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THE PYTHAGORAS TREE AS A JULIA SET

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Various properties of the so-called Pythagoras tree are considered, especially with respect to iterated mappings, self-similarity and Julia sets. The approach is based on binary representations of real numbers and on the use of complex variables.

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1. Introduction

Forty years ago in the dark days of the second world war the Dutch engineer A. Bosman constructed the so-called Pythagoras tree reproduced here in fig. 1.1. It must taken him many, many hours at the drawingboard. But now with a personal computer and a plotter a nice tree can be formed within an hour and generalizations can be made to order.

This research did actually start when we tried to determine the set of the infinitesimally small squares. Let J denote the closure of this set then J is a continuous curve which is invariant with respect to two similarity transformations A and B. Coordinates can be chosen in such a way that in complex notation

$$\begin{cases} A: \ z \to 1+(1+i)z/2, \\ B: \ z \to 1+(1-i)z/2. \end{cases} \tag{1.1}$$

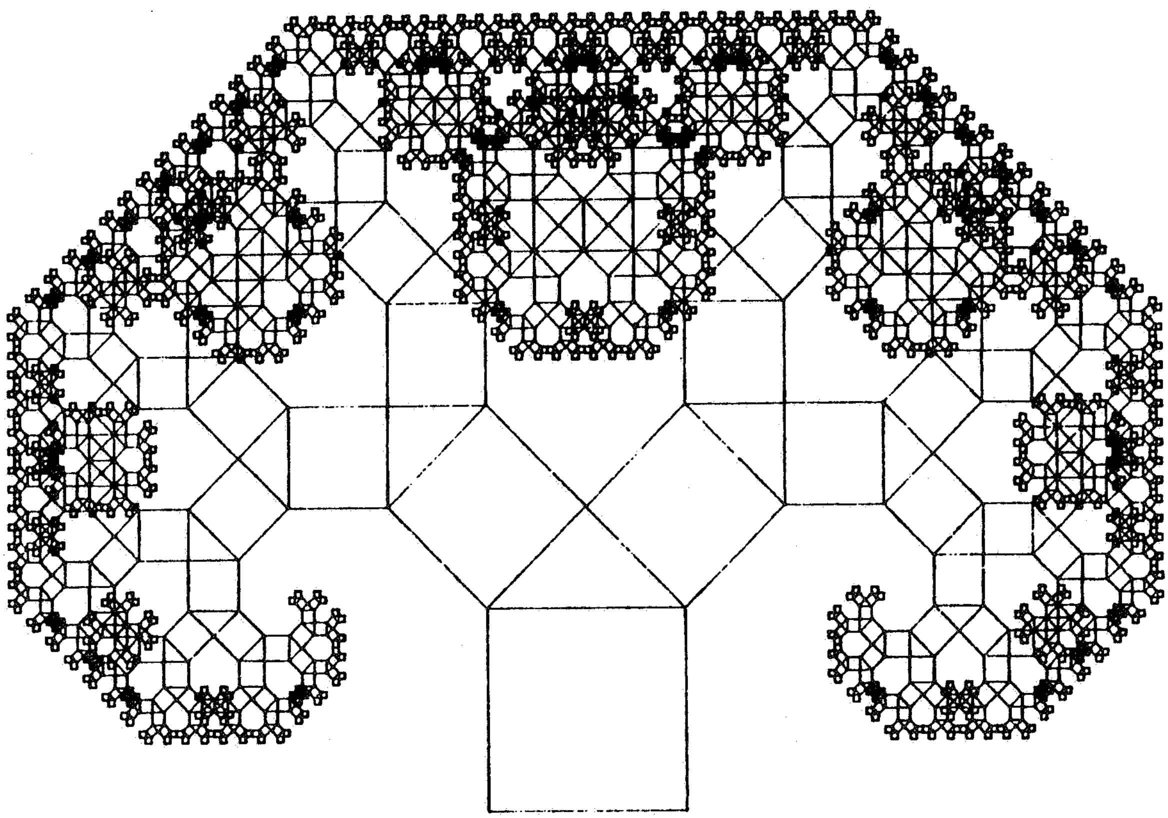

Fig. 1.1 The Pythagoras tree

We see that A has $1+i$ as its centre of rotation, the reduction factor $1/\sqrt{2}$ and the rotation angle $\pi/4$. For B the centre is at $1-i$ with factor $1/\sqrt{2}$ and angle $-\pi/4$. Both centres $1 \pm i$ are elements of J.

More points of J can be obtained from them by subjecting them to a random sequence of A and B. In this way fig. 1.2 has been obtained as the "blossom" of the Pythagoras tree.

J is a continuous image of the unit interval $0 \leqslant r \leqslant 1$. Let $r (r < 1)$ have the binary expansion

$$r = 0.r_1 r_2 r_3 \cdots \tag{1.2}$$

and define for $k \geqslant 1$

$$S_k = (1 - r_k)a + r_k b \tag{1.3}$$

with

$$a = (1+i)/2 \quad , \quad b = (1-i)/2 . \tag{1.4}$$

This means that S_k is either a or b according to the values 0 or 1 of the kth binary number. Then to r we may associate the following point of J

Fig. 1.2 The limit set of the Pythagoras tree.
(The Pythagoras tree in spring).

$$z = 1 + c_1 + c_2 + c_3 + \cdots \tag{1.5}$$

with for $k \geqslant 1$ and with $c_0 = 1$

$$c_k = S_k \, c_{k-1}. \tag{1.6}$$

Thus to $r = 0$ there corresponds the point

$$1+a+a^2+a^3+\cdots = 1/(1-a) = 1+i\ ,$$

the fixed point of A, and for $r = 1/3$ we would have

$$1+b+ab+ab^2+a^2b^2+a^2b^3+\cdots = 3-i.$$

On J the actions of A and B are then translated into

$$\begin{cases} A : r \to r/2, \\ B : r \to (1+r)/2. \end{cases} \tag{1.7}$$

Thus a random sequence of transformation A and B corresponds to a uniform distribution of numbers in (0,1) and accordingly to a uniform distribution of points on J. The obvious generalization is to give a and b arbitrary complex values with $|a|<1$ and $|b|<1$. The problem under what conditions J can be considered as the limit set of some Pythagoras tree will be taken up in the next section. The overall situation is reminiscent of the inverse logistic map in its complex form as studied by Mandelbrot [1].

$$z' = \pm \sqrt{z} + \mu. \tag{1.8}$$

For suitable values of μ this two-valued map has a Julia set as the collection of limit points of random iterative sequences. The fixed points are $p/2$ and $1-p/2$ where $\mu = (p^2-2p)/4$ in the usual notation of the logistic map as $x \to px(1-x)$. In fact, both the more general version of (1.1).

$$z' = 1+az \quad \text{or} \quad z' = 1+bz, \tag{1.9}$$

and (1.8) can be considered as the members of a family of quadratic (2-2)- maps described by a relation of the form

$$F(z', z) = 0, \tag{1.10}$$

where F is a quadratic polynomial of its arguments. In particular the blossom of the Pythagoras tree and the San Marco attractors can be interpreted as Julia sets of the map (1.10). However, the theory of iterated analytic maps is only fully developed for the case that $z'(z)$ (or its inverse) is a single-valued meromorphic analytic function. The examples given here may give rise to extending the theory to algebraic functions of the kind (1.10).

2. The Pythagoras tree

In fig. 2.1 the initial part of a tree with the two basic branches a and b is given. The endpoints P_k can be labelled by complex numbers z_k in such a way that

$$z_0 = 0,\ z_1 = 1,\ z_2 = 1+a,\ z_3 = 1+b\ ,$$

$$z_4 = 1+a+a^2,\ z_5 = 1+a+ab, \text{ etcetera.}$$

For $k = 50$, which is 110010 in binary representation, we have e.g.

$$z_{50} = 1 + b + ab + a^2b + a^2b^2 + a^3b^2.$$

The tree is transformed into itself by either similarity transformation

$$\begin{cases} A : z \to 1+az, \\ B : z \to 1+bz. \end{cases} \tag{2.1}$$

An endpoint with index k is transformed into an endpoint with a higher index. In particular

$$A\ z_{50} = z_{82}\ ,\ B\ z_{50} = z_{114}.$$

The general rule is as follows. Let $2^m \leqslant k < 2^{m+1}$ then symbolically

$$A(k) = k + 2^m\ ,\ B(k) = k + 2^{m+1}.$$

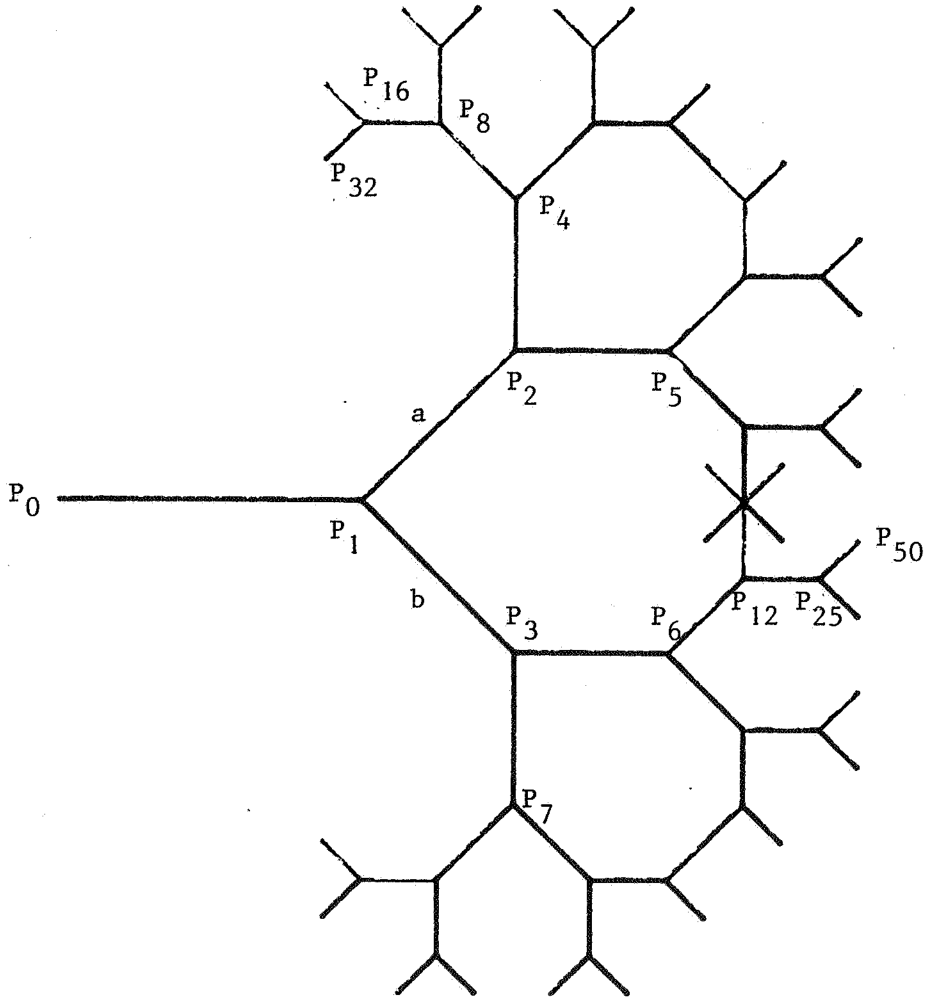

Fig. 2.1 The reduced Pythagoras tree. (The Pythagoras tree in winter).

The question arises whether such a tree can be blown up into some Pythagoras tree. The basic pattern is given in fig. 2.2. Let $UU'V'V$ be an arbitrary quadrangle such that $U' = A(U)$ and $V' = B(V)$. Then there exists a single point W which is both the A-image of V as well as the B-image of U. Labelling U and V by complex numbers u and v this gives the condition $1 + bu = 1 + av$ or

$$bu = av.$$

Thus we may introduce an arbitrary complex number λ and write u and v as

$$u = \lambda a \quad , \quad v = \lambda b\ . \tag{2.2}$$

Each choice of λ will then lead to a Pythagoras tree with, however, similar quadrangles instead of squares.

Let us require that - say from an artistic point of view - the quandrangles are parallelograms. This gives the condition that the vectors UU' are equal :

$$1 + (a-1)u = 1 + (b-1)\,v.$$

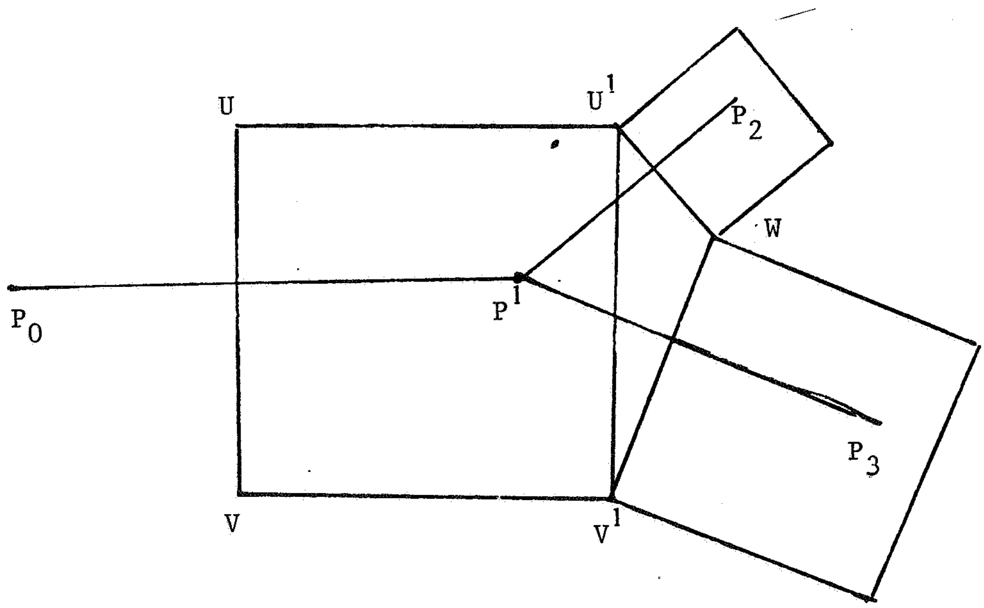

Fig. 2.2 The basis of a generalized Pythagoras tree.

Substitution of (2.2) gives

$$a + b = 1\ , \tag{2.3}$$

a condition depending only on the similarity transformation.

If this condition is fulfilled we write

$$a = (1 + ic)/2\ ,\ b = (1-ic)/2\ , \tag{2.4}$$

where c is an arbitrary complex number. The quadrangle $UU'V'V$ is a square if

$$1 = (a-1)u = i(v-u).$$

Substitution of (2.2) and (2.4) gives the unique solution

$$\lambda = \frac{4}{c^2+4c+1} \tag{2.5}$$

So unless $c = -2\pm\sqrt{3}$ a unique generalized Pythagoras tree can be constructed. A normal oblique Pythagoras tree where the triangle $U'V'W$ is rectangular gives a further reduction of the number of free parameters. A simple calculation shows that this requires that

$$\begin{cases} a = \cos^2\alpha + i\ \sin\alpha\ \cos\alpha\ , \\ b = \sin^2\alpha - i\ \sin\alpha\ \cos\alpha, \end{cases} \tag{2.6}$$

with

$$c = -ie^{2id},$$

and

$$\lambda = \frac{ie^{2id}}{1+\sin\alpha\ \cos\alpha}\ .$$

The corresponding geometric situation is given in fig. 2.3.

A "full" oblique Pythagoras tree with angles of 54° and 36° is given in fig. 2.4. The limit set of the infinitesimally small squares, its blossoms, is given in fig. 2.5. It is obtained as the invariant set of the similarity transformations

$$z' = 1 + az \quad \text{and} \quad z' = 1 + bz \ ,$$

with a and b given by (2.6). The computer program starts with the fixed point $1/(1-a)$ or $1/(1-b)$ which is then subjected to a random sequence of similarity transformations.

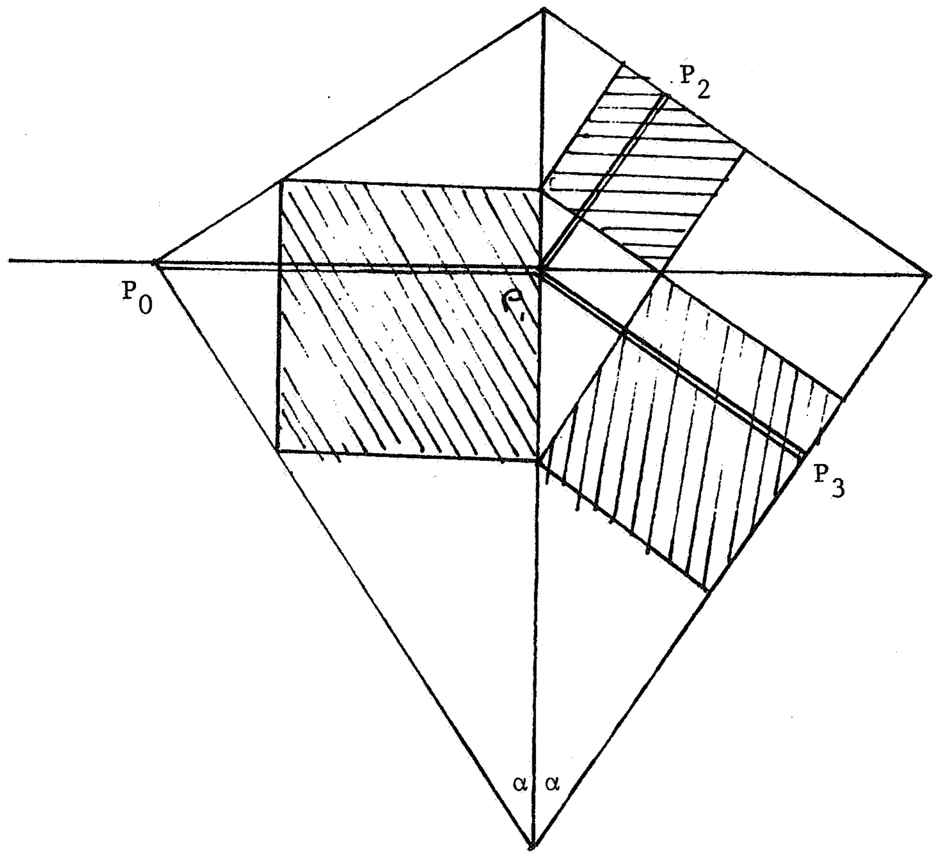

Fig. 2.3 The basis of an oblique Pythagoras tree.

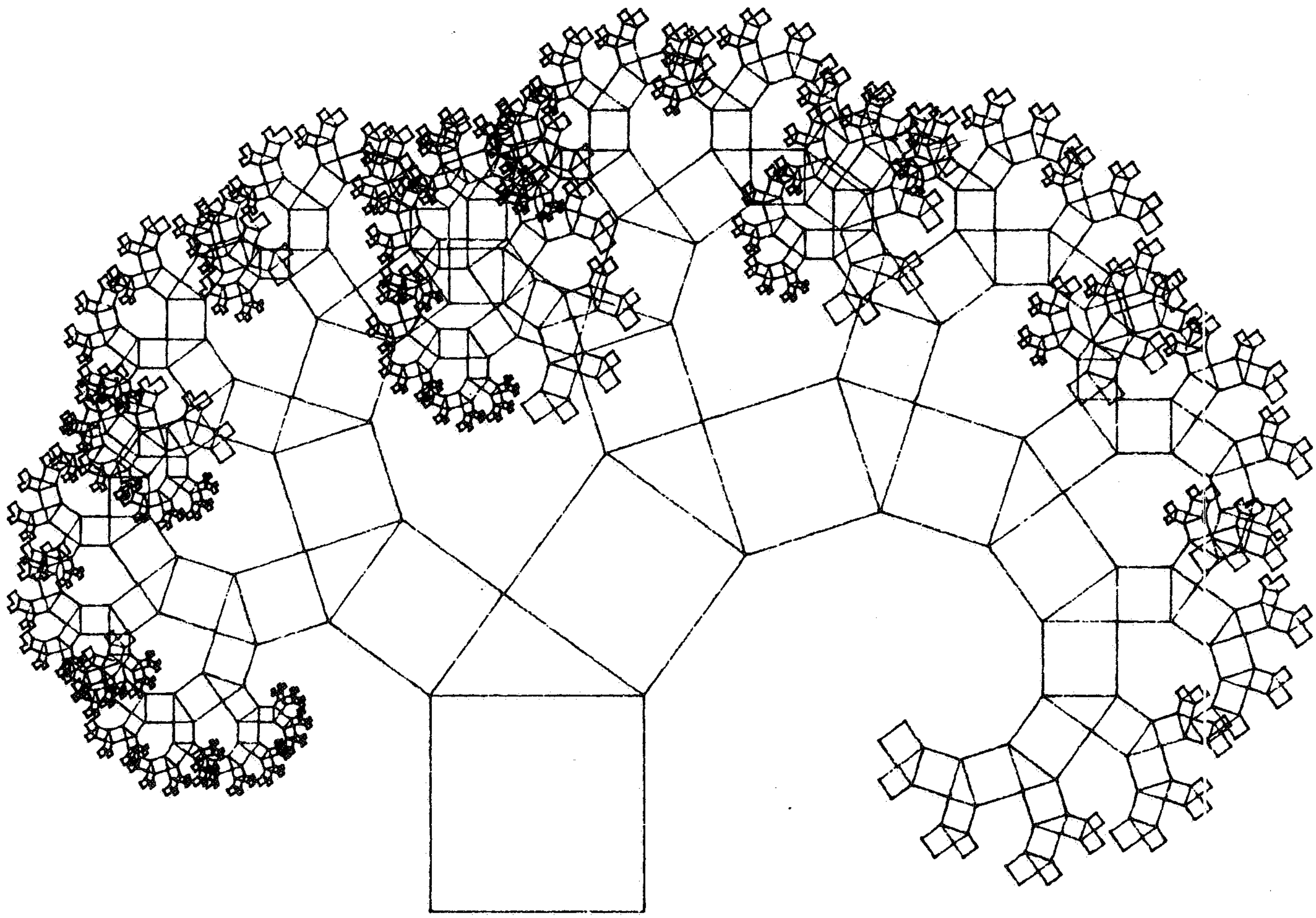

Fig. 2.4 An oblique Pythagoras tree with angles 54° and 36°

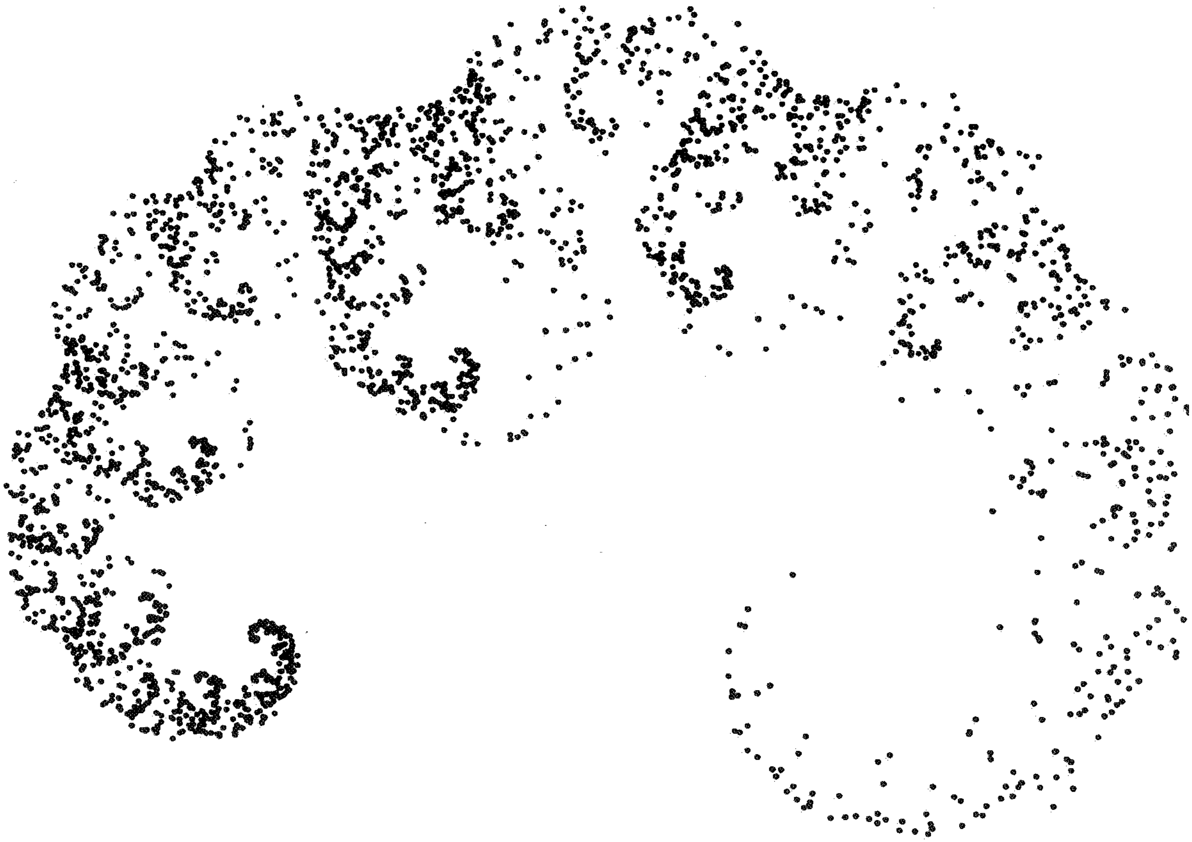

Fig. 2.5 The limit set of an oblique Pythagoras tree.

3. Julia sets

It is well known that Julia sets of iterated rational functions can be obtained as attractors of the inverse multi-valued maps. In particular the quadratic map

$$z \rightarrow pz(1-z),$$

or equivalently

$$z \rightarrow z^2 - \mu, \tag{3.1}$$

where $\mu = p(p-2)/4$ has been studied in great detail by Mandelbrot and many others. For bibliographic details the reader is advised to consult the very recent survey paper by Blanchard [2]. In particular for $p = 3$ ($\mu = 3/4$) the Julia set J has an interesting shape nicknamed the San Marco attractor by Mandelbrot. It is given in fig. 3.1. The Julia set is densely covered by the preimages of any of its points. Figure 3.1 is accordingly obtained by starting from an unstable fixed point of (3.1) and subjecting it to a random sequence of transformations

$$\begin{cases} A : z \rightarrow \sqrt{\mu + z}, \\ B : z \rightarrow -\sqrt{\mu + z}. \end{cases}$$

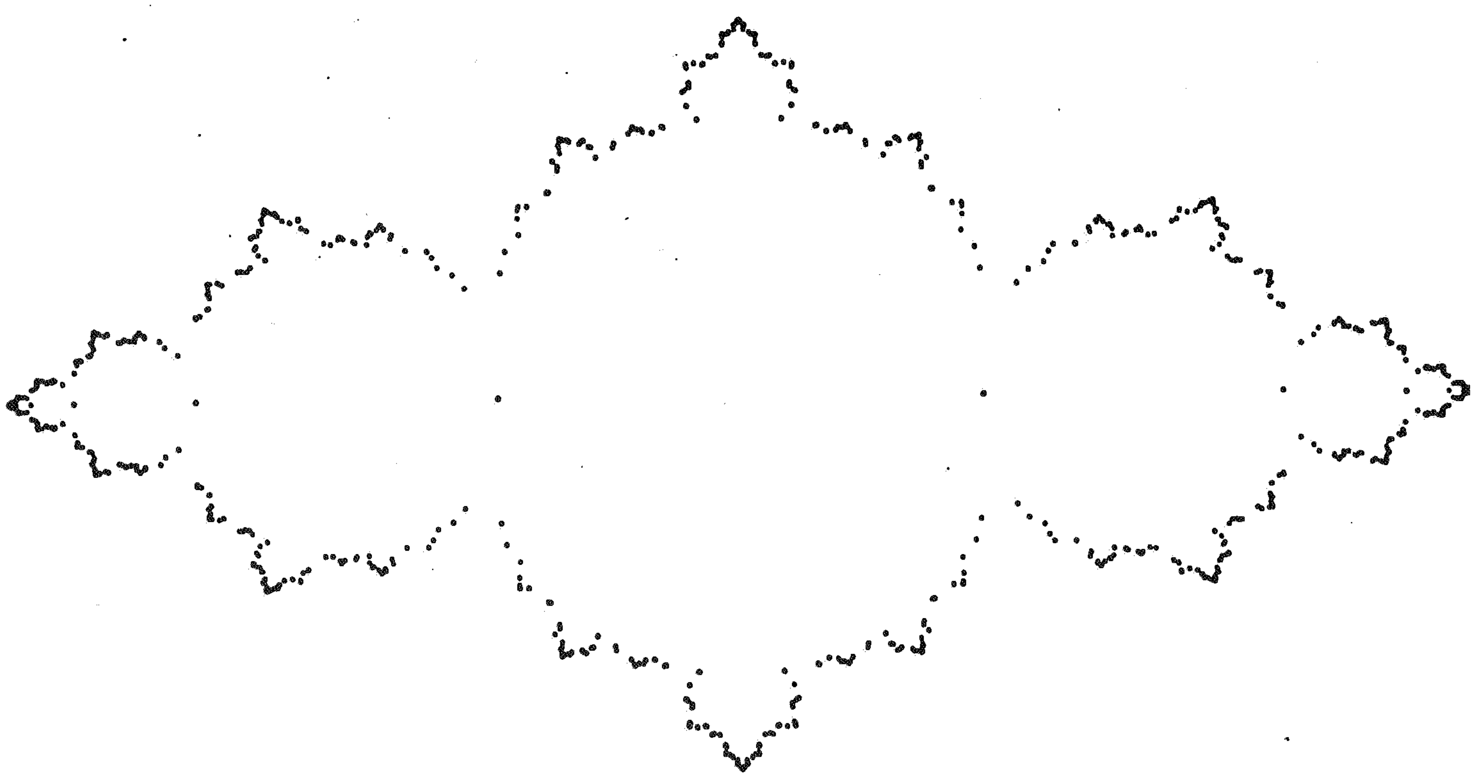

Fig. 3.1 The San Marco attractor, the Julia set of $z \rightarrow z^2 - 3/4$.

Points of J can be labelled as in the first section by a real fraction (1.2) i.e.

$$r = 0.r_1r_2r_3\cdots \tag{3.3}$$

Let σ_k be the sign function corresponding to r_k in the following way

$$\sigma_k = 1-2r_k, \tag{3.4}$$

then

$$z = \sigma_1\sqrt{\mu+\sigma_2\sqrt{\mu+\sigma_3\sqrt{\mu+\sigma_4\sqrt{\mu+\cdots}}}} \tag{3.5}$$

is the point of J associated to r. In particular for the elementary case $\mu = 0$ $(a = 2)$ this reduces to

$$z = \exp(r_1+r_2/2+r_3/4+r_4/8\cdots)\pi i \ ,$$

i.e. simply

$$z = \exp 2\pi ri.$$

The actions of A and B on J can be translated into corresponding transformations of the unit interval :

$$z \to z^2-\mu \quad \Leftrightarrow \quad r \to 2r \quad (mod\, 1),$$

$$z \to \sigma_0\sqrt{\mu+z} \quad \Leftrightarrow \quad r \to (r_0+r)/2 \ ,$$

or

$$0.r_1r_2r_3\cdots \quad \to \quad 0.r_0r_1r_2r_3\cdots .$$

The striking similarity of these properties with those of the Pythagoras tree suggests the independent study of a map in which both features are combined. We introduce the following (2-2) - complex map

$$F(w,z) = 0\ , \tag{3.6}$$

where F is a quadratic polynomial. It is assumed that ± 1 are fixed points with multipliers dw/dz equal to a and b. Then F has single complex parameter c left and it can be written as

$$(w-az-1+a)(w-bz+1-b) + c(w-z)^2 = 0. \tag{3.7}$$

For $c = 0$ this reduces to the "Pythagoras map"

$$\begin{cases} w-1 = a(z-1)\ , \\ w+1 = b(z+1) \end{cases}$$

and for $c = -ab = -\frac{1}{2}(a+b)$ the inverse logistic map is obtained in the form

$$\tfrac{1}{2}\sqrt{(1+c)/c}\ (w^2-1) + w-2=0, \tag{3.8}$$

which is seen to be equivalent to (3.1) with $c = 1/(4\mu)$. In particular the San Marco attractor corresponds to the case $c = 1/3$, i.e.

$$w^2+w-1 = z. \tag{3.9}$$

Generally for all complex values of a,b and c (of course with $|a|<1$, $|b|<1$) a Julia set of (3.7) can be obtained as shown above. By way of illustration we consider the case with the multipliers

$$a = (1+i)/2\ ,\ b = (1-i)/2 \tag{3.10}$$

which are those of the normal Pythagoras tree. Then (3.7) can be written as

$$2(1+c)w = (1+2c)z-i\pm\sqrt{(1+2c)(1-z^2)+2iz}. \tag{3.11}$$

The special value $c=0$ gives the Pythagoras tree. For $c=-1/2$ we have the logistic map with $\mu=-1/2$. The latter case is illustrated in fig. 3.2. It is known to be an entirely disconnected Julia set. The intermediate case $c=-1/4$ is illustrated in fig. 3.3. Indeed, this Julia set looks like the blossom of some Pythagoras tree but it also has the cauliflower structure of the logistic Julia set of fig. 3.2. Shown are the 512 preimages of the point $z=1$ which is a fixed point of (3.11). The Julia set appears to have a very regular self-similar structure each lobe being a replica of half the previous one.

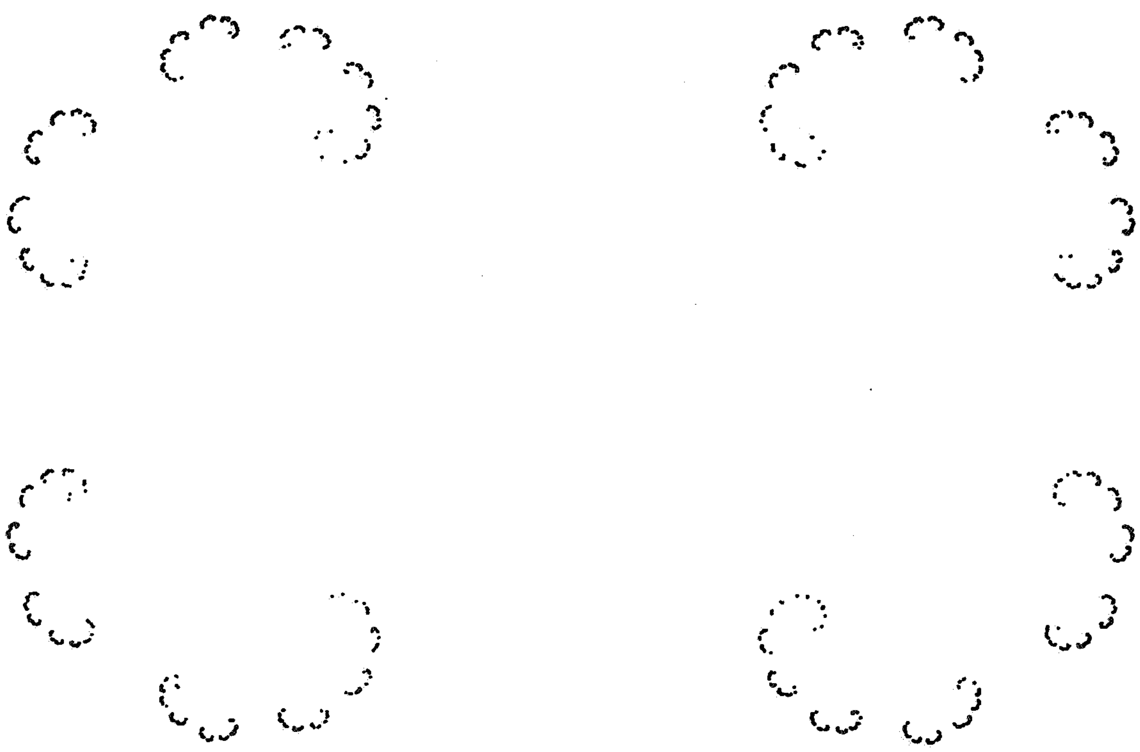

Fig. 3.2 The Julia set of $z \rightarrow z^2+1/2$

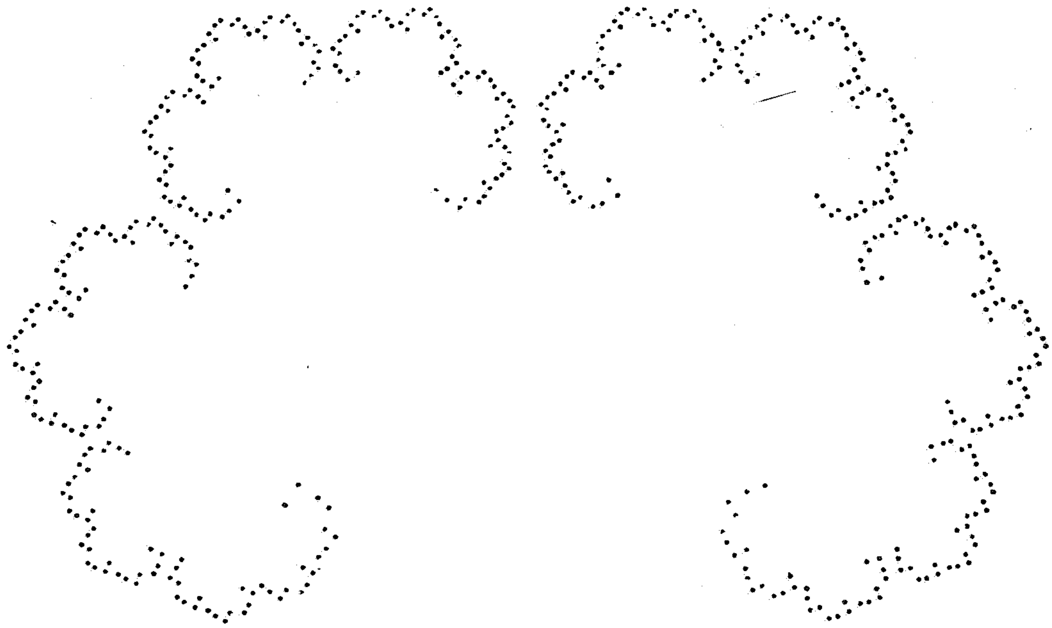

Fig. 3.3 A mixed Pythagoras-logistic Julia set.

REFERENCES

[1] B. MANDELBROT, *Fractal aspects of the iteration of* $z \to \lambda z(1-z)$. Ann. New York Acad. Sci. **357** (1980), 249-259.

[2] P. BLANCHARD, *Complex analytic dynamics.* Bull. Am. Math. Soc. **11** (1984), 85-141.