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P.J. van der Houwen

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SPATIAL DISCRETIZATION OF HYPERBOLIC EQUATIONS WITH PERIODIC SOLUTIONS

P.J. van der Houwen
Centre for Mathematics and Computer Science
Amsterdam

We investigate the Cauchy problem for hyperbolic equations for which the frequencies of the main Fourier components in the solution are located in a given frequency interval. Difference formulas for the spatial derivatives are constructed that are tuned to the given intervals of frequencies. Numerical examples illustrating these special discretizations are given both for linear and nonlinear problems.

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1. Finite difference approximation

We consider the space discretization by finite differences in partial differential equations of the form

$$\partial w/\partial t = A\partial w/\partial x_1 + B\partial w/\partial x_2 + g, \quad x := (x_1,x_2)^T \in \Omega \tag{1.1}$$

where A and B are symmetric matrices, and g is a vector function; A, B and g may depend on (t,x,w). In particular we will study symmetric discretizations: let the spatial mesh sizes and the discretization weights be denoted by

$$\Delta x_1, \quad \Delta x_2, \quad \epsilon_j^{(l)}, \quad \eta_j^{(l)},$$

respectively, and define the shift operators

$$E_j f(x) = f(x+\Delta x_j), \quad j = 1, 2,$$

then we approximate the spatial derivatives by the formulas

$$\begin{aligned} \partial/\partial x_1 \sim D_1 &:= \frac{1}{2\Delta x_1} \sum_{l=0}^{k} \sum_{j=1}^{k} \epsilon_j^{(l)} (E_1^{+j} - E_1^{-j})(E_2^{+l} + E_2^{-l}) \\ \partial/\partial x_2 \sim D_2 &:= \frac{1}{2\Delta x_2} \sum_{l=0}^{k} \sum_{j=1}^{k} \eta_j^{(l)} (E_2^{+j} - E_2^{-j})(E_1^{+l} + E_1^{-l}) \end{aligned}, \tag{1.2}$$

The discretization weights determine the accuracy of these difference formulas. In the following the summation indices j and l are assumed to range from 1 until k and from 0 until k, respectively.

The difference operators (1.2) can be expressed as

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$$D_1 = X_1(\Delta x_1 \partial/\partial x_1, \Delta x_2 \partial/\partial x_2)\partial/\partial x_1, \quad X_1(u,v) := 2\sum_{j,1}^{k} \epsilon_j^{(1)} \frac{\sinh(ju)}{u} \cosh(lv)$$

(1.3)

$$D_2 = X_2(\Delta x_1 \partial/\partial x_1, \Delta x_2 \partial/\partial x_2)\partial/\partial x_2, \quad X_2(u,v) := 2\sum_{j,1}^{k} \eta_j^{(1)} \frac{\sinh(jv)}{v} \cosh(lu)$$

We will always assume that

$$X_1(0,0) = 1 + O(\Delta^2 x), \quad X_2(0,0) = 1 + O(\Delta^2 x), \quad \Delta x := \max\{\Delta x_1, \Delta x_2\},$$

or, equivalently,

$$\sum_{j,1}^{k} j\epsilon_j^{(1)} = 1/2 + O(\Delta^2 x), \quad \sum_{j,1}^{k} j\eta_j^{(1)} = 1/2 + O(\Delta^2 x). \tag{1.4}$$

These conditions ensure that the difference formulas are second order accurate.

2. The truncation error

Let W(t,x) be a function satisfying the equation

$$L(W) := \partial W/\partial t - AD_1 W - BD_2 W - g = 0$$

$$x \in \Omega_\Delta := \{x: x = (j\Delta x_1, l\Delta x_2),\ j,l = 0, \pm 1, \pm 2, \ldots\} \tag{2.1}$$

Here, A, B and g are evaluated at (t,x,W).

Suppose that it is known that the solution of (1.1) is approximately of the form

$$w_\theta := \sum_{r=1}^{R} w_\theta^{(r)}(t) \exp(if^{(r)}.x), \quad f^{(r)} := (f_1^{(r)}, f_2^{(r)})^T. \tag{2.2}$$

then, in order to exploit this extra information, we proceed as follows:

(i) The discretization weights are determined in such a way that the function (2.2) satisfies (2.1) with the lowest possible error.

(ii) The equation (2.1) is solved by a high order ODE integrator.

The resulting numerical solution method possesses a small truncation error provided that the exact solution is dominated by the function (2.2).

The requirement (i) leads us to consider the truncation error left when (2.2) is substituted into (2.1), i.e. the quantity

$$L(w_\theta) = \partial w/\partial t - i\sum_{r=1}^{R} [A(t,x,w_\theta)\delta_1^{(r)} + B(t,x,w_\theta)\delta_2^{(r)}]w_\theta^{(r)} e^{if^{(r)}.x} - g(t,x,w_\theta), \quad x \in \Omega_\Delta, \tag{2.3}$$

where

$$i\delta_j^{(r)} = f_j^{(r)} X_j(if_1^{(r)}\Delta x_1, if_2^{(r)}\Delta x_2), \quad j=1,2. \tag{2.4}$$

are the eigenvalues of the difference operators defined in (1.2). Since (2.2) is supposed to be dominating in the solution of (1.1), we may write

$$g(t,x,w_\theta) \sim \partial w_\theta/\partial t - i\sum_{r=1}^{R} [A(t,x,w_\theta)f_1^{(r)} + B(t,x,w_\theta)f_2^{(r)}]w_\theta^{(r)} e^{if^{(r)}.x}, \tag{2.5}$$

so that, on substitution into (2.3), we obtain the approximation

$$\begin{aligned} L(w_\theta) &\sim i\sum_{r=1}^{R} \Phi^{(r)}(t,x,w_\theta)\, w_\theta^{(r)} e^{if^{(r)}.x} \\ \Phi^{(r)}(t,x,w_\theta) &:= [A(t,x,w_\theta)(f_1^{(r)}-\delta_1^{(r)}) + B(t,x,w_\theta)(f_2^{(r)}-\delta_2^{(r)})] \end{aligned} \tag{2.6}$$

Using (2.4) we may write

$$\Phi^{(r)} := f_1^{(r)} a(\mu^{(r)})A + f_2^{(r)} b(\mu^{(r)})B, \tag{2.7}$$

$$\begin{aligned} \mu^{(r)} &:= (\mu_1,\mu_2) := (f_1^{(r)}\Delta x_1, f_2^{(r)}\Delta x_2) \\ a(\mu) &:= 1 - X_1(i\mu_1, i\mu_2) \\ b(\mu) &:= 1 - X_2(i\mu_1, i\mu_2) \end{aligned} \tag{2.8}$$

showing that the truncation error can be reduced in magnitude by minimizing the functions a and b over the region of dominant frequencies.

Alternatively, if the dominant frequencies are known with sufficient accuracy, one may choose the zeros of a and b at the points corresponding to these dominant frequencies.

We shall investigate the general case of **two-dimensional** waves where the frequency vectors are arbitrary, and the special case of **one-dimensional** waves where the frequency vectors are all in the same direction.

3. Two-dimensional waves

3.1 Four-point line discretizations

Let

$$k = 2, \quad \epsilon_j^{(1)} = \eta_j^{(1)} = 0 \quad \text{for } 1 \neq 0,$$

then the coefficient a in (2.7) assumes the form

$$(3.1) \qquad a(\mu) = \bar{a}(\mu_1) := 1 - 2\,\frac{\sin(\mu_1)}{\mu_1}\,[\epsilon_1 + 2\epsilon_2\cos(\mu_1)],$$

where the index 1 in the discretization weights is omitted. A similar expression can be derived for the coefficient b.

Suppose that the dominant frequencies are located in the frequency interval

$$\underline{f}_1 \leqslant f_1^{(r)} \leqslant \overline{f}_1,$$

then we want to minimize the function $|a(z)|$ in the interval

$$\underline{\mu}_1 := \underline{f}_1 \Delta x_1 \leqslant z \leqslant \overline{f}_1 \Delta x_1 =: \overline{\mu}_1,$$

If $\bar{a}(z)$ would be a polynomial (and since $\bar{a}(z)$ is even it would be a polynomial in z^2), then the minimax problem indicated above is solved by identifying $a(z)$ with the shifted Chebyshev polynomial

$$S_{2q}(z) := cT_q\left(\frac{2z^2 - \overline{\mu}_1^2 - \underline{\mu}_1^2}{\overline{\mu}_1^2 - \underline{\mu}_1^2}\right), \quad c \text{ constant}.$$

This suggests the identification of the zeros of $\bar{a}(z)$ with those of such a polynomial. Since there are only two free parameters available in (3.1), we set q=2 to obtain the two (double) zeros

$$(3.2a) \qquad z_\pm := \Delta x_1\left[\tfrac{1}{2}(\overline{f}_1^2 + \underline{f}_1^2) \pm \tfrac{1}{4}\sqrt{2}(\overline{f}_1^2 - \underline{f}_1^2)\right]^{1/2},$$

and we determine the free discretization weights by requiring

$$(3.2b) \qquad \bar{a}(z_+) = 0, \quad \bar{a}(z_-) = 0.$$

These equations are solved by

$$(3.3)\qquad \epsilon_2 = \frac{\frac{z_+}{\sin(z_+)} - \frac{z_-}{\sin(z_-)}}{4[\cos(z_+) - \cos(z_-)]}, \quad \epsilon_1 = \frac{z_+}{2\sin(z_+)} - 2\epsilon_2\cos(z_+).$$

We remark that the case of line discretizations using more than four points can be treated in the same way. Furthermore, in the special case where we know that the solution is dominated by two frequencies, suppose given by the end points of the frequency interval mentioned above, then we should choose in (3.3):

$$z_+ = \underline{f}_1\Delta x_1, \quad z_- = \overline{f}_1\Delta x_1.$$

We shall derive approximations to the discretization weights and the coefficient a for small values of the mesh parameter. A straightforward calculation yields

$$(3.4)\qquad \epsilon_2 = 1/12 - (\overline{f}_1^2 + \underline{f}_1^2)\Delta^2 x_1/60 + O(\Delta^4 x_1), \quad \epsilon_1 = 1/2 - 2\epsilon_2 + O(\Delta^4 x_1).$$

These expressions are easily verified to satisfy the order condition (1.4). Since the limiting values of the discretization weights correspond to the conventional fourth-order discretization, it can easily be shown that (3.3) also defines a fourth-order discretization of the spatial derivatives (cf. Theorem 3.1 below).

Next we consider the order of the extreme value of the coefficient a in the interval of dominant frequencies. By virtue of (3.4) we find

$$\begin{aligned}(3.5)\qquad \max_{[\underline{\mu}_1,\bar{\mu}_2]} |\bar{a}(z)| &= \bar{a}(\bar{\mu}_1) = \bar{a}(\bar{\mu}_1) - \bar{a}(z_+)\\ &= 2\epsilon_1\left(\frac{\sin(z_+)}{z_+} - \frac{\sin(\bar{\mu}_1)}{\bar{\mu}_1}\right) + 4\epsilon_2\left(\frac{\sin(2z_+)}{2z_+} - \frac{\sin(2\bar{\mu}_1)}{2\bar{\mu}_1}\right)\\ &= \frac{1}{3}(\bar{\mu}_1^2 - z_+^2)(\epsilon_1+8\epsilon_2) - \frac{1}{60}(\bar{\mu}_1^4 - z_+^4)(\epsilon_1+32\epsilon_2) + O(\Delta^6 x_1)\\ &= \frac{1}{240}(\overline{f}_1^2 - \underline{f}_1^2)^2\Delta^4 x_1 + O(\Delta^6 x_1).\end{aligned}$$

Thus, we arrive at the following result:

Theorem 3.1. Let

$$(f_1^{(r)}, f_2^{(r)}) \in [\underline{f}_1, \bar{f}_1] * [\underline{f}_2, \bar{f}_2],$$

and let the discretization weights be defined by (3.3). Then the truncation error corresponding to (2.2) is given by

$$(3.6) \qquad L(w) = \frac{i}{240} \sum_{r=1}^{R} [c_a f_1^{(r)} \Delta^4 x_1 A + c_b f_2^{(r)} \Delta^4 x_2 B] w_0^{(r)} \exp(i f^{(r)}.x) + O(\Delta^6 x),$$

where the order constants satisfy the inequalities

$$|c_a| \leq (\underline{f}_1^2 - \bar{f}_1^2)^2 , \quad |c_b| \leq (\underline{f}_2^2 - \bar{f}_2^2)^2 .$$

In the conventional case that arises for

$$(3.7) \qquad z_+ = z_- = 0 \quad \Rightarrow \quad \epsilon_1 = 2/3, \quad \epsilon_2 = -1/12,$$

a similar theorem holds, however the order constants now satisfy the inequalities

$$(3.8) \qquad |c_a| \leq 8\bar{f}_1^4, \quad |c_b| \leq 8\bar{f}_2^4,$$

showing the considerable gain factor when the discretization (3.3) is used.

Example 3.1. Suppose that the exact solution consists of R Fourier components with frequencies

$$f_j^{(r)} = r f_j \quad , \; j = 1, 2; \quad r = 1, \ldots, R.$$

Putting

$$\underline{f}_j = f_j, \quad \bar{f}_j = R f_j,$$

the order constants in (3.6) satisfy

$$|c_a| < (R^2 - 1)^2 f_1^4, \qquad |c_b| < (R^2 - 1)^2 f_2^4,$$

whereas in the conventional case (3.7), we deduce from (3.8) the inequalities

$$|c_a| < 8R^4 f_1^4, \qquad |c_b| < 8R^4 f_2^4.$$

These inequalities indicate that

$$\text{Trunc. error (3.3)} = \frac{(R^2 - 1)^2}{8R^4} \text{ trunc. error (3.7).}$$

This relation shows that the gain factor varies from 14 for R=2 to 8 for R large.

Example 3.2. If the mesh sizes are not small then Theorem 3.1 does not apply and we have to consider the function a(z) defined by (3.1) itself and not its asymptotic expansion (3.5). In Table 3.1 we list the maximum value of |a(z)| in a number of intervals $[0,\bar{\mu}_1]$ for the discretizations (3.3) and (3.7):

Table 3.1. Maximum values of the coefficient function a(z)

$[0,\bar{\mu}_1]$	(3.1)&(3.7)	(3.1)&(3.3)	(3.5)&(3.3)
[0,.1]	.0000033	.00000042	.00000042
[0,.2]	.000053	.0000067	.0000067
[0,.3]	.00027	.000034	.000034
[0,.4]	.00084	.00011	.00011
[0,.5]	.0020	.00027	.00026
[0,.6]	.0041	.00057	.00054
[0,.7]	.0075	.0011	.0010
[0,.8]	.013	.0019	.0017
[0,.9]	.020	.0030	.0027
[0,1.0]	.030	.0048	.0042

These figures show that the approximation (3.5) is rather accurate also for larger mesh sizes.

3.2. Other discretizations

One may wonder whether it is advantageous to use all direct neighbours of a particular grid point in the definition of the difference operators (1.2), rather then extending the discretization molecule in one direction as we did in the preceding subsection. We shall show that this is not the case.

Let k=1 in (1.2), then the coefficient a in the truncation error of the corresponding six-point discretization is given by

$$(3.9) \qquad a(\mu) = 1 - 2\epsilon_1^{(0)} \frac{\sin(\mu_1)}{\mu_1} - 2\epsilon_1^{(1)} \frac{\sin(\mu_1)}{\mu_1} \cos(\mu_2).$$

For sufficiently small frequency intervals this function is monotonic in the region

$$M = [\underline{\mu}_1, \bar{\mu}_2] * [\underline{\mu}_1, \bar{\mu}_2],$$

so that its extreme value on this region is assumed at one of the corner points. From this it follows that the extreme value of a is minimized if

$$a(\underline{\mu}_1, \underline{\mu}_2) = -a(\bar{\mu}_1, \underline{\mu}_2), \quad a(\underline{\mu}_1, \bar{\mu}_2) = -a(\bar{\mu}_1, \bar{\mu}_2),$$

which results into the discretization weights

$$(3.10) \qquad \epsilon_1^{(0)} = \left[\frac{\sin(\underline{\mu}_1)}{\underline{\mu}_1} + \frac{\sin(\bar{\mu}_1)}{\bar{\mu}_1} \right]^{-1}, \quad \epsilon_1^{(1)} = 0.$$

Thus, the optimal six-point dicretization is in fact a two-point discretization, so that, apparently, there is no advantage in using 'non-line' molecules.

The maximal value of a in the region M satisfies

$$(3.11) \qquad \max_M |a(\mu)| = \left[\frac{\sin(\bar{\mu}_1)}{\bar{\mu}_1} - \frac{\sin(\underline{\mu}_1)}{\underline{\mu}_1}\right] / \left[\frac{\sin(\bar{\mu}_1)}{\bar{\mu}_1} + \frac{\sin(\underline{\mu}_1)}{\underline{\mu}_1}\right]$$

$$= (\bar{f}_1^2 - \underline{f}_1^2)\Delta^2 x_1 \left[1 + (\bar{f}_1^2 + \underline{f}_1^2)\Delta^2 x_1 / 30\right] / 12 + O(\Delta^6 x_1),$$

whereas the conventional two-point dicretization yields

$$(3.12) \qquad \max_M |a(\mu)| = 1 - \frac{\sin(\bar{\mu}_1)}{\bar{\mu}_1} = \bar{f}_1^2 \Delta^2 x_1 \left[1 - \bar{f}_1^2 \Delta^2 x_1 / 20\right] / 6 + O(\Delta^6 x_1).$$

4. One-dimensional waves

Suppose that the one-dimensional solution components in (2.2) represent waves all traveling in the same direction in the x-plane, i.e.

$$(4.1) \qquad f_2^{(r)} = c f_1^{(r)}, \quad c = \text{constant}, \quad r = 1, \ldots, R.$$

Then, by choosing

$$(4.2) \qquad \Delta x_2 = \Delta x_1 / c, \quad \eta_j^{(1)} = \epsilon_j^{(1)},$$

it follows from (2.8) that

$$(4.3) \qquad a(\mu) = b(\mu) = \bar{a}(\mu_1) := 1 - 2 \sum_{j=1}^{k} \sum_{l=0}^{k} \epsilon_j^{(l)} \frac{\sin(j\mu_1)}{\mu_1} \cos(l\mu_1),$$

so that, in (2.6)

$$(4.4) \qquad \Phi^{(r)} = a(\mu_1^{(r)}) [f_1^{(r)} A + f_2^{(r)} B].$$

Thus, there is only one function that needs to be minimized on an interval of frequencies. A related minimax problem was considered in [6]; according to the analysis given there it can be shown that not all parameters in (4.3) are independent, free parameters. In fact, there are only 2k degrees of freedom. We shall now consider a few special cases.

First we consider the four-point line discretization molecule of Section 3.1. It is easily verified that (4.3) is identical to (3.1) so that the optimal discretization is also given by (3.3).

Secondly, consider the general k=1 formula, leading to six-point discretizations of the derivatives. The corresponding coefficient function a is given by

$$(4.5) \qquad \bar{a}(\mu_1) = 1 - 2 \frac{\sin(\mu_1)}{\mu_1} [\epsilon_1^{(0)} + \epsilon_1^{(1)} \cos(\mu_1)].$$

Writing

$$\epsilon_1^{(0)} = \epsilon_1, \quad \epsilon_1^{(1)} = 2\epsilon_2$$

we see that this function is identical to (3.1), so that, again, all considerations of Section 3.1 apply.

Finally, we consider the 16-point molecules which arise for

$$k=2, \quad \epsilon_2^{(2)} = 0.$$

The function a now assumes the form

$$\bar{a}(\mu_1) = 1 - 2\,\frac{\sin(\mu_1)}{\mu_1}\,\left[\epsilon_1 + \epsilon_2\cos(\mu_1) + \epsilon_3\cos^2(\mu_1)\right]$$

(4.6)

$$\epsilon_1 := \epsilon_1^{(0)} - \epsilon_1^{(2)}, \quad \epsilon_2 := 2\epsilon_2^{(0)} + \epsilon_1^{(1)}, \quad \epsilon_3 := 2(\epsilon_2^{(1)} + \epsilon_1^{(2)})$$

showing that (only) three degrees of freedom are available for minimizing a. Proceeding as in Section 3.1, this minimax problem can approximately be solved by identifying the zeros of a with those of a shifted Chebyshev polynomial. For details we refer to [6] where a similar minimax problem is discussed.

5. Numerical experiments

We confine our experiments to the integration of a scalar, one-dimensional equation of the form

$$\partial w/\partial t = A(t,x,w)\partial w/\partial x + g(t,x,w), \quad 0 \leqslant t \leqslant T, \quad 0 \leqslant x \leqslant 2\pi \tag{5.1}$$

with periodic boundary conditions. The functions A and g, the endpoint T, and the initial condition are specified in the tables of results.

The spatial derivative is approximated by (1.3) and (3.3); the zeros of the coefficient function a(z) will equal the two frequencies we want to eliminate. These frequencies are also be specified in the tables of results. By using the fourth-order standard Runge-Kutta method with a relatively small step for the time integration we achieve that the space error is dominating. Choosing

$$\Delta x = 2\pi\Delta t, \tag{5.3}$$

the time discretization error was insignificant in all experiments presented here.

The accuracy obtained is measured by

$$cd(\Delta t,T) := -\log(\text{maximal absolute error at } t=T), \tag{5.4}$$

i.e., by the minimal number of correct digits at the endpoint. In order to interpret the various results we observe that

$$\text{computational effort} = \frac{\text{constant}}{\Delta^2 t} \sim \text{constant}.10^{cd/2} \text{ as } \Delta t \to 0. \tag{5.5}$$

By means of this relation we can predict the computational effort corresponding to given numbers of correct digits.

In the following subsections we describe experiments with and without known solution.

5.1. Problems with known solutions

The problem specified in Table 5.1 has the exact solution

$$w = \sin(t+x) + \cos(t+x)/2, \tag{5.6}$$

and may be considered as a linear model problem; it is of the form (2.2) with R=2 and with frequencies 1 and 2. By eliminating both frequencies from the truncation error (case (1,2)), we obtain an accuracy of more than four correct digits, whereas the conventional method (case (0,0)) produces less than a half correct digit. By eliminating only one frequency the accuracy is hardly better than in the conventional case.

Table 5.1. A=1, g=0, w=sin(x)+cos(2x)/2

Eliminated frequencies	(0,0)	(1,1)	(1,2)	(2,2)
Correct digits: cd(1/10,5)	0.45	0.67	4.16	0.47

Table 5.2. A=1, g=0, w=sin(x)+cos(2x)/2+sin(3x)/30

Eliminated frequencies	(0,0)	(1,2)	(1,3)	(2,3)
Correct digits:cd(1/10,5)	0.45	1.19	0.27	-0.09
Correct digits:cd(1/17,5)	1.19	1.93		

Now we add a third Fourier component to the solution (5.6):

(5.7) $$w = \sin(t+x) + \cos(t+x)/2 + \sin(3t+3x)/30.$$

This function represents the exact solution of the problem defined in Table 5.2. The elimination of the frequencies 1 and 2 has now considerably less effect than in the preceding case, because the truncation error contains a Fourier component with frequency 3. However we still save a lot of computation time. To see this we applied the conventional method with a smaller time step in order to get the same accuracy as the elimination method with step 1/10. The step 1/17 yielded the desired accuracy, but requiring almost three times as much computation time (we note that the asymptotic relation (5.5) predicts the optimistic factor 2.34 because the step is relatively large).

The Tables 5.3 and 5.4 give two further examples. The exact solutions of these problems are respectively

(5.8) $$\sin(\sin(t+x)),$$
(5.9) $$\tan(\sin(t+x)).$$

In both cases we have a Fourier expansion with more than two components. Apparently, the first three components are dominating as can be concluded from these tables (we remark that by eliminating the first and third frequency the second frequency is strongly damped, cf. Theorem 3.1).

Table 5.3. A=1, g=0, w=sin(sin(x))

Eliminated frequencies	(0,0)	(1,1)	(1,2)	(1,3)
Correct digits: cd(1/10,5)	1.07	1.12	1.13	3.99

Table 5.4. A=1, g=0, w=tan(sin(x))

Eliminated frequencies	(0,0)	(1,1)	(1,2)	(1,3)
Correct digits: cd(1/10,5)	0.61	0.53	0.53	2.50

Sofar we only considered linear problems. In Table 5.5 results for a **nonlinear model problem** are presented of which the exact solution is given by (5.6). We used this nonlinear model to illustrate the sensitivity of the method to deviations in estimating the frequency of the solution. The results show that the method is rather sensitive indeed: a 5% deviation from the true frequency causes a loss of accuracy of almost 3 digits. This is reminiscent to a similar phenomenon displayed by the Gautschi modification of linear multistep methods and by the exponential fitting techniques, introduced by Liniger and Willoughby [8], for integrating ODEs with periodic solutions of known frequency [4,9]; here too, an accurate estimate of the frequency is crucial. Nevertheless, when compared with the conventional method (the case (0,0)), the elimination method is considerably more efficient, even when the estimated frequency is 10% incorrect. Moreover, the above results concern a nonlinear problem showing that the discretization (3.3) is of value in nonlinear problems too.

Table 5.5. A=w*w, g=(1-w*w)[cos(t+x)-sin(2t+2x)], w=sin(x)+cos(2x)/2

Eliminated frequencies	(0,0)	(1.1,2.2)	(1.05,2.1)	(1.2)
Correct digits: cd(1/8,1)	.76	1.57	1.87	4.56

In the next experiment we show the sensitivity of the method to the presence of a third Fourier component in the solution. The exact solution of the problem specified in Table 5.6 is given by (cf. (5.7))

(5.10) $$w = \sin(t+x) + \cos(2t+2x)/2 + c\,\sin(3t+3x).$$

As we should expect, the conventional method hardly notices the introduction of the third Fourier component, whereas the efficiency of the elimination method decreases rapidly.

Table 5.6. A=w*w, g=(1-w*w)[cos(t+x)-sin(2t+2x)+ccos(3t+3x)], w=sin(x)+cos(2x)/2+ccos(3x)/3

Eliminated frequencies	(0,0)	(1,2)
c=0, cd(1/8,1)	.76	4.56
c=1/1000, cd(1/8,1)	.76	3.10
c=1/100, cd(1/8,1)	.75	2.10
c=1/10, cd(1/8,1)	.65	1.09

Finally, we consider the nonlinear problem given in Table 5.7 of which the exact solution is defined by (5.8). A comparison with the results obtained for the linear problem of Table 5.3 possessing the same exact solution, indicates that the elimination method is more effective for linear problems.

Table 5.7. A=w*w, g=(1-w*w)cos(sin(t+x))cos(t+x), w=sin(sin(x))

Eliminated frequencies	(0,0)	(1,1)	(1,2)	(1,3)
Correct digits: cd(1/10,5)	1.58	1.61	1.71	3.30

5.2. Problems without known solution

We start with a simple linear problem (see Table 5.8) of which the approximate solution (obtained by numerical integration with a relatively small integration step) is given by

$$
\begin{aligned}
&w(0) = -w(\pi) = 1.30116844\\
&w(\pi/4) = -w(5\pi/4) = 1.89712646\\
&w(\pi/2) = -w(3\pi/2) = 1.38177352\\
&w(3\pi/4) = -w(7\pi/4) = 0.05699639
\end{aligned}
\qquad (5.11)
$$

The results in Table 5.8 indicate that this solution is mainly determined by the first two Fourier components.

Table 5.8. A=1, g=sin(x), w=sin(x)

Eliminated frequencies	(0,0)	(0,1)	(0,2)	(1,2)
Correct digits: cd(1/8,1)	1.76	4.45	1.13	5.58

Finally, we consider the nonlinear problem in Table 5.9 whose exact solution is approximately given by

$$
\begin{aligned}
&w(0) = .84211776, \quad w(\pi) = -.84178511\\
&w(\pi/4) = .97632179, \quad w(5\pi/4) = -.97711709\\
&w(\pi/2) = .54085051, \quad w(3\pi/2) = -.54014878\\
&w(3\pi/4) = -.21271215, \quad w(7\pi/4) = .21220369
\end{aligned}
\qquad (5.12)
$$

The results show that this solution mainly consists of one Fourier component.

Table 5.9. A=w, g=cos(x+t)-sin(2(t+x))/2+sin(3(t+x))/1000

Eliminated frequencies	(0,0)	(1,1)	(1,2)	(1,3)
Correct digits: cd(1/8,1)	2.11	3.30	3.29	3.21

6. Concluding remarks

The experiments in the preceding section with the four-point line dicretization (3.3) clearly show that a considerable saving of computing time can be obtained provided that the solution consists of only a few dominating Fourier components with known frequency. The increased efficiency is particularly apparent in linear problems, but is still substantial in nonlinear problems.

Since the discretization (3.3) is one-dimensional, it can be used in problems with an arbitrary number of spatial dimensions (this is also true for the discretization (3.9), but not in the case of the two-dimensional discretizations based on (4.5) and (4.6)).

For the sake of completeness we remark that in the case of periodic behaviour in time it is also possible to improve the accuracy by employing special time discretizations. For second-order ODEs this has been studied in, e.g, [1], [2], [3], [4], [5], [7] and [10], and can be used for the time integration of semidiscrete, second-order hyperbolic problems with periodic solutions. Similar techniques can be used for the time integration of semidiscrete, first-order hyperbolic problems of the form (2.1) (cf. [7]).

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