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Recursive Constructions of Mutually Orthogonal Latin Squares

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0. Introductory definitions

Let S be fixed set (of 'symbols') of cardinality n. We say that two $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ with entries in S are *orthogonal* if for all $(s,t) \in S \times S$ there is a unique position (i,j) such that $a_{ij} = s$ and $b_{ij} = t$. In this chapter we shall be interested in constructing large sets of pairwise orthogonal matrices.

The connection with Latin squares is as follows: given a set $A^{(k)}$ $(k=1,...,r)$ of pairwise orthogonal matrices we may (after permuting the n^2 positions) assume that $A^{(1)}$ has constant rows and $A^{(2)}$ has constant columns. Now each $A^{(k)}$ $(k=3,...,r)$ is a Latin square, and we have found $r-2$ mutually orthogonal Latin squares.

Conversely, given $r-2$ mutually orthogonal Latin squares we can add two orthogonal matrices, one with constant rows and one with constant columns and get a set of r pairwise orthogonal matrices.

Clearly, for the concept of orthogonality the matrix structure does not play a rôle, that is, we might as well talk about orthogonal vectors of length n^2. If we define an *orthogonal array* $OA(n,r)$ of order n and depth r to be an $r \times n^2$ matrix over S such that any two rows are orthogonal, then an $OA(n,r)$ is equivalent to a set of r pairwise orthogonal matrices of order n. (Similarly one might consider orthogonal arrays *of strength* t, that is $r \times n^t$ matrices A over S such that for any t rows $i_1,...,i_t$ and any t symbols $s_1,...,s_t$ there is a unique column j such that $A_{i_k j} = s_k (1 \leq k \leq t)$. Unless the contary is explicitly mentioned, all our orthogonal arrays will have strength 2.)

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A more geometric picture is obtained by regarding the columns of an orthogonal array as the lines of a geometry. If R is the set of rows of the orthogonal array A, then take as point set the set $R \times S$, and let the line L_j corresponding to the $j-$th column be the set $L_j = \{(i,A_{ij}) \mid i \in R\}$. What we get is called a *transversal design.*

1. Pairwise balanced designs - definitions

A *pairwise balanced design* is a set X (of *points*) together with a set $\mathfrak{B}$ of subsets of X (called *blocks*) such that for some integer λ each 2-subset $\{x,y\}$ of X is contained in precisely λ blocks. The number λ is called the *index* of the design, and unless specified otherwise we shall always assume $\lambda = 1$. (In particular we need not worry about the possibility of repeated blocks.) When $\lambda = 1$ two blocks have at most one point in common, and the blocks are also called *lines* and the design a *linear space* (not to be confused with the linear spaces from linear algebra).

More generally, a *partial linear space* is a set X (of *points*) together with a set $\mathfrak{L}$ of subsets of X (called *lines*) such that two points are joined by at most one line.

A set of blocks $\mathcal{C}$ of a pairwise balanced design is called a *clear set* if the elements of $\mathcal{C}$ are pairwise disjoint. It is called a *parallel class* if $\mathcal{C}$ is a partition of X.

If ∞ is an element not in X then there is a natural $1-1$ correspondence between pairwise balanced designs (of index unity) on $X \cup \{\infty\}$ and pairwise balanced designs with designated parallel class $\mathcal{C}$ on X : if the latter has block set $\mathfrak{B}$ then the former has block set $(\mathfrak{B} \setminus \mathcal{C}) \cup \mathcal{C}^*$ where $\mathcal{C}^* = \{C \cup \{\infty\} \mid C \in \mathcal{C}\}$

If a pairwise balanced design $(X,\mathfrak{B})$ has several pairwise disjoint parallel classes $\mathcal{C}_j$ $(1 \leq j \leq k)$ then one obtains a new pairwise balanced design by "adding points at infinity" : find k new points $\infty_1,...,\infty_k$ and set $\overline{X} = X \cup \{\infty_1,...,\infty_k\}$, $\overline{\mathfrak{B}} = (\mathfrak{B} \setminus \bigcup_{j=1}^{k} \mathcal{C}_j) \cup \bigcup_{j=1}^{k} \mathcal{C}_j^* \cup \{\{\infty_1,...,\infty_k\}\}$, where $\mathcal{C}_j^* = \{C \cup \{\infty_j\} \mid C \in \mathcal{C}_j\}$.
(Example: construct the projective plane from the affine plane by adding 'a line at infinity').
A pairwise balanced design $(X,\mathfrak{B})$ is called *resolvable* if $\mathfrak{B}$ can be partitioned into parallel classes.

A *group divisible design* is a set X (of *points*), a partition $\mathcal{G}$ of X (the elements of which are called *groups*) and a collection $\mathfrak{B}$ of subsets of X (the *blocks*) such that $(X, \mathfrak{B} \cup \mathcal{G})$ is a pairwise balanced design of index unity. (In other words, we have a pairwise balanced design with designated parallel class, and decide to call the elements of this parallel class groups instead of blocks. There should be no confusion with the algebraic concept of group). (Many other concepts of group deivisible design exist. A fairly standard definition says that a group divisible design with indices λ_1 and λ_2 is a set X, a partition $\mathcal{G}$ of X and a collection $\mathcal{B}$ of blocks such that if $x,y \in X$ are two points in the same group $G \in \mathcal{G}$ then $\{x,y\}$ is in λ_1 blocks, otherwise $\{x,y\}$ is in λ_2 blocks. HANANI [H 1975a] takes $\lambda_1 = 0, \lambda_2 = \lambda$. We take $\lambda_1 = 0, \lambda_2 = 1$. For the general concept and, more generally, for partially balanced incomplete block designs see RAGHAVARAO [RA].)

A pairwise balanced design with blocks of size k on $|X| = v$ points is called a $B(k;v)$. A group divisible design with blocks of size k and groups of size m on v points is called a $GD(k,m\ ;\ v)$. A *transversal design* $TD(r;n)$ is a group divisible design $GD(r,n;rn)$.
If several blocks sizes may occur, we write $B(K;v)$ when each occurring block size is a member of K, and similarly $GD(K,M;v)$.
(For a study of transversal designs with index $\lambda > 1$ see HANANI [H 1975]).

2. SIMPLE CONSTRUCTIONS FOR TRANSVERSAL DESIGNS

As we have seen, the concepts of set of $(r-2)$ mutually orthogonal Latin squares and orthogonal array (of depth r) and transversal design (with r groups) are equivalent. We shall mostly use the language of transversal designs. Let $TD(r)$ be the set of all n such that a $TD(r;n)$ exists.

A. $0,1 \in TD(r)$ *for all* r. *If* $1 < n \in TD(r)$ *then* $n \geq r-1$, *and* $r-1 \in TD(r)$ *if and only if there exists a projective plane of order* $r-1$.

PROOF A $TD(r;0)$ is a design with no points and no blocks. A $TD(r;1)$ is a design with r points, all in unique block. If $n > 1$ and $r \geq 2$ then let B be a fixed block, G a fixed group and x a point not in $B \cup G$. The $r-1$ blocks on x meeting B meet G in distinct points, so $n = |G| \geq r-1$. If $n = r-1$ then any two bloccks meet, and adding a point at infinity to the parallel class formed by the groups produces a projective plane with lines of size r.
Conversely, given a projective plane with lines of size r, removal of a point yields a $TD(r;r-1)$. □

B. [MACNEISH, BUSH] *If* $m,n \in TD(r)$ *then* $mn \in TD(r)$.

PROOF Given r pairwise orthogonal matrices $A^{(i)}$ over a symbol set S and r pairwise orthogonal matrices $B^{(i)}$ over a symbol set T $(1 \leq i \leq r)$, the r matrices $A^{(i)} \times B^{(i)}$ over the symbol set $S \times T$ will be pairwise orthogonal. □

B1. COROLLARY *Let* $n = \prod_i p_i^{e_i}$ *be the factorization of* n *into prime powers. Then* $n \in TD(r)$ *for* $r = \min_i (p_i^{e_i} + 1)$.

PROOF If q is a prime power then there exists a projective plane of order q and hence $q \in TD(r)$ for $r \leqslant q+1$. □

C. [PARKER, BOSE & SHRIKHANDE] Let $(X, \mathfrak{B})$ be a pairwise balanced design such that for each $B \in \mathfrak{B}$ we have $|B| \in TD(r+1)$. Then $|X| \in TD(r)$.

PROOF If R is an r-set then construct a transversal design with point set $R \times X$ and groups $\{y\} \times X$ for $y \in R$ with subdesigns $TD(r;|B|)$ on $R \times B$ for each $B \in \mathfrak{B}$, taking care that each of these subdesigns contains the blocks $R \times \{b\}$ for $b \in B$. This will yield the required design. The $TD(r;|B|)$ with parallel class that we need are obtained from the given $TD(r+1;|B|)$ by throwing away one group and taking as parallel class the blocks that used to contain a fixed point of this thrown-away group. □

This construction can be strengthened in many ways. First of all one can weaken the hypothesis "$|B| \in TD(r+1)$" to "there exists a $TD(r;|B|)$ with a parallel class", and strengthen the conclusion to "there exists a $TD(r;|X|)$ with a parallel class".

Let us call a transversal design $TD(r;n)$ with e pairwise disjoint parallel classes a $TD_e(r;n)$. Adding points at infinity shows that a $TD_n(r;n)$ exists if and only if a $TD(r+1;n)$ exists.

A second variation on Theorem C requires $|B| \in TD_1(r)$ for all $B \in \mathfrak{B} \setminus \mathcal{C}$ where $\mathcal{C}$ is a clear set of blocks, and $|B| \in TD(r)$ for $B \in \mathcal{C}$. (Now the conclusion is $|X| \in TD(r)$). If even suffices to ask that $\mathcal{C}$ be an *almost clear* set, that is, that for each $C \in \mathcal{C}$ there is at most one $x \in C$ such that x is member of more than one block of $\mathcal{C}$.

A third version is the following.

D. [PARKER, BOSE & SHRIKHANDE] Let $(X, \mathfrak{B})$ be a pairwise balanced design such that $\mathfrak{B}$ has a partition $\{\mathfrak{B}_j\}_j$, where each family $\mathfrak{B}_j$ has blocks of constant size k_j and is either a partition of X or a symmetric 1-design on X. (Such a design is called *separable*).
Assume that $|B| \in TD(r)$ for each $B \in \mathfrak{B}$. Then $|X| \in TD(r)$.

PROOF By the previous we have $|X| \in TD(r-1)$. We shall show that a $TD(r-1;n)$ (where $n=|X|$) can be constructed so as to possess n pairwise disjoint parallel classes ; then adding points at infinity will show that $n \in TD(r)$. Indeed, if B_j is a partition of X then for each $B \in \mathfrak{B}_j$ construct a transveral design $TD_k(r-1;k)$ (where $k=k_j$) with pointset $R \times B$ and groups $\{y\} \times B, y \in R$ where R is a fixed $(r-1)$-set. If we number the parallel classes of each of these designs from 1 to k (and make sure that the 'verticals' $R \times \{b\}$ belong to parallel class 1 for all b) then the union of the parallel classes with a given number is a parallel class on $R \times X$.
On the other hand, if $\mathfrak{B}_j$ is a symmetric 1-design on X then we cannot

construct the transversal designs on $R \times B$ $(B \in \mathfrak{B}_j)$ independently. Instead, let N be the point-block incidence matrix of $(X,\mathfrak{B}_j)$ and write N as the sum of $k=k_j$ permutation matrices N_t $(1 \leqslant t \leqslant k)$. We may regard N_t as a $1-1$ correspondence $\phi_t : \mathfrak{B}_j \rightarrow X$.
Let B_0 be a fixed block in $\mathfrak{B}_j$ and construct a $TD_1(r-1,k)$ on $R \times B_0$ containing the verticals. For each non-vertical block T of this design construct a parallel class $\{T_B \mid B \in \mathfrak{B}_j\}$ with $T_{B_0} = T$ and such that for each $r \in R$ the transversal T_B contains the point $(r,\phi_t B)$, where t is determined by $(r,\phi_t B_0) \in T$.
In this way we 'transport' the transversal design on $R \times B_0$ and construct isomorphic copies on $R \times B$ for all $B \in \mathfrak{B}_j$, but in such a way that the entire collection of blocks is resolvable into parallel classes.
Taking all the blocks found in this way, and the groups $\{y\} \times X$, $y \in R$ yields the required design. □

E. As a modification to the previous idea of transporting a transversal design around a symmetric 1-design, suppose $(X,\mathfrak{B})$ is as in D. and that $\mathfrak{B}_j$ is a symmetric 1-design. This time construct a $TD,(r-1,k+1)$ on $R \times (B_0 \cup \infty)$ where ∞ is a new element. Repeating the previous construction we find for each point (r,∞) k almost parallel classes of transversals (disjoint apart from the common point (r,∞)) ; label this point now (r,∞_i) $(1 \leqslant i \leqslant k)$ so that different almost parallel classes have different points (r,∞_i) in common. This yields:

Let $(X,\mathfrak{B})$ be a pairwise balanced design such that $\mathfrak{B}$ contains pairwise disjoint families $\mathfrak{B}_j$ such that $(X,\mathfrak{B}_j)$ is a symmetric 1-design with block size k_j.
Suppose that $|B| \in TD(r+1)$ for $B \in \mathfrak{B} \setminus \bigcup_j \mathfrak{B}_j$ and that $|B+1| \in TD(r+1)$ for $B \in \bigcup_j \mathfrak{B}_j$. Finally suppose that $\sum_j k_j \in TD(r)$. Then $|X| + \sum_j k_j \in TD(r)$.

I shall call this contruction "adding points at infinity to symmetric 1-design". Note that this terminology is misleading: we do not construct a pairwise balanced design on $n+k$ points, but only a transversal design with groups of that size.

2A. Examples

Let $N(v)$ be the maximum number of mutually orthogonal Latin squares of order v. We have $N(0) = N(1) = \infty$, $N(q) = q-1$ for prime powers q and $N(v) \leqslant v-1$ for arbitrary v. The statements $v \in TD(r)$ and $N(v) \geqslant r-2$ are equivalent.

(i) We may apply $C1$ with a projective plane as design $(X,\mathfrak{B})$. This yields for prime powers q that $N(q^2+q+1) \geqslant N(q+1)$. Usually this bound is bad, but when $q+1$ is also a prime power we get $N(q^2+q+1) \geqslant q$.
Examples: $N(21) \geqslant 4$, $N(57) \geqslant 7$, $N(273) \geqslant 16$, $N(993) \geqslant 31$.

(ii) If q is a prime power then there exists a $2-(q^3+1,q+1,1)$ design (a 'unital', the isotropic points and hyperbolic lines in the projective plane $PG(2,q^2)$ with a unitary polarity). This design is resolvable with q^2 parallel classes, and adding q^2 points at infinity yields a pairwise balanced design $B(\{q+2,q^2\}\ ;\ q^3+q^2+1)$. In case also $q+2$ is a prime power, this yields $N(q^3+q^2+1)\geqslant q$.
EXAMPLE: $N(393)\geqslant 7$.

(iii) If q is an even prime power then there exists a resolvable $2-(\frac{1}{2}q(q-1), \frac{1}{2}q, 1)$ design (where points and blocks are the exterior lines and points of a hyperoval in $PG(2,q)$) with $q+1$ parallel classes. Thus we find a $B(\{\frac{1}{2}q, \frac{1}{2}q+1, x\}\ ;\ \frac{1}{2}q(q-1)+x)$ by adding x points at infinity $(0\leqslant x\leqslant q+1)$, where blocksize $\frac{1}{2}q+1$ does not occur for $x=0$ and blocksize $\frac{1}{2}q$ not for $x=q+1$.

EXAMPLES:	$N(120)\geqslant 7$	($q=16$, design resolvable),
	$N(136)\geqslant 7$	($q=16$, $x=16$, one parallel class of blocks of size 8),
	$N(504)\geqslant 7$	($q=32$, $x=8$),
	$N(528)\geqslant 15$	($q=32$, $x=32$, one parallel class of blocks of size 16),
	$N(2016)\geqslant 31$	($q=64$, design resolvable).

(iv) Useful pairwise balanced designs can often be constructed from a projective or affine plane by throwing away a suitably chosen set of points.
Throwing away one point from $PG(2,q)$ we find a $B(\{q,q+1\}\ ;\ q^2+q)$ where the blocks of size q from a parallel class. If $q+1$ is a prime power then it follows that $N(q^2+q)\geqslant q-1$. Examples: $N(20)\geqslant 3$, $N(72)\geqslant 7$, $N(272)\geqslant 15$, $N(992)\geqslant 30$.
Starting with $AG(2,q)$ instead we find (if $q-1,q$ are prime powers) $N(q^2-1)\geqslant q-2$. Examples: $N(24)\geqslant 3$, $N(63)\geqslant 6$, $N(80)\geqslant 7$, $N(288)\geqslant 15$, $N(1023)\geqslant 30$.

Throwing away x points from one line we find a $B(\{q+1-x,q,q+1\}\ ;\ q^2+q+1-x)$ or $B(\{q-x,q-1,q\}\ ;\ q^2-x)$. In this way one gets
$N(54)\geqslant 4$ ($q=7$, $x=3$), $N(280)\geqslant 7$ ($q=17$, $x=9$),
$N(264)\geqslant 7$, $N(265)\geqslant 8$, $N(267)\geqslant 10$ ($q=16, x=9,8,6$),
$N(285)\geqslant 12$ ($q=17$, $x=4$),
$N(993-x)\geqslant 31-x$ ($q=31$, $x=3,5,7,13,15,19,21,23$),
$N(1024-x)\geqslant 31-x$ ($q=32$, $x=7,9,13,16,24$).

If $q\equiv 0$ or 1 (mod 3) then $PG(2,q)$ contains a subconfiguration isomorphic to $AG(2,3)$, and removing that yields a $B(\{q-2,q,q+1\}\ ;\ v-9)$.
For $q=31$ this shows $N(984)\geqslant 27$.

Throwing away x points from a (hyper)oval in $PG(2,q)$ or $AG(2,q)$ yields a $B(\{q-1,q,q+1\}\ ;\ q^2+q+1-x)$ or $B(\{q-2,q-1,q\}\ ;\ q^2-x)$ for $x \leqslant q+1$ (or $x \leqslant q+2$ if q is even). Since 7,8,9 are three consecutive prime powers we find with $q=8$: $N(66) \geqslant 5$, $N(68) \geqslant 5$, $N(69) \geqslant 6$, $N(70) \geqslant 6$, and with $q=9$: $N(74) \geqslant 5$, $N(75) \geqslant 5$, $N(76) \geqslant 5$, $N(78) \geqslant 6$.
Note that the blocks of size 7 form a clear set in $B(\{7,8,9\}\ ;\ 70)$ and $B(\{7,8,9\}\ ;\ 78)$ and an almost clear set in $B(\{7,8,9\}\ ;\ 69)$. (After writing this I found that L. ZHU (1984) had made the same observation).

(v) Continuing in this vein we note that $PG(2,q^2)$ has a partition into Baer subplanes, and taking t of those produces a $B(\{t,q+t\}\ ;\ t(q^2+q+1))$ where the collection of blocks of size $q+t$ forms a symmetric 1-design and the collection of blocks of size t is resolvable into $q^2-q+1-t$ parallel classes.
This yields many useable pairwise balanced designs

EXAMPLES:	$N(189) \geqslant 8$	($q=4$, $t=9$),
	$N(253) \geqslant 12$	($q=4$, $t=12$, add one point at infinity to get an almost clear set of blocks of size 13),
	$N(357) \geqslant 9$	($q=5$, $t=11$, add 16 points at infinity to the symmetric 1-design).

[For more details, see BROUWER [Br 1980]].

(vi) Adding $q+1$ points at infinity to the symmetric 2-design $PG(2,q)$ we find if both $q+1$ and $q+2$ are prime powers : $N((q+1)^2+1) \geqslant q$. Examples: $N(10) \geqslant 2$, $N(65) \geqslant 7$.

(vii) From a Singer difference set we find a separable subdesign $B(\{9,13,16\}\ ;\ 469)$ in $PG(2,37)$. It follows that $N(469) \geqslant 8$. [Again, see BROUWER, [Br 1980]].

In these examples I have listed virtually all instances I know of where the pairwise balanced design construction yields the best known bound on $N(v)$, and where the pairwise balanced design was not a (truncated) transversal design itself. (In fact, under (iv) it was a truncated transversal design). In the next section we shall see that one can do better with a transversal design as ingredient than with a general pairwise balanced design as starting point.

3. WILSON'S CONSTRUCTION

Applying the PBD construction to a transversal design $TD(m+1\ ;\ t)$ of which one group has been truncated to size h (so that we have a $GD(\{m,m+1\},\{t,u\}\ ;\ mt+h)$) we find

- *If* $N(t) \geqslant m-1$ *then for* $0<h<t$:
  $N(mt+h) \geqslant \min\{N(t),\ N(h),\ N(m)-1,\ N(m+1)-1\}$

- *If* $N(t) \geqslant m-2$ *then* $N(mt) \geqslant \min\{N(m)-1, N(t)\}$.

Clearly the second bound is worse than MacNeish's bound. The first one is always worse (or at least : not better) than Wilson's bound [WILSON, 1974 Thm. 2.3]
F. - *If* $0 < h < t$ *then*
$N(mt+h) \geqslant \min\{N(m), N(m+1), N(t)-1, N(h)\}$.
(For: if $m-1 \leqslant N(t)$ then $N(m)-1 < N(t)$, so in the first bound the minimum cannot be $N(t)$).
This bound, together with Wojtas's bound [WOJTAS, 1977]
G. - *If* $0 < h < t$ *then*
$N(mt+h) \geqslant \min\{N(m), N(m+1), N(m+h), N(t)-h\}$
account for the majority of the best lower bounds for $N(v)$ known. Both bounds follow from special cases of Wilson's construction, which we shall now describe.

Construction Ingredients: (1) A transversal design $TD(k+l\,;\,t)$ of which l groups have been truncated, so that k groups have size t and the remaining groups size h_i $(1 \leqslant i \leqslant l)$ where clearly $0 \leqslant h_i \leqslant t$. Denote the union of the l truncated groups by H (so that $h := |H| = \sum_{i=1}^{l} h_i$). (2) Transversal designs $TD(k\,;\,h_i)$ for $1 \leqslant i \leqslant l$. (3) Transversal designs $TD(k\,;\,m+|B \cap H|)$ for each block B from the $TD(k+l;t)$ with $|B \cap H|$ pairwise disjoint blocks.
We construct a $TD(k;mt+h)$ in the obvious way (given ingredients and result):
Let the $TD(k+l;t)$ have groups $G_1,...,G_k$, $H_1,...,H_l$, then the constructed design will have groups $(G_j \times M) \cup (H \times \{j\})$ $(j=1,2,..,k)$, all of size $mt+h$ (M is an arbitrary set of cardinality m) ; put ingredients (2) on $H_i \times K$ $(1 \leqslant i \leqslant l)$ where $K = \{1,2,...,k\}$; for each block B from ingredient (1) the set $(B \setminus H) \times M \ \cup (B \cap H) \times K$ has cardinality $k(m+|B \cap H|)$; put on this set ingredient (3) in such a way that the groups of this design are subsets of the design to be constructed and for each $b \in B \cap H$ the set $\{b\} \times K$ is a block.
It is straightforward to check that this works. □

Bound F is obtained by taking $l=1$, $h_1=h$. Bound G is obtained by taking $l=h$, $h_i=1$ $(i \leqslant i \leqslant l)$. Taking $l=2$, $h_1=u$, $h_2=v$ one gets
H. - *If* $0<u,v<t$ *then*
$N(mt+u=v) \geqslant \min\{N(m), N(m+1), N(m+2), N(u), N(v), N(t)-2\}$.
[WILSON 1974, Thm. 2.4]

3A. EXAMPLES

(i) $N(95) \geqslant 6$ follows from the PBD construction using a truncated $TD(9;11)$ since $95=8.11+7$.

(ii) $N(33) \geqslant 3$ follows from F. since $33=4.8+1$.
$N(84) \geqslant 6$ follows from F. since $84=7.11+7$.

(iii) $N(91) \geqslant 7$ follows from G. since $91=8.11+3$.

(iv) $N(94) \geqslant 6$ follows from H. since $94=7.11+8+9$.

(v) $N(90) \geqslant 6$ [WOJTAS 1980 a] follows since $90=6.11+8+8+8$ and we can truncate a $TD(9;11)$ in such a way that each block meets the set H in at least one point. (In general with $l=3$ one can obtain the condition that $B \cap H \neq \emptyset$ for all B certainly when $h_1 \leqslant h_2$ and $(t-h_1)(t-h_2) < h_3$). Another example is $N(796) \geqslant 7$ since $796 = 70.11+8+8+8$.

(vi) $N(135) \geqslant 7$ [BROUWER 1978] follows since $135 = 8.16+7$ and we can truncate a $TD(15;16)$ in such a way that each block meets the set H in 0,1 or 3 points, andd $h_i = 1$ $(1 \leqslant i \leqslant 7)$ - in fact we may take H to be a Fano subplane of $PG(2,16)$.

(vii) $N(164) \geqslant 6$ follows since $164 = 7.23+3$ and we can take $h=3$, $h_1=h_2=h_3=1$, $|B \cap U| \leqslant 2$. In fact, for $h \leqslant t$ and t a prime power we can take H to be part of an oval in $PG(2,t)$ and obtain
$N(mt+h) \geqslant \min\{N(m), N(m+1), N(m+2), N(t)-h\}$.
[WILSON 1974, thm 2.5 - BROUWER 1979]

(viii) Continuing the previous construction: if we take v points, no 3 on a line, all on different groups and $t > \binom{v}{2}$ then we can add w points all on one group and get
$N(mt+w+v) \geqslant \min\{N(m), N(m+1), N(m+2), N(w), N(t)-v-1\}$
for $t \geqslant w+\binom{v}{2}$. [VAN REES]

EXAMPLES:

n	m	t	w	v	lower bound for $N(n)$
1554	81	19	13	2	8
1884	81	23	19	2	8
2046	81	25	19	2	8
2298	99	23	19	2	8
2694	99	27	19	2	8
4622	271	17	13	2	12
4776	207	23	13	2	8

I know of no examples with $v \neq 2$ where this construction yields the best known lower bound.

4. Weighting and holes

As was noted by Wojtas [1980] and Stinson [1979 a] in certain special cases, and by Brouwer & van Rees [1982] in general, one may generalize Wilson's construction by giving weights to the points of H.
In this way one constructs a transversal design $TD(k;mt+\sum_{h\in H} m_h)$, where m_h is the weight of $h(h\in H)$. Ingredient (1) is unchanged, and (2) and (3) now read:
(2′) Transversal designs $TD(k;\sum_{h\in H_i} m_h)$ for $1 \leqslant i \leqslant l$.
(3′) For each block B from the $TD(k+l;t)$ a transversal design $TD(k;m+\sum_{h\in B\cap H} m_h)$ with pairwise disjoint subdesigns $TD(k;m_h)$ $(h\in B\cap H)$.
(The construction is entirely analogous to that in Section 3.)

But one may go further: all one needs the subdesigns in (3′) for, is to throw them out in order not to cover certain pairs twice ; in other words, what actually is needed is a transversal designs with holes
(3″) $TD(k;m+\sum_{h\in B\cap H} m_h) - \sum_{h\in B\cap H} TD(k;m_h)$
and (3″) may well exist while (3) does not.

Let us formally define the concept of 'transversal design with holes' - the above discussion shows that what we have in mind looks like a transversal design from which a collection of pairwise disjoint subdesigns has been removed.
A *transversal design with holes* $TD(k;v) - \sum_{i=1}^{r} TD(k;u_i)$ consists of a set X of cardinality kv (the set of *points*), a partition $\mathcal{G}$ of X into k *groups* of v elements each, pairwise disjoint subsets Y_i of X $(1 \leqslant i \leqslant r)$ of cardinality ku_i (the *holes*) such that $|Y_i\cap G| = u_i$ for each $G\in\mathcal{G}$ and each i, $1 \leqslant i \leqslant r$, and a collection $\mathcal{B}$ of subsets of X of cardinality k (the *blocks*) such that no block meets a group or a hole in more than one point, and any two points not in the same group or hole are in a unique block.

If follows that $|\mathcal{B}| = v^2 - \sum_{i=1}^{r} u_i^2$. For $r=0$ the concept 'transversal design with zero holes' coincides with the usual transversal design. In case $u_i = 1$ for all i, $1 \leqslant i \leqslant r$, then $TD(k;v)-rTD(k;1)$ (in an obvious extension of the notation) exists iff a $TD(k;v)$ with r pairwise disjoint blocks exists - showing that (3″) generalizes (3). If $TD(k;u_i)$ exists we may put it on Y_i and thus 'plug' the hole Y_i, obtaining a transversal design with $r-1$ holes. Conversely, if a transversal design (with holes) has a subdesign (disjoint from all the holes) we can unplug it and obtain a transversal design with $r+1$ holes. Not all holes can be filled: Horton [1974], who introduced the concept 'transversal design with one hole' under the name 'incomplete array', constructs a $TD(4;6) - TD(4;2)$, but neither $TD(4;6)$ nor $TD(4;2)$ exist. (Also, Brouwer [1978] constructs $TD(6;10) - TD(6;2)$, while not even $TD(5;10)$ is known.)
As most important special case we find (with $l=1$) [Brouwer, 1979]

I. - *If* $t = \sum_{j=1}^{p} h_j$ *and* $TD(k+1;t)$, $TD(k;\sum_{j=1}^{p} m_j h_j)$ *and (for* $j=1,...,p$*)* $TD(k;m+m_j) - TD(k;m_j)$ *all exist, then also a* $TD(k;mt+\sum_{j=1}^{p} m_j h_j)$ *exists.*

Instead of making holes $TD(k;m_h)$ in the ingredients (3″) corresponding to all blocks B on the point $h \in H$ we may leave one such ingredient alone and make a hole in the ingredient (2′) corresponding to the group containing h. For the general formulation of this construction see BROUWER & VAN REE [1982], Theorem 1.2. The most important special case is [BROUWER, 1980 a]

J. - *If* $w = \sum_{i=1}^{l} w_i$ *and* $TD(k+l;t)$, $TD(k;m)$, $TD(k;m+w)$ *and (for* $j=1,...,l$*)* $TD(k;m+w_i) - TD(k;w_i)$ *all exist, then also* $TD(k;mt+w)$ *exists.*

Fore more details about construction of transversal designs with holes, see BROUWER & VAN REES [1982].

4A. EXAMPLES

(i) We show $N(5467) \geq 15$. The construction uses a distribution of holes as discussed above before J. Noting that $5467 = 19.271+289+29$ we apply the construction with $k=17$, $t=19$, $m=271$, $l=2$, $h_1=17$, $h_2=13$; $289=17.17$: the points in H_1 all get weight 17; $29 = 1.17+12.1$: one point x_0 in H_2 gets weight 17, the twelve others weight 1. We need the following ingredients:

(i) $TD(19;19)$ exists since 19 is prime.

(2) $TD(17;289)-17TD(17;17)$ exists, e.g. by the PBD construction on the affine plane $AG(2,17)$.
$TD(17;29)$ exists since 29 is prime.

(3) $TD(17;271)$ exists since 271 is prime.
$TD(17;272)-TD(17;1)$ exists by MacNeish : $272=16.17$.
$TD(17;288)-TD(17;17)$ exists since Wilson's construction for $TD(17;288)$ using $288=16.17+16$ yields a design with subdesign $TD(17;17)$.
$TD(17;289) - TD(17;17) - TD(17;1)$ exists, and is found from $AG(2,17)$.
$TD(17;305) - TD(17;17)$ exists since Wilson's construction for $TD(17;305)$ using $305=16.19+1$ yields a design with subdesign $TD(17;17)$.

For the standard distribution of holes we would have needed $TD(17;305) - 2TD(17;17)$, but it is not obvious how to obtain this ingredient. Therefore we cover the pairs in the km_h -subsets corresponding to points $h \in H_1$ in the designs corresponding to the (unique) block B containing h and x_0. This yields the required $TD(17;5467)$.

(ii) We show $N(4738) \geqslant 8$. (This was the largest unknown value for 8 squares; it follows that $n_8 \leqslant 4242$).
$4738 = 271.17 + (125 = 7*17 + 6) + 6\times 1$
Apply the construction with $k=10$, $t=17$, $m=271$, $l=7$, $h_1=13$, $h_2=h_3=h_4=h_5=h_6=h_7=1$; give in H_1 seven points weight 17 and six points weight 1. Give all other points in H weight 1. Choose the six points on $H \setminus H_1$ on a single block B where $B \cap H_1 = \varnothing$.

(iii) We show $N(10618) \geqslant 15$ and $N(10632) \geqslant 15$. (These were the largest unknown values for 15 squares ; it follows that $n_{15} < 10000$).
$10618 = 435.23 + (293=2*16+9*29) + (320=20*16)$
$10632 = 435.23 + (128=8*16) + (499=4*16+15*29)$

Ingredients:	23,128,293,499 are prime powers.
$320 = 16.19 + 16$	shows $N(320) \geqslant 15$.
$435 = 16.27 + 3\times 1$	shows $N(435) \geqslant 15$.
$451 = 16.27 + 19$	shows the existence of $TD(17;451)-TD(17;16)$.
$464 = 16.29$	shows the existence of $TD(17;464)-TD(17;29)$.
$467 = 16.29 + 3\times 1$	shows the existence of $TD(17;467)-2TD(17;16)$.

5. Asymptotic results

Chowla, Erdös & Straus [CES] showed that $\lim_{v\to\infty} N(v)=\infty$. Consequently we may define

$$n_r := \max\{v \mid N(v)<r\} \quad (\text{for } r\geqslant 2).$$

In fact they showed that $n_r < (3r)^{91}$, a result that was improved by Rogers [Ro] to $n_r<r^{42}$, by Wang Yuan [WY] to $n_r<r^{26}$, by Wilson [W 1974] to $n_r<r^{17}$ and by Beth [Be] to $n_r<r^{14.8}$, all for sufficiently large r.
For small values of r explicit upper bounds for n_r have been obtained. The current state of affairs is:

$n_2=6$ (Bose, Shrikhande & Parker [BSP]),
$n_3\leqslant 14$ (Wang & Wilson [WaW]),
$n_4\leqslant 52$ (Guérin [G]),
$n_5\leqslant 62$ (Hanani [H 1979]),
$n_6\leqslant 76$ (Wojtas [Wo 1980a]),
$n_7\leqslant 780$, $n_9\leqslant 5842$, $n_{10}\leqslant 7222$ (Brouwer & van Rees [Br vR]),
$n_8\leqslant 4216$, $n_{11}\leqslant 7222$, $n_{12}\leqslant 7286$, $n_{13}\leqslant 7288$,
$n_{14}\leqslant 7874$, $n_{15}\leqslant 8360$ (Brouwer, unpublished),
$n_{30}\leqslant 52502$, (Brouwer, unpublished, cf. [Br 1980a]).

The proofs are by the constructions given above (together with some explicit constructions for small v) coupled with some number theory (trivial for fixed r, sieve methods for large r) required to show that sufficiently large numbers can be written in a suitable form.

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