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E.P. van den Ban

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ON THE HOLOMORPHIC CONTINUATION OF THE IWASAWA AND A RELATED DECOMPOSITION

E.P. VAN DEN BAN
Centre for Mathematics and Computer Science, Amsterdam

Let G be a real semisimple adjoint Lie group, G_c its complexification. In this paper we study the holomorphic continuation to G_c of a decomposition of G which essentially generalizes the Iwasawa decomposition. The results are of interest for the analysis on a semisimple symmetric space.

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0. Introduction

Let $\mathfrak{g}$ be a real semisimple Lie algebra, G its adjoint group, and G = KAN an Iwasawa decomposition for G. Let G_c be the adjoint group of the complexified Lie algebra $\mathfrak{g}_c$. In [1] we studied the global holomorphic continuation of the maps κ, h, ν : G $\to$ K, A, N determined by

$$x = \kappa(x)\ h(x)\ \nu(x).$$

The main result of [1, Chapter 1] is as follows. Let $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$ be the Lie algebras of K, A, N and let K_c, A_c, N_c be the connected analytic subgroups of G_c with the complexifications $\mathfrak{k}_c$, $\mathfrak{a}_c$, $\mathfrak{n}_c$ as their respective Lie algebras. Put

$$S = G_c - K_c A_c N_c$$

(the set theoretic difference). Then we have

<u>Theorem 0.1.</u> The set S is algebraic and the maps κ, h, ν extend to multi-valued holomorphic maps $G_c - S \longrightarrow K_c, A_c, N_c$. Moreover, h^2 and ν are rational (hence single valued). If $\{x_n\}$ is a sequence in $G_c - S$ converging to a point $x \in S$, then the set $\{h^2(x_n)\ ;\ n \in \mathbb{N}\}$ is not relatively compact in A_c.

<u>Remarks.</u> (i) For the terminology of multi-valued holomorphic maps the reader is referred to the appendix.

(ii) Observe that S can be identified with the union of the lower dimensional K_c-orbits on the flag manifold

G_c/P_c, where P_c is the parabolic in G_c with Lie algebra centralizer($\mathfrak{a}_c$) $\oplus$ $\mathfrak{n}_c$ (cf. also [11, 13]).

We used the above theorem in the study of the asymptotic behaviour of elementary spherical functions on a real semisimple Lie group (see [1] , see also [2] for more general results).

In this paper we prove a generalization of Theorem 0.1 which has a similar importance for the harmonic analysis on semisimple symmetric spaces. As an application we derive a result which is of crucial importance in the paper [4] , where a generalization of Kostant's convexity theorem to semisimple symmetric spaces is given.

1. The main result

Let τ be an involution of $\mathfrak{g}$. Then there exists a Cartan involution θ of $\mathfrak{g}$ which commutes with τ (cf. [5]). Let $\mathfrak{k}$, $\mathfrak{h}$ denote the +1 eigenspaces and $\mathfrak{p}$, $\mathfrak{q}$ the -1 eigenspaces of θ and τ respectively. Then we have the joint eigenspace decomposition

$$\mathfrak{g} = \mathfrak{k}\cap\mathfrak{q} \oplus \mathfrak{k}\cap\mathfrak{h} \oplus \mathfrak{p}\cap\mathfrak{q} \oplus \mathfrak{p}\cap\mathfrak{h} . \tag{1}$$

Let K be the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$, and let $\mathfrak{a}_{pq}$ be a maximal abelian subspace of $\mathfrak{p}\cap\mathfrak{q}$. The set $\Delta = \Delta(\mathfrak{g}, \mathfrak{a}_{pq})$ of restricted roots of $\mathfrak{a}_{pq}$ in $\mathfrak{g}$ is a (possibly non-reduced) root system (cf. [13]). Fix a choice Δ^+ of positive roots for Δ.

The Lie algebra $\mathfrak{l}$ of the centralizer L of $\mathfrak{a}_{pq}$ in G admits the direct sum decomposition

$$\mathfrak{l} = \mathfrak{l}_{kq} \oplus \mathfrak{l}_{kh} \oplus \mathfrak{a}_{pq} \oplus \mathfrak{l}_{ph}$$

subordinate to (1). Since L normalizes the Killing ortho-complement $\mathfrak{l}_0 = (\mathfrak{l}\cap\mathfrak{k}) \oplus \mathfrak{l}_{ph}$ of $\mathfrak{a}_{pq}$ in $\mathfrak{l}$, it follows that

$$L_0 = (L\cap K)\exp(\mathfrak{l}_{ph})$$

is a closed subgroup of L with Lie algebra $\mathfrak{l}_0$. Moreover, writing $A_{pq} = \exp(\mathfrak{a}_{pq})$, we have the direct product

$$L \simeq L_0 \times A_{pq} . \tag{2}$$

Let $\mathfrak{a}_{ph}$ be a maximal abelian subspace of $\mathfrak{l}_{ph}$ and put

$$\mathfrak{a}_p = \mathfrak{a}_{ph} \oplus \mathfrak{a}_{pq}.$$

Then $\mathfrak{a}_p$ is maximal abelian in $\mathfrak{p}$. Let Δ_p^+ be a choice of positive roots for $\Delta_p = \Delta(\mathfrak{g}, \mathfrak{a}_p)$, compatible with Δ^+, and put

$$\mathfrak{n} = \sum_{\alpha \in \Delta_p^+} \mathfrak{g}^\alpha \quad , \qquad N = \exp \mathfrak{n}.$$

Then

$$\mathfrak{n}_Q = \sum_{\alpha \in \Delta^+} \mathfrak{g}^\alpha$$

is a subalgebra of $\mathfrak{n}$, normalized by $\mathfrak{l}$. In fact, writing $\mathfrak{n}_L = \mathfrak{n} \cap \mathfrak{l}$, we have $\mathfrak{n} = \mathfrak{n}_L \oplus \mathfrak{n}_Q$ as a semi-direct product of Lie algebras. It follows that $N_L = N \cap L$ normalizes the group $N_Q = \exp(\mathfrak{n}_Q)$, and we have the semi-direct product

$$N = N_L N_Q. \tag{3}$$

Moreover, the parabolic subgroup $Q = LN$ has the Levi decomposition

$$Q = L N_Q.$$

<u>Proposition 1.1.</u> If $x \in G$, then there exist unique $\lambda(x) \in (L_0 \cap K)\backslash L_0$, $h_q(x) \in A_{pq}$, $\nu_Q(x) \in N_Q$ such that

$$x \in K \lambda(x) h_q(x) \nu_Q(x). \tag{4}$$

Moreover, the maps λ, h_q and ν_Q are real analytic.

<u>Proof.</u> This is an easy consequence of the Iwasawa decomposition $G = KAN$ (where $A = \exp(\mathfrak{a}_p)$), the decompositions (2) and (3), and the fact that A_{pq} centralizes N_L. Observe that $h_q(x)$ is the A_{pq}-part of $h(x)$ in the direct product decomposition $A = A_{ph}A_{pq}$.

The main result of this paper is a generalization of Theorem 0.1 for the decomposition (4). Let L_c be the centralizer of $\mathfrak{a}_{pq}$ in G_c, Q_c the normalizer of $\mathfrak{l}_c + \mathfrak{n}_{Qc}$ in G_c and $N_{Qc} = \exp(\mathfrak{n}_{Qc})$. Then it is well known that L_c, Q_c and N_{Qc} are algebraic and connected. Moreover, Q_c is a parabolic subgroup with Levi decomposition $Q_c = L_cN_{Qc}$.

Let K_c, A_c, A_{pqc} , L_{0c} be the connected analytic subgroups of G_c with Lie algebras $\mathfrak{k}_c$, $\mathfrak{a}_c$, $\mathfrak{a}_{pqc}$, $\mathfrak{l}_{0c}$. Then we have

<u>Proposition 1.2.</u> The groups K_c, A_c, A_{pqc}, L_{0c} are the identity components (for the usual topology) of algebraic subgroups of G_c.

<u>Remark.</u> If we speak about connected components, it will always be with respect to the usual (i.e. non-Zariski) topology.

<u>Proof.</u> The holomorphic involutions of G_c whose differentials at the identity are θ and τ , are denoted by the same symbols. Define

$$'K_c = \{x \in G_c \; ; \quad \theta x = x\} ,$$

$$'A_c = \{x \in G_c \; ; \quad \theta x = x^{-1}, \; x|\mathfrak{g}^{\alpha} \in \mathbb{C}\mathrm{Id}(\mathfrak{g}^{\alpha}) \text{ for } \alpha \in \Delta_p\}.$$

$$'A_{pqc} = \{x \in A_c ; \quad \tau(x) = x^{-1}\}.$$

Then K_c, A_c, A_{pqc} are the identity components of the algebraic subgroups $'K_c$, $'A_c$, $'A_{pqc}$. As for the last assertion, we claim that L_{0c} is the identity component of

$$'L_{0c} = \{x \in L_c ; \quad \det(x \mid \mathfrak{g}^{\alpha}) = 1 \text{ for } \alpha \in \Delta^{+}\} .$$

To prove this it suffices to show that $\mathfrak{l}_0$ equals

$$'\mathfrak{l}_0 = \{X \in \mathfrak{l} ; \quad \mathrm{tr}(\mathrm{ad}(X) \mid \mathfrak{g}^{\alpha}) = 0 \text{ for } \alpha \in \Delta^{+}\} .$$

Since $\bigcap \{\ker \alpha ; \alpha \in \Delta^{+}\} = 0$, we have $'\mathfrak{l}_0 \cap \mathfrak{a}_{pq} = 0$. Hence it suffices to show that $\mathfrak{l}_0 \subset {'\mathfrak{l}_0}$.

If $\alpha \in \Delta$, we write $\Delta_p(\alpha) = \{\beta \in \Delta_p ; \beta \mid \mathfrak{a}_{pq} = \alpha\}$. Thus, if $\alpha \in \Delta$, then

$$\mathfrak{g}^{\alpha} = \sum_{\beta \in \Delta_p(\alpha)} \mathfrak{g}^{\beta} .$$

Let $t_{\alpha}(X) = \mathrm{tr}(\mathrm{ad}(X) \mid \mathfrak{g}^{\alpha})$, for $X \in \mathfrak{l}$. Since $\mathfrak{l} \cap \mathfrak{k}$ acts by skew symmetric transformations on $\mathfrak{g}^{\alpha}$, it follows that $t_{\alpha} = 0$ on $\mathfrak{l} \cap \mathfrak{k}$. Moreover, for $X \in \mathfrak{a}_{ph}$ we have

$$t_{\alpha}(X) = \sum_{\beta \in \Delta_p(\alpha)} \beta(X) \dim(\mathfrak{g}^{\beta}).$$

Since $\Delta_p(\alpha) = -\Delta_p(-\alpha)$, it follows that $t_{\alpha} = -t_{-\alpha}$ on $\mathfrak{a}_{ph}$. On the other hand, if $X \in \mathfrak{a}_{ph}$, then $\tau X = X$, so that $t_{\alpha}(X) = t_{\alpha}(\tau X) = \mathrm{tr}(\tau \circ \mathrm{ad}(X) \circ \tau^{-1} \mid \mathfrak{g}^{\alpha}) = \mathrm{tr}(\mathrm{ad}(X) \mid \mathfrak{g}^{-\alpha})$ $= t_{-\alpha}(X)$. Hence $t_{\alpha} = 0$ on $\mathfrak{a}_{ph}$. Since obviously $t_{\alpha}(k \cdot X) =$ $= t_{\alpha}(X)$ for $X \in \mathfrak{l}$, $k \in L \cap K$, this implies that $t_{\alpha} = 0$ on $(L \cap K) \cdot \mathfrak{a}_{ph} = \mathfrak{l}_{ph}$, hence on $\mathfrak{l}_0$. We conclude that $\mathfrak{l}_0 \subset {'\mathfrak{l}_0}$.

We now have the following generalization of Theorem 0.1.

Let S_Q be the complement of $K_c Q_c$ in G_c. Then clearly $S_Q \subset S$, and we may identify S_Q with the union of the lower dimensional K_c-orbits on the flag manifold G_c/Q_c.

<u>Theorem 1.3.</u> The set S_Q is algebraic. The maps λ, h_q and ν_Q extend to multi-valued holomorphic maps $G_c - S_Q \longrightarrow$ $\longrightarrow (K_c \cap L_{Oc}) \backslash L_{Oc}$, A_{pqc}, N_{Qc}. The map ν_Q is rational and there exists an integer $m > 0$ such that h_q^m is rational. Moreover, if $\{x_k\}$ is a sequence in $G_c - S_Q$ converging to a point $x \in S_Q$, then $\{h_q^m(x_k);\ k \in \mathbb{N}\}$ is not relatively compact in A_{pqc}.

Loosely said, the line of proof is as follows. Suppose $x \in K_c \ell an$. Then $(\theta x)^{-1} x = (\theta n)^{-1} (\theta \ell)^{-1} \ell\, a^2 n$. Now ℓ, a, n can be solved from this by using properties of the $\overline{N}_{Qc} L_{Oc} A_{pqc} N_{Qc}$-decomposition (here $\overline{N}_{Qc} = \theta(N_{Qc})$). The latter decomposition is studied as follows. First we construct an embedding of G_c in the matrix group $GL(n,\mathbb{C})$ (here $n = \dim \mathfrak{g}_c$). Then, in the next section, we generalize certain matrix computations which go back to [8 , Ch. 2, § 8].

The proof is completed in Section 3.

Let $<$ be the ordering of $\mathfrak{a}^*_{pq}$ which is lexicographic in the coordinates relative to the simple roots of Δ^+, and let $\alpha_s < \ldots < \alpha_1$ be the corresponding enumeration of the elements of Δ^+. For every $1 \leq j \leq s$, we put $\mathfrak{g}_j = \mathfrak{g}^{\alpha_j}$. Moreover, we write $\mathfrak{g}_{s+1} = \ell_0$, $\mathfrak{g}_{s+2} = \mathfrak{a}_{pq}$, $\mathfrak{g}_{s+2+j} = \theta \mathfrak{g}_{s+1-j}$ for $1 \leq j \leq s$. Now let $(.,.)$ be the positive definite inner product on $\mathfrak{g}$ defined by $(X,Y) = -B(X, \theta Y)$ for $X,Y \in \mathfrak{g}$. Then

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_t \tag{5}$$

(where $t = 2s+2$) is an orthogonal direct sum decomposition. Select an orthonormal basis $(e_i;\ 1 \leqslant i \leqslant n)$ of $\mathfrak{g}$ which is subordinate to (5) and such that the ordering $e_1, \ldots, e_n$ of its elements is compatible with the ordering of the sum in (5).

If $1 \leqslant j \leqslant t$, let $d_j = \dim(\mathfrak{g}_j)$, and let P_j denote the orthogonal projection $\mathfrak{g} \longrightarrow \mathfrak{g}_j$. In the sequel we shall identify real linear maps with their complex linear extensions. Also, given a linear endomorphism $X \in \mathrm{End}(\mathfrak{g}_c)$, we let X_{ij} denote the $d_i \times d_j$-matrix of the linear map $(P_i \cdot X)|\mathfrak{g}_j$ from $\mathfrak{g}_j$ into $\mathfrak{g}_i$ and we identify X with the matrix of blocks $(X_{ij};\ 1 \leqslant i,j \leqslant t)$. With these notations the composition of endomorphisms corresponds to matrix multiplications in the usual way:

$$(XY)_{ik} = \sum_{1 \leqslant j \leqslant t} X_{ij} Y_{jk},$$

for $X,Y \in \mathrm{End}(\mathfrak{g}_c)$, $1 \leqslant i,k \leqslant t$.

Now let

$$\underline{\mathfrak{n}}_Q = \{ X \in \mathrm{End}(\mathfrak{g});\quad X_{ij} = 0 \text{ for } 1 \leqslant j \leqslant i \leqslant t \},$$

$$\underline{\bar{\mathfrak{n}}}_Q = \{ X \in \mathrm{End}(\mathfrak{g});\quad X_{ij} = 0 \text{ for } 1 \leqslant i \leqslant j \leqslant t \},$$

$$\underline{\mathfrak{l}} = \{ X \in \mathrm{End}(\mathfrak{g});\quad X_{ij} = 0 \text{ for } i \neq j \}.$$

Then clearly $\mathrm{End}(\mathfrak{g}) = \bar{\mathfrak{n}}_Q \oplus \mathfrak{l} \oplus \mathfrak{n}_Q$. Moreover, $\mathfrak{l} =$
$= \mathfrak{l}_0 \oplus \mathfrak{a}_{pq}$, where

$$\mathfrak{l}_0 = \{ X \in \mathfrak{l}\ ; \quad tr(X_{ii}) = 0 \text{ for } 1 \leqslant i \leqslant t \} ,$$

$$\mathfrak{a}_{pq} = \{ X \in \mathfrak{l}\ ; \quad X_{jj} \in \mathbb{C} \cdot I_j \quad \text{for } 1 \leqslant j \leqslant t \} .$$

Here we have written I_j for the identity matrix of size $d_j \times d_j$.

Consequently

$$\mathrm{End}(\mathfrak{g}) = \bar{\mathfrak{n}}_Q \oplus \mathfrak{l}_0 \oplus \mathfrak{a}_{pq} \oplus \mathfrak{n}_Q. \tag{6}$$

<u>Proposition 1.4.</u> Let $\mathfrak{b}$ be any of the algebras $\bar{\mathfrak{n}}_Q$, $\mathfrak{l}_0$, $\mathfrak{a}_{pq}$, $\mathfrak{n}_Q$. Then $\mathrm{ad}(\mathfrak{b}) = \mathrm{ad}(\mathfrak{g}) \cap \mathfrak{b}$.

<u>Proof.</u> The inclusions $\mathrm{ad}(\mathfrak{b}) \subset \mathrm{ad}(\mathfrak{g}) \cap \mathfrak{b}$ are obvious (see also the proof of Proposition 1.2). Therefore the direct sum decomposition $\mathrm{ad}(\mathfrak{g}) = \mathrm{ad}(\bar{\mathfrak{n}}_Q) \oplus \mathrm{ad}(\mathfrak{l}_0) \oplus$ $\oplus\ \mathrm{ad}(\mathfrak{a}_{pq}) \oplus \mathrm{ad}(\mathfrak{n}_Q)$ is compatible with (6). The latter sum being direct, the inclusions must be equalities.

Now let $\underline{G} = GL(\mathfrak{g})$, $\underline{G}_c = GL(\mathfrak{g}_c)$, and put

$$\underline{N}_{Qc} = \{ x \in \underline{G}_c ; \quad x - I \in \mathfrak{n}_{Qc} \} ,$$

$$\bar{\underline{N}}_{Qc} = \{ x \in \underline{G}_c ; \quad x - I \in \bar{\mathfrak{n}}_{Qc} \} ,$$

$$\underline{L}_c = \{ x \in \underline{G}_c ; \quad x_{ij} = 0 \quad \text{if } i \neq j \} ,$$

$$\underline{L}_{0c} = \{ x \in \underline{L}_c ; \quad \det(x_{jj}) = 1 \quad \text{for } 1 \leqslant j \leqslant t \} ,$$

$$\underline{A}_{pqc} = \{ x \in \underline{L}_c ; \quad x_{jj} \in \mathbb{C} \cdot I_j \quad \text{for } 1 \leqslant j \leqslant t \} .$$

These are algebraic subgroups of $\underline{G}_c$ with Lie algebras

$\underline{n}_{Qc}$, $\underline{\bar{n}}_{Qc}$, $\underline{l}_c$, $\underline{l}_{Oc}$, $\underline{a}_{pqc}$ respectively. The following corollary is now immediate.

<u>Corollary 1.5.</u> Let B be any of the groups N_Q, $\bar{N}_Q$, L_O or A_{pq}. Then $B_c = (G_c \cap \underline{B}_c)^o$. In particular, B_c is the identity component (with respect to the usual topology) of an algebraic subgroup of $GL(\mathfrak{g}_c)$.

2. Decompositions in $GL(\mathfrak{g}_c)$

If $1 \leq k \leq t$ we define the polynomial function $D_k : \underline{G}_c \to \mathbb{C}$ by

$$D_k(x) = \det(x_{ij};\ 1 \leq i,j \leq k),$$

for $x \in \underline{G}_c$. Moreover, we let $D_0 \equiv 1$ and $D = D_1 \cdot \ldots \cdot D_t$. We now have the following result.

Proposition 2.1. Let $1 \leq k \leq t$. If $x \in \underline{G}_c$, $\bar{n} \in \underline{\bar{N}}_{Qc}$, $\ell \in \underline{L}_c$, $n \in \underline{N}_{Qc}$, then

$$D_k(\bar{n}x\ell n) = D_k(\bar{n}\ell x n) = D_k(x)\ D_k(\ell).$$

Proof. If $1 \leq k \leq t$, $x \in \mathrm{End}(\mathfrak{g}_c)$, let $\mathfrak{M}_k(x)$ denote the matrix $(x_{ij};\ 1 \leq i,j \leq k)$. Then an easy matrix computation yields $\mathfrak{M}_k(\bar{n}x\ell n) = \mathfrak{M}_k(\bar{n})\mathfrak{M}_k(x)\mathfrak{M}_k(\ell)\mathfrak{M}_k(n)$ and $\mathfrak{M}_k(\ell x) = \mathfrak{M}_k(\ell)\mathfrak{M}_k(x)$. The assertion now follows by taking determinants.

If $1 \leq k \leq t$, we define the subgroup G_k of $\underline{G}_c$ by

$$G_k = \{ x \in \underline{G}_c;\ \ x_{ij} = 0 \text{ for } 1 \leq j < k,\ i > j \}.$$

Thus $G_1 = \underline{G}_c$ and $G_t = \underline{L}_c \underline{N}_{Qc}$. Moreover, we define the subgroup $\bar{N}_k$ of $\underline{\bar{N}}_{Qc}$ by

$$\bar{N}_k = \{ x \in \underline{\bar{N}}_{Qc};\ \ x_{ij} = 0 \text{ for } j \neq i,k \}.$$

Proposition 2.2. Let $1 \leq k < t$, $y \in G_k$, $D_k(y) \neq 0$. Then there exists a unique $W_k(y) \in \overline{N}_k$ such that $W_k(y)y \in G_{k+1}$. Moreover, the map $y \longmapsto D_k(y)W_k(y)$ is polynomial (in the entries of y).

<u>Proof.</u> The uniqueness follows from the fact that $\overline{N}_k \cap G_{k+1} = \{I\}$.

The existence is proved by sweeping the k-th column $(y_{\cdot k})$ of y. This amounts to left multiplication by an element of $\overline{N}_k$. More precisely, let E_k be the space of linear maps from $\mathfrak{g}_{kc}$ into $\mathfrak{g}_{k+1c} \oplus \dots \oplus \mathfrak{g}_{tc}$. If $a \in E_k$, we put $a_j = P_j \circ a$ for $k+1 \leqslant j \leqslant t$ and identify a with its matrix

$$\begin{pmatrix} a_{k+1} \\ \vdots \\ a_t \end{pmatrix}$$

Also, we let $w_k(a)$ denote the element of $\overline{N}_k$ whose k-th column x is given by $x_j = 0$ for $1 \leqslant j < k$, $x_k = I_k$, $x_j = a_j$ for $k < j \leqslant t$. If E_k is viewed as an abelian group for the addition, then the map $w_k : E_k \longrightarrow \overline{N}_k$ thus defined is a group isomorphism.

If $y \in G_k$, $D_k(y) \neq 0$, then clearly $\det(y_{kk}) \neq 0$. Put

$$\alpha_k(y) = - \begin{pmatrix} y_{k+1\;k} \\ \vdots \\ y_{t\;k} \end{pmatrix} \cdot (y_{kk}^{-1}) .$$

Then $w_k(\alpha_k(y))\, y \in G_{k+1}$. Hence $W_k(y) = w_k(\alpha_k(y))$. Since $D_k(y)\det(y_{kk})^{-1} = D_{k-1}(y)$, it follows that $D_k(y)W_k(y)$ is polynomial in the entries of y.

<u>Corollary 2.3.</u> Let $y \in \underline{G}_c$, $D(y) \neq 0$. Then there exist unique $U(y) \in \overline{\underline{N}}_{Qc}$, $\mathcal{L}(y) \in \underline{L}_c$ and $V(y) \in \underline{N}_{Qc}$ such that $y = U(y)\,\mathcal{L}(y)V(y)$. The maps U, $\mathcal{L}$ and V are rational.

<u>Proof.</u> In view of Proposition 2.1, the polynomial function D is left $\overline{\underline{N}}_{Qc}$-invariant. Therefore we may apply Proposition 2.2 repeatedly and infer that for $y \in \underline{G}_c - D^{-1}(0)$ there exists a $W(y) \in \overline{\underline{N}}_{Qc}$ such that $W(y)y \in \underline{L}_c\underline{N}_{Qc}$. It is unique because $\overline{\underline{N}}_{Qc} \cap \underline{L}_c\underline{N}_{Qc} = \{I\}$. Clearly $W(y)$ is rational in the entries of y and therefore $U(y) = W(y)^{-1}$ is. The proof is completed by the easy observation that the map $\underline{L}_c \times \underline{N}_{Qc} \longrightarrow \underline{L}_c\underline{N}_{Qc}$, $(\ell, n) \longmapsto \ell n$ is a diffeomorphism with rational inverse.

We end this section with a proposition which will be needed in the next section. If $1 \leqslant j \leqslant t$, $d_j \neq 0$, let the function $\lambda_j : \underline{A}_{pqc} \longrightarrow \mathbb{C}^*$ be defined by

$$x \,|\, \mathfrak{I}_j = \lambda_j(x) \cdot I_j$$

for $x \in \underline{A}_{pqc}$. It might occur that $d_j = 0$ for some j. This only happens when $\mathcal{I}_0 = 0$, $j = s+1$. In that case we define $\lambda_{s+1} \equiv 1$. Observe that the latter equality holds in any case.

<u>Proposition 2.4.</u> If $x = u\ell bv$, with $u \in \overline{\underline{N}}_{Qc}$, $\ell \in \underline{L}_{0c}$, $b \in \underline{A}_{pqc}$, $v \in \underline{N}_{Qc}$, then

$$\lambda_j(b)^{d_j} = D_j(x)/D_{j-1}(x),$$

for $1 \leqslant j \leqslant t$.

Proof. In view of Proposition 2.1 and the definition of $\underline{L}_{0c}$, we have $D_j(x) = D_j(b)D_j(\ell) = D_j(b)$. But obviously

$$D_k(b) = \prod_{1 \leqslant j \leqslant k} \lambda_j(b)^{d_j} \qquad (1 \leqslant k \leqslant t),$$

from which the assertion follows (recall that $D_0 \equiv 1$).

3. Proof of the main result.

In this section we complete the proof of Theorem 1.3. We start with some results on the $\overline{N}_{Qc}L_{Oc}A_{pqc}N_{Qc}$-decomposition.

Proposition 3.1. The map $p: L_{Oc} \times A_{pqc} \longrightarrow L_c$, $(\ell, a) \longmapsto \ell a$ is a finite covering.

Proof. By a standard computation of differentials, the map p is seen to be a submersion. Moreover, since A_{pqc} is central in L_c, p is a group homomorphism. Hence its image $L_{Oc}A_{pqc}$ is an open subgroup of L_c. The latter group being connected, it follows that p is an epimorphism of Lie groups. Since L_{Oc} and A_{pqc} are connected, whereas $L_{Oc} \cap A_{pqc}$ is discrete, it follows that p is a covering with fibre $p^{-1}(e) =$ $= L_{Oc} \cap A_{pqc}$.

From Proposition 2.4 we infer that $\underline{L}_{Oc} \cap \underline{A}_{pqc}$ consists of elements $b \in \underline{A}_{pqc}$ with

$$\lambda_j(b)^{d_j} = 1$$

for $1 \leqslant j \leqslant t$. Hence $\underline{L}_{Oc} \cap \underline{A}_{pqc}$ is finite. In view of Corollary 1.5, $p^{-1}(e)$ is contained in $\underline{L}_{Oc} \cap \underline{A}_{pqc}$, hence finite. Consequently, p is a finite covering.

Lemma 3.2. The map $\gamma : \overline{N}_{Qc} \times L_{Oc} \times A_{pqc} \times N_{Qc} \longrightarrow G_c - D^{-1}(0)$, $(n, \ell, a, n) \longmapsto n\ell an$ is a finite covering.

Proof. By Corollary 2.3 the map $\psi : (\bar{n},\ell,n) \longmapsto \bar{n}\ell n$ from $\mathfrak{M} = \bar{N}_{Qc} \times L_c \times N_{Qc}$ onto $G_c - D^{-1}(0)$ is a diffeomorphism. Since D is not identically zero on G_c, $G_c - D^{-1}(0)$ is connected. In view of Proposition 3.1 it therefore suffices to prove that ψ^{-1} maps $G_c - D^{-1}(0)$ onto $\mathfrak{M} = \bar{N}_{Qc} \times L_c \times N_{Qc}$. Now clearly $\psi^{-1}(G_c - D^{-1}(0)) \supset \mathfrak{M}$. Since ψ is a diffeomorphism, it follows by comparison of dimensions that there exists an open neighbourhood U of (e,e,e) in $\mathfrak{M}$, such that $V = \psi(U)$ is an open neighbourhood of e in $G_c - D^{-1}(0)$. Hence ψ^{-1} maps V into $\mathfrak{M}$. By analytic continuation, the holomorphic map ψ^{-1} maps the connected complex analytic manifold $G_c - D^{-1}(0)$ into the Zariski closure $\mathcal{C}$ of $\mathfrak{M}$. By connectedness, $\psi^{-1}(G_c - G^{-1}(0))$ is contained in the identity component $\mathcal{C}^o$ of the linear algebraic group $\mathcal{C}$ (with respect to the usual topology). Finally, by Corollary 1.5, $\mathcal{C}^o = \mathfrak{M}$, so that $\psi^{-1}(G_c - D^{-1}(0)) \subset \mathfrak{M}$.

Let (u,ℓ,b,v) denote the multi-valued holomorphic inverse of the covering γ with base points e and (e,e,e,e) (for the terminology used here, we refer the reader to the appendix).

Proposition 3.3. Let $1 \leqslant j \leqslant t$. Then the map $\lambda_j^{d_j} \circ b$: $G_c - D^{-1}(0) \longrightarrow \mathbb{C}^*$, $y \longmapsto \lambda_j(b(y))^{d_j}$ is rational. In fact, if $y \in G_c - D^{-1}(0)$, then

$$\lambda_j(b(y))^{d_j} = D_j(y)/D_{j-1}(y).$$

Proof. This follows immediately from Proposition 2.4.

<u>Corollary 3.4.</u> Let μ be the least common multiple of $d_1,\ldots,\widehat{d}_{s+1},\ldots,d_t$. Then $\mu>0$, and the maps $v\colon G_c - D^{-1}(0) \longrightarrow N_{Qc}$ and $b^{\mu}\colon G_c - D^{-1}(0) \longrightarrow A_{pqc}$ are rational. Moreover, if $\{y_k\}$ is a sequence in $G_c - D^{-1}(0)$ converging to a point $y \in D^{-1}(0)$, then the set $\{b^{\mu}(y_k);\ k \in \mathbb{N}\}$ is not relatively compact in A_{pqc}.

<u>Proof.</u> Obviously v is the restriction of V to $G_c - D^{-1}(0)$, hence rational (see Corollary 2.3). Since $\underline{A}_{pqc}$ centralizes $\mathcal{L}_{0c}$, we have $\lambda_{s+1} \equiv 1$. Hence the rationality of the map b^{μ} follows from Proposition 3.3.

Now let j be the lowest index among $1,\ldots,t$ such that $D_j(y) = 0$. Then $D_{j-1}(y) \neq 0$ (recall that $D_0 \equiv 1$) and by Proposition 3.3 it follows that we must have $d_j \neq 0$ and

$$\lambda_j(b^{\mu}(y_k)) = \lambda_j(b(y_k))^{d_j\cdot \mu/d_j} \longrightarrow 0$$

as $k \longrightarrow \infty$. Hence $\{\lambda_j(b^{\mu}(y_k));\ k \in \mathbb{N}\}$ is not relatively compact in the subset $\lambda_j(A_{pqc})$ of $\mathbb{C}\setminus\{0\}$, so that the last assertion follows.

Before proceeding, we recall some facts that can essentially be found in [10, Thm. II.1.3 and Proof of Prop. IV.4.4]. Let B be any connected Lie group and σ an involution of B. Then B^{σ} denotes the fixed point set of σ. The set $\mathcal{T} = \{x \in B;\ \sigma(x) = x^{-1}\}$ is a smooth submanifold of B. Now B acts on $\mathcal{T}$ according to the rule $b\cdot x = \sigma(b)xb^{-1}$. By a computation of differentials one may check that all B-orbits are open in $\mathcal{T}$. Hence the connected identity component $\mathcal{S}_{\sigma}(B)$ of $\mathcal{T}$ is equal to the B-orbit through e:

$$\mathcal{S}_\sigma(B) = \{ \sigma(b)b^{-1};\ b \in B \}.$$

The manifold $\mathcal{S}_\sigma(B)$ is called the space of symmetric elements in B. The map $B \longrightarrow \mathcal{S}_\sigma(B)$, $b \longmapsto \sigma(b)^{-1}b$ induces a B-equivariant diffeomorphism $B^\sigma \backslash B \xrightarrow{\simeq} \mathcal{S}_\sigma(B)$. If C is any open subgroup of B^σ, than $|C\backslash B^\sigma| < \infty$ (cf. [10, Thm. IV.3.4]) and the above map $B \longrightarrow \mathcal{S}_\sigma(B)$ induces a finite covering $C\backslash B \longrightarrow \mathcal{S}_\sigma(B)$.

Applying the above to G_c and L_{0c} together with the holomorphic continuation of the Cartan involution θ, we obtain finite coverings

$$X \longrightarrow \mathcal{S}, \qquad X_L \longrightarrow \mathcal{S}_L,$$

where $X = K_c\backslash G_c$, $X_L = (K_c \cap L_{0c})\backslash L_{0c}$, $\mathcal{S} = \mathcal{S}_\theta(G_c)$, $\mathcal{S}_L = \mathcal{S}_\theta(L_{0c})$.

Let us now return to the proof of the main theorem. If $x \in G_c$, we put $x' = (\theta x)^{-1}$. In view of Lemma 3.2 the map $(\ell, b, n) \longmapsto n'\ell bn$ maps $\mathcal{S}_L \times A_{pqc} \times N_{Qc}$ into $\mathcal{S} - D^{-1}(0)$.

<u>Proposition 3.5.</u> The map $\varepsilon : \mathcal{S}_L \times A_{pqc} \times N_{Qc} \longrightarrow \mathcal{S} - D^{-1}(0)$, $(\ell, b, n) \longmapsto n'\ell bn$ is a finite covering.

<u>Proof.</u> Consider the finite covering γ of Lemma 3.2. One easily checks that $\gamma^{-1}(\mathcal{S} - D^{-1}(0))$ equals the smooth submanifold

$$\mathcal{T} = \{(\bar{n}, \ell, b, n) \in \bar{N}_{Qc} \times L_{0c} \times A_{pqc} \times N_{Qc};\ n = n',\ \ell = \ell'\}.$$

Let $\mathcal{S}$ be the connected component of $\mathcal{T}$ which contains (e,e,e,e). Then $\gamma | \mathcal{S} : \mathcal{S} \longrightarrow \mathcal{S}(G_c) - D^{-1}(0)$ is a finite

covering. Moreover, the map $i : \mathcal{S}_L \times A_{pqc} \times N_{Qc} \longrightarrow$
$\longrightarrow \bar{N}_{Qc} \times L_{Oc} \times A_{pqc} \times N_{Qc}$, $(\ell, b, n) \longmapsto (n', \ell, b, n)$
maps $\mathcal{S}_L \times A_{pqc} \times N_{Qc}$ diffeomorphically onto S. Since
$\varepsilon = (\gamma | S) \circ i$, the proposition follows.

Define the map

$$\delta : X_L \times A_{pqc} \times N_{Qc} \longrightarrow \mathcal{S}_L \times A_{pqc} \times N_{Qc}$$

by $\delta(\bar{\ell}, a, n) = (\ell'\ell, a^2, n)$ (here $\bar{\ell}$ denotes the coset of ℓ).
Then clearly δ is a finite covering.

Consider the map $\vartheta : G_c \longrightarrow \mathcal{S}$, $x \longmapsto x'x$, and define the polynomial function $F : G_c \longrightarrow \mathbb{C}$ by

$$F(x) = D(\vartheta x) = D(x'x).$$

Then F is left K_c-invariant, hence can be viewed as a function on X. Similarly, $F^{-1}(0)$ can be viewed as a subset of X. As such it is the preimage of $D^{-1}(0)$ under the finite covering $\bar{\vartheta} : X \longrightarrow \mathcal{S}$ induced by ϑ. Being the complement of an analytic null set, $X - F^{-1}(0)$ is connected, so that the restriction of $\bar{\vartheta}$ to $X - F^{-1}(0)$ is a finite covering

$$\eta : X - F^{-1}(0) \longrightarrow \mathcal{S} - D^{-1}(0).$$

Finally, if we define the map $\varphi : X_L \times A_{pqc} \times N_{Qc} \longrightarrow X$ by $\varphi(\bar{\ell}, a, n) = K_c \ell a n$, then $\bar{\vartheta} \circ \varphi = \varepsilon \circ \delta$, where ε is the map of Proposition 3.5. Hence $\mathrm{im}(\varphi) \subset \bar{\vartheta}^{-1}(\mathrm{im}\, \varepsilon) =$
$= X - F^{-1}(0)$ and the following diagram commutes:

$$\begin{array}{ccc}
X_L \times A_{pqc} \times N_{Qc} & \xrightarrow{\ \varphi\ } & X - F^{-1}(0) \\
\delta \downarrow & & \\
\mathcal{L}_L \times A_{pqc} \times N_{Qc} & \swarrow \eta & \\
\varepsilon \downarrow & & \\
\mathcal{L} - D^{-1}(0) & &
\end{array}$$

Since δ , ε and η are finite coverings, we now have the following result.

<u>Lemma 3.6.</u> The map $\varphi : (K_c \cap L_{0c}) \setminus L_{0c} \times A_{pqc} \times N_{Qc} \longrightarrow$ $\longrightarrow K_c \setminus G_c - F^{-1}(0)$, $(\bar{\ell}, a, n) \longmapsto K_c \ell a n$ is a finite covering with base points $(\bar{e}, e, e)$ and $\bar{e}$.

<u>Proof of Theorem 1.3.</u> Let $\pi : G_c - F^{-1}(0) \longrightarrow X - F^{-1}(0)$ be the restriction of the canonical map $G_c \longrightarrow X$. Moreover, let $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ be the multi-valued holomorphic inverse of the covering φ with base points $(\bar{e}, e, e)$ and $\bar{e}$. In view of Proposition 1.1, locally at $\bar{e}$ the basic branch $\zeta_{\bar{e}}$ of ζ is given by $\zeta_{\bar{e}} \circ \pi =$ $= (\lambda, h_q, \nu_Q)$. Consequently $\zeta_1 \circ \pi$, $\zeta_2 \circ \pi$, $\zeta_3 \circ \pi$ are the multi-valued holomorphic extensions of λ , h_q, ν_Q to $G_c - F^{-1}(0)$. Moreover, $\pi^{-1}(X - F^{-1}(0)) = \pi^{-1}(\mathrm{im}\, \varphi) =$ $= K_c L_{0c} A_{pqc} N_{Qc} = K_c Q_c$, and therefore $S_Q = G_c - K_c Q_c = F^{-1}(0)$ is algebraic.

As for the last three assertions, by holomorphic continuation it follows that

$$x'x = \nu_Q(x)' \, \lambda(x)' \, \lambda(x) \, h_q^2(x) \, \nu_Q(x)$$

for all $x \in G_c - D^{-1}(0)$. Consequently, with the notations preceding Proposition 3.3,

$$h_q^{2\mu}(x) = b(x'x)^{\mu} , \tag{7}$$

$$\nu_Q(x) = v(x'x). \tag{8}$$

Now put $m = 2\mu$. Then the last assertions readily follow by application of Proposition 3.4.

We end this section with two related propositions, which will be useful in the next section.

Proposition 3.7. $G_c - S_Q = G_c^{\theta} Q_c$.

Proof. From [9 , Proposition 1] it follows that $G_c^{\theta} \subset K_c L_c$. Hence $G_c^{\theta} Q_c = K_c Q_c$.

Proposition 3.8. Let $(a,n) \in A_{pqc} \times N_{Qc}$ and assume that $x \in G_c^{\theta} L_{Oc} an$. Then

$$a^{2\mu} = h_q^{2\mu}(x),$$

$$n = \nu_Q(x).$$

Proof. It follows that $x'x \in n' \mathcal{S}_L a^2 n$. Hence, with the notations preceding Proposition 3.3, we have $a^{2\mu} = b(x'x)^{\mu}$ and $n = v(x'x)$. The assertion now follows by comparison of these two formulas with (7 , 8).

4. An application to reductive symmetric spaces

In this section we will apply Theorem 1.3 to obtain the following result which is basic for [4].

We assume that G is a real reductive group of the Harish-Chandra class (class $\mathcal{H}$) with Lie algebra $\mathfrak{g}$, that τ is an involution of G, and that H is an open subgroup of G^{τ}. The space $H\backslash G$ is called a reductive symmetric space of class $\mathcal{H}$ (see also [3]). There exists a Cartan involution θ of G which commutes with τ (cf. [3]). Its fixed point set K is a maximal compact subgroup of G. We may now introduce groups A_{pq}, N_Q, L, L_0 by the same definitions as in the case of an adjoint group in Section 1. By a standard computation of differentials one may check that the map $H \times L_0 \times A_{pq} \times N_Q \longrightarrow G$, $(h, \ell, a, n) \longmapsto h\ell an$ is a submersion onto an open subset Ω of G. The main result of this section is:

<u>Lemma 4.1.</u> If $x \in \Omega$, then there exist unique $\ell(x) \in (H \cap L_0)\backslash L_0$, $a_{pq}(x) \in A_{pq}$, $n_Q(x) \in N_Q$ such that

$$x \in H\,\ell(x)\,a_{pq}(x)\,n_Q(x). \tag{9}$$

The corresponding maps ℓ, a_{pq} and n_Q are real analytic. Moreover, if $\{x_k\}$ is any sequence in Ω converging to a point $x \in \partial\Omega$, then the set $\{a_{pq}(x_k);\ k \in \mathbb{N}\}$ is not relatively compact in A_{pq}.

We split the proof of this lemma into two parts, the first being:

Reduction to the adjoint case. Let X(G) denote the group of all continuous homomorphisms $G \longrightarrow \mathbb{R}^*$ and put $^{o}G = \bigcap \{ \ker |\chi| \ ; \ \chi \in X(G) \}$. Then ^{o}G is a closed subgroup of class $\mathcal{H}$. Let $\mathfrak{v} = \mathfrak{p} \cap \text{centre}(\mathfrak{g})$ and put $V = \exp \mathfrak{v}$. Then V is a closed vector subgroup of G and we have the direct product $G \simeq {}^{o}G \times V$ (see also [14, p. 196]). Moreover, putting $\mathfrak{v}_h = \mathfrak{v} \cap \mathfrak{h}$ and $\mathfrak{v}_q = \mathfrak{v} \cap \mathfrak{q}$, we have $\mathfrak{v} = \mathfrak{v}_h \oplus \mathfrak{v}_q$ and a direct product $V \simeq V_h \times V_q$, where $V_h = \exp(\mathfrak{v}_h)$, $V_q = \exp(\mathfrak{v}_q)$. Combined with the above this yields the direct product

$$G \simeq V_h \times {}^{o}G \times V_q. \qquad (10)$$

Now clearly $V_q \subset A_{pq}$ and $V_h \subset H \cap L_0$. Therefore, putting $^{o}A_{pq} = {}^{o}G \cap A_{pq}$, $^{o}L_0 = {}^{o}G \cap L_0$ and $^{o}H = {}^{o}G \cap H$, we have direct products $A_{pq} \simeq {}^{o}A_{pq} \times V_q$, $L_0 \simeq V_h \times {}^{o}L_0$ and $H \simeq V_h \times {}^{o}H$. Moreover, $N_Q \subset {}^{o}G$, and so the set $^{o}\Omega = \Omega \cap {}^{o}G$ equals $^{o}H\,^{o}L_0\,^{o}A_{pq}N_Q$, and Ω admits the decomposition $\Omega \simeq V_h \times {}^{o}\Omega \times V_q$ subordinate to (10). Thus the $HL_0A_{pq}N_Q$-decomposition of Ω is compatible with the decomposition (10) of G and $\partial\Omega \simeq V_h \times \partial({}^{o}\Omega) \times V_q$. Therefore, it suffices to prove Lemma 4.1 for the group ^{o}G together with the decomposition $^{o}\Omega = {}^{o}H\,^{o}L_0\,^{o}A_{pq}N_Q$.

Thus we may as well assume that $G = {}^{o}G$. In that case, the the centre Z(G) of G is contained in K. Assume now that the lemma is valid for the image $Ad_G(G)$ of G in the adjoint group G_c of $\mathfrak{g}_c$ under the adjoint representation Ad_G of G in $\mathfrak{g}_c$. We claim that it then holds for G as well. In fact, if $(\ell_i, a_i, n_i) \in ((H \cap L_0)\backslash L_0) \times A_{pq} \times N_Q$ $(i = 1,2)$ and $H\ell_1a_1n_1 = H\ell_2a_2n_2$, then it follows that $Ad_G(a_1) = Ad_G(a_2)$, $Ad_G(n_1) = Ad_G(n_2)$, whence $a_1 = a_2$, $n_1 = n_2$ and $H\ell_1 = H\ell_2$.

Hence $(H \cap L_0)\ell_1 = (H \cap L_0)\ell_2$, the uniqueness statement holds and the maps ℓ , a_{pq} and n_Q are well defined on Ω .

By a standard computation of differentials it now follows that the real analytic map $(\bar{\ell},a,n) \longmapsto H\ell an$ maps $((H\cap L_0)\backslash L_0) \times A_{pq} \times N_Q$ diffeomorphically onto the canonical image $\underline{\Omega}$ of Ω in H\G. Its inverse is a real analytic diffeomorphism $\zeta : \underline{\Omega} \longrightarrow ((H\cap L_0)\backslash L_0) \times A_{pq} \times N_Q$. Now let $\pi : \Omega \rightarrow \underline{\Omega}$ be the restriction of the canonical map $G \longrightarrow H\backslash G$ to Ω . Then (ℓ, a_{pq}, n_Q) equals $\zeta \circ \pi$, hence is real analytic.

Finally, $\ker(Ad_G) = Z(G)$ is contained in K, hence in $K \cap L \subset L_0$. It follows that $\Omega = \Omega Z(G) = Ad_G^{-1}(Ad_G(\Omega))$, whence $\partial Ad_G(\Omega) = Ad_G(\partial\Omega)$. Therefore, if $\{x_k\}$ is a sequence in Ω converging to a point $x \in \partial\Omega$, then by applying Lemma 4.1 to the sequence $\{Ad_G(x_k)\}$ in $Ad_G(G)$, we infer that $\{Ad_G(a_{pq}(x_k));\ k \in \mathbb{N}\}$ is not relatively compact in $Ad_G(A_{pq})$. It follows that $\{a_{pq}(x_k);\ k \in \mathbb{N}\}$ is not relatively compact in A_{pq}, and the lemma's validity for G has been established.

From the above we see that it suffices to prove Lemma 4.1 under the following assumption (A) which we assume to hold from now on.

(A) $\mathfrak{g}$ is a real semisimple Lie algebra and G is an open subgroup of the normalizer $G_{\mathbb{R}}$ of $\mathfrak{g}$ in the adjoint group G_c of $\mathfrak{g}_c$.

The next idea is to exploit the duality introduced by Berger [5] (and also used by [6,7,12]).

The space

$$\mathfrak{g}^d = i(\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{q}) \oplus i(\mathfrak{p} \cap \mathfrak{h})$$

is a subalgebra of $\mathfrak{g}_c$, called the real form dual to $\mathfrak{g}$. The restriction θ^d of the (complex) involution τ to $\mathfrak{g}^d$ is a Cartan involution for $\mathfrak{g}^d$, with associated eigenspace decomposition

$$\mathfrak{g}^d = \mathfrak{k}^d \oplus \mathfrak{p}^d,$$

where $\mathfrak{k}^d = \mathfrak{h}_c \cap \mathfrak{g}^d$, $\mathfrak{p}^d = \mathfrak{q}_c \cap \mathfrak{g}^d$ (read this as: the $\mathfrak{k}$ in the dual situation, etc.). Similarly, we put $\tau^d = \theta | \mathfrak{g}^d$, $\mathfrak{h}^d = \mathfrak{k}_c \cap \mathfrak{g}^d$, $\mathfrak{q}^d = \mathfrak{p}_c \cap \mathfrak{g}^d$. Let G^d, K^d, H^d be the connected analytic subgroups of G_c with Lie algebras $\mathfrak{g}^d$, $\mathfrak{k}^d$ and $\mathfrak{h}^d$ respectively. Moreover, let $\mathfrak{a}^d_{pq} = \mathfrak{a}_{pq}$, $A^d_{pq} = A_{pq}$, $L^d = L_c \cap G^d$, $\mathfrak{n}^d_Q = \mathfrak{n}_{Qc} \cap \mathfrak{g}^d$, $N^d_Q = \exp(\mathfrak{n}^d_Q)$, and define $L^d_0 = (K^d \cap L^d)\exp(\mathfrak{p}^d \cap \mathfrak{h}^d \cap \mathfrak{l}^d)$. Then according to Proposition 1.1, we have a decomposition $G^d = K^d L^d_0 A^d_{pq} N^d_Q$ with corresponding maps λ^d, h^d_q, ν^d_Q: $G^d \longrightarrow (L^d_0 \cap K^d)\backslash L^d_0$, A^d_{pq}, N^d_Q determined by

$$x \in K^d\, \lambda^d(x)\, h^d_q(x)\, \nu^d_Q(x). \tag{11}$$

The idea is now to view (9) and (11) as different real forms of the same multi-valued holomorphic decomposition.

Let H_c be the connected analytic subgroup of G_c with Lie algebra $\mathfrak{h}_c$. Set $S^d_Q = G_c - H_c L_c N_{Qc}$. Then according to Theorem 1.3, the maps λ^d, h^d_q, ν^d_Q have multi-valued holomorphic extensions to maps $G_c - S^d_Q \longrightarrow (H_c \cap L^d_{Oc})\backslash L^d_{Oc}$,

A_{pqc}, N_{Qc}.

To complete the proof of Lemma 4.1, we need the following.

Proposition 4.2. Under the assumption (A), the set Ω is a union of connected components of $G-S^d_Q$.

Proof. The group $H \times Q$ acts on G_c according to the rule $(h,q)\cdot x = hxq^{-1}$, for $h \in H$, $q \in Q$, $x \in G_c$. In view of Proposition 3.7, this action leaves $G_c - S^d_Q = G_c^{\tau} Q_c$ invariant. Moreover, by an easy computation of differentials at points of $G_c^{\tau} Q_c$, it follows that all $H \times Q$-orbits in $G_c - S^d_Q$ are submanifolds of real dimension $\dim(G)$. Hence $G-S^d_Q$ is a union of open $H \times Q$-orbits. Now Ω is just the $H \times Q$-orbit through e, hence open and closed in $G-S^d_Q$.

End of proof of Lemma 4.1. Let $\ell_1, \ell_2 \in L_0$, $a_1, a_2 \in A_{pq}$, $n_1, n_2 \in N_Q$ and assume that $H\ell_1 a_1 n_1 = H\ell_2 a_2 n_2$. Then $G_c^{\tau}\ell_1 a_1 n_1 = G_c^{\tau}\ell_2 a_2 n_2$ and using Proposition 3.8 we infer that $n_1 = n_2$ and $a_1^{2\mu} = a_2^{2\mu}$. The map $\exp\colon \mathfrak{a}_{pq} \longrightarrow A_{pq}$ being a diffeomorphism it follows that $a_1 = a_2$. Hence $H\ell_1 = H\ell_2$ from which it is immediate that $(H \cap L_0)\ell_1 = (H \cap L_0)\ell_2$. This proves uniqueness and the maps ℓ, a_{pq}, n_Q are well defined by (9). By the same argument as in the reduction part of the proof it now follows that these maps are real analytic.

Finally, by Proposition 3.8 we have

$$a(x)^{2\mu} = h^d_q(x)^{2\mu} ,$$

for $x \in \Omega$. By Proposition 4.2, $\partial\Omega$ is contained in S_Q^d and so the last assertion of the lemma follows from the corresponding assertion of Theorem 1.3.

Appendix: Multi-valued holomorphic maps

Let X be a connected complex analytic manifold. A covering $p:E \longrightarrow X$ together with points $e \in E$, $a \in X$ such that $p(e) = a$ is called a covering with base points of X. We write $p: (E,e) \longrightarrow (X,a)$ for such a covering. Let a point $a \in X$ be fixed from now on.

Fix a universal covering $\pi : (\tilde{X}, \alpha) \longrightarrow (X,a)$ with base points of X. A holomorphic map f from $\tilde{X}$ into a complex analytic manifold Y is called a multi-valued holomorphic map from X into Y. Let us denote the germ of a holomorphic map F at a point x by F_x. Then $f_a = f_\alpha \circ (\pi_\alpha)^{-1}$ is the germ of a holomorphic map at a. We call f_a the basic branch of f at the base point a; it determines f uniquely, and f is called the multi-valued holomorphic extension of f_a to X.

If we work with multi-valued holomorphic maps this will always be done with respect to a base point. Thus a multi-valued holomorphic map will always be viewed as the multi-valued holomorphic extension of a holomorphic germ at the base point.

Single valued maps. A holomorphic map $G: \tilde{X} \longrightarrow Y$ will be called single valued on X iff $G_\xi \circ (\pi_\xi)^{-1} = G_\eta \circ (\pi_\eta)^{-1}$ for all $\xi, \eta \in \tilde{X}$ with $\pi(\xi) = \pi(\eta)$. Clearly G is single valued if and only if there exists a holomorphic map $F: X \longrightarrow Y$ such that $G = F \circ \pi$. We often identify a holomorphic map $F: X \longrightarrow Y$ with the associated single valued map $F \circ \pi$, viewed as a multi-valued map $X \longrightarrow Y$.

<u>Composition.</u> Let Y, Z be complex analytic manifolds and f a multi-valued holomorphic map $X \to Y$. If $g: Y \to Z$ is a holomorphic map then $g \circ f$ is a well defined multi-valued holomorphic map $X \to Z$. If Y is connected and $g: Y \to Z$ a multi-valued holomorphic map with base point $b = f(\alpha)$, then $g \circ f$ is defined as follows.

By definition, f actually is a holomorphic map $\tilde{X} \to Y$. Now let $p: (\tilde{Y}, \beta) \to (Y,b)$ be the universal covering with base points of Y. Since X is simply connected, there exists a unique holomorphic map $\tilde{f}: \tilde{X} \to \tilde{Y}$ with $f(\alpha) = \beta$, $f = p \circ \tilde{f}$, called the lifting of f. The map $g \circ \tilde{f}: \tilde{X} \to Z$ is a multi-valued holomorphic map $X \to Z$, called the composition of f and g. We also denote it by $g \circ f$. Its basic branch is given by $(g \circ f)_a = g_b \circ f_a$. Observe that these definitions are compatible with the identification of holomorphic and single valued maps described above.

<u>The inverse of a covering.</u> Let $\varphi : (Y,b) \to (X,a)$ be a covering with base points. Then there exists a unique holomorphic map $\psi : \tilde{X} \to Y$ with $\psi(\alpha) = b$ and $\varphi \circ \psi = \pi$. We call ψ the multi-valued holomorphic inverse of φ. Observe that its basic branch is given by $\psi_a = (\varphi_b)^{-1}$.

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