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# On the Theory of Topographic Vorticity Production by Tidal Currents

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Starting from the shallow water equations of a homogeneous rotating fluid we derive the equation describing the evolution of vorticity induced by a fluctuating bottom topography. By a twofold expansion in a small parameter, it is shown that nonlinear vorticity advection can be reduced to a quasi-linear form in the limit of small amplitude topography and for the topographic horizontal length scale being much smaller than the length of a tidal gravity wave. Topographic vorticity is then produced by two mechanisms, viz. planetary vortex stretching and differential bottom friction. For both mechanisms, the vorticity response functions at the basic forcing frequency and all of its higher harmonics, as well as the residual components, are shown to be given by sums of products of Bessel functions. We discuss the asymptotic behaviour of these response functions and compare them with other approximate methods of solving the vorticity equation, particularly the method of harmonic truncation. These results are used in deriving the exact shape of the residual velocity field for a one-dimensional step-like bottom topography. We show that in the absence of vorticity diffusion the residual velocity vanishes exactly outside a region of twice the tidal excursion length scale, in contrast to the result derived by the method of harmonic truncation. Inside that region the residual velocity profile can be expressed in Legendre functions, or equivalently as a hypergeometric function, and in an integral over Legendre functions. This result is finally used to calculate the shape of the residual velocity profile numerically.

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## 1. Introduction

When a long inertio-gravity wave propagates over a depth-varying sea bed, the horizontal length scale of which is much smaller than the wave-length, it has been shown that in the limit of small-amplitude topography a vorticity field is produced that contains, besides the fundamental driving frequency, also a residual (time-independent) part and higher harmonics due to nonlinear vorticity advection (Huthnance, 1981; Robinson, 1983; Zimmerman, 1978, 1980). The small amplitude limit is a convenient means of dealing with this process mathematically because it allows a quasi linearization of the problem and because it is possible to represent bottom topographies of any shape by their spatial Fourier transform, solving subsequently for the Fourier transformed vorticity field. The back-transformation then contains weighting functions by which the Fourier components of the bottom topography are multiplied. In this paper we derive these weighting functions, starting from the shallow water equations of a homogeneous rotating fluid. The solution is derived in two steps. First the vorticity field is solved in a frame of reference moving with the basic "unperturbed" flow, giving the so-called quasi-Lagrangean vorticity field. Secondly, transforming back to a reference system fixed to the sea bottom

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we obtain the so-called Eulerian vorticity field. We present a complete discussion of the asymptotic properties of the weighting functions so derived, which are of importance because of the existence of a "resonance peak" for residual velocity at topographic length scales of the order of the tidal excursion (Zimmerman, 1978, 1980). Finally we discuss the Fourier back-transform for a given Fourier representation of the bottom topography. Here we are particularly interested in shape of the residual velocity field for a one-dimensional step-like topography. We give here, for the first time, the exact results for a topography in which the shape approaches the Dirac delta function. This exact result is compared with the approximations obtained by harmonic truncation of the problem concerned.

## 2. Basic equation

We start from the two-dimensional shallow water equations for a homogeneous fluid in a uniformly rotating medium (Pedlosky, 1979):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + r\frac{u}{D+Z} = -g\frac{\partial Z}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] , \tag{1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + r\frac{v}{D+Z} = -g\frac{\partial Z}{\partial y} + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] , \tag{2}$$

$$\frac{\partial Z}{\partial t} + \frac{\partial}{\partial x}[(D+Z)u] + \frac{\partial}{\partial y}[(D+Z)v] = 0 . \tag{3}$$

Here $u$ and $v$ are the $x$-and $y$ components of the velocity vector and $Z$ is the disturbance of sea level from its equilibrium position. Friction is represented by a linear friction coefficient, $r$, parametrizing vertical momentum transfer and by a horizontal eddy viscosity, $\mu$. The Coriolis parameter is denoted by $f$, the accelaration of gravity by $g$.

For studying the perturbations induced by a spatially varying topography, the depth, $D$, below the undisturbed sea level is written as

$$D(x,y) = H + h(x,y) . \tag{4}$$

Let $\bar{h}$ be a characteristic amplitude of the topography, then we assume $\bar{h} << H$, the latter inequality representing the limit of small amplitude topography, which enables us to approach the solutions of eqs, (1) - (3) perturbatively. Let $\delta = \bar{h} / H$, and write

$$\begin{aligned} u &= u_0 + \delta u_1 + \delta^2 u_2 + \ldots , \\ v &= v_0 + \delta v_1 + \delta^2 v_2 + \ldots , \\ Z &= Z_0 + \delta Z_1 + \delta^2 Z_2 + \ldots . \end{aligned} \tag{5}$$

To zeroth order in $\delta$ we then have:

$$\begin{aligned} &\frac{\partial u_0}{\partial t} - v_0\left(\frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} + f\right) + \frac{r}{H+Z_0}u_0 = \\ &-\frac{\partial}{\partial x}\left(gZ_0 + \frac{u_0^2+v_0^2}{2}\right) + \mu\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}\right) , \end{aligned} \tag{6}$$

$$\begin{aligned} &\frac{\partial v_0}{\partial t} + u_0\left(\frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y} + f\right) + \frac{r}{H+Z_0}v_0 = \\ &-\frac{\partial}{\partial y}\left(gZ_0 + \frac{u_0^2+v_0^2}{2}\right) + \mu\left(\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_0}{\partial y^2}\right) , \end{aligned} \tag{7}$$

$$\frac{\partial Z_0}{\partial t} + \frac{\partial}{\partial x}[(H+Z_0)u_0] + \frac{\partial}{\partial y}[(H+Z_0)v_0] = 0 , \tag{8}$$

where we have recasted the nonlinear advection terms such that they appear as vortex force and dynamic pressure gradient. Since the bottom is assumed to be flat at this order no topographic length scale enters the problem here. Supposing the wave described by (6) - (8) to be a small amplitude wave it is standard practice to linearize (6) - (8) by assuming the following inequalities to apply:

$$\frac{U}{fL_0} \ll 1 \quad , \quad \frac{U}{(gH)^{1/2}} = \frac{a}{H} \ll 1 \tag{9}$$

where $U$ is a representative velocity scale, $L_0$ the wave length and $a$ a representative scale for the amplitude of sea level variations which, from the continuity equation, (8), must be of order $UTH / L_0$, $T$ denoting the wave period, from which the equality in the second part of (9) is derived. The first inequality, stating a small Rossby number, linearizes the vortex force in (6) and (7). The second inequality in (9), stating a small Froude number, linearizes the dynamic pressure gradient and friction term in (6) and (7) and the shallow water terms in (8). Finally we assume

$$\frac{\mu T}{L_0^2} \ll 1 \, , \tag{10}$$

making lateral viscous friction unimportant at this order.

What remains is the linearized set:

$$\frac{\partial u_0}{\partial t} - f v_0 + \frac{r}{H} u_0 = -g \frac{\partial Z_0}{\partial x} \, , \tag{11}$$

$$\frac{\partial v_0}{\partial t} + f u_0 + \frac{r}{H} v_0 = -g \frac{\partial Z_0}{\partial y} \, , \tag{12}$$

$$\frac{\partial Z_0}{\partial t} + H[\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}] = 0 \, . \tag{13}$$

For simple harmonic time dependence (11) - (13) describe free propagating damped long inertio-gravity waves. We are not concerned here with the structure of these waves over length scales comparable with the wave-length. As will become clear later on, in fact all we need is a "local" value for the velocity vector $(u_o, v_0)$ for a prescribed local pressure gradient. Since a uniformly rotating fluid is an isotropic medium we may without loss of generality prescribe:

$$-g \frac{\partial Z_0}{\partial x} = Pe^{i\sigma t} \quad , \quad -g \frac{\partial Z_0}{\partial y} = 0 \ , \ \sigma = \frac{2\pi}{T} \, . \tag{14}$$

It is then easily shown that from (11) and (12) we have:

$$u_0 = \frac{[f^2 + (\frac{r}{H})^2 + \sigma^2](\frac{r}{H}) + i\sigma[f^2 - (\frac{r}{H})^2 - \sigma^2]}{[f^2 + (\frac{r}{H})^2 - \sigma^2]^2 + 4\sigma^2(\frac{r}{H})^2} Pe^{i\sigma t} \, , \tag{15}$$

$$v_0 = -\frac{[f^2 + (\frac{r}{H})^2 - \sigma^2] f - 2i\sigma f \frac{r}{H}}{[f^2 + (\frac{r}{H})^2 - \sigma^2]^2 + 4\sigma^2(\frac{r}{H})^2} Pe^{i\sigma t} \, , \tag{16}$$

describing an elliptically polarized velocity vector. It should be noted that for $f = 0$ this ellipse degenerates to a line:

$$u_0 = \frac{(r / H) - i\sigma}{(\frac{r}{H})^2 + \sigma^2} Pe^{i\sigma t} \, , \tag{17}$$

$$v_0 = 0 \, , \tag{18}$$

whereas for $f^2 = (\frac{r}{H})^2 + \sigma^2$ we have:

$$|v_0| = |u_0| = \frac{P}{2(r/H)} \quad . \tag{19}$$

We now proceed with treating (1) - (3) perturbatively in $\delta = \bar{h}/H$ going to first order in $\delta$. Using (9) we then have:

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_0}{\partial x} + u_0\frac{\partial u_1}{\partial x} + v_1\frac{\partial u_0}{\partial y} + v_0\frac{\partial u_1}{\partial y} - fv_1 + \frac{r}{H}u_1 - \frac{r}{H^2}hu_0 = -g\frac{\partial Z_1}{\partial x} + \mu(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2}) \ , \tag{20}$$

$$\frac{\partial v_1}{\partial t} + u_1\frac{\partial v_0}{\partial x} + u_0\frac{\partial v_1}{\partial x} + v_1\frac{\partial v_0}{\partial y} + v_0\frac{\partial v_1}{\partial y} + fu_1 + \frac{r}{H}v_1 - \frac{r}{H^2}hv_0 = -g\frac{\partial Z_1}{\partial y} + \mu(\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2}) \ , \tag{21}$$

$$\frac{\partial Z_1}{\partial t} + \frac{\partial}{\partial x}(Hu_1 + hu_0) + \frac{\partial}{\partial y}(Hv_1 + hv_0) = 0 \ . \tag{22}$$

These equations can now be further reduced by the crucial assumption of having a topography with an intrinsic length scale that is much smaller than any length scale inherent in the zeroth order solution of (11) - (13). The latter length scales can either be the wave-length $(gH)^{1/2}T$, the Rossby deformation radius $(gH)^{1/2}/f$ or a frictional damping length scale $\frac{r}{H}T$. Let these length scales (or at least the smallest one) be denoted by $L_0$ and let the length scale over which the bottom topography varies be $L$. We now assume:

$$\epsilon = L/L_0 \ll 1 \ . \tag{23}$$

This suggest that we may regard (20) - (22) as a multiple scale problem. Introducing the "slow" variables $X$ and $Y$, related to the "fast" variables $x$ and $y$ by

$$X = \epsilon x \ , \quad Y = \epsilon y \ , \tag{24}$$

so that

$$\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial x} + \epsilon\frac{\partial}{\partial X} \ , \quad \frac{\partial}{\partial y} \rightarrow \frac{\partial}{\partial y} + \epsilon\frac{\partial}{\partial Y} \ , \tag{25}$$

and

$$u_0 = u_0(X,Y,t) \ , \ v_0 = v_0(X,Y,t) \ , \ Z_0 = Z_0(X,Y,t) \ , \tag{26}$$

$$h = h(x,y) \ , \tag{27}$$

$$\left.\begin{aligned} u_1 &= u_1(x,y,X,Y,t) + \mathcal{O}(\epsilon), \\ v_1 &= v_1(x,y,X,Y,t) + \mathcal{O}(\epsilon), \\ Z_1 &= Z_1(x,y,X,Y,t) + \mathcal{O}(\epsilon), \end{aligned}\right\} \tag{28}$$

we have to zeroth order in $\epsilon$:

$$\frac{\partial u_1}{\partial t} + u_0\frac{\partial u_1}{\partial x} + v_0\frac{\partial u_1}{\partial y} - fv_1 + \frac{ru_1}{H} - \frac{rhu_0}{H^2} = -g\frac{\partial Z_1}{\partial x} + \mu\left[\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2}\right] \ , \tag{29}$$

$$\frac{\partial v_1}{\partial t} + u_0\frac{\partial v_1}{\partial x} + v_0\frac{\partial v_1}{\partial y} + fu_1 + \frac{rv_1}{H} - \frac{rhv_0}{H^2} = -g\frac{\partial Z_1}{\partial y} + \mu\left[\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2}\right] \ , \tag{30}$$

$$\frac{\partial Z_1}{\partial t} + H\left\{\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right\} + u_0\frac{\partial h}{\partial x} + v_0\frac{\partial h}{\partial y} = 0 \ . \tag{31}$$

Further insight can be obtained by making these equations dimensionless by means of the following scaling:

$$\left.\begin{aligned} &(x^*,y^*) = \frac{1}{L}(x,y),\ t^* = t\,/\,T, \\ &(u^*,v^*) = \frac{1}{U}(u,v),\ \ Z^* = Z\,/\,a = ZgT\,/\,UL_0, \\ &h^* = h\,/\,H. \end{aligned}\right\} \tag{32}$$

Then (29) - (31) become:

$$\begin{aligned} &\epsilon\left[\frac{\partial u_1^*}{\partial t^*} + \lambda u_0^*\frac{\partial u_1^*}{\partial x^*} + \lambda v_0^*\frac{\partial u_1^*}{\partial y^*} - fTv_1^* + \frac{rT}{H}u_1^* - \frac{rT}{H}h^*u_0^*\right] = \\ &\qquad -g\frac{\partial Z_1^*}{\partial x^*} + \epsilon\frac{\mu T}{L^2}\left[\frac{\partial^2 u_1^*}{\partial x^{*2}} + \frac{\partial^2 u_1^*}{\partial y^{*2}}\right] , \end{aligned} \tag{33}$$

$$\begin{aligned} &\epsilon\left[\frac{\partial v_1^*}{\partial t} + \lambda u_0^*\frac{\partial v_1^*}{\partial x^*} + \lambda\, v_0^*\frac{\partial v_1^*}{\partial y^*} + fTu_1^* + \frac{rT}{H}v_1^* - \frac{rT}{H}h^*v_0^*\right] = \\ &\qquad -g\frac{\partial Z_1^*}{\partial y^*} + \epsilon\,\frac{\mu T}{L^2}\left[\frac{\partial^2 v_1^*}{\partial x^{*2}} + \frac{\partial^2 v_1^*}{\partial y^{*2}}\right] , \end{aligned} \tag{34}$$

$$\epsilon^2\frac{\partial Z_1^*}{\partial t^*} + \frac{\partial h^*}{\partial x} + \frac{\partial h^*}{\partial y} + \frac{\partial u_1^*}{\partial x^*} + \frac{\partial v_1^*}{\partial y^*} = 0 \ , \tag{35}$$

where $\lambda = \frac{UT}{L}$, being the ratio of the tidal excursion to the topographic length scale, is the crucial parameter later on. From (35) it is evident that for $\epsilon << 1$ in first approximation we have

$$\frac{\partial}{\partial x^*}(h^* + u_1^*) + \frac{\partial}{\partial y^*}(h^* + v_1^*) = 0 \ , \tag{36}$$

so that to this order in $\epsilon$ the perturbation field to first order in $\delta$ behaves as if it has a rigid upper surface. As to the momentum balance it appears from (34) and (35) that $Z_1 = const = 0$ to zeroth order in $\epsilon$, pressure gradients only entering the dynamic balance in higher orders.

Returning now to the dimensional equations we are in a position for the final step, taking the curl of (29) and (30). Introducing the perturbation vorticity field to first order in $\delta$:

$$\eta(x,y,X,Y,t) = \frac{\partial v_1}{\partial x} - \frac{\partial u_1}{\partial y} \ , \tag{37}$$

separating fast and slow variables, we obtain to zeroth order in $\epsilon$:

$$\begin{aligned} &\frac{\partial\eta}{\partial t} + u_0\frac{\partial\eta}{\partial x} + v_0\frac{\partial\eta}{\partial y} + f(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}) + \frac{r}{H}\eta + \frac{r}{H^2}\left[u_0\frac{\partial h}{\partial y} - v_0\frac{\partial h}{\partial x}\right] = \\ &\qquad \mu\left[\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2}\right] . \end{aligned} \tag{38}$$

Making use of (36) in its dimensional form, (38) reads:

$$\frac{\partial\eta}{\partial t} + u_0\frac{\partial\eta}{\partial x} + v_0\frac{\partial\eta}{\partial y} + \frac{r}{H}\eta - \mu(\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2}) =$$

$$\frac{f}{H}(u_0\frac{\partial h}{\partial x} + v_0\frac{\partial h}{\partial y}) - \frac{r}{H^2}(u_0\frac{\partial h}{\partial y} - v_0\frac{\partial h}{\partial x}) \ , \tag{39}$$

where we have put the two basic vorticity generation mechanisms in the right-hand side. Evidently the assumptions of small amplitude topography and of multiple length scales have reduced the non-linear starting point, eqs. (1) - (3), to the readily solvable quasi-linear equation (39) in which the velocity vector $(u_0,v_0)$ is prescribed in its "local" form as discussed in deriving (15) and (16).

## 3. Fourier transformed solution

In vector form (39) reads:

$$\frac{\partial \eta}{\partial t} + \vec{u}_0.\nabla\eta + \frac{r}{H}\eta - \mu\nabla^2\eta = \frac{f}{H}\vec{u}_0.\nabla h - \frac{r}{H^2}(\vec{u}_0\times\nabla h).\vec{j} \ , \tag{40}$$

where $\vec{j}$ is the vertical unit vector.

For given $\vec{u}_0(t)$, independent of the 'fast' coordinates $x$ and $y$ (the local approach), we solve (40) by introducing the Fourier transformations:

$$\eta(\vec{x},t) = \int\int \hat{\eta}(\vec{k},t)e^{-i\vec{k}.\vec{x}}d\vec{k} \ , \tag{41}$$

$$h(\vec{x}) = \int\int \hat{h}(k)e^{-i\vec{k}.\vec{x}}d\vec{k} \ , \tag{42}$$

where the integrals are over all of the wave number space. Of course, for a specific topography, $h(\vec{x})$ is prescribed as well. Substituting (41) and (42) in (40) gives

$$\frac{\partial \hat{\eta}}{\partial t} - (i\vec{k}.\vec{u}_0 - \frac{r}{H} - \mu k^2)\hat{\eta} = -\frac{i\hat{h}}{H}\{f\vec{u}_0.\vec{k} - \frac{r}{H}(\vec{k}\times\vec{u}_0).\vec{j}\} \ . \tag{43}$$

Let now $\vec{u}_0$ be given as an elliptically polarized current

$$\vec{u}_0 = \begin{bmatrix} U\sin\sigma t \\ \pm\,\epsilon U\cos\sigma t \end{bmatrix} \ , \tag{44}$$

where $\epsilon$ is the ellipticity, which should not be confused with the perturbation parameter introduced earlier. Transforming to polar coordinates such that

$$\vec{k} = \begin{bmatrix} k\cos\theta \\ k\sin\theta \end{bmatrix} \ , \tag{45}$$

(43) reads:

$$\frac{\partial \hat{\eta}}{\partial t} - [ikU\{(\epsilon^2-1)\sin^2\theta+1\}^{1/2}\sin(\sigma t\pm\alpha_1) - \frac{r}{H} - \mu k^2]\hat{\eta} = \tag{46}$$

$$\frac{iUk\hat{h}}{H}[-f\{(\epsilon^2-1)\sin^2\theta+1\}^{1/2}\sin(\sigma t\pm\alpha_1)\pm\frac{r}{H}\{(1-\epsilon^2)\sin^2\theta + \epsilon^2\}^{1/2}\cos(\sigma t\pm\alpha_2)] \ ,$$

where

$$\alpha_1 = \operatorname{arctg}(\epsilon\,\operatorname{tg}\theta) \ , \ \alpha_2 = \operatorname{arctg}\left[\frac{\operatorname{tg}\theta}{\epsilon}\right] \ . \tag{47}$$

Introducing the following scaling and dimensionless parameters:

$$\left.\begin{aligned} &\hat{\eta} \to \hat{\eta}/f,\ \hat{h} \to \hat{h}/H,\ t \to \sigma t \pm \alpha_1 \ , \\ &k \to \frac{kU}{\sigma}\{(\epsilon^2-1)\sin^2\theta+1\}^{1/2}, \\ &\tau_1 = \frac{r}{H\sigma} \ , \ \tau_2 = \frac{r}{Hf} \ , \ \tau_3 = \frac{\mu\sigma}{U^2\{(\epsilon^2-1)\sin^2\theta+1)\}} \end{aligned}\right\} \tag{48}$$

(46) can be written as:

$$\frac{\partial \hat{\eta}}{\partial t} - (ik \sin t - \tau_1 - \tau_3 k^2)\hat{\eta} = i\hat{h}k\{ -\sin t \pm \tau_2 F(\theta,\epsilon)\cos(t \pm \alpha_2 \pm (-\alpha_1))\} \ , \tag{49}$$

where all variables are understood as being scaled according to (48). In (49)

$$F(\theta,\epsilon) = \left\{ \frac{(1-\epsilon^2)\sin^2\theta + \epsilon^2}{(\epsilon^2-1)\sin^2\theta + 1} \right\}^{1/2} \ . \tag{50}$$

Both terms in the right-hand side of (49) still describe the two basic vorticity production mechanisms, viz. planetary vortex-stretching and differential bottom friction (Zimmerman, 1978, 1980). Mathematically it is more convenient to regard the right-hand side of (49) as the sum of a contribution proportional to sin $t$ and one proportional to cos $t$, in which case, generally, both production mechanisms are mixed up depending on the values of $\alpha_1$ and $\alpha_2$. In the case of a circularly polarized velocity vector $\vec{u}_0$, however, we have a convenient mathematical and physical separation in the right-hand side of (49) since then

$$\epsilon = 1 \ , \ F(\theta,t) = 1 \ , \ \alpha_1 = \alpha_2 = \theta, \tau_3 = \frac{\mu\sigma}{U^2} \ . \tag{51}$$

We also drop the distinction between left and right oriented current in (44). Its effect can always be traced by giving $\tau_2$ in (49) and further on a different sign. Choosing in (44) for the minus sign, we obtain for (49)

$$\frac{\partial \hat{\eta}}{\partial t} - (ik \sin t - \tau_1 - \tau_3 k^2)\, \hat{\eta} = -i\hat{h}k(\sin t + \tau_2 \cos t) \ . \tag{52}$$

It is easy to see that a solution of (52) can readily be transformed to one of the more general equation (49), so that from here on we pursue with the case of a circularly polarized "undisturbed" current velocity vector. We now solve for the non-transient solution of (52) in two steps.

### 3.1. *Quasi-Lagrangean solution*

Equation (52) suggests the following transformation:

$$\hat{\eta}(k,t) = \hat{\omega}(k,t)e^{-ik\cos t} \ . \tag{53}$$

Since this is equivalent to introducing a coordinate transformation

$$\vec{x}' = \vec{x} + \int \vec{u}_0 \, dt \tag{54}$$

in physical space (i.e. in eq. (40)) we call $\hat{\omega}(k,t)$ the quasi-Lagrangean (Fourier transformed) vorticity, as it describes the vorticity seen by an observer moving with the "undisturbed" flow. Then (52) reads:

$$\frac{d\hat{\omega}}{dt} + b(k)\,\hat{\omega} = a(k)\,e^{ik\cos t}\{\sin t + \tau_2 \cos t\} \ , \tag{55}$$

where

$$a(k) = -ik\hat{h} \ , \ b(k) = \tau_1 + \tau_3 k^2 \ . \tag{56}$$

Writing $\hat{\omega}^{(1)}$ for the vorticity driven by the term sin $t$ of the nonhomogeneous term in (55) and $\hat{\omega}^{(2)}$ for the term cos $t$, employing the Fourier series

$$e^{ik\cos t} = \sum_{n=-\infty}^{\infty} i^n J_n(k)e^{int} \ , \tag{57}$$

where $J_n$ is a Bessel function of the first kind of order $n$, the nontransient solutions for $\hat{\omega}^{(1)}$ and $\hat{\omega}^{(2)}$

read:

$$\hat{\omega}^{(1)} = a(k) \sum_{n=-\infty}^{\infty} \phi_n^{(1)}(k,b)e^{int} \quad , \tag{58}$$

$$\phi_n^{(1)}(k,b) = -\frac{i^n n}{b(k)+in} \frac{J_n(k)}{k} \quad , \tag{59}$$

$$\hat{\omega}^{(2)} = a(k)\tau_2 \sum_{n=-\infty}^{\infty} \phi_n^{(2)}(k,b)e^{int} \quad , \tag{60}$$

$$\phi_n^{(2)}(k,b) = \frac{i^{n-1}}{b(k)+in} J_n'(k) \; . \tag{61}$$

Note that the quasi - Lagrangean vorticity field all higher harmonics of the fundamental frequency are present. It can also be seen that for $n = 0$, the residual vorticity field $\hat{\omega}^{(1)}$ vanishes whereas $\hat{\omega}^{(2)}$ will be different from zero. Hence topographic stretching of planetary vortex lines does not produce a Lagrangean residual current up to this order in $\delta = \bar{h} / H$. Any Lagrangean residual field is due to differential bottom friction.

*3.2. Eulerian solution*

Combining (58) - (61) with (53) gives the Fourier transformed Eulerian vorticity field. After some rearranging of terms we have $\hat{\eta} = \hat{\eta}^{(1)} + \hat{\eta}^{(2)}$, where

$$\hat{\eta}^{(1)} = a(k) \sum_{n=-\infty}^{\infty} \psi_n^{(1)}(k,b)e^{-int} \quad , \tag{62}$$

$$\psi_n^{(1)}(k,b) = -\frac{i^{-n}}{k} \sum_{m=-\infty}^{\infty} \frac{m}{b(k)+im} J_m(k)J_{m+n}(k), \tag{63}$$

$$\hat{\eta}^{(2)} = a(k)\tau_2 \sum_{n=-\infty}^{\infty} \psi_n^{(2)}(k,b)e^{-int}, \tag{64}$$

$$\psi_n^{(2)}(k,b) = i^{-(n+1)} \sum_{m=-\infty}^{\infty} \frac{J_m'(k)J_{m+n}(k)}{b(k)+im}. \tag{65}$$

Note that in contrast to the quasi-Lagrangean solution (58)-(59), the Eulerian solution connected with topographic planetary vortex stretching, (62)-(63), now contains a nonvanishing residual component for $n=0$. This means that if we regard the quasi-Lagrangean residual solution as the Eulerian one plus a correction, connected with the "Stokes drift" (Zimmerman, 1979), the latter completely offsets the Eulerian residual field in the quasi-Lagrangean frame of reference.

In the next chapter we provide a further analysis of the solutions (58) - (61) and (62) - (65) by looking at the various asymptotic regimes of the transfer functions (59), (61), (63) and (65).

## 4. Asymptotic regimes

In this chapter we give a detailed description of the asymptotic behaviour of the transfer functions (59), (61), (63) and (65), for limiting values of the parameters $k$ and $b$. First we write (63) and (65) as an explicit sum in terms of Bessel functions. By using well-known properties of these functions the approximations are easily obtained.

It is convenient to consider the absolute values of $\psi_n^{(j)}$ of (63), (65). Using $J_{-n}(z) = (-1)^n J_n(z)$, $n = 0,1,\cdots$, we infer that $|\psi_n^{(j)}| = |\psi_{-n}^{(j)}|$, $j = 1,2$. Hence we introduce

$$G_n^{(j)}(k,b) = |\psi_{\pm n}^{(j)}(k,b)| \; , \; j = 1,2, \; n = 0,1,\cdots . \tag{66}$$

We suppose throughout that $k,b \in [0,\infty)$ with $b$ given in (56).

*4.1. Summing a series of Bessel functions*

To give an explicit summation for the series in (63), (65) it is useful to consider a basic series of the form

$$R_m(\alpha,k) = \sum_{n=-\infty}^{\infty} \frac{J_n(k)J_{n+m}(k)}{n+\alpha}, \tag{67}$$

where $m \in \mathbb{Z}, \alpha \notin \mathbb{Z}$.

We have the following result:

$$R_m(\alpha,k) = \frac{\pi}{\sin\alpha\pi} \cdot \begin{cases} J_\alpha(k)\, J_{m-\alpha}(k) & m \geqslant 0 \ , \\ (-1)^m J_{\alpha-m}(k) J_{-\alpha}(k) & m \leqslant 0 \ . \end{cases} \tag{68}$$

A proof of a more general result can be found in Newberger (1982), with a correction in Bakker & Temme (1984). A direct proof of (68) can be obtained by considering the integral

$$\frac{1}{2\pi i} \int_L \frac{J_{-\nu}(k)J_{m+\nu}(k)}{\sin(\pi\nu).(\nu+\alpha)}\, d\nu \ ,$$

with Im $\alpha \neq 0$ , $m \geqslant 0$, and $L = L_+ \cup L_-$ with

$$L_\pm = \{\nu = \mu \pm i\nu_0 | -\infty < \mu < \infty \ , \ 0 < \nu_0 < |Im \ \alpha|\}.$$

By shifting $L_+(L_-)$ upwards (downwards) we pass the pole at $\nu = -\alpha$. When $\nu_0 \rightarrow \infty$ the contributions from $L_\pm$ vanish, which follows from

$$J_\nu(z) \sim (\tfrac{1}{2}z)^\nu / \ \Gamma(\nu+1), \quad J_{-\nu}(z) \sim \frac{1}{\pi}(\tfrac{1}{2}z)^{-\nu}\Gamma(\nu) \ , \ \nu \rightarrow \infty, \ |arg\, \nu| < \pi \ , \ z \text{ fixed.}$$

So we are left with residues at $\nu = -\alpha$, $\nu = n$ , $n \in \mathbb{Z}$, and the result (68,$m \geqslant 0$) easily follows. The above restriction |Im $\alpha| \neq 0$ can be replaced by $\alpha \notin \mathbb{Z}$ (by using the principle of analytic continuation). The result for negative $m$-values in (68) follows from (67) by using

$$R_{-m}(\alpha,k) = \sum_{n=-\infty}^{\infty} \frac{J_{-n-m}(k)J_{-n}(k)}{-n+\alpha} = (-1)^{m+1} R_m(-\alpha,k),$$

where we used

$$J_{-n}(k) = (-1)^n J_n(k).$$

To express (63), (65) in terms of $R_m$ we write

$$\psi_n^{(1)}(k,b) = \frac{i^{-n}}{k} \sum_{m=-\infty}^{\infty} [1 + \frac{ib}{m-ib}] J_m(k) J_{m+n}(k) = \frac{i^{1-n}}{k} [\delta_{0,n} + ib \ R_n(-ib,k)],$$

where $\delta_{m,n}$ is Kronecker's symbol and where we used

$$\sum_{m=-\infty}^{\infty} J_m(k) J_{m+n}(k) = \delta_{0,n} \ ,$$

(Watson, 1944, p. 30).

By using $J_n'(k) = \frac{1}{2}[J_{n+1}(k) - J_{n-1}(k)]$, (65) can be written as

$$\psi_n^{(2)}(k,b) = -\tfrac{1}{2} i^{-n} [R_{n+1}(1-ib,k) - R_{n-1}(-1-ib,k)].$$

Expressing the functions $R_m$ in terms of Bessel functions, by means of (68), we arrive at the following results:

$$\psi_0^{(1)}(k,b) = \frac{i}{k}[1-\frac{\pi b}{\sinh(\pi b)}\, J_{ib}(k)\, J_{-ib}(k)] \,,$$
$$\psi_n^{(1)}(k,b) = \frac{-i^{1-n}}{k}\, \frac{\pi b}{\sinh(\pi b)}\, J_{n+ib}(k)\, J_{-ib}(k),\;\; n \geqslant 1 \,,$$
$$\psi_0^{(2)}(k,b) = \frac{\pi}{2i\sinh(\pi b)}\, \frac{d}{dk}\, [J_{ib}(k)\, J_{-ib}(k)] \qquad (69)$$
$$= \frac{i\pi}{\sinh(\pi b)}\, \mathrm{Re}[J_{ib}(k)\, J_{1-ib}(k)] \,,$$
$$\psi_n^{(2)}(k,b) = \frac{\pi i^{-n-1}}{\sinh(\pi b)}\, J_{n+ib}(k)\, J'_{-ib}(k),\; n \geqslant 1 \,.$$

So we obtain for the functions defined in (66) :

$$G_0^{(1)}(k,b) = \frac{1}{k}|1-\frac{\pi b}{\sinh(\pi b)} J_{ib}(k)\, J_{-ib}(k)| \,,$$
$$G_0^{(2)}(k,b) = \frac{\pi}{\sinh(\pi b)}|\, \mathrm{Re}\{J_{1-ib}(k)\, J_{ib}(k)\}| \,, \qquad (70)$$

and for

$$n = 1,2,\cdots :$$

$$G_n^{(1)}(k,b) = \frac{\pi b}{k\sinh(\pi b)}\, |\, J_{ib}(k)\, J'_{n+ib}(k)\, | \,,$$
$$G_n^{(2)}(k,b) = \frac{\pi}{\sinh(\pi b)}\, |\, J'_{ib}(k)\, J_{n+ib}(k)| \,. \qquad (71)$$

*4.2. Asymptotic expansions for $k \to 0$*

This case is rather elementary. By using the well-known series expansion of the Bessel-function $J_\nu(z)$, viz.

$$J_\nu(z) = (\tfrac{1}{2}z)^\nu \sum_{n=0}^{\infty} \frac{(-\frac{1}{4}z^2)^n}{n!\Gamma(n+\nu+1)} \,, \qquad (72)$$

the following results are easily obtained. We recall that $b$ in (56) is defined by

$$b(k) = \tau_1 + \tau_3 k^2. \qquad (73)$$

For the functions introduced in (59), (61) we have

$$|\phi_{\pm n}^{(j)}(k,b)| = (\tfrac{1}{2}k)^{n-1}\, \frac{n}{n!}\, \frac{1}{\sqrt{n^2+\tau_1^2}}\, [1+\mathcal{O}(k^2)], \quad n = 1,2,\cdots, j = 1,2. \qquad (74)$$

The function $\phi_0^{(1)}$ vanishes identically and for $\phi_0^{(2)}$ we have

$$|\phi_0^{(2)}(k,b)| = \frac{k}{2\tau_1}[1+\mathcal{O}(k^2)]. \qquad (75)$$

For the functions defined in (66) we have by using (70), (71) and (72):

$$G_0^{(1)}(k,b) = \frac{\frac{1}{2}k}{1+\tau_1^2}\, [1+\mathcal{O}(k^2)] \,,\; G_0^{(2)}(k,b) = \frac{\frac{1}{2}k}{\tau_1(1+\tau_1^{2)}}\, [1+\mathcal{O}(k^2)] \,, \qquad (76)$$

and for $j = 1,2$ , $n = 1,2,3,\cdots$ , we have

$$G_n^{(j)}(k,b) = \tfrac{1}{2}(\tfrac{1}{2}k)^{n-1}[(1+\tau_1^2)\cdots(n^2+\tau_1^2)]^{-\frac{1}{2}}[1+\mathcal{O}(k^2)]\,. \tag{77}$$

Evidently this implies for small $k$ - i.e. a large topographic length scale compared to the tidal excursion - the residual vorticity is proportional to the first derivative of the bottom slope.

### *4.3. Asymptotic expansions for $k \to \infty$.*

It is necessary to distinquish between the cases $\tau_3 > 0$ and $\tau_3 = 0$. When $\tau_3 > 0$ the order of the Bessel functions $J_{n+ib}(k)$, etc., appearing in (70), (71), (72) is large as well, see (73). Since the ratio order/argument is large, (72) can be used if $\tau_3 > 0$; if $\tau_3 = 0$ the well-known asymptotic expansions of the Bessel functions should be used. The latter case is somewhat more complicated the first one.

#### *4.3.1.* $\tau_3 > 0$

For a first approximation we replace $b$ of (73) by $\tau_3 k^2$. The following results follow from (76), (77) just by replacing $\tau_1$ by $\tau_3 k^2$ and $\tau_1^2+n^2$ by $\tau_3^2 k^4$:

$$\begin{aligned} G_0^{(1)}(k,b) &= \tfrac{1}{2}\tau_3^{-2}k^{-3}[1+\mathcal{O}(k^{-2})]\ , \\ G_0^{(2)}(k,b) &= \tfrac{1}{2}\tau_3^{-3}k^{-5}[1+\mathcal{O}(k^{-2})]\ , \\ G_n^{(j)}(k,b) &= 2^{-n}\tau_3^{-n}\,k^{-n-1}[1+\mathcal{O}(k^{-2})]\ ,\quad n \geqslant 1,\ j=1,2. \end{aligned} \tag{78}$$

For the functions of (59), (61) we need

$$J_\nu(z) = [2/(\pi z)]^{\frac{1}{2}}\ \{P(\nu,z)\cos\chi - Q(\nu,z)\sin\chi\}\ , \tag{79}$$

where

$$\chi = z-(\tfrac{1}{2}\nu+\tfrac{1}{4})\pi\,,\quad P(\nu,z) = 1+\mathcal{O}(z^{-2})\,,\quad Q(\nu,z) = \mathcal{O}(z^{-1})\ \text{as}\ z\to\infty.$$

Hence we obtain

$$\begin{aligned} \phi_n^{(1)}(k,b) &= \frac{n}{\tau_3 k^3}\ \mathcal{O}(k^{-\frac{1}{2}})\,, \\ \phi_n^{(2)}(k,b) &= \frac{1}{\tau_3 k^2}\ \mathcal{O}(k^{-\frac{1}{2}})\ . \end{aligned} \tag{80}$$

#### *4.3.2.* $\tau_3 = 0$

By using (79) with $b = \tau_1$, we have

$$J_{n+ib}(k)\,J_{-ib}(k) \sim \frac{1}{\pi k}\ [\cos(\tfrac{1}{2}\pi n + b\pi i) + \sin(2k - \tfrac{1}{2}\pi n)]\,.$$

It follows that

$$\begin{aligned} G_0^{(1)}(k,b) &\sim \frac{1}{k}[1-\frac{b\,\{\cosh(\pi b)+\sin(2k)\}}{k\sinh(\pi b)}\ ], \\ G_0^{(2)}(k,b) &\sim |\frac{\cos(2k)}{k\pi}|\,, \end{aligned} \tag{81}$$

and

$$G_1^{(1)}(k,b) \sim \frac{b}{k^2\sinh(\pi b)}\ [\cos(2k) + \sinh^2(\pi b)]^{\frac{1}{2}}\ ,$$

$$G_1^{(2)}(k,b) \sim \frac{1}{k\sinh(\pi b)}\,[\cosh(\pi b) - \sin(2k)]\,, \tag{82}$$

$$G_2^{(1)}(k,b) \sim \frac{b}{k^2\sinh(\pi b)}\,[\cosh(\pi b) + \sin(2k)]\,,$$

$$G_2^{(2)}(k,b) \sim \frac{1}{k\sinh(\pi b)}\,[\cos^2(2k) + \sinh^2(\pi b)]^{\frac{1}{2}}.$$

For the $\phi_n^{(j)}-$ functions we have similar results as in (80), with $\tau_3 k^2$ replaced by $\tau_1$. Comparing both results for the $\phi-$functions or comparing (78) and (81) , we see that the vorticity response functions tend to zero, irrespective of lateral viscosity being introduced or not ($\tau_3 > 0$ or $\tau_3 = 0$), but of course more strongly so if lateral viscosity is present ($\tau_3 > 0$).

*4.4. Asymptotic expansion for $\tau_1 \to 0,\infty$ ($\tau_3 = 0$).*
In this case we keep $k$ fixed; for $b$ of (73) we have $b = \tau_1 = r/(H\sigma), \tau_2 = r/(Hf)$, see (48), and we consider $b$, or $\tau_1$, as an asymptotic parameter.

*4.4.1. $b \to 0$*
From the representations (70), (71) we obtain

$$G_0^{(1)}(k,b) = \frac{1}{k}[1 - J_0^2(k)] + \mathcal{O}(b)\,,$$

$$G_0^{(2)}(k,b) = \frac{1}{b}|J_0(k)J_1(k)| + \mathcal{O}(1)\,, \tag{83}$$

$$G_n^{(1)}(k,b) = \frac{1}{k}|J_0(k)J_n(k)| + \mathcal{O}(b)\,,$$

$$G_n^{(2)}(k,b) = \frac{1}{b}|J_0'(k)\,J_n(k)| + \mathcal{O}(1)\,.$$

The results for $\phi_n^{(j)}$ are readily obtained from their definitions (59), (60):

$$|\phi_0^{(2)}(k,b)| = \frac{1}{b}\ |J_0'(k)| + \mathcal{O}(1)\,,$$

$$|\phi_n^{(1)}(k,b)| = \frac{1}{k}\ \ |J_n(k)| + \mathcal{O}(b)\,, \tag{84}$$

$$|\phi_n^{(2)}(k,b)| = \frac{1}{n}\ \ |J_n'(k)| + \mathcal{O}(b)\,.$$

Evidently the $G_n^{(1)}$ functions have a nonzero finite value for the bottom friction going to zero. In fact the same applies to the vorticity described by the $G_n^{(2)}$ functions, as can be seen in (64) where $\tau_2$ multiplies $1/b$ in (83). As both $b = \tau_1$ and $\tau_2$ are proportional to the bottom friction parameter $r$ (see (48)), the final result for $\hat{\eta}^{(2)}$ remains finite (and nonzero).

*4.4.2. $b \to \infty$ .*
As in section 4.2 the starting point is (72). For fixed $k$ it gives the asymptotic expansion of $J_{ib}(k)$, etc. The results of section 4.2 can be copied just by stressing the role of $\tau_1 = b$ as a large parameter. We have

$$G_0^{(1)}(k,b) = \tfrac{1}{2}kb^{-2}\,[1 + \mathcal{O}(b^{-2})]\,,$$

$$G_0^{(2)}(k,b) = \tfrac{1}{2}kb^{-3}\,[1 + \mathcal{O}(b^{-2})]\,, \tag{85}$$

$$G_n^{(j)}(k,b) = \tfrac{1}{2}(k/2)^{n-1}b^{-n}[1+\mathcal{O}(b^{-2})]\,,\; j = 1,2,\; n = 1,2,\cdots .$$

Hence, as could be expected, for increasing bottom friction all vorticity response functions tend to zero, irrespective of whether they represent vorticity production by vortex stretching, the $G_n^{(1)'}s$, or by differential bottom friction, the $G_n^{(2)'}s$.

### 4.5. *Asymptotic regimes, primitive perturbation series and harmonic truncation*

The asymptotic analysis of the preceeding section clearly shows that the response functions (63) and (65) have a maximum for $k = \mathcal{O}(1)$. We now compare these results for the exact solution with approximate solutions of (52).

#### 4.5.1. *Primitive perturbations series*

Suppose that one regards the (linearized) advection term, $-ik\ \hat{\eta}\ \sin t$, in the left-hand side of (52) as a perturbation as one would do in the case of a full nonlinear problem. In that case it is easily seen that to zeroth order the nontransient solution of

$$\frac{\partial \hat{\eta}_0}{\partial t} + b(k)\hat{\eta}_0 = a(k)\{\ \sin t\ +\ \tau_2 \cos t\ \} \tag{86}$$

reads

$$\hat{\eta}_0^{(1)} = \frac{a(k)}{1+b^2(k)}\ \{b(k)\sin t\ -\ \cos t\}\ , \tag{87}$$

$$\hat{\eta}_0^{(2)} = \frac{\tau_2 a(k)}{1+b^2(k)}\{\sin t\ +\ b\ \cos t\}\ , \tag{88}$$

whereas to first order the solution of

$$\frac{\partial \hat{\eta}_1^{(1)}}{\partial t} + b(k)\hat{\eta}_1^{(1)} = \frac{ika(k)}{1+b^2(k)}\ \{b(k)\sin t\ -\ \cos t\} \tag{89}$$

and

$$\frac{\partial \hat{\eta}_1^{(2)}}{\partial t} + b(k)\hat{\eta}_1^{(2)} = \frac{ik\,\tau_2 a(k)}{1+b^2(k)}\left\{\sin t + b(k)\cos t\right\} \tag{90}$$

reads:

$$\hat{\eta}_1^{(1)} = \frac{ika(k)}{1+b^2(k)}\ [\tfrac{1}{2}+\tfrac{1}{2}(b^2(k)+2)\ \cos 2t\ -(\tfrac{1}{2}b(k)+1)\sin 2t]\ , \tag{91}$$

$$\hat{\eta}_1^{(2)} = \frac{\tau_2 ika(k)}{1+b^2(k)}[\tfrac{1}{2}+\tfrac{3}{2}(b(k)\ \cos 2t\ +(\tfrac{1}{2}b^2(k)-1)\sin 2t]\ . \tag{92}$$

If now $b^2(k) =$ constant , i.e. $\tau_3 = 0$, then for $k \rightarrow 0$ the asymptotic regime for $\hat{\eta}_1^{(1,2)}$ is the same as for (62) - (65). However for $k \rightarrow \infty$ we now have

$$\frac{\eta_1^{(1,2)}}{a(k)} \sim k\ , \tag{93}$$

up to a multiplicative constant, in contrast to the asymptotic regime of (63) and (65) for $n = 0$, for which we found in (81), $\sim k^{-1}$. Evidently the approach of a primitive perturbation series leads to a continuous increase in response for $k \rightarrow \infty$, in contrast to the exact result which locates the optimum response at $k = \mathcal{O}(1)$.

*4.5.2. Harmonic truncation*

Another approach to solving (52), used for instance by Loder (1980), is to write the solution for $\hat{\eta}$ first as a harmonic series:

$$\hat{\eta} = \hat{\eta}_0 + \hat{\eta}_1 e^{it} + \hat{\eta}_2 e^{2it} + \ldots\ldots, \tag{94}$$

and then to truncate the series at same order, in which case (52) leads to a set of $n$ coupled linear equations that can subsequently be solved. As one is usually interested in the residual vorticity $\hat{\eta}_0$ in (94), the lowest non-trivial truncation is

$$\hat{\eta} = \hat{\eta}_0 + \hat{\eta}_1 e^{it} \ . \tag{95}$$

Substituting this in (52), neglecting terms proportional to $e^{2it}$, one then finds for the residual vorticity, again subdivided in $\hat{\eta}_0^{(1)}$ and $\hat{\eta}_0^{(2)}$:

$$\hat{\eta}_0^{(1)} = \tfrac{1}{2} \frac{ika(k)}{1+b^2(k)+\tfrac{1}{2}k^2} = \hat{\eta}_0^{(2)} / \tau_2 \ . \tag{96}$$

As can be seen, taking $b(k)=const$, now both for $k \rightarrow 0$ and $k \rightarrow \infty$ the asymptotic behaviour of $\hat{\eta}_0^{(1,2)} / a(k)$ is the same as for (63) and (65), albeit with the absence of oscillatory behaviour. This means that at first sight harmonic truncation, producing the right response peak for $k = \mathcal{O}(1)$, is a much better approximation to the exact result than a primitive perturbation series. This is particularly encouraging for situations where an exact result cannot be obtained, for instance in the case of finite amplitude topography (Loder, 1980). However we shall show in the next chapter, on the Fourier back-transformation, that the asymptotic agreement between (96) and (76), (81) is still no guarantee that for specific bottom profiles there is agreement between the shapes of the residual velocity field after back-transformation.

## 5. Fourier back-transformation for a step profile

Having obtained the exact solution of (52) for $\hat{\eta}(k,t)$ we obtain the vorticity field $\eta(x,t)$ in physical space by applying (41), assuming $\hat{h}(k)$ to be known. We shall discuss the back-transformation for a specific one-dimensional profile that approaches the Heaviside step function if a parameter approaches infinity. The relevance of such a profile is evident from the work of Loder (1980) on residual currents at a shelf break. Let $h(x)$ be given as

$$h(x) = h_* \arctan \lambda x \ , \tag{97}$$

where $h_*$ is the dimensionless step height. If $\lambda \rightarrow \infty$, $h(x)$ becomes the step function. Evidently

$$\frac{\partial h}{\partial x} = \frac{\lambda h_*}{\lambda^2 x^2 + 1} \ , \tag{98}$$

so that

$$a(k) = -ik\hat{h}(k) = \tfrac{1}{2} h_* e^{-k/\lambda} \ . \tag{99}$$

From hereon our interest is mainly in the shape of the *residual* velocity profile $v(x)$ over the step, rather than the residual vorticity. Dropping the 0 subscripts, the relationship between $v$ and $\eta$ is simply

$$\eta = \frac{\partial v}{\partial x} \ , \tag{100}$$

hence

$$\hat{\eta}(k) = -ik\hat{v}(k) \ . \tag{101}$$

Since

$$\eta^{(j)}(k) = a(k)\psi_0^{(j)}(k) \ , \tag{102}$$

we have

$$v^{(j)}(k) = ia(k)k^{-1}\psi_0^{(j)}(k) \ , \tag{103}$$

where the superscript has the same meaning as in (62), (64). Thus after back-transformation the residual velocity profile of the current along the isobaths is:

$$v^{(j)}(x) = i\int_{-\infty}^{\infty} a(k)k^{-1}\psi_0^{(j)}(k)e^{ikx}dk \tag{104}$$

$$= i\int_{-\infty}^{\infty} h'(\xi)F^{(j)}(x-\xi)d\xi \ , \tag{105}$$

where

$$F^{(j)}(x) = \int_{-\infty}^{\infty} k^{-1}\,\psi_0^{(j)}(k)e^{ikx}dk \ . \tag{106}$$

Zimmerman (1981) discussed two limits of (105) for $\lambda \to 0$ and $\lambda \to \infty$. It can easily be seen that for $\lambda \to 0$ (up to a multiplication constant)

$$v^{(j)}(x) \sim \frac{\lambda h_*}{\lambda^2 x^2 + 1} \ , \tag{107}$$

which follows from (105) and the asymptotic behaviour of $k^{-1}\psi_0^{(i\to j)}(k)$ for $k \to 0$. In this limit, which physically means that the tidal excursion is much smaller than the stepwidth, the residual velocity profile is simply the mirror image of the bottom slope. More interesting is the other extremum, $\lambda \to \infty$, which means a step-width much smaller than the tidal excursion. In that case

$$\frac{\partial h}{\partial x} \to \pi h_* \delta(x) \ , \ a(k) \to \tfrac{1}{2}h_* \ , \tag{108}$$

hence

$$v^{(j)}(x) = i\pi h_* F^{(j)}(x) \ . \tag{109}$$

Therefore, in this case, we have to deal with the Fourier back-transform defined by (106). This is the subject of the next sections.

### 5.1. *Quasi-Langrangean residual velocity profile*

As in chapter 3 we solve for the Eulerian residual velocity field after first having obtained the quasi-Lagrangean residual velocity profile at the step. Analogously to (100) - (103), using (58) - (61), and taking the limit $\lambda \to \infty$ the quasi Lagrangean residual velocity denoted by $w(x)$ is given by:

$$w^{(j)}(x) = \frac{i}{2}h_* \int_{-\infty}^{\infty} k^{-1}\phi_0^{(j)}(k)e^{ikx}dk \ . \tag{110}$$

As from (59), $\phi_0^{(1)} = 0$, we are left with

$$w^{(2)}(x) = -\tfrac{1}{2}h_*\tau_2 \int_{-\infty}^{\infty} \frac{J_0'(k)}{kb(k)}\, e^{ikx}dk \ , \tag{111}$$

which for $b(k) = const\,(\tau_3 = 0)$ reads

$$w^{(2)}(x) = h_* \tau_2 b^{-1} \int_{-\infty}^{\infty} J_1(k)\, k^{-1} e^{ikx} dk \quad ,$$

or

$$w^{(2)}(x) = \begin{cases} 2h_* \tau_2 b^{-1} \sqrt{1-x^2}\ , & |x| \leqslant 1\ , \\ 0 & |x| > 1\ . \end{cases} \tag{112}$$

Evidently by excluding vorticity diffusion ($\tau_3 = 0$) the quasi Langrangean residual velocity field has a finite width which for a step topography is exactly equal to the tidal excursion. We shall see that this structure is also present in the Eulerian residual velocity field. For $\tau_3 \neq 0$ vorticity diffusion evenly spreads the velocity profile, which for $|x| > 1$ can then be shown to read

$$w^{(2)}(x) = \pi h_* \frac{\tau_2}{\tau_1} e^{-|x|\sqrt{\tau_1/\tau_3}} I_1(\sqrt{\tau_1/\tau_3}), \quad x \geqslant 1\ ,$$

where $I_1(z)$ is the modified Bessel function of the first kind, order 1. This result follows from residue calculus by considering the integral

$$\int_C \frac{J_1(z) e^{izx}}{z(\tau_1 + \tau_3 z^2)} dz \quad ,$$

where $C$ is composed of the interval $[-R,R]$ and the semi-circle $Re^{i\theta}(0 \leqslant \theta \leqslant \pi)$, with $R$ large. If $x > 1$ the function $e^{izx} J_1(z)$ is exponentially small in the upper half plane. So the contribution to the integral due to the semi-circle will vanish when $R \to \infty$. For $x = 1$ the proof is also valid and for $x < -1$ we need a semi-circle in the lower half plane.

If $|x| < 1$ we are not able to express $w^{(2)}(x)$ in terms of known special functions. For fixed $x \in (-1,1)$ we have for small values of $\tau_3$ the expansion

$$w^{(2)}(x) = 2h_* \frac{\tau_2}{\tau_1} \sqrt{1-x^2}\, [1 - \frac{\tau_3}{\tau_1} (1-x^2)^{-2} + \mathcal{O}(\tau_3^2)].$$

This formally follows from (111) by expanding $1/b$ in powers of $\tau_3$. It can be justified by a more detailed analysis, which will not be given here.

### *5.2. Eulerian residual velocity field at a step*

In the next subsections we evaluate the Fourier transforms of $\psi_n^{(j)}(k,b)/k$ for the special case $\tau_3 = 0$, i.e., $b(k) = \tau_1 = constant$. We only consider $n = 0$. The functions

$$F^{(j)}(x) = \frac{1}{i} \int_{-\infty}^{\infty} \psi_0^{(j)}(k,b)\, k^{-1} e^{ikx}\, dk\ , \quad j = 1,2, \tag{113}$$

are expressed in terms of Legendre functions. We also give a representation in terms of hypergeometric functions. First we will show that the functions in (113) vanish identically outside the $x$-interval $(-2,2)$.

To prove this consider the integral

$$\int_{C_r} \psi_0^{(j)}(z,b) z^{-1} e^{izx}\, dz \quad ,$$

where $C_r$ consists of the following parts: $C_r = C_r^{(1)} \cup C_r^{(2)}$, with

$$C_r^{(1)} = \{z \,|\, -r \leqslant z \leqslant r\}\ ,\ C_r^{(2)} = \{z \,|\, z = re^{i\theta}, 0 \leqslant \theta \leqslant \pi\},$$

and where $r$ is a positive real number. Now we use (81) with $k$ replaced by the complex number $z \in C_r^{(2)}(r$ large $)$ . It follows that the function $e^{izx}$ dominates $|\psi_0^{(j)}(z,b)|$ in the upper half plane

Im$z>0$ when $x>2$. For $x<-2$ we can use a contour in the lower half plane. So, if $x>2$, the $z$-integral over $C_r$ (which vanishes identically since there is no singularity in the finite domain with boundary $C_r$) equals the integral over $C_r^{(1)}$ in the limit $r\to\infty$. Also if $x = \pm 2$ this is true, due to

$$\psi_0^{(j)}(z,b)\, z^{-1}\, e^{izx} = \mathcal{O}(z^{-2}), \quad z\in C_r^{(2)}.$$

So we obtain

$$F^{(j)}(x) = 0 \quad , \; x\leqslant -2 \text{ or } x\geqslant 2\,, \tag{114}$$

having the same physical explanation as for (112).

In the remaining subsections we consider (113) for $x\in(-2,2)$.

### 5.2.1. *Elaboration of $F^{(2)}(x)$*

Since $\psi_0^{(j)}(k,b)/k$ are even functions of $k$, the Fourier transform (110) is a cosine transform. Using (69) we obtain

$$\frac{d}{dx}F^{(2)}(x) = -\frac{\pi x}{\sinh(\pi b)}\int_0^\infty J_{ib}(k)J_{-ib}(k)\cos kx\; dk.$$

From Gradsthein & Ryzhik (1965, p. 732) we have

$$\frac{d}{dx}\,F^{(2)}(x) = -\frac{\pi x}{2\sinh(\pi b)}\,P_{ib-\frac{1}{2}}\,(\tfrac{1}{2}x^2-1).$$

Using the well-known relation

$$\int_1^z P_\nu(t)dt = -(1-z^2)^{\frac{1}{2}}\,P_\nu^{-1}(z), \quad z\in(-1,1),$$

we can integrate the equation for $dF^{(2)}(x)/dx$ into

$$F^{(2)}(x) = \frac{\pi}{2\sinh(\pi b)}\,[x^2-\tfrac{1}{4}x^4]^{\frac{1}{2}}\,P_{ib-\frac{1}{2}}^{-1}(\tfrac{1}{2}x^2-1).$$

The Legendre function is associated with the so-called conical functions. It is real, although one of the parameters is complex. For numerical computations it is convenient to have a representation in the form of a series. This can be obtained by writing the Legendre function as a hypergeometric function (Gradshtein & Ryzhik, 1965, p. 999) :

$$F^{(2)}(x) = \frac{\pi(1-\frac{1}{4}x^2)}{\sinh(\pi b)}\;{}_2F_1(\tfrac{1}{2}-ib,\tfrac{1}{2}+ib;2;1-\tfrac{1}{4}x^2).$$

With the familiar expansion of the ${}_2F_1$-function we can write this as

$$F^{(2)}(x) = \frac{\pi(1-\frac{1}{4}x^2)}{\sinh(\pi b)}\sum_{n=0}^{\infty}\frac{(\frac{1}{2}-ib)_n(\frac{1}{2}+ib)_n}{(2)_n\;\; n!}(1-\tfrac{1}{4}x^2)^n, \tag{115}$$

where $(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)$, $n = 0,1,\cdots$. This series is useful for values of $x$ near $\pm 2$, but it converges for all $x\in[-2,2]$. For $x = 0$ we have by using

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b)>0 \quad ,$$

the value of $F^{(2)}(0)$, i.e.,

$$F^{(2)}(0) = \frac{4\coth(\pi b)}{4b^2+1} \quad . \tag{116}$$

From (115) we infer that $\lim_{|x|\uparrow 2} F^{(2)}(x) = 0$, so $F^{(2)}(x)$ is continuous at $x = \pm 2$.

For $x \to 0$ we can use a transformation for the hypergeometric function (Abramowitz & Stegun, 1964, p. 559, formula 15.3.11), which gives in our case:

$$F^{(2)}(x) = F^{(2)}(0) + \tfrac{1}{4}x^2\coth(\pi b) \sum_{n=0}^{\infty} \frac{(\frac{3}{2}+ib)_n(\frac{3}{2}-ib)_n}{n!(n+1)!}(\tfrac{1}{4}x^2)^n \times \tag{117}$$

$$\times [\ln\tfrac{1}{4}x^2 - \psi(n+1) - \psi(n+2) + \psi(\tfrac{3}{2}+n+ib) + \psi(\tfrac{3}{2}+n-ib)].$$

If $\psi(\frac{3}{2}+ib) + \psi(\frac{3}{2}-ib)$ is available, then the remaining terms are easily computed by recursion.

Representations (115) and (117) are convenient starting points for numerical evaluation of $F^{(2)}(x)$. The graph of the function (for $b = 1$) is shown in figure 5.1.

*5.2.2. Elaboration of $F^{(1)}(x)$*

The function $F^{(1)}(x)$ defined in (113) cannot be expressed in terms of a single Legendre function or a Gauss hypergeometric function, as was possible in the previous case. We express $F^{(1)}(x)$ in terms of an integral of $F^{(2)}(x)$, from which representation numerical evaluation is easily performed.

By using the relations in (69) we obtain

$$x\frac{d^2}{dx^2}F^{(1)}(x) = -2b\frac{d}{dx}F^{(2)}(x).$$

Hence

$$-2bF^{(2)}(x) = x\frac{d}{dx}F^{(1)}(x) - F^{(1)}(x) + F^{(1)}(0) - 2bF^{(2)}(0), \tag{118}$$

where we use $\lim_{x\to 0^+} x\frac{d}{dx}F^{(1)}(x) = 0$; note that $F^{(j)}(x)$ is even in $x$.

The value $F^{(2)}(0)$ is given in (116). The value of $F^{(1)}(0)$ follows from (113), (69), giving

$$F^{(1)}(0) = 4\sum_{m=1}^{\infty}\frac{m^2}{b^2+m^2}\int_0^{\infty} k^{-2}J_m^2(k)\,dk.$$

The integrals are special cases of the Weber-Schafheitlin integrals. From Gradshteyn & Ryhzik (1965, p. 692) we obtain

$$F^{(1)}(0) = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{m^2}{b^2+m^2}\frac{1}{m^2-\frac{1}{4}} = \frac{8b\coth(\pi b)}{1+4b^2}.$$

It follows that the constant $F^{(1)}(0) - 2bF^{(2)}(0)$ in (118) equals zero, so we are left with the equation

$$\frac{d}{dx}\frac{F^{(1)}(x)}{x} = -\frac{2b}{x^2}F^{(2)}(x). \tag{119}$$

Integration gives

$$F^{(1)}(x) = 2bx\int_x^2 \xi^{-2}F^{(2)}(\xi)d\xi, \tag{120}$$

which is the desired relation between $F^{(1)}(x)$ and $F^{(2)}(x)$. If we integrate (119) with initial value $x = 0$, we first write it as

$$\frac{d}{dx}\frac{F^{(1)}(x) - F^{(1)}(0)}{x} = -2\frac{b}{x^2}[F^{(2)}(x) - F^{(2)}(0)],$$

which gives as a final result

$$F^{(1)}(x) = F^{(1)}(0) + 2bx \int_0^x \xi^{-2}[F^{(2)}(0) - F^{(2)}(\xi)]\, d\xi + cx,$$

where $c$ follows from $F^{(1)}(2) = 0$. A graph of $F^{(1)}(x)$ is shown in figure 5.2 ($b = 1$).

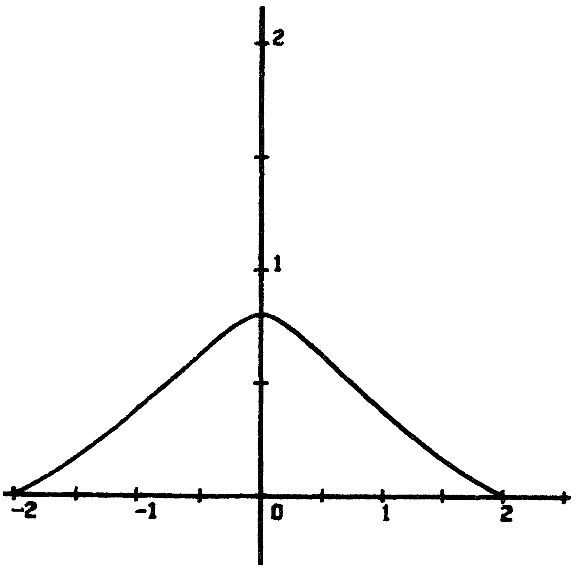


Fig. 5.1. Graph of $F^2(x)$

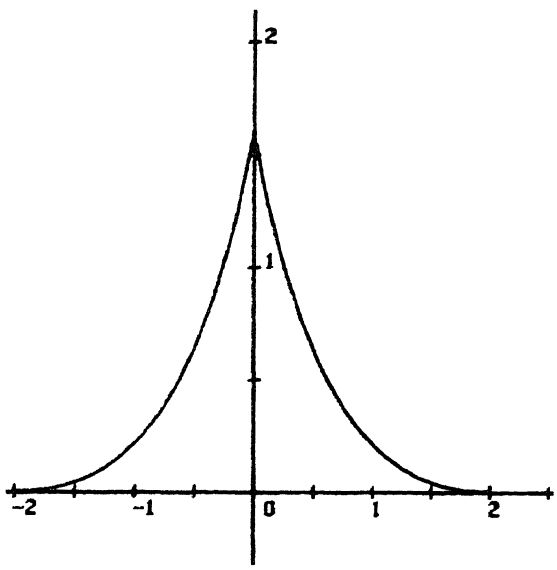


Fig. 5.2. Graph of $F^{(1)}(x)$

*5.3. Harmonic truncation and back-transformation*

Using (97) the residual velocity profile in the limit $\lambda \to \infty$ for the approximate solution of (52) by harmonic truncation as discussed in chapter 4.5.2. reads:

$$v^{(j)}(x) = \frac{\pi h_*}{2} \int_{-\infty}^{\infty} \frac{e^{ikx}}{1 + b^2 + \frac{1}{2}k^2}\, dk = \pi h_* \delta^{\frac{1}{2}} e^{-|x|/\Delta^{\frac{1}{2}}},$$

where

$$\Delta = \frac{1}{(1 + b^2)}\ .$$

It is interesting to compare this result with (114). The exact residual velocity profile (for $\tau_3 = 0$) vanishes for $|x| > 2$, whereas the result obtained by harmonic truncation gives a residual velocity field that vanishes at infinity. Evidently the fact that $\hat{\eta}^{(j)}$ has the right asymptotic behaviour if harmonic truncation is applied is no guarantee for a sound residual velocity profile after back-transformation. In fact the method of harmonic truncation induces a spurious diffusive effect in its approximate solution.

## 6. Conclusions

We have shown that by an appropriate quasi linearization of the vorticity equation, the exact shape of the residual current velocity profile at a step like bottom slope can be obtained in the limit of small amplitude topography. The profile vanishes outside a region given by twice the tidal excursion, centered at the step, if lateral vorticity diffusion is excluded. This result is contrary to the approximate results obtained by the method of harmonic truncation in which case the residual velocity profile falls off exponentially at both sides of the step (Loder, 1980). It is shown here that these differences in behaviour can be reduced to the different forms of the vorticity response functions in wave number space in both cases. Whereas these are simple rational functions if harmonic truncation is used, these are shown here to be intricate sums of products of Bessel functions. Even though the asymptotic behaviour of both kind of response functions is similar, our results demonstrate that in calculating the shape of the residual current velocity profile for a specific bottom topography one has to use the exact response functions in terms of Bessel function products if one wishes to avoid a spurious spreading out of the velocity profile in physical space.

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