ in $K\cap H$.

With the above notations we can formulate our main result. We say that H is essentially connected if

$$H = Z_{K\cap H}(\mathfrak{a}_{pq})\, H^o, \tag{1.2}$$

where H^o denotes the identity component of H.

<u>Theorem 1.1.</u> Let G be a connected real semisimple Lie group with finite centre, τ an involution of G, and H an essentially connected open subgroup of G^τ. If $a \in A_{pq}$, then

$$\mathrm{im}(F_a) = \mathrm{conv}(\, W_{K\cap H} \cdot \log a\,) + \Gamma(\Delta_-^+).$$

2. Some notes on the induction procedure

In the proof of Theorem 1.1 (see Section 6), induction via centralizers in G will be used. Therefore, we need Theorem 1.1 to be valid under the somewhat more general assumption that G is a reductive group of the Harish-Chandra class (class $\mathcal{H}$), τ an involution of G and H an open subgroup of G^{τ} , satisfying condition (1.2). All definitions of Sections 0 and 1 make sense in the context of a group of class $\mathcal{H}$. Instead of the Killing form we use a Ad(G)-invariant non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$, which is positive definite on $\mathfrak{k}$, negative definite on $\mathfrak{p}$, and for which the decomposition (0.1) is orthogonal. For the basic theory of a reductive symmetric space G/H of class $\mathcal{H}$, we refer the reader to [4].

<u>Lemma 2.1.</u> Let G be a group of class $\mathcal{H}$, τ an involution of G, and H an essentially connected open subgroup of G^{τ} . Then Theorem 1.1 holds for G, H if it holds for $Ad(G)^{o}$, $Ad(H)^{o}$.

<u>Proof.</u> Let $\mathfrak{v} = \text{centre}(\mathfrak{g}) \cap \mathfrak{p}$. Then $V = \exp(\mathfrak{v})$ is a closed vector subgroup of G, and we have a direct product

$$G = {}^{o}G\, V,$$

where ${}^{o}G = \bigcap \{\ker|\chi| ;\ \chi : G \to \mathbb{R}\setminus\{0\}$ a homomorphism $\}$ (cf. e.g. [20 , p. 196]). Obviously ${}^{o}G$ and V are τ-invariant, so that

$$H = (H \cap {}^{o}G)\ (H \cap V).$$

Now clearly $E_{pq} \circ \mathcal{H}$ is right $H \cap V$ -invariant, and if $a \in {}^oG \cap A_{pq}$, $a' \in V \cap A_{pq}$, then

$$E_{pq} \circ \mathcal{H}(a'ah) = E_{pq} \circ \mathcal{H}(ah) + \log a'$$

for all $h \in H$. It thus easily follows that we may reduce the proof to the case that $G = {}^oG$. Moreover, $E_{pq} \circ \mathcal{H}$ is right $Z_{K \cap H}(\mathfrak{a}_{pq})$-invariant, so that by (1.2) we may reduce the proof to the case that H is connected. But then we may as well assume that G is connected. Finally, the observation that $E_{pq} \circ \mathcal{H}$ is right centre(G)-invariant completes the proof.

For the remainder of this section, let G be a group of class $\mathcal{H}$.

Let $W(\Delta_+)$ denote the reflection group of the root system Δ_+ defined by

$$\Delta_+ = \{\alpha \in \Delta \; ; \quad \mathfrak{g}_+^\alpha \neq 0\}$$

(cf. (1.1)). Since Δ_+ can also be viewed as the root system of $\mathfrak{a}_{pq}$ in $\mathfrak{g}_+$, it follows from standard semisimple theory, applied to $[\mathfrak{g}_+, \mathfrak{g}_+]$, that

$$W(\Delta_+) \simeq W_{K \cap H^o} . \tag{2.1}$$

<u>Proposition 2.2.</u> Let H be an open subgroup of G^τ . Then the following conditions are equivalent.

(i) H is essentially connected,

(ii) $W(\Delta_+) \simeq W_{K \cap H}$.

<u>Proof.</u> In view of (2.1) the assertion follows straightforwardly from the fact that

$$H = N_{K\cap H}(\mathfrak{a}_{pq})\, H^o. \tag{2.2}$$

Now this is seen as follows. H and H^o are both θ-invariant (cf. [4]), hence admit the Cartan decompositions $H = (K\cap H)\exp(\mathfrak{p}\cap\mathfrak{h})$ and $H^o = (K\cap H^o)\exp(\mathfrak{p}\cap\mathfrak{h})$. From this we see that $(K\cap H)^o = K\cap H^o$. Moreover, (2.2) will follow from $K\cap H = N_{K\cap H}(\mathfrak{a}_{pq})\,(K\cap H)^o$. Thus let $k\in K\cap H$. Then $Ad(k^{-1})\,\mathfrak{a}_{pq}$ is maximal abelian in $\mathfrak{p}\cap\mathfrak{q}$. By standard semi-simple theory applied to $[\mathfrak{g}_+, \mathfrak{g}_+]$ it follows that there exists a $k_1\in(K\cap H)^o$ such that $Ad(k_1^{-1}k^{-1})\,\mathfrak{a}_{pq} = \mathfrak{a}_{pq}$. Hence $kk_1\in N_{K\cap H}(\mathfrak{a}_{pq})$ and we are done.

In the proof of Theorem 1.1 we shall use induction via centralizers of elements $Z\in\mathfrak{a}_{pq}$. The following result guarantees that the class of pairs (G,H) under consideration is stable under this induction. If $\mathfrak{b}$ is a subalgebra (or subspace) of $\mathfrak{g}$, we let $\mathfrak{b}_Z$ denote the centralizer of the element $Z\in\mathfrak{a}_{pq}$ in $\mathfrak{b}$. Similarly, if B is a subgroup of G (or a group acting on $\mathfrak{a}_{pq}$), we let B_Z denote the centralizer of Z in B.

<u>Proposition 2.3.</u> Let $Z\in\mathfrak{a}_{pq}$. Then G_Z is of class $\mathcal{H}$ and τ-stable. Moreover, if H is essentially connected then the same holds for H_Z.

<u>Proof.</u> The first assertion is standard (cf. [17, p. 286]). The second follows immediately from $\tau(Z) = -Z$.

Clearly, $\Delta_+(Z) = \{\alpha \in \Delta_+ ;\ \alpha(Z) = 0\}$ is the root system of $\mathfrak{a}_{pq}$ in $\mathfrak{g}_+ \cap \mathfrak{g}_Z$. In view of (2.1) we have a commutative diagram of natural monomorphisms

$$\begin{array}{ccc} W_{K\cap H_Z} & \xrightarrow{\ \varphi\ } & W_{K\cap H} \\ f \uparrow & & g \uparrow \\ W(\Delta_+(Z)) & \xrightarrow{\ \psi\ } & W(\Delta_+). \end{array}$$

Here the map g is an isomorphism onto because H is essentially connected (see Proposition 2.2). Obviously φ maps $W_{K\cap H_Z}$ into $(W_{K\cap H})_Z$, and it is well known that $\mathrm{im}(\psi) = W(\Delta_+)_Z$. Since g is compatible with the natural actions of $W_{K\cap H}$ and $W(\Delta_+)$ on $\mathfrak{a}_{pq}$ it follows that $g(W(\Delta_+)_Z) = (W_{K\cap H})_Z$, and we infer that f is surjective. By Proposition 2.2 this implies that H_Z is essentially connected.

From now on we assume again that G is connected and semisimple. In Section 6 we will prove Theorem 1.1 under the assumption that it has already been established for centralizers G_Z, $Z \in \mathfrak{a}_{pq}$. In view of the results of this section, this induction procedure is legitimate.

3. Some properties of the map F_a

Let L be the centralizer of $\mathfrak{a}_{pq}$ in G, $\mathfrak{l}$ its Lie algebra. The parabolic subgroup Q = LN of G has the Levi decomposition

$$Q = L N_Q ,$$

where $N_Q = \exp(\mathfrak{n}_Q)$,

$$\mathfrak{n}_Q = \sum_{\alpha \in \Delta^+} \mathfrak{g}^\alpha .$$

Let $\mathfrak{n}_L = \mathfrak{n} \cap \mathfrak{l}$, $N_L = N \cap L$. Then L normalizes N_Q and we have the semidirect product

$$N = N_L N_Q. \tag{3.1}$$

The $\langle .,. \rangle$ -orthocomplement $\mathfrak{l}_0$ of $\mathfrak{a}_{pq}$ in $\mathfrak{l}$ decomposes as

$$\mathfrak{l}_0 = \mathfrak{l}_{kq} \oplus \mathfrak{l}_{kh} \oplus \mathfrak{l}_{ph} ,$$

where we have written $\mathfrak{l}_{kq} = \mathfrak{l} \cap \mathfrak{k} \cap \mathfrak{q}$, etc.. $\mathfrak{l}_0$ is the Lie algebra of the closed subgroup $L_0 = (K \cap L) \exp(\mathfrak{l}_{ph})$ of L. Moreover, we have a direct product

$$L = L_0 A_{pq} .$$

<u>Proposition 3.1.</u> Let $x \in G$. Then there exist unique $a \in A_{pq}$, $n \in N_Q$ such that

$$x \in K L_0 a n. \tag{3.2}$$

Moreover, $\log a = E_{pq} \circ \mathcal{H}(x)$.

Proof. Write $x = k\, a_1\, n_1$, where $a_1 \in A_p$, $n_1 \in N$. Then $a_1 = a_0\, a$, $n_1 = n_0\, n$, with $a_0 \in A_{ph}$, $a \in A_{pq}$, $n_0 \in N_L$, $n \in N_Q$. It follows that $x = k\, a_0 (a n_0 a^{-1})\, a\, n$, whence (3.2) and the last assertion is obvious. The uniqueness follows easily from the uniqueness for the decompositions (0.2), (3.1) and $A_p = A_{ph}\, A_{pq}$.

Corollary 3.2. The map $E_{pq} \circ \mathcal{H}$ is right L_0-invariant.

Proof. Use that L_0 normalizes N_Q and centralizes A_{pq}.

In particular, if $a \in A_{pq}$, then $F_a : H \to \mathfrak{a}_{pq}$, defined by (0.3), naturally induces a map $\underline{F}_a : H/H \cap L_0 \longrightarrow \mathfrak{a}_{pq}$.

Lemma 3.3. Let $\mathcal{A}$ be a compact subset of A_{pq}. Then the map $\mathcal{A} \times H/H\cap L_0 \longrightarrow \mathfrak{a}_{pq}$, $(a,h) \mapsto \underline{F}_a(h)$ is proper.

We prove this lemma by comparing $\underline{F}_a$ with another map. Using the direct sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{l}_q \oplus \mathfrak{n}_Q$ (cf. [3]), one easily checks that the map $H \times L \times N_Q \longrightarrow G$, $(h,l,n) \longmapsto hln$ is a submersion onto an open subset Ω of G (see also [14], [16]).

Lemma 3.4. If $x \in \Omega$ then there exist unique $a_{pq}(x) \in A_{pq}$ and $n_Q(x) \in N_Q$ such that

$$x \in H\, L_0\, a_{pq}(x)\, n_Q(x) .$$

Moreover, the maps $a_{pq} : \Omega \longrightarrow A_{pq}$ and $n_Q : \Omega \longrightarrow N_Q$ are real analytic, and if $\{x_n\}$ is a sequence in Ω converging to a boundary point $x \in \partial\Omega$, then $\{a_{pq}(x_n)\}$ is not relatively compact in A_{pq}.

<u>Remarks.</u> (i) We will prove this lemma in another paper ([5]).

(ii) If $\mathfrak{a}_{pq}$ is maximal abelian in $\mathfrak{q}$ then a_{pq} can be viewed as a branch of the Iwasawa projection associated with the Iwasawa decomposition $\mathfrak{g}^d$ = $\mathfrak{h}_c \cap \mathfrak{g}^d \oplus \mathfrak{a}_{pq,c} \cap \mathfrak{g}^d \oplus \mathfrak{n}_{Q,c} \cap \mathfrak{g}^d$ of the dual real form

$$\mathfrak{g}^d = i(\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{q}) \oplus i(\mathfrak{h} \cap \mathfrak{p}). \qquad (3.3)$$

Here the subscripts c indicate that the complexification of a linear space in the complexified Lie algebra $\mathfrak{g}_c$ is taken. Lemma 3.4 is now an easy consequence of the results of [2 , Chapter 1] . In particular, the last assertion follows from [2 , Lemma 1.9] .

(iii) If the involution τ arises from a signature on Δ_p (cf. [17]), then $\mathfrak{a}_p = \mathfrak{a}_{pq}$, $\mathfrak{a}_{pq}$ is maximal abelian in $\mathfrak{q}$, and the above result is contained in [17].

In view of Lemma 3.4 we have $H \cap LN = H \cap L = H \cap L_0$, so that the inclusion $H \hookrightarrow G$ induces an embedding

$$i \; : \; H/H\cap L \longrightarrow G/Q$$

of $H/H\cap L$ onto an open subset $\underline{\Omega}$ of G/Q (the underlining indicates that $\underline{\Omega}$ is the canonical image of Ω in G/Q). If $a \in A_{pq}$, we let λ_a or $\lambda(a)$ denote left multiplication by a on G/Q, and put $\underline{\Omega}_a = \lambda(a^{-1})\underline{\Omega}$. Let

$$j : K/K\cap L \longrightarrow G/Q$$

be the natural diffeomorphism, and set

$$\underline{\Omega}_{K,a} = j^{-1}(\underline{\Omega}_a).$$

Then $\underline{\Omega}_{K,a}$ is the canonical image of $K\cap a^{-1}\Omega$ in $K/K\cap L$. Since L_0 centralizes A_{pq} and normalizes N_Q, the map a_{pq} is right L_0-invariant. Moreover, $K\cap L_0 = K\cap L$ by definition of L_0, so that for $a\in A_{pq}$ we can define a map $P_a: \underline{\Omega}_{K,a} \longrightarrow \mathfrak{a}_{pq}$ by

$$P_a(\,k(K\cap L)\,) = \log\circ a_{pq}(ak). \tag{3.4}$$

If $\mathcal{A}$ is a subset of A_{pq}, we define the subset $\underline{\Omega}_{K,\mathcal{A}}$ of $\mathcal{A}\times K/K\cap L$ by

$$\underline{\Omega}_{K,\mathcal{A}} = \{\,(a,k)\in\mathcal{A}\times K/K\cap L;\quad k\in\underline{\Omega}_{K,a}\}.$$

Proposition 3.5. If $\mathcal{A}$ is a compact subset of A_{pq}, then the map $P: \underline{\Omega}_{K,\mathcal{A}}\times\mathfrak{a}_{pq}$, $(a,k)\longmapsto P_a(k)$ is proper.

Proof. Clearly, it suffices to prove that the map a_{pq} restricted to $\Omega_{K,\mathcal{A}} = \{(a,k)\in\mathcal{A}\times K;\quad ak\in\Omega\}$ is proper. Let $\mathcal{C}$ be a compact subset of A_{pq}. We claim that the set $T = a_{pq}^{-1}(\mathcal{C})\cap\Omega_{K,\mathcal{A}}$ is compact. For assume not; then it is not closed in $\mathrm{cl}(\Omega_{K,\mathcal{A}})$. Hence there exists a point $(a,k)\in\mathcal{A}\times K$ such that $ak\in\mathrm{cl}(\Omega)\setminus\Omega = \partial\Omega$, and a sequence $\{(a_n,k_n)\}$ in T such that $a_nk_n\to ak$. By Lemma 3.4 the set $\{a_{pq}(a_nk_n)\}$ is not relatively compact in A_{pq}, contradicting the assumption on $\mathcal{C}$. Hence T is compact.

Lemma 3.6. Let $a \in A_{pq}$. Then

$$\underline{F}_a = - P_{a^{-1}} \circ j^{-1} \circ \lambda_a \circ i .$$

Proof. From the definitions it is evident that $j^{-1} \circ \lambda_a \circ i$ is a diffeomorphism of $H/H \cap L$ onto $\underline{\Omega}_{K, a^{-1}}$.
Let $h \in H$, and set $ah = kl \exp Y n$, with $(k, l, Y, n) \in K \times L_0 \times \mathfrak{a}_{pq} \times N_Q$. Then $Y = E_{pq} \circ \mathcal{H}(ah) = \underline{F}_a(h(H \cap L))$. On the other hand, $a^{-1}k = hl^{-1}\exp(-Y)\, n'$, where $n' = l \exp Y\, n^{-1}(l \exp Y)^{-1} \in N_Q$, so that $- Y = \log a_{pq}(a^{-1}k)$. This proves the lemma.

Proof of Theorem 3.3. The map $\mathcal{A} \times H/H \cap L \longrightarrow \underline{\Omega}_{K, \mathcal{A}^{-1}}$, $(a,h) \longmapsto j^{-1} \circ \lambda_a \circ i\, (a,h)$ is easily seen to be a diffeomorphism. Thus the assertion follows from Proposition 3.5 and Lemma 3.6.

Corollary 3.7. If $a \in A_{pq}$, then the set $E_{pq} \circ \mathcal{H}(aH)$ is closed in $\mathfrak{a}_{pq}$.

Observe that in view of Lemma 3.6 the following is an equivalent formulation of Theorem 1.1.

Theorem 3.8. Let H be essentially connected, and let $a \in A_{pq}$. Then

$$\mathrm{im}(P_a) = \mathrm{conv}(W_{K \cap H} \cdot \log a) + (- \Gamma(\Delta_-^+)).$$

We now come to the second main result of this section. It deals with a first restriction on the location of the set $E_{pq} \circ \mathcal{H}(aH)$. Put

$$\mathfrak{a}^+_{pq}(\Delta^+) = \{U \in \mathfrak{a}_{pq};\ \alpha(U) > 0 \text{ for } \alpha \in \Delta^+\}.$$

Lemma 3.9. Let $a \in A_{pq}$, $X \in \mathrm{cl}(\mathfrak{a}^+_{pq}(\Delta^+))$. Then the function $F_{a,X}: H \longrightarrow \mathbb{R}$ defined by

$$F_{a,X}(h) = \langle X , F_a(h) \rangle ,$$

for $h \in H$, is bounded from below.

Remarks. (i) If τ arises from a signature on Δ_p (cf. [17] , see also Remark (iii) following Lemma 3.4), then Lemma 3.9 is a consequence of [17 , Prop. 3.8] .

(ii) The proof presented below is based on a comparison with matrix coefficients of finite dimensional representations. This idea goes back to [12], and plays a main role in [17] as well.

Proof. The map $E_{pq} \circ \mathcal{H}$ is right centre(G)-invariant, so by factoring out the centre we may reduce the proof to the case that G is the adjoint group of $\mathfrak{g}$. Moreover, if x_w is a representative of $w \in W_{K\cap H}$ in $N_{K\cap H}(\mathfrak{a}_{pq})$, then

$$E_{pq} \circ \mathcal{H}(a\, x_w\, h) = E_{pq} \circ \mathcal{H}(a^{w^{-1}} h).$$

In view of (2.2) we may therefore reduce the proof to the case that H is connected.

For the rest of this section, we assume that G is the adjoint group of the semisimple algebra $\mathfrak{g}$, and that H is connected.

In view of Lemma 3.6 it suffices to show that the function

$$P_{a,X} = \langle X , P_a \rangle$$

is bounded from above on its domain $\Omega_{K,a}$. We shall prove this in a series of propositions.

Let $\mathfrak{a}_{kq}$ be maximal abelian in $\mathfrak{l}_{kq}$, and put

$$\mathfrak{a}_q = \mathfrak{a}_{kq} \oplus \mathfrak{a}_{pq} .$$

Consider the dual real form $\mathfrak{g}^d$ defined by (3.3), and put $\mathfrak{k}^d = \mathfrak{h}_c \cap \mathfrak{g}^d$, $\mathfrak{a}_p^d = \mathfrak{a}_{q,c} \cap \mathfrak{g}^d$ (this may be read as: "the $\mathfrak{k}$ in the dual form", etc.). Let $\Delta_q = \Delta(\mathfrak{g}^d, \mathfrak{a}_p^d) = \Delta(\mathfrak{g}_c, \mathfrak{a}_{q,c})$, select a system Δ_q^+ of positive roots compatible with Δ^+, and put

$$\mathfrak{n}^d = \sum_{\alpha \in \Delta_q^+} (\mathfrak{g}^d)^\alpha .$$

Let G^d, K^d, A_p^d, N^d be the connected analytic subgroups of the complex adjoint group G_c with Lie algebras $\mathfrak{g}^d$, $\mathfrak{k}^d$, $\mathfrak{a}_p^d$, $\mathfrak{n}^d$ respectively. Then G^d is a closed subgroup of G_c with the Iwasawa decomposition

$$G^d = K^d A_p^d N^d .$$

We denote the associated Iwasawa maps $G^d \longrightarrow K^d, \mathfrak{a}_p^d, N^d$ by κ^d, $\mathcal{H}^d$, ν^d respectively. Now let K_c^d, $A_{p,c}^d$, N_c^d be the connected analytic subgroups of G_c with Lie algebras $\mathfrak{k}_c^d = \mathfrak{h}_c$, $\mathfrak{a}_{p,c}^d = \mathfrak{a}_{q,c}$, $\mathfrak{n}_c^d$, and put

$$S^d = G_c \setminus K_c^d A_{p,c}^d N_c^d$$

(here $\setminus$ denotes the set theoretic difference). In [2] we proved that the Iwasawa maps extend to multi-valued holomorphic maps κ^d, $\mathcal{H}^d$, ν^d : $G_c \setminus S^d \longrightarrow K_c^d$, $\mathfrak{a}_{p,c}^d$, N_c^d.

Let L^d, L_c be the centralizers of $\mathfrak{a}_{pq}$ in G^d, G_c and let L_0^d, $L_{0,c}$, $A_{pq,c}$ be the connected analytic subgroups of G_c with Lie algebras $\mathfrak{l}_0^d = \mathfrak{l}^d \cap \mathfrak{a}_{pq}^{\perp}$, $\mathfrak{l}_{0,c}$ and $\mathfrak{a}_{pq,c}$ respectively. Then L_c is connected

and $L_{0,c}A_{pq,c}$ is an open subgroup of L_c, hence equal to L_c.

The maps κ^d and ν^d map L^d into L^d. Now L_c is defined by global polynomial equations. By holomorphic continuation the images of the multi-valued holomorphic extensions κ_L^d, ν_L^d of κ^d, ν^d to $L_c \setminus S^d$ must satisfy these equations. Hence κ_L^d and ν_L^d map $L_c \setminus S^d$ into $K_c^d \cap L_c$ and $N_{L,c}^d = N_c^d \cap L_c$ respectively. Similarly, the multi-valued holomorphic extension $\mathcal{H}_L^d$ of $\mathcal{H}^d$ to $L_c \setminus S^d$ maps $L_{0,c} \setminus S^d$ into $\mathfrak{a}_{kq,c}$. This implies

<u>Proposition 3.10.</u> $L_{0,c} \setminus S^d \subsetneq (K_c^d \cap L_c) \exp(\mathfrak{a}_{kq,c}) N_{L,c}^d$.

The set S^d is algebraic in G_c (cf. [2 , Lemma 1.8]). Since $e \notin S^d$, its defining polynomials do not vanish identically on G, so that $\Omega \setminus S^d$ is an open dense subset of Ω. Let $\{\Omega_i;\ i \in I\}$ be the set of components of $\Omega \setminus S^d$. Fix $i \in I$. Then $\Omega_i \cap L_0 \neq \phi$ because S^d is left H- and right $A_{pq}N_Q$-invariant. So we may select an element ℓ_i of $\Omega_i \cap L_0$. In view of Proposition 3.10, ℓ_i can be written as a product

$$\ell_i = h_i \exp Y_i \, n_i,$$

with $h_i \in K_c^d \cap L_c = H_c \cap L_c$, $Y_i \in \mathfrak{a}_{kq,c}$, $n_i \in N_{L,c}^d$.

Let $\mathcal{K}_{i,0}$, $\mathcal{H}_{i,0}$, $\nu_{i,0}$,be the local branches of $\mathcal{K}^d$, $\mathcal{H}^d$, ν^d at ℓ_i, determined by $\mathcal{K}_{i,0}(\ell_i) = h_i$, $\mathcal{H}^d_{i,0}(\ell_i) = Y_i$, $\nu^d_{i,0}(\ell_i) = n_i$. These local branches extend to multi-valued real analytic maps $\mathcal{K}^d_i$, $\mathcal{H}^d_i$, $\nu^d_i : \Omega_i \longrightarrow$ $\longrightarrow H_c$, $\mathfrak{a}^d_{p,c}$, N^d_c.

Proposition 3.11. Let E_{pq} denote the $\langle .,. \rangle$ -orthogonal projection of $\mathfrak{a}^d_{p,c} = \mathfrak{a}_{q,c}$ onto $\mathfrak{a}_{pq,c}$. Then $E_{pq} \circ \mathcal{H}^d_i$ is single valued and equals $\log \circ a_{pq}$ on Ω_i.

Proof. Fix a simply connected open neighbourhood U of ℓ_i in Ω_i. Then $\mathcal{K}^d_{i,0}$, $\mathcal{H}^d_{i,0}$, $\nu^d_{i,o}$ extend holomorphically to U. Select a connected open neighbourhood $U_1 \times U_2 \times U_3 \times U_4$ of $(e,e,0,e)$ in $H \times L_0 \times \mathfrak{a}_{pq} \times N_Q$ such that $U_1 U_2 \exp(U_3)\, U_4 \subset U$. In the proof of Proposition 3.10 we saw that $\mathcal{H}^d_{i,0}(U_2) \subset \mathfrak{a}_{kq,c}$. Moreover, the equivariance properties of $\mathcal{H}^d : G^d \longrightarrow \mathfrak{a}^d_p$ extend holomorphically, so that

$$\mathcal{H}^d_{i,0}(\, h\ell \exp Y\, n\,) = \mathcal{H}^d_{i,0}(\ell) + Y$$

for $(h,\ell,Y,n) \in U_1 \times U_2 \times U_3 \times U_4$. Therefore $E_{pq} \circ \mathcal{H}^d_{i,0} = \log \circ a_{pq}$ in a neighbourhood of ℓ_i. By analytic continuation this completes the proof.

Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}_q$, and Φ^+ a choice of positive roots for the root system Φ of $\mathfrak{a}_c$ in $\mathfrak{g}_c$, compatible with Δ^+_q.

Proposition 3.12. Let Λ be a dominant integral weight for $(\mathfrak{a}_c\, , \Phi^+)$ and put $\lambda = 2\Lambda|\mathfrak{a}_{q,c}$. Then for some

$m \in \mathbb{N}\setminus\{0\}$, the function $x \mapsto \exp m\lambda \mathcal{H}^d(x)$, $G^d \longrightarrow \mathbb{C}$ extends holomorphically to G_c.

Proof. By [21, Lemma 8.5.8] there exists an irreducible representation π of G^d in a finite dimensional complex linear space E with a highest $\mathfrak{a}_p^d$-weight λ and a K^d-fixed vector. Fix a highest weight vector $e_\lambda \in E\setminus\{0\}$. It is well known that the space E_K^* of K^d-fixed vectors for the contragredient π^* of π is one dimensional. Moreover, $\eta \in E_K^*$ can be normalized so that $\eta(e_\lambda) = 1$. Thus

$$\exp \lambda\mathcal{H}^d(x) = \eta(\pi(x)e_\lambda),$$

for $x \in G^d$.

Now let $p\colon \tilde{G}_c \longrightarrow G_c$ be the (finite) universal covering of G_c, and fix $\tilde{e} \in \tilde{G}_c$ with $p(\tilde{e}) = e$. Then π lifts to $\tilde{G}_c$, that is, there exists a unique holomorphic representation $\tilde{\pi}$ of $\tilde{G}_c$ in E such that

$$\tilde{\pi} = \pi \circ p$$

on the connected component $\tilde{G}^d$ of $p^{-1}(G^d)$ containing $\tilde{e}$. Put $\tilde{K}^d = \tilde{G}^d \cap p^{-1}(K^d)$. Then E_K^* is also the space of $\tilde{K}^d$-fixed vectors for the contragredient $\tilde{\pi}^*$ of $\tilde{\pi}$. Being contained in centre(G_c), the finite group $p^{-1}(e)$ leaves E_K^* invariant. Since $\dim(E_K^*) = 1$, there exists a multiplicative homomorphism $\chi : p^{-1}(e) \longrightarrow \mathbb{C}^*$ such that $\tilde{\pi}^*(x) = \chi(x)\cdot I$ on E_K^*, for $x \in p^{-1}(e)$. Let $m = |p^{-1}(e)|$. Then $\chi^m = 1$. It follows that the function $\tilde{G}_c \longrightarrow \mathbb{C}$, $x \longmapsto \eta(\tilde{\pi}(x)e_\lambda)^m$ is constant on the fibres of p, hence is the pull back of a holomorphic function $G_c \longrightarrow \mathbb{C}$, which in turn restricts to $\exp m\lambda\mathcal{H}^d$

on G^d.

Let $\mu \in \mathfrak{a}^*_{pq}$ be the element determined by

$$\langle X,Y \rangle = \mu(Y) \qquad (Y \in \mathfrak{a}_{pq}).$$

Via the Killing form, we view $\mathfrak{a}^*_{pq}$ and $\mathfrak{a}^*_q$ as subspaces of $\mathfrak{a}^*_c$. Then, if $\langle .,. \rangle$ also denotes the dual of the Killing form on $\mathfrak{a}^*_c$,

$$\langle \mu, \alpha \rangle \geq 0 \qquad (\alpha \in \Phi^+).$$

Thus, if $\Lambda_1, \ldots, \Lambda_r$ are the fundamental weights for $(\mathfrak{a}_c, \Phi^+)$, then $\mu = \sum_{1 \leq j \leq r} \tilde{\varepsilon}_j \Lambda_j$ for certain $\tilde{\varepsilon}_j \geq 0$. In view of Proposition 3.12, there exists a positive integer m such that if $\lambda_j = m\Lambda_j \mid \mathfrak{a}_{q,c}$ $(1 \leq j \leq r)$ then the functions

$$\varphi_j = \exp \lambda_j \circ \mathcal{H}^d$$

extend holomorphically to G_c. But since $\mu \in \mathfrak{a}^*_{pq}$, we must have

$$\mu = \sum_{1 \leq j \leq r} \varepsilon_j \lambda_j , \qquad (3.5)$$

with $\varepsilon_j = \frac{1}{m} \tilde{\varepsilon}_j \quad (1 \leq j \leq r)$.

<u>Proposition 3.13.</u> If $x \in \Omega$, then

$$\exp \langle X, \log a_{pq}(x) \rangle = \prod_{1 \leq j \leq r} |\varphi_j(x)|^{\varepsilon_j} . \qquad (3.6)$$

<u>Proof.</u> By continuity of both sides of (3.6) it suffices to prove the equality on a fixed component Ω_i

of the dense open subset $\Omega \setminus S^d$ of Ω. Therefore, let us fix $i \in I$.

If $\mathcal{H}_0^d$ is any branch of $\mathcal{H}^d$ at a point $x_0 \in G_c \setminus S^d$, then locally at x_0 we have

$$\varphi_j = \exp \circ \lambda_j \circ \mathcal{H}_0^d \; . \tag{3.7}$$

From Proposition 3.11 it follows that $\exp \circ \mu \circ \mathcal{H}_i^d$ is single valued and equals the left hand side of (3.6) on Ω_i. Moreover, by (3.5) it follows that on Ω_i we have

$$\mu \circ \mathcal{H}_i^d = \mathrm{Re}(\mu \circ \mathcal{H}_i^d) =$$

$$= \sum_{1 \leqslant j \leqslant r} \varepsilon_j \, \mathrm{Re}(\lambda_j \circ \mathcal{H}_i^d) \; .$$

Taking exponentials and using (3.7), we find that (3.6) holds on Ω_i, whence the result.

<u>End of the proof of Lemma 3.9.</u> The right hand side of (3.6) extends continuously to G_c, hence is bounded on the compact set $aK \cap \mathrm{cl}(\Omega)$. This implies that the function $\exp P_{a,X}$ is bounded on the canonical image $\underline{\Omega}_{K,a}$ of $K \cap a^{-1}\Omega$ in $K/K \cap L$. Hence $P_{a,X}$ is bounded from above.

4. Critical points of the functions $F_{a,X}$

In this section we let $a \in A_{pq}$ and $X \in \mathfrak{a}_{pq}$ be fixed and determine the critical set of the function $F_{a,X}: H \longrightarrow \mathbb{R}$ defined by

$$F_{a,X}(h) = \langle X , \mathcal{H}(ah) \rangle ,$$

for $h \in H$. Moreover, in the next section we shall compute Hessians of $F_{a,X}$ at points of this set. As it turns out, the computations are highly analogous to those in [8], and so are the results. As in [8], the critical set is a finite union of smooth submanifolds depending only on the subsets $\{\alpha \in \Delta_+ ;\ \alpha(\log a) = 0\}$ and $\{\alpha \in \Delta ;\ \alpha(X) = 0\}$ of Δ. Moreover, the Hessian of $F_{a,X}$ at a critical point is non-degenerate transversally to the critical manifold through it. Though such results hold for general $a \in A_{pq}$, we shall only prove them for $a \in A'_{pq}$, this being sufficient for our purposes. Here $A'_{pq} = \exp(\mathfrak{a}'_{pq})$, with

$$\mathfrak{a}'_{pq} = \{Z \in \mathfrak{a}_{pq} ;\ \alpha(Z) \neq 0 \text{ for } \alpha \in \Delta_+\} .$$

If $u \in U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}_c$, we let R_u or $R(u)$ denote the infinitesimal right regular action of u on smooth vector valued functions on G. If f is such a function, we also write

$$f(x;u) = (R_u f)(x) \qquad (x \in G).$$

In view of the Poincaré-Birkhoff-Witt theorem, the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a}_p \oplus \mathfrak{n}$ gives rise to a direct sum decomposition

$$U(\mathfrak{g}) = (\mathfrak{k}\, U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}) \oplus U(\mathfrak{a}_p).$$

Let $E_{\mathfrak{a}}$ denote the corresponding projection $U(\mathfrak{g}) \longrightarrow U(\mathfrak{a}_p)$. If $v \in U(\mathfrak{a}_p)$, we denote its homogeneous component of degree m by v_m. This makes sense because $\mathfrak{a}_p$ is abelian, so that $U(\mathfrak{a}_p) \simeq S(\mathfrak{a}_p)$, the symmetric algebra of $\mathfrak{a}_{p,c}$. This being said, we have the following result (cf. [8 , Lemma 5.1]).

<u>Lemma 4.1.</u> Let $x \in G$, $u \in U(\mathfrak{g})\mathfrak{g}$. Then

$$\mathcal{H}(x;u) = (E_{\mathfrak{a}}(u^{t(x)}))_1.$$

Here $u^{t(x)}$ denotes the image of u under the adjoint action of $t(x) = \exp \mathcal{H}(x)\cdot \nu(x)$, the "triangular part" of x, and the suffix 1 indicates that the homogeneous component of degree 1 is taken.

Let $F_X\colon G \longrightarrow \mathbb{R}$ be defined by

$$F_X(x) = \langle X , \mathcal{H}(x) \rangle \qquad (x \in G).$$

Then the following corollary holds (cf. [8 , Corollary 5.2]).

<u>Corollary 4.2.</u> If $x \in G$, $U \in \mathfrak{g}$, then

$$F_X(x;U) = \langle U^{t(x)}, X \rangle = \langle U , X^{\nu(x)^{-1}} \rangle .$$

<u>Proposition 4.3.</u> $h \in H$ is a critical point for $F_{a,X}$ if and only if $ah \in KA_pN_X$.

<u>Proof.</u> Write $ah = kbn$, with $k \in K$, $b \in A_p$, $n \in N$. Then

$F_{a,X}(h;U) = \langle X^{n^{-1}}, U \rangle$, so h is a critical point iff

$$\mathrm{Ad}(n^{-1})X \perp U \quad \text{for all } U \in \mathfrak{h}.$$

The last statement is equivalent to $\mathrm{Ad}(n^{-1})X \in \mathfrak{q}$, and since $\mathrm{Ad}(n^{-1})X \equiv X \bmod \mathfrak{n}$, this in turn is equivalent to $\mathrm{Ad}(n^{-1})X - X \in \mathfrak{n} \cap \mathfrak{q}$. Since τ maps $\mathfrak{n}$ onto $\bar{\mathfrak{n}} = \theta(\mathfrak{n})$, we have $\mathfrak{n} \cap \mathfrak{q} = 0$ and the proof is complete.

Lemma 4.4. Let H be essentially connected, $a \in A'_{pq}$, $X \in \mathfrak{a}_{pq}$. Then the critical set of $F_{a,X}$ equals the set

$$\mathcal{C}_X = \bigcup_{w \in W(\Delta_+)} wH_X$$

Remark. By Proposition 2.1, $W(\Delta_+) \simeq N_{K\cap H}(\mathfrak{a}_{pq})/Z_{K\cap H}(\mathfrak{a}_{pq})$. Since $Z_{K\cap H}(\mathfrak{a}_{pq}) \subset H_X$, the notation wH_X $(w \in W(\Delta_+))$ makes sense.

Proof of Lemma 4.4. If x_w is a representative of $w \in W(\Delta_+)$ in $N_{K\cap H}(\mathfrak{a}_{pq})$, and $h \in H_X$, then $\nu(ax_w h) = \nu(a^{w^{-1}} h) = \nu(h) \in N_X$. Hence $x_w h$ is a critical point for $F_{a,X}$ (Prop. 4.3).

Conversely, let h be a critical point for $F_{a,X}$, and write $ah = kbn$ as in the proof of Proposition 4.3. Then $n \in N_X$ and it follows that $k^{-1}ah = bn \in G_X$. Write $h = h_1h_2$, with $h_1 \in H \cap K$, $h_2 \in \exp(\mathfrak{p} \cap \mathfrak{h})$. Then $k^{-1}ah = k^{-1}h_1(h_1^{-1}ah_1)h_2$, where $k^{-1}h_1 \in K$, $h_1^{-1}ah_1 \in \exp(\mathfrak{p} \cap \mathfrak{q})$, $h_2 \in \exp(\mathfrak{p} \cap \mathfrak{h})$. Using uniqueness properties of the decomposition $G = K \exp(\mathfrak{p} \cap \mathfrak{q}) \exp(\mathfrak{p} \cap \mathfrak{h})$ and of the analogous decomposition of G_X (cf. [9, Thm 4.1]), we infer that $k^{-1}h_1 \in K_X$, $h_1^{-1}ah_1 \in \exp(\mathfrak{p} \cap \mathfrak{q}_X)$, $h_2 \in \exp(\mathfrak{p} \cap \mathfrak{h}_X)$. By standard semisimple theory, applied to $[\mathfrak{g}_+ \cap \mathfrak{g}_X, \mathfrak{g}_+ \cap \mathfrak{g}_X]$ it follows that there exists a $\ell \in K \cap H_X^o$ such that $\ell^{-1}h_1^{-1}ah_1\ell \in \exp(\mathfrak{a}_{pq})$. Thus, a being regular for the root system $\Delta_+ = \Delta(\mathfrak{g}_+, \mathfrak{a}_{pq})$, it follows that $\mathrm{Ad}(h_1\ell)$ normalizes $\mathfrak{a}_{pq}$. Hence $h_1 \in N_{K\cap H}(\mathfrak{a}_{pq})(K \cap H_X^o)$, so that $h \in N_{K\cap H}(\mathfrak{a}_{pq})H_X^o$. In view of Proposition 2.1 and the assumption on H this implies that $h \in \mathcal{E}_X$.

Observe that $\mathcal{E}_X$ is a finite union of disjoint smooth manifolds. Moreover, if $y \in \mathcal{E}_X$, then

$$T_y\mathcal{E}_X = d\lambda_y(e)\ (\mathfrak{h}_X),$$

where $T_y\mathcal{E}_X$ denotes the tangent space at y, $d\lambda_y(e)$ the derivative of the map $\lambda_y : G \to G$, $x \mapsto yx$ at e.

5. Hessians of the functions $F_{a,X}$

As in the previous section, we fix $a \in A_{pq}$ and $X \in \mathfrak{a}_{pq}$. In addition, we assume that H is essentially connected. Following [8], we write $E_{\mathfrak{k}}$, $E_{\mathfrak{a}}$, $E_{\mathfrak{n}}$ for the projections $\mathfrak{g} \longrightarrow \mathfrak{k}, \mathfrak{a}_p, \mathfrak{n}$ according to the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a}_p \oplus \mathfrak{n}$. Observe that this definition of $E_{\mathfrak{a}}$ is compatible with the definition of the map $E_{\mathfrak{a}} : U(\mathfrak{g}) \longrightarrow U(\mathfrak{a}_p)$ preceding Lemma 4.1.

<u>Lemma 5.1.</u> Let $x \in G$, $U, V \in \mathfrak{g}$. Then

$$F_X(x;UV) = B_X(U^{t(x)}, V^{t(x)}),$$

where the bilinear form B_X on $\mathfrak{g} \times \mathfrak{g}$ is given by

$$\begin{aligned} B_X(U,V) &= \langle E_{\mathfrak{a}}(UV)_1 , X \rangle \\ &= \langle [U , E_{\mathfrak{k}}(V)] , X \rangle . \end{aligned}$$

<u>Proof.</u> See [8, Lemma 6.1].

Motivated by the above formula we first study the map $V \longmapsto E_{\mathfrak{k}} \circ \mathrm{Ad}(t(x))(V)$, $\mathfrak{h} \longrightarrow \mathfrak{k}$ in more detail. Given $x \in A_{pq}$, let Θ_x be the map $H \to K$ defined by

$$\Theta_x(h) = \kappa(xh).$$

Then, writing

$$\dot{\Theta}_x(h) = d\lambda_{\Theta_x(h)}(e)^{-1} \circ d\Theta_x(h) \circ d\lambda_h(e),$$

we have the following result.

Proposition 5.2. Let $x \in A_{pq}$, $h \in H$. Then

$$\dot{\Theta}_x(h) = E_{\mathfrak{k}} \circ \mathrm{Ad}(\ t(xh)\ ).$$

Proof. Let $V \in \mathfrak{h}$, and set $xh = kt$, with $k \in K$, $t \in A_pN$. For s sufficiently close to zero, we may write

$$xh\ \exp(sV) = k\ \exp K(s)\ \exp A(s)\ t\ \exp N(s), \qquad (5.1)$$

with $K(s) \in \mathfrak{k}$, $A(s) \in \mathfrak{a}_p$, $N(s) \in \mathfrak{n}$ smoothly depending on s. Clearly $\dot{K}(0) = \dot{\Theta}_x(h)V$. Multiplying both sides of (5.1) by k^{-1} from the left and by t^{-1} from the right and differentiating at $s = 0$, we infer that $\dot{K}(0) = E_{\mathfrak{k}}(\mathrm{Ad}(t)V)$.

Lemma 5.3. The map Θ_a maps every coset $h(H \cap L)$ into a coset $\mathcal{K}(ah)(K \cap L)$ and induces a diffeomorphism $\underline{\Theta}_a$ of $H/H\cap L$ onto the open subset $\underline{\Omega}_{K,a}$ of $K/K\cap L$. Moreover, for each $w \in W(\Delta_+)$, it maps the submanifold wH_X into wK_X.

Proof. Fix $h \in H$, $\ell \in H \cap L$. We may write $ah = \mathcal{K}(ah)\ell_1 a_1 n_1$, with $\ell_1 \in L_0$, $a_1 \in A_{pq}$, $n_1 \in N_Q$. Thus $ah\ell = \mathcal{K}(ah)\ell_1 \ell\, a_1 \ell^{-1} n_1\, \ell$ $= \mathcal{K}(ah)\,\mathcal{K}(\ell_1\ell)\,[\exp \mathcal{H}(\ell_1\ell)\,\nu(\ell_1\ell)\; a_1 \ell^{-1} n\, \ell]$. The expression between brackets is easily checked to be contained in A_pN, so that $\mathcal{K}(ah\ell) = \mathcal{K}(ah)\,\mathcal{K}(\ell_1\ell)$. Since $\mathcal{K}$ maps L into $K \cap L$ this implies that $\Theta_a(\ h(H \cap L)\ ) \subset \mathcal{K}(ah)(H\cap L)$. The induced map $\underline{\Theta}_a : H/H\cap L \longrightarrow K/K\cap L$ is just $j^{-1} \circ \lambda_a \circ i$ (see Section 3), hence maps $H/H\cap L$ diffeomorphically onto $\underline{\Omega}_{K,a}$. Finally, the last assertion follows from the fact that that $\mathcal{K}$ maps G_X into K_X.

Let $\mathfrak{h}^c$, $\mathfrak{k}^c$ denote the orthocomplements of $\mathfrak{l}$ in $\mathfrak{h}$

and $\mathfrak{k}$ respectively. Then $\mathfrak{h} = \mathfrak{h}^c \oplus (\mathfrak{h} \cap \mathfrak{l})$ and $\mathfrak{k} = \mathfrak{k}^c \oplus (\mathfrak{k} \cap \mathfrak{l})$, and the maps

$$\mathfrak{n}_Q \longrightarrow \mathfrak{h}^c, \qquad U \longmapsto U + \tau U, \qquad \text{and}$$

$$\mathfrak{n}_Q \longrightarrow \mathfrak{k}^c, \qquad U \longmapsto U + \theta U$$

are linear isomorphisms. They map $\mathfrak{n}_Q \cap \mathfrak{g}_X$ onto $\mathfrak{h}^c \cap \mathfrak{g}_X$ and $\mathfrak{k}^c \cap \mathfrak{g}_X$ respectively. We now have the following.

<u>Proposition 5.4.</u> If $h \in H$, then the map $\dot{\Theta}_a(h) : \mathfrak{h} \to \mathfrak{k}$ maps $\mathfrak{l} \cap \mathfrak{h}$ into $\mathfrak{l} \cap \mathfrak{k}$ and $\mathfrak{h}_X$ into $\mathfrak{k}_X$. Moreover, the induced maps $\mathfrak{h}/\mathfrak{h} \cap \mathfrak{l} \longrightarrow \mathfrak{k}/\mathfrak{k} \cap \mathfrak{l}$ and $\mathfrak{h}_X/\mathfrak{h} \cap \mathfrak{l} \to \mathfrak{k}_X/\mathfrak{k} \cap \mathfrak{l}$ are bijective.

<u>Proof.</u> The first two assertions follow immediately from Lemma 5.3 and the fact that $d\lambda_h(e)^{-1}T_h(H \cap L) = \mathfrak{h} \cap \mathfrak{l}$, $d\lambda_{\Theta_a(h)}(e)^{-1}T_{\Theta_a(h)}(K \cap L) = \mathfrak{k} \cap \mathfrak{l}$, etc. Moreover, by the same lemma the induced map $\dot{\Theta}_a(h) : \mathfrak{h}/\mathfrak{h} \cap \mathfrak{l} \longrightarrow \mathfrak{k}/\mathfrak{k} \cap \mathfrak{l}$ must be a linear isomorphism. It maps the canonical image of $\mathfrak{h}_X$ into that of $\mathfrak{k}_X$. In view of the remarks above Proposition 5.4, the last assertion now follows for dimensional reasons.

We now return to the Hessian of $F_{a,X}$.

<u>Lemma 5.5.</u> Let $a \in A_{pq}$, $X \in \mathfrak{a}_{pq}$. Then for any $h \in H$, $U, V \in \mathfrak{h}$, we have:

$$F_{a,X}(h;UV) = \langle U , L_{a,X,h}(V) \rangle ,$$

where $L_{a,X,h}$ is the linear map $\mathfrak{h} \to \mathfrak{h}$ given by

$$L_{a,X,h} = - \mathrm{Ad}(h^{-1}) \circ \pi_{\mathfrak{h}} \circ \mathrm{Ad}(a^{-1}) \cdot \mathrm{Ad}\, \Theta_a(h) \circ \mathrm{ad}\, X \circ \dot{\Theta}_a(h).$$

Here $\pi_{\mathfrak{h}}$ denotes the projection $\mathfrak{g} \to \mathfrak{h}$ according to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$.

Proof. By Lemma 5.1 we have $F_{a,X}(h;UV) = - \langle U , \pi_{\mathfrak{h}} \circ \mathrm{Ad}(t(ah)^{-1}) \circ \mathrm{ad}\, X \circ E_{\mathfrak{k}}(\mathrm{Ad}(t(ah))V)\rangle$. Now $ah = \Theta_a(h)\, t(ah)$, so that $t(ah)^{-1} = h^{-1}a^{-1}\, \Theta_a(h)$, and the assertion follows from Proposition 5.2 and the observation that $\mathrm{Ad}(h^{-1})$ commutes with $\pi_{\mathfrak{h}}$.

Lemma 5.6. For each $a \in A'_{pq}$, $X \in \mathfrak{a}_{pq}$ the Hessian of $F_{a,X}$ at any critical point is transversally non-degenerate to the critical set of $F_{a,X}$.

Proof. Let $h = x_w h'$ be a critical point for $F_{a,X}$. Here x_w is a representative of $w \in W(\Delta_+)$ in $N_{K \cap H}(\mathfrak{a}_{pq})$, and $h' \in H_X$. It is obvious that $d\lambda_h(e)^{-1} T_h(hH_X) = \mathfrak{h}_X$. The bilinear form $\beta(U,V) = F_{a,X}(h;UV)$ on $\mathfrak{h}$ is symmetric. Since $F_{a,X}$ is locally constant on hH_X, we therefore have that $\beta = 0$ on $\mathfrak{h} \times \mathfrak{h}_X$ and on $\mathfrak{h}_X \times \mathfrak{h}$. We must show that the induced bilinear form on $\mathfrak{h}/\mathfrak{h}_X$ is non-degenerate. The Killing form being non-degenerate on $\mathfrak{h}$, this comes down to showing that the map $L_{a,X,h}$ of Lemma 5.5 has kernel $\mathfrak{h}_X$. Now $L_{a,X,h} = L_{a',X,h'}$, where $a' = a^{w^{-1}}$ still belongs to A'_{pq}. Therefore we may restrict ourselves to the case that $h \in H_X$. But then $\Theta_a(h) \in K_X$ (Lemma 5.3), so $\mathrm{Ad}(\Theta_a(h))$ and $\mathrm{ad}\, X$ commute. Hence an element $V \in \mathfrak{h}$ belongs to $\ker(L_{a,X,h})$ iff

$$\mathrm{Ad}(a^{-1}) \circ \mathrm{ad}\, X \circ \mathrm{Ad}(\Theta_a(h)) \cdot \dot{\Theta}_a(h)\, V \in \mathfrak{q}\,. \tag{5.2}$$

Now $\mathrm{ad}(X) \cdot \mathrm{Ad}(\Theta_a(h)) \circ \dot{\Theta}_a(h)$ maps $\mathfrak{h}$ into $\mathfrak{p}$ and if $U \in \mathfrak{p}$, then $\mathrm{Ad}(a^{-1})U \in \mathfrak{q}$ iff $U \in \mathfrak{a}_{pq}$ (see the proposition below). So $V \in \ker(L_{a,X,h})$ iff $\mathrm{ad}(X) \circ \mathrm{Ad}(\Theta_a(h)) \cdot \dot{\Theta}_a(h)\, V \in \mathfrak{a}_{pq}$. Now $\mathrm{Ad}(\Theta_a(h)) \circ \dot{\Theta}_a(h)$ maps $\mathfrak{h}$ into $\mathfrak{p}$, and an easy root space calculation shows that (5.2) is equivalent to

$$\mathrm{Ad}(\Theta_a(h)) \circ \dot{\Theta}_a(h)\, V \in \mathfrak{k}_X.$$

Since $\Theta_a(h) \in K_X$, $\mathrm{Ad}(\Theta_a(h))$ maps $\mathfrak{k}_X$ bijectively onto itself. Moreover, by Lemma 5.4, $\dot{\Theta}_a(h)$ induces an isomorphism $\mathfrak{h}/\mathfrak{h}_X \longrightarrow \mathfrak{k}/\mathfrak{k}_X$, and we conclude that (5.2) is equivalent to $V \in \mathfrak{h}_X$.

<u>Proposition 5.7.</u> If $a \in A'_{pq}$, $U \in \mathfrak{p}$, then $\mathrm{Ad}(a^{-1})U \in \mathfrak{q}$ if and only if $U \in \mathfrak{a}_{pq}$.

<u>Proof.</u> The if part is obvious. For the converse, suppose that $U \in \mathfrak{p}$. Using the decompositions (1.1), we may write

$$U = U_L + \sum_{\alpha \in \Delta^+} (U_+^\alpha - \theta U_+^\alpha) + (U_-^\alpha - \theta U_-^\alpha)\,,$$

with $U_L \in \mathfrak{l} \cap \mathfrak{p} \cap \mathfrak{q} = \mathfrak{a}_{pq}$, $U_+^\alpha \in \mathfrak{q}_+^\alpha$, $U_-^\alpha \in \mathfrak{q}_-^\alpha$. Using that $\tau = \theta$ on $\mathfrak{q}_+$, whereas $\tau = -\theta$ on $\mathfrak{q}_-$, we find

$$\mathrm{Ad}(a^{-1})U = U_L + \sum (a^{-\alpha} U_+^\alpha - a^\alpha \tau U_+^\alpha) + (a^{-\alpha} U_-^\alpha + a^\alpha \tau U_-).$$

Since $a^\alpha \neq a^{-\alpha}$ for all $\alpha \in \Delta_+^+$, $\mathrm{Ad}(a^{-1})U \in \mathfrak{q}$ implies $U_+^\alpha = U_-^\alpha = 0$ for all $\alpha \in \Delta^+$. Hence $U \in \mathfrak{a}_{pq}$.

Corollary 5.8. Let $a \in A'_{pq}$, $X \in \mathfrak{a}_{pq}$, $w \in W(\Delta_+)$. Then at all points of wH_X the value of $F_{a,X}$ and the signature and rank of its Hessian stay constant.

Proof. From Corollary 5.6 it follows by continuity that the statement is true on $x_w H_X^o$ $(w \in W(\Delta_+))$. In view of Proposition 2.3 we have $H_X = H_X^o Z_{K\cap H}(\mathfrak{a}_{pq})$. Moreover, by Corollary 3.2 the function $F_{a,X}$ is right $Z_{K\cap H}(\mathfrak{a}_{pq})$-invariant, and the proof is complete.

Corollary 5.9. Let $a \in A'_{pq}$, $X \in \mathfrak{a}_{pq}$, $w \in W(\Delta_+)$. Then $F_{a,X}$ has a local maximum at the critical point $h \in wH_X$ if and only if:

$$\alpha(X)\,\alpha(w^{-1} \log a) \geq 0 \qquad \text{for all} \quad \alpha \in \Delta_+^+ , \tag{5.3}$$

$$\alpha(X) \leq 0 \qquad \text{for all} \quad \alpha \in \Delta_-^+ . \tag{5.4}$$

Proof. Because of Corollary 5.8, $F_{a,X}$ has a local maximum at $h \in wH_X$ iff its Hessian at a representative x_w of w in $N_{K\cap H}(\mathfrak{a}_{pq})$ is negative definite transversally to wH_X. For this it is necessary and sufficient that all its eigenvalues are ≤ 0 (use Lemma 5.6).

By Lemma 5.5, the Hessian of $F_{a,X}$ at x_w is given by $F_{a,X}(x_w;UV) = \langle U , L_{a,X,x_w}(V)\rangle = \langle U , L'(V)\rangle$, where $L' = L_{a',X,e}$, $a' = a^{w^{-1}}$. In view of Proposition 5.2 we have

$$L'(V) = -\pi_{\mathfrak{h}} \circ \mathrm{Ad}(a^{w^{-1}})^{-1} \circ \mathrm{ad}(X) \circ E_{\mathfrak{k}}(V),$$

for $V \in \mathfrak{h}$. If $\alpha \in \Delta^+$, we put $\mathfrak{h}_+^\alpha = \{U + \theta U;\ U \in \mathfrak{g}_+^\alpha\}$ and $\mathfrak{h}_-^\alpha = \{U - \theta U;\ U \in \mathfrak{g}_-^\alpha\}$. Then

$$\mathfrak{h} = \mathfrak{h} \cap \mathfrak{k} \oplus \sum_{\alpha \in \Delta^+}^{\oplus} (\mathfrak{h}_+^\alpha \oplus \mathfrak{h}_-^\alpha).$$

We claim that L' diagonalizes over this decomposition.

Indeed, it is obvious that $L' = 0$ on $\mathfrak{h} \cap \mathfrak{k}$. Moreover, if $\alpha \in \Delta^+_-$, $U \in \mathfrak{g}_-^\alpha$, then $E_{\mathfrak{k}}(U - \theta U) = E_{\mathfrak{k}}(2U - (U + \theta U)) =$ $= -(U + \theta U)$. Also, $Ad(a^{w^{-1}})^{-1} \circ ad(X)\ (U + \theta U) =$ $= \alpha(X)\ (a^{-w\alpha}U - a^{w\alpha}\theta U) = \alpha(X)\ (a^{-w\alpha}U + a^{w\alpha}\tau U)$. Since $a^{-w\alpha}U + a^{w\alpha}\tau U = p(U + \tau U) + q(U - \tau U)$, with $p = \frac{1}{2}(a^{w\alpha} + a^{-w\alpha})$, $q = \frac{1}{2}(a^{-w\alpha} - a^{w\alpha})$, it follows that

$$L'(U - \theta U) = \alpha(X) \cosh \alpha(w^{-1} \log a)\ (U - \theta U),$$

for $U \in \mathfrak{g}_-^\alpha$. A similar computation yields:

$$L'(U + \theta U) = \alpha(X) \sinh \alpha(w^{-1} \log a)\ (U + \theta U),$$

for $U \in \mathfrak{g}_+^\alpha$, whence the claim.

Taking into account that the Killing form is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, we infer that the Hessian has all eigenvalues $\leqslant 0$ iff (5.3,4), thereby completing the proof.

6. Proof of the convexity theorem

We prove Theorem 1.1 by induction on the rank $rk(\Delta)$ of Δ. If $rk(\Delta) = 0$, then $E_{pq} \cdot \mathcal{H} = 0$, and the theorem evidently holds. So let us assume that $rk(\Delta) > 0$, and that the theorem has been proved already for groups of lower rank. In Section 2 we saw that this hypothesis implies that the theorem is also valid for lower rank groups of the Harish-Chandra class.

If $X \in \mathfrak{a}_{pq}$, we write $\Delta(X) = \{\alpha \in \Delta ; \alpha(X) = 0\}$, $\Delta^+(X) = \Delta^+ \cap \Delta(X)$, etc.. Moreover, $W(\Delta_+(X))$ denotes the reflection group generated by the reflections in roots $\alpha \in \Delta_+(X)$. Put

$$\mathfrak{a}(X,Z) = \text{conv}(W(\Delta_+(X)) \cdot Z) + \Gamma(\Delta_-^+(X)),$$

for $X, Z \in \mathfrak{a}_{pq}$. Then the assertion of Theorem 1.1 can be reformulated as

$$\text{im}(F_a) = \mathfrak{a}(0, \log a). \tag{6.1}$$

We shall first prove (6.1) for $a \in A'_{pq}$. As a first step we have:

Proposition 6.1. Let $a \in A'_{pq}$. Then

$$\text{im}(F_a) \subset \mathfrak{a}(0, \log a).$$

Proof. By Lemma 4.4 the map $F_a : H \longrightarrow \mathfrak{a}_{pq}$ is submersive except at points of

$$\mathcal{E} = \bigcup_{w \in W(\Delta_+)} \bigcup_{X \in \mathfrak{a}_{pq} \setminus \{0\}} wH_X .$$

Being a finite union of lower dimensional closed submanifolds of H, $\mathcal{E}$ has a complement which is open and dense in H. Therefore $\mathrm{im}(F_a)$ has dense interior. Moreover, $\mathrm{im}(F_a)$ being closed (Corollary 3.7), a point Z of the boundary $\partial\,\mathrm{im}(F_a)$ of $\mathrm{im}(F_a)$ must be the image $F_a(h)$ of some $h \in \mathcal{E}$. Write $h = x_w h'$, with x_w a representative of $w \in W(\Delta_+)$ in $N_{K\cap H}(\mathfrak{a}_{pq})$, and $h' \in H_X$, $X \in \mathfrak{a}_{pq}\setminus\{0\}$. Then $E_{pq}\circ\mathcal{H}(ah) = E_{pq}\circ\mathcal{H}(a^w h')$ which by the induction hypothesis is contained in $\mathfrak{a}(X, w^{-1}(\log a))$ (cf. Section 2). Now put

$$\mathcal{B} = \bigcup_{w\in W(\Delta_+)} \bigcup_{X\in\mathfrak{a}_{pq}\setminus\{0\}} \mathfrak{a}(X, w^{-1}(\log a)).$$

Then from the above reasoning we infer that

$$\partial\,\mathrm{im}(F_a) \subset F_a(\mathcal{E}) \subset \mathcal{B} .$$

It follows that every component of $\mathfrak{a}_{pq}\setminus\mathcal{B}$ must be entirely contained in the set $F_a(H)$, or have empty intersection with it. Now clearly $\mathcal{B} \subset \mathfrak{a}(0, \log a)$. In view of Lemma 3.9, $\mathrm{im}(F_a)$ does not contain the connected set $\mathfrak{a}_{pq}\setminus\mathfrak{a}(0, \log a)$. Therefore $\mathrm{im}(F_a)\setminus\mathfrak{a}(0, \log a) = \emptyset$ and the assertion follows.

<u>Proposition 6.2.</u> Let $a \in A'_{pq}$, $X \in \mathfrak{a}_{pq}$. If $F_{a,X}$ has a local maximum at $h \in H$, then $\langle U, X\rangle \leq F_{a,X}(h)$ for all $U \in \mathfrak{a}(0, \log a)$.

<u>Proof.</u> Suppose $F_{a,X}$ has a local maximum in $h \in H$. Then h is a critical point, hence of the form $x_w h'$, with x_w

a representative of $w \in W(\Delta_+)$ in $N_{K\cap H}(\mathfrak{a}_{pq})$, and $h' \in H_X$. Moreover, by Corollary 5.9 we must have

$$\alpha(X)\,\alpha(w^{-1}\log a) \geqslant 0 \qquad \text{for all } \alpha \in \Delta_+^+ ,$$

$$\alpha(X) \leqslant 0 \qquad \text{for all } \alpha \in \Delta_-^+ .$$

In Proposition 6.3 below we deduce that the first statement implies that $\langle X,Z\rangle \leqslant \langle X, w^{-1}(\log a)\rangle$ for all $Z \in \mathrm{conv}(\, W(\Delta_+)\cdot \log a\,)$. Moreover, the second statement implies that $\langle X,Y\rangle \leqslant 0$ for all $Y \in \Gamma(\Delta_-^+)$. Hence

$$\langle X , U\rangle \leqslant \langle X , w^{-1}(\log a)\rangle ,$$

for every $U \in \mathfrak{a}(0, \log a)$. Since $F_{a,X}(x_w h') = F_{a,X}(x_w) =$ $= \langle X , w^{-1}(\log a)\rangle$, the assertion now follows.

<u>Proposition 6.3.</u> Let $X,Y \in \mathfrak{a}_{pq}$ be such that $\alpha(X)\,\alpha(Y) \geqslant 0$ for all $\alpha \in \Delta_+$. Then $\langle X , uY\rangle \leqslant \langle X , Y\rangle$ for all $u \in W(\Delta_+)$.

<u>Proof.</u> Let E be the subspace of $\mathfrak{a}_{pq}$ spanned by H_α, $\alpha \in \Delta_+$. Then $\Delta_+ = \Delta(\mathfrak{g}_+, \mathfrak{a}_{pq})$ is a (possibly non-reduced) root system on E. Moreover, since $W(\Delta_+)$ leaves E invariant and acts trivially on $E^\perp$, it suffices to prove the statement for $X,Y \in E$. But then it is well known that the hypothesis implies the existence of a closed Weyl chamber C such that $X,Y \in C$. The proposition now follows.

<u>Proposition 6.4.</u> If $a \in A'_{pq}$ then $\partial\, \mathrm{im}(F_a) \subset \partial \mathfrak{a}(0,\log a)$.

<u>Proof.</u> Given $X \in \mathfrak{a}_{pq}$, write

$$\mathfrak{a}(X) = \sum_{\alpha \in \Delta(X)} \mathbb{R}\cdot H_\alpha .$$

Then for every $Z \in \mathfrak{a}_{pq}$, we have

$$\mathfrak{a}(X,Z) \subset Z + \mathfrak{a}(X).$$

By regularity of a, the set $\mathfrak{a}(X, w^{-1}(\log a))$ $(w \in W(\Delta_+))$ has non-empty interior $\mathring{\mathfrak{a}}(X, w^{-1}(\log a))$ in $w^{-1}(\log a) + \mathfrak{a}(X)$. Put $'\mathfrak{a}_{pq} = \{X \in \mathfrak{a}_{pq};\ \operatorname{rk} \Delta(X) = \operatorname{rk} \Delta - 1\}$. Then clearly

$$\mathcal{B} = \bigcup_{w \in W(\Delta_+)} \bigcup_{X \in {}'\mathfrak{a}_{pq}} \mathfrak{a}(X, w^{-1}(\log a)).$$

Moreover,

$$\mathring{\mathcal{B}} = \bigcup_{w \in W(\Delta_+)} \bigcup_{X \in {}'\mathfrak{a}_{pq}} \mathring{\mathfrak{a}}(X, w^{-1}(\log a))$$

is dense in $\mathcal{B}$. Since $\operatorname{im}(F_a)$ is the closure of a union of connected components of $\mathfrak{a}_{pq} \setminus \mathcal{B}$, it follows that $\partial \operatorname{im}(F_a) \cap \mathring{\mathcal{B}}$ is dense in $\partial \operatorname{im}(F_a)$. Therefore it suffices to show that $\partial \operatorname{im}(F_a) \cap \mathring{\mathcal{B}} \subset \partial \mathfrak{a}(0, \log a)$.

Let $Z \in \partial \operatorname{im}(F_a) \cap \mathring{\mathcal{B}}$. Then there exist $w \in W(\Delta_+)$ and $X \in {}'\mathfrak{a}_{pq}$ such that $Z \in \mathring{\mathfrak{a}}(X, w^{-1}(\log a))$. Moreover, by the induction hypothesis there exists a $h \in H_X$ such that $Z = E_{pq}\circ\mathcal{H}(a^{w^{-1}}h) = F_a(x_w h)$. Multiplying X by -1 if necessary, we can arrange that X is an outward normal to $F_a(H)$. Thus, $F_{a,X}$ attains a local maximum at $x_w h$. By Proposition 6.2 it now follows that $Z = F_a(x_w h) \in \partial \mathfrak{a}(0, \log a)$.

<u>Corollary 6.5.</u> If $a \in A'_{pq}$, then $\operatorname{im}(F_a) = \mathfrak{a}(0, \log a)$.

<u>Completion of the proof.</u> Let $a \in A_{pq} \setminus A'_{pq}$, and select a sequence $\{a_n\}$ in A'_{pq} which converges to a. Then $\mathcal{A} = \{a\} \cup \{a_n\}$ is a compact subset of A_{pq}.

Let $h \in H$. Then $E_{pq} \circ \mathcal{H}(a_n h) = U_n + V_n$, where $U_n \in \mathrm{conv}(\ W(\Delta_+) \cdot \log a_n\ )$ and $V_n \in \Gamma(\Delta_-^+)$. Clearly U_n varies in a compact subset of $\mathfrak{a}_{pq}$, and so does $E_{pq} \circ \mathcal{H}(a_n h)$. It follows that $\{V_n\}$ is relatively compact in $\Gamma(\Delta_-^+)$. Passing to a subsequence if necessary, we may therefore assume that the sequences $\{U_n\}$ and $\{V_n\}$ converge, to say U and V respectively. Clearly $U \in \mathrm{conv}(\ W(\Delta_+) \cdot \log a)$, $V \in \Gamma(\Delta_-^+)$. On the other hand, $U + V = E_{pq} \circ \mathcal{H}(ah)$, and we have shown that $E_{pq}(\mathcal{H}(aH)) \subset \mathfrak{a}(0, \log a)$.

For the converse, let $W \in \mathfrak{a}(0, \log a)$ and write $W = U + V$, with $U \in \mathrm{conv}(\ W(\Delta_+) \cdot \log a\ )$ and $V \in \Gamma(\Delta_-^+)$. Then there exists a sequence $\{U_n\}$ in $\mathfrak{a}_{pq}$ which converges to U, and such that $U_n \in \mathrm{conv}(\ W(\Delta_+) \cdot \log a_n)$ for all n. By Corollary 6.5 there exists a sequence $\{h_n\}$ in H such that $E_{pq} \circ \mathcal{H}(a_n h_n) = U_n + V$, and by Lemma 3.3 the set $\{h_n\}$ must be relatively compact in H. Passing to a subsequence if necessary, we may therefore assume that h_n converges to a point $h \in H$. It follows that $E_{pq} \circ \mathcal{H}(ah) =$ $= \lim E_{pq} \circ \mathcal{H}(a_n h_n) = \lim (U_n + V) = U + V$ and the proof is complete.

Appendix: the group case

Let G be a connected real semisimple Lie group with finite centre. It may be viewed as a symmetric space in the following way. Let $'G = G \times G$, $'\tau : 'G \longrightarrow 'G$ the involution given by $'\tau(x,y) = (y,x)$. Then $'H = d(G)$, the diagonal in $G \times G$, and the map $G \times G \longrightarrow G$, $(x,y) \longmapsto xy^{-1}$ induces a diffeomorphism $'G/'H \simeq G$.

In this appendix we reformulate Theorem 1.1 for the symmetric pair $('G,'H)$ in terms of the structure of G. If not specified, our notations have an obvious meaning.

Let θ be a Cartan involution for G. Then $'\theta = \theta \times \theta$ is a Cartan involution for $'G$ which commutes with $'\tau$. Thus, on the Lie algebra level we have $'\mathfrak{p} = \mathfrak{p} \times \mathfrak{p}$, $'\mathfrak{q} = \delta(\mathfrak{g})$, $'\mathfrak{p} \cap {}'\mathfrak{q} = \delta(\mathfrak{p})$, where we have used the notation $\delta(\mathfrak{g})$ for the subset $\{(X,-X);\ X \in \mathfrak{g}\}$ of $'\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}$, etc..

Let $\mathfrak{a}_p$ be maximal abelian in $\mathfrak{p}$ and put $'\mathfrak{a}_p = \mathfrak{a}_p \times \mathfrak{a}_p$ and $'\mathfrak{a}_{pq} = \delta(\mathfrak{a}_p)$. Let $j : \mathfrak{a}_p \longrightarrow {}'\mathfrak{a}_{pq}$ be the linear isomorphism given by $j(X) = (X,-X)$. Then the projection $'E_{pq}: {}'\mathfrak{a}_p \longrightarrow {}'\mathfrak{a}_{pq}$ is given by

$$'E_{pq}(X,Y) = j\left(\tfrac{1}{2}(X - Y) \right).$$

Moreover, with obvious notations, $'\Delta_{pq} = j^{*\,-1}(\Delta_p)$, and if π_i denotes the projection of $'\mathfrak{a}_p$ on the i-th coordinate (i=1,2), then $'\Delta_p = \pi_1^* \Delta_p \cup \pi_2^* \Delta_p$. Let Δ_p^+ be a choice of positive roots for Δ_p,

$$G = K A_p N$$

the associated Iwasawa decomposition and $\mathcal{H} : G \longrightarrow \mathfrak{a}_p$ the corresponding Iwasawa projection. Then $'\Delta^+_{pq} = j^{*-1}(\Delta^+_p)$ and $'\Delta^+_p = \pi^*_1(\Delta^+_p) \cup \pi^*_2(-\Delta^+_p)$ are compatible choices of positive roots. The associated Iwasawa decomposition for 'G is $'G = 'K'A_p'N$, where $'K = K \times K$, $'A_p = A_p \times A_p$, $'N = N \times \bar{N}$. The associated projection $'\mathcal{H} : 'G \longrightarrow '\mathfrak{a}_p$ is given by $'\mathcal{H}(x,y) = (\mathcal{H}(x), -\mathcal{H}(\theta y))$, so that

$$'E_{pq} \circ '\mathcal{H}(x,y) = \tfrac{1}{2} j(\mathcal{H}(x) + \mathcal{H}(\theta y)).$$

It is now straightforward to derive the following equivalent formulation of Theorem 1.1 in terms of G's structure. Let W denote the Weyl group of $\mathfrak{a}_p$ in $\mathfrak{g}$. If $\alpha \in \Delta^+_p$, we let H_α denote the element of $\mathfrak{a}_p \cap (\ker \alpha)^\perp$ with $\alpha(H_\alpha) = 1$, and write

$$\Gamma(\Delta^+_p) = \sum_{\alpha \in \Delta^+_p} \mathbb{R}_+ H_\alpha .$$

<u>Theorem A.1.</u> Let G be a connected real semisimple Lie group with finite centre, $G = KA_pN$ an Iwasawa decomposition for G, and $\mathcal{H} : G \longrightarrow \mathfrak{a}_p$ the corresponding projection. If $a \in A_p$, then the image of the map $\Psi_a : G \longrightarrow \mathfrak{a}_p$ given by

$$\Psi_a(x) = \tfrac{1}{2}(\mathcal{H}(ax) + \mathcal{H}(a\,\theta x))$$

is equal to

$$\mathrm{im}(\Psi_a) = \mathrm{conv}(W \cdot \log a) + \Gamma(\Delta^+_p).$$

In particular, putting $a = e$, and using the Iwasawa decomposition $G = KA_pN$, one easily finds

$$\mathcal{H}(\bar{N}) = \Gamma(\Delta^+_p).$$

Moreover, Lemma 3.3 implies that the map $\mathcal{H} : \bar{N} \longrightarrow \mathfrak{a}_p$ is proper. Now these facts can be checked independently as follows.

By [10] (cf. also [19]), there exists a diffeomorphism $\Phi : \times_{\alpha \in P} \bar{N}_\alpha \longrightarrow \bar{N}$, such that

$$\mathcal{H} \circ \Phi ((\bar{n}_\alpha)_{\alpha \in P}) = \sum_{\alpha \in P} \mathcal{H}(\bar{n}_\alpha) . \qquad (A.1)$$

Here the Cartesian product extends over the set P of indivisible roots in Δ_p^+. Moreover, $\bar{N}_\alpha = \bar{N} \cap G_\alpha$, where G_α is a closed semisimple subgroup of G, whose Lie algebra is the real rank one algebra generated by $\mathfrak{g}^{-2\alpha}$, $\mathfrak{g}^{-\alpha}$, $\mathfrak{g}^{\alpha}$ $\mathfrak{g}^{2\alpha}$. The Iwasawa decomposition of G induces the Iwasawa decompositions $G_\alpha = K_\alpha A_{p,\alpha} N_\alpha$ with $K_\alpha = K \cap G_\alpha$, etc.. Thus we see that by (A.1) the above statements for the map $\mathcal{H} : \bar{N} \longrightarrow \mathfrak{a}_p$ reduce to the corresponding statements for the maps $\mathcal{H} : \bar{N}_\alpha \rightarrow \mathfrak{a}_{p,\alpha}$. The latter statements can be checked to be true from the explicit formula for the Iwasawa projection of a real rank one group (cf. [14] , [19]).

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