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A group theoretic interpretation of Wilson polynomials

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Racah and Wilson polynomials figure at the top level of Askey's scheme of hypergeometric orthogonal polynomials. The first family of polynomials has a group theoretic interpretation as Racah coefficients, but for Wilson polynomials such an interpretation was not known. The paper presents a new group theoretic interpretation of Racah polynomials in connection with O(p)xO(q)xO(r)-invariant spherical harmonics on $S^{p+q+r-1}$ and next, by analytic continuation, a group theoretic interpretation of Wilson polynomials in connection with O(p)xO(q)xO(r)-invariant harmonics on the hyperboloid O(p+q,r)/O(p+q,r-1). This is a preliminary report not containing full proofs.

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0. Introduction. This paper is the second part (after [9]) of an informal account of a research activity which started with the observation of a curiosity (namely two explicit orthogonal bases mapped onto each other by the Jacobi function transform), but which grew out into a research program to complement Askey's scheme of hypergeometric orthogonal polynomials with group theoretic interpretations and with further orthogonal systems of hypergeometric nature but of nonpolynomial type. Here I will deal with a group theoretic interpretation of Wilson polynomials as kernels connecting with each other two canonical bases of harmonics on a hyperboloid satisfying a certain invariance condition. This is preceded by a similar interpretation of Racah polynomilas in connection with spherical harmonics. These main results can be found in §4,5. The earlier sections are of introductory nature.

1. Jacobi and Wilson polynomials mapped onto each other by the Jacobi function transform. *Hermite polynomials* H_n are orthogonal polynomials of degree n on the interval $(-\infty,\infty)$ with respect to the weight function $x \to \exp(-x^2)$. It is well-known that the functions $t \to H_n(t)\exp(-\frac{1}{2}t^2)$ form an orthogonal basis for $L^2(\mathbb{R})$ of eigenfunctions of the Fourier transform with eigenvalues i^{-n}:

$$(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} H_n(t)e^{-\frac{1}{2}t^2} e^{-i\lambda t}dt = i^{-n}H_n(\lambda)e^{-\frac{1}{2}\lambda^2} . \tag{1.1}$$

A similar set of eigenfunctions exists for the *Hankel transform* pair

$$\left.\begin{aligned} g(\lambda) &= \int_0^{\infty} f(t)J_\alpha(\lambda t)tdt \\ f(t) &= \int_0^{\infty} g(\lambda)J_\alpha(\lambda t)\lambda d\lambda \end{aligned}\right\} , \tag{1.2}$$

where

$$J_\alpha(x) \ := \ (\tfrac{1}{2}x)^\alpha \ {}_0F_1(\alpha+1;-\tfrac{1}{4}x^2)/\Gamma(\alpha+1) \tag{1.3}$$

denotes a *Bessel function*. An orthogonal basis for $L^2(\mathbb{R}_+,tdt)$ of eigenfunctions of the Hankel transform with eigenvalue $(-1)^n$ is given by the functions $t \mapsto L_n^\alpha(t^2)t^\alpha \exp(-\tfrac{1}{2}t^2)$, where the *Laguerre polynomials* L_n^α are orthogonal polynomials of degree n on $(0,\infty)$ with respect to the weight function $x \mapsto x^\alpha e^{-x}(\alpha>-1)$:

$$\int_0^\infty L_n^\alpha(t^2)t^\alpha e^{-\frac{1}{2}t^2} J_\alpha(\lambda t)tdt = (-1)^n L_n^\alpha(\lambda^2)\lambda^\alpha e^{-\frac{1}{2}\lambda^2} , \tag{1.4}$$

cf. [4,8.9(3)].

Let us next consider an analogue of (1.1) and (1.4) for the Jacobi function transform. Let $\alpha > -1$, $\beta \in \mathbb{R}$,

$$\Delta(t) = \Delta_{\alpha,\beta}(t) \ := \ (2\mathrm{sh}t)^{2\alpha+1}(2\mathrm{ch}t)^{2\beta+1}, \quad t > 0, \tag{1.5}$$

$L = L_{\alpha,\beta}$ a differential operator defined by

$$(Lu)(t) \ := \left(\frac{d^2}{dt^2} + \frac{\Delta'(t)}{\Delta(t)}\frac{d}{dt}\right)u(t), \ t \in \mathbb{R}. \tag{1.6}$$

Let the Jacobi function $\phi_\lambda = \phi_\lambda^{(\alpha,\beta)}$ be the unique solution u of

$$L_{\alpha,\beta}u = (-\lambda^2-(\alpha+\beta+1)^2)u \tag{1.7}$$

which is C^∞, even and satisfies $u(0) = 1$. It can be expressed as a hypergeometric function:

$$\phi_\lambda^{(\alpha,\beta)}(t) = {}_2F_1(\tfrac{1}{2}(\alpha+\beta+1+i\lambda),\tfrac{1}{2}(\alpha+\beta+1-i\lambda);\alpha+1;-\mathrm{sh}^2t) \tag{1.8}$$

$$= (\mathrm{ch}t)^{-\alpha-\beta-1-i\lambda}\,{}_2F_1(\tfrac{1}{2}(\alpha+\beta+1+i\lambda),\tfrac{1}{2}(\alpha-\beta+1+i\lambda);\alpha+1;\mathrm{th}^2t). \tag{1.9}$$

The transform $f \mapsto \hat{f}$ defined by

$$\hat{f}(\lambda) := \int_0^\infty f(t)\phi_\lambda(t)\Delta(t)dt \tag{1.10}$$

is called the *Jacobi function transform.*
Noteworthy special cases are the Fourier-cosine transform ($\alpha=\beta=-\frac{1}{2}$) and the Mehler-Fock transform ($\alpha=\beta=0$). See for instance [7] and the survey [8] for details and background of this transform. Group theoretic interpretations of (1.10) highly contribute to its significance. In particular, the spherical Fourier transform on a Riemannian rank one symmetric space of the noncompact type can be written in the form (1.10).

For the inversion of (1.10) consider a second solution Φ_λ of (1.7) which is, for $\text{Im}\,\lambda > 0$, uniquely determined by the asymptotic behaviour

$$\Phi_\lambda(t) = e^{(i\lambda-\alpha-\beta-1)t}(1+o(1)) \text{ as } t\to\infty \tag{1.11}$$

and which, for $\lambda \in \mathbb{C}\backslash\{-i,-2i,\ldots\}$, is defined by analytic continuation with respect to λ. Then

$$\phi_\lambda = c(\lambda)\Phi_\lambda + c(-\lambda)\Phi_{-\lambda}, \quad \lambda \notin i\mathbb{Z}, \tag{1.12}$$

where

$$c(\lambda) = c_{\alpha,\beta}(\lambda) = \frac{2^{\alpha+\beta+1-i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(i\lambda+\alpha+\beta+1))\Gamma(\frac{1}{2}(i\lambda+\alpha-\beta+1))}. \tag{1.13}$$

If $f \in \mathcal{D}_{\text{even}}$ (even C^∞-functions with compact support on $\mathbb{R}$) then (1.10) can be inverted as

$$f(t) = (2\pi)^{-1}\int_0^\infty \hat{f}(\lambda)\phi_\lambda(t)|c(\lambda)|^{-2}d\lambda, \tag{1.14}$$

provided $|\beta| \le \alpha+1$, otherwise we have to add a finite sum

$$\sum_\lambda \gamma(\lambda)\hat{f}(\lambda)\phi_\lambda(t)$$

to the right hand side of (1.14), where λ runs over the poles of $\lambda \mapsto (c(\lambda))^{-1}$ in the upper half plane, all lying on the positive imaginary axis, and the positive constants $\gamma(\lambda)$ are expressed as certain residues.

For convenience, we will further assume that $|\beta| \leq \alpha+1$.

There is a Paley-Wiener type theorem statying that $f \mapsto \hat{f}$ maps $\mathcal{D}_{\text{even}}$ one-to-one onto the space of even entire analytic functions of exponential type, rapidly decreasing on $\mathbb{R}$, which is dense in $L^2(\mathbb{R}_+;(2\pi)^{-1}|c(\lambda)|^{-2}d\lambda)$. There is a Plancherel formula

$$\int_{-\infty}^{\infty} |f(t)|^2 \Delta(t)dt = (2\pi)^{-1} \int_0^{\infty} |\hat{f}(\lambda)|^2 |c(\lambda)|^{-2} d\lambda, \tag{1.15}$$

which first can be derived for $f \in \mathcal{D}$. The formula shows that the transform $f \mapsto \hat{f}$ uniquely extends to an isometry of the Hilbert space $L^2(\mathbb{R}_+;\Delta(t)dt)$ onto the Hilbert space $L^2(\mathbb{R}_+;(2\pi)^{-1}|c(\lambda)|^{-2}d\lambda)$.

Since $\phi_\lambda(t)$ depends on λ and t in quite different ways, we cannot expect to find eigenfunctions for the Jacobi function transform as in (1.1), (1.4). But it is possible to give nice explicit orthogonal bases of $L^2(\mathbb{R}_+;\Delta(t)dt)$ and $L^2(\mathbb{R}_+;(2\pi)^{-1}|c(\lambda)|^{-2}d\lambda)$ which are mapped onto each other by the Jacobi function transform. For this we need two other families of orthogonal polynomials.

Jacobi polynomials are orthogonal polynomials of degree n on (-1,1) with respect to the weight function $x \mapsto (1-x)^\alpha (1+x)^\beta$ and with normalization $P_n^{(\alpha,\beta)}(1) = (\alpha+1)_n/n!$. Important formulas are

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$$
$$= \frac{(\alpha+1)_n}{n!} \, {}_2F_1(-n,n+\alpha+\beta+1;\alpha+1;\tfrac{1}{2}(1-x)). \tag{1.16}$$

Wilson polynomials were introduced by Wilson [14], [15]. In the notation of J. Labelle's poster [10] they are given by

$$W_n(x^2;a,b,c,d) := (a+b)_n(a+c)_n(a+d)_n$$
$$\cdot {}_4F_3\left(\left.\begin{matrix}-n,n+a+b+c+d-1,a+ix,a-ix\\ a+b,a+c,a+d\end{matrix}\right| 1\right), \quad n = 0,1,2,\dots . \tag{1.17}$$

Note that the ${}_4F_3$-hypergeometric function is a sum running over $k = 0,1,\dots,n$, the k^{th} term containing the factor

$$(a+ix)_k(a-ix)_k = (a^2+x^2)((a+1)^2+x^2)\dots((a+k-1)^2+x^2),$$

which is a polynomial of degree k in x^2. It can be shown that W_n is symmetric in the four parameters a,b,c,d. If they are all real or if one or both pairs of them consist of complex conjugates, a possibly remaining pair being real, then W_n is real-valued. If, moreover, a,b,c,d have positive real parts then the functions $x \mapsto W_n(x^2)$ are complete and orthogonal on $\mathbb{R}_+$ with respect to the weight function

$$x \mapsto \left| \frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)} \right|^2 . \tag{1.18}$$

The desired orthogonal systems mapped onto each other by the Jacobi function transform are now given by the following formula:

$$\int_0^\infty (\mathrm{ch}\,t)^{-\alpha-\beta-\delta-i\mu-2} P_n^{(\alpha,\delta)}(1-2\mathrm{th}^2 t)\,\phi_\lambda^{(\alpha,\beta)}(t)\Delta_{\alpha,\beta}(t)dt$$
$$= \frac{2^{2\alpha+2\beta+1}\Gamma(\alpha+1)(-1)^n\Gamma(\frac{1}{2}(\delta+i\mu+1+i\lambda))\Gamma(\frac{1}{2}(\delta+i\mu+1-i\lambda))}{n!\,\Gamma(\frac{1}{2}(\alpha+\beta+\delta+i\mu+2)+n)\Gamma(\frac{1}{2}(\alpha-\beta+\delta+i\mu+2)+n)} \tag{1.19}$$
$$\cdot W_n(\tfrac{1}{4}\lambda^2;\tfrac{1}{2}(\delta+i\mu+1),\tfrac{1}{2}(\delta-i\mu+1),\tfrac{1}{2}(\alpha+\beta+1),\tfrac{1}{2}(\alpha-\beta+1)),$$

$$\beta,\lambda,\mu \in \mathbb{R}, \quad \alpha,\delta > -1.$$

This formula was derived for $\mu = 0$ in [8,(9.4)] and in full generality in [9,(3.3)]. A decisive hint for finding (1.19) was given by the paper of Boyer & Ardalan [2], where the special case $\alpha = \frac{1}{2}p-3/2$, $\beta = \delta = -\frac{1}{2}$ is obtained in the group theoretic setting of spherical principal series representations of the group $SO_0(1,p)$. It is curious that Wilson polynomials were not yet known at the time [2] was published.

It follows from the orthogonality relations for Jacobi and Wilson polynomials that the functions

$$t \mapsto (\mathrm{ch}\,t)^{-\alpha-\beta-\delta-i\mu-2} P_n^{(\alpha,\delta)}(1-2\mathrm{th}^2 t)$$

are orthogonal on $\mathbb{R}_+$ with respect to the weight function $\Delta_{\alpha,\beta}$, and the functions at the right hand side of (1.19) orthogonal on $\mathbb{R}_+$ with respect to the weight function $\lambda \mapsto (2\pi)^{-1}|c(\lambda)|^{-2}$, provided $|\beta| \leq \alpha+1$. For other values of β the polynomials $x \mapsto W_n(x^2)$ remain orthogonal, but with discrete masses supported at the positive imaginary axis added to the orthogonality measure, compatible with the added terms to (1.14), (1.15) in the case $|\beta| > \alpha+1$.

2. Racah coefficients and polynomials. Wilson [14],[15] obtained his Wilson polynomials as a kind of analytic continuation of the orthogonality relations for Racah polynomials. These latter orthogonality relations naturally follow from their group theoretic setting as Racah coefficients. Let us briefly explain this.

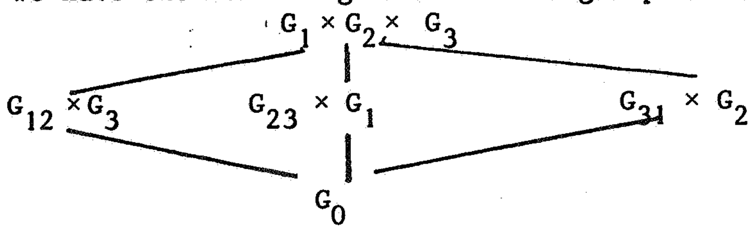
Let SU(2) be the group of 2×2 unitary matrices of determinant 1. Write

$$G_1 \times G_2 \times G_3 := SU(2)\times SU(2)\times SU(2),$$

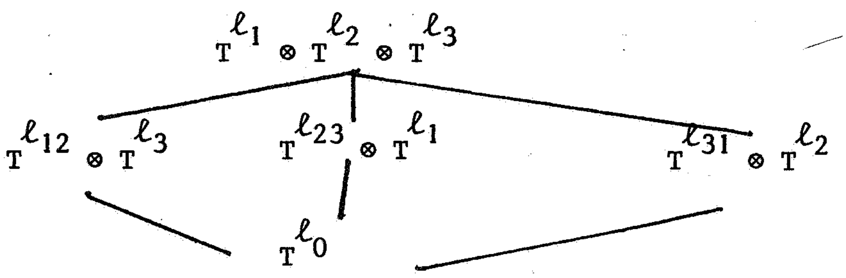
$$G_{ij} := \mathrm{diag}(G_i\times G_j) \qquad (i\neq j),$$

$$G_0 = \mathrm{diag}(G_1\times G_2\times G_3).$$

Then we have the following scheme of subgroup inclusions.



Let $\ell = 0,\frac{1}{2},1,\ldots$ and let T^{ℓ} be the (up to equivalence unique) irreducible unitary representation of SU(2) of dimension $2\ell+1$. (See Vilenkin [13,Ch.3] for an account of the representation theory of SU(2).) In general a representation T^{ℓ_0} of G_0 will be contained with multiplicity higher than one in a representation $T^{\ell_1} \otimes T^{\ell_2} \otimes T^{\ell_3}$ of $G_1\times G_2\times G_3$. But we can decompose this multiple of T^{ℓ_0} into irreducible representations by using the irreducible representations of any of the intermediate subgroups in the above scheme, as we indicate in the following scheme.



Now each representation in the scheme occurs with multiplicity at most 1 in a representation occurring on a line immediately above it. So, if $H(T)$ denotes the subspace of the representation space of $T^{\ell_1} \otimes T^{\ell_2} \otimes T^{\ell_3}$ consisting of all vectors behaving according to the representation T of some subgroup then we have

$$H(T^{\ell_0}) = \begin{cases} \bigoplus_{\ell_{12}} H(T^{\ell_{12}} \otimes T^{\ell_3}) \cap H(T^{\ell_0}), \\ \bigoplus_{\ell_{23}} H(T^{\ell_{23}} \otimes T^{\ell_1}) \cap H(T^{\ell_0}), \\ \bigoplus_{\ell_{31}} H(T^{\ell_{31}} \otimes T^{\ell_2}) \cap H(T^{\ell_0}), \end{cases} \tag{2.1}$$

and each of the three decompositions is into subspaces irreducible under G_0, all behaving according to T^{ℓ_0}.

In general, if H is the representation space of the n-fold direct sum of an irreducible unitary representation T of a compact group G and if $H = \oplus_{j=1}^{n} V_j$ and $H = \oplus_{j=1}^{n} W_j$ are two orthogonal decompositions into irreducible subspaces then there are intertwining isometries $A_{ij}: V_i \to W_j$ which are compatible in the sense that $A_{kj}^{-1} A_{ij}$ is independent of j. By Schur's lemma two such choices A_{ij} and B_{ij} differ at most by a factor $\exp(\sqrt{-1}(\phi_i+\psi_j))$ for certain real $\phi_1,\ldots,\phi_n,\psi_1,\ldots,\psi_n$. Now there is a unique n×n matrix (c_{ij}) such that

$$v = \sum_{j=1}^{n} c_{ij} A_{ij} v, \quad v \in V_i.$$

Of course, the coefficients c_{ij} satisfy a row orthogonality

$$\sum_{j=1}^{n} c_{ij} \overline{c_{kj}} = \delta_{ik} \tag{2.2}$$

and a similar column orthogonality. Now apply this to the first two decompositions in (2.1). For fixed ℓ_1,ℓ_2,ℓ_3 and ℓ_0 we will obtain a unitary matrix (c_{ij}) with ℓ_{12} and ℓ_{23} as row and column indices. Racah [12] (see also Biedenharn & van Dam [1]) computed the matrix coefficients as elementary factors times terminating ${}_4F_3$-hypergeometric series of unit arguments, which should satisfy the orthogonality relations (2.2). These coefficients are called *Racah coefficients or 6j-symbols.*

Next Wilson [14], [15] made the observation that the above ${}_4F_3$'s can be viewed as polynomials, which become orthogonal polynomials in view of the orthogonality for the Racah coefficients. By analytic interpolation between the discrete values of the parameters $\ell_0,\ell_1,\ell_2,\ell_3$ a big class of orthogonal polynomials was obtained: the *Racah polynomials.*

Racah polynomials are defined by

$$R_n(x(x+\gamma+\delta+1);\ \alpha,\beta,\gamma,\delta) = {}_4F_3\left(\begin{matrix}-n,n+\alpha+\beta+1,-x,x+\gamma+\delta+1\\ \alpha+1,\beta+\delta+1,\gamma+1\end{matrix}\middle|\,1\right) \tag{2.3}$$

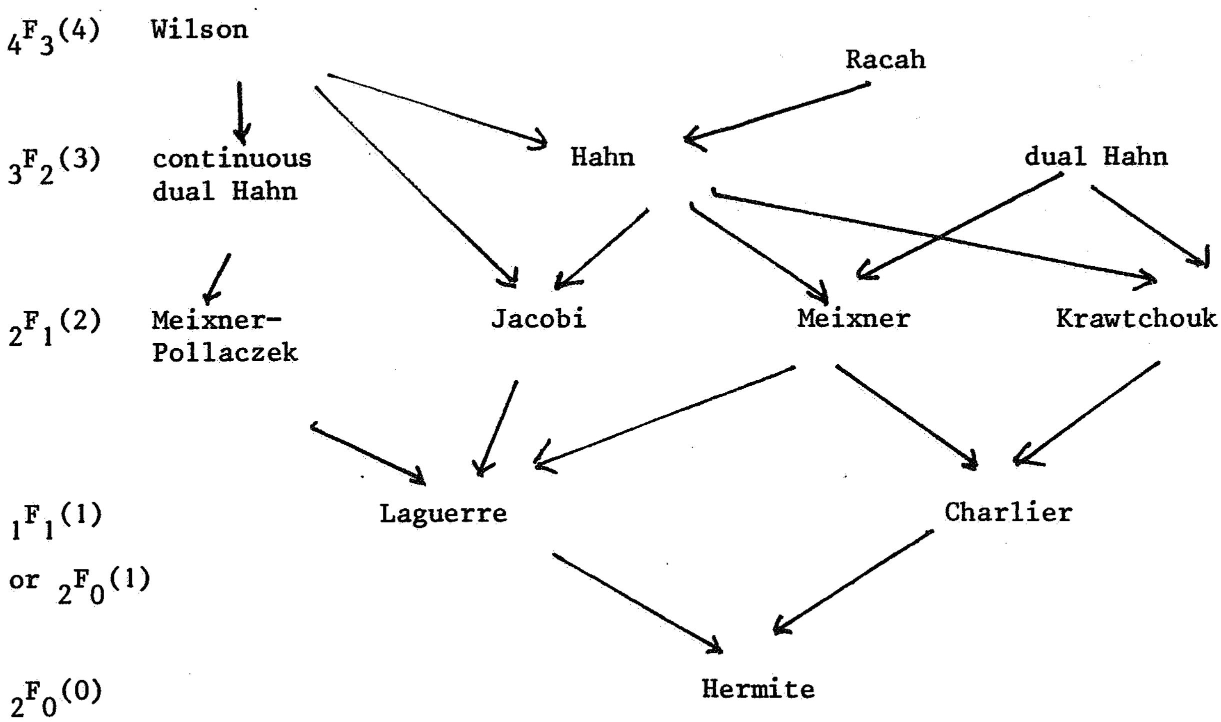
where $\alpha+1$ or $\beta+\delta+1$ or $\gamma+1$ equals a nonpositive integer $-N$. Let n run over $0,1,2,\ldots,N$ and let the hypergeometric series in (2.3) terminate with the n^{th} term. Since $(-x)_k(x+\gamma+\delta+1)_k$ is a polynomial of degree k in $x(x+\gamma+\delta+1)$, R_n is indeed a polynomial of degree n in $x(x+\gamma+\delta+1)$. The polynomials R_n satisfy a discrete orthogonality relation

$$\sum_{x=0}^{N} R_n(x(x+\gamma+\delta+1))R_m(x(x+\gamma+\delta+1))w_x = 0,\ n \neq m, \tag{2.4}$$

where the weights w_x can be given explicitly.

<u>3. The Askey scheme of hypergeometric orthogonal polynomials.</u> The orthogonality relation (2.4) can be obtained by taking residues in an orthogonality relation for ${}_4F_3$-polynomials along a complex contour. This last relation has another real form, which yields the orthogonality for Wilson polynomials. Racah and Wilson polynomials are, for the moment, the culmination of a scheme of hypergeometric orthogonal polynomials, which are related by limit transitions (the arrows below). This scheme is generally

ascribed to Askey, cf. Labelle [10];



The left column denotes the type of hypergeometric function and, in brackets, the number of parameters on which the family depends. The Wilson, continuous dual Hahn, Meixner-Pollaczek, Jacobi, Laguerre and Hermite polynomials have an absolutely continuous orthogonality measure, the other ones a discrete measure. (For simplicity, the continuous symmetric Hahn polynomials are omitted in the scheme.)

Let me formulate some problems associated with the Askey scheme:

1. Find group theoretic interpretations of all the families of polynomials in the scheme.
2. Find also group theoretic interpretations of the limit transitions.
3. Extend the Askey scheme with nonpolynomial families of orthogonal functions of the hypergeometric type (possibly orthogonality in the generalized sense).

In this paper I only consider problem 1 for the case of the Wilson polynomials. Section 2 suggests to look for this in some noncompact real form of $SL(2,\mathbb{C})xSL(2,C)xSL(,\mathbb{C})$. In this I did not yet succeed, but I will present in §4 a different group theoretic interpretation of Racah polynomials, which admits more easily analytic continuation to a noncompact case.

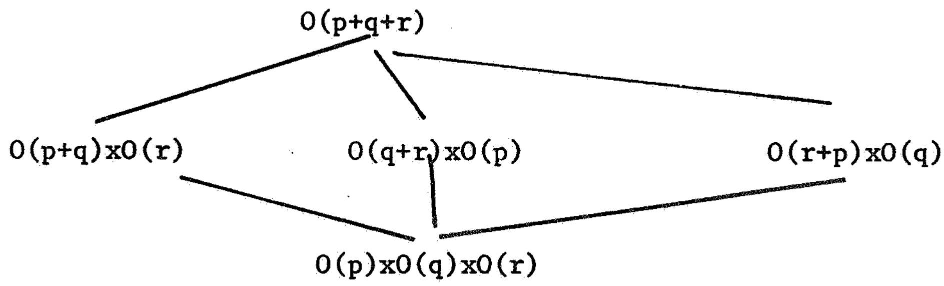
4. Racah polynomials, spherical harmonics and orthogonal polynomials on the triangle. Let H_n^p denote the space of *spherical harmonics* of degree n on the unit sphere S^{p-1} in $\mathbb{R}^p$, i.e. of the restrictions to S^{p-1} of homogeneous harmonic polynomials of degree n on $\mathbb{R}^p$. See for instance Müller [11] for the theory of spherical harmonics. The group O(n) of real orthogonal n×n matrices acts irreducibly on H_n^p, unitarily under the inner product from $L^2(S^{n-1})$. Denote this representation by π_n^p.

Lemma 4.1 (cf. [6 , Theorem 4.2]). *Let* $f \in H_{2n}^{p+q}$, *write elements of* $\mathbb{R}^{p+q}$ *as* $(x,y) \in \mathbb{R}^p \times \mathbb{R}^q$. *Then* f *behaves according to the representation* $\pi_{2m}^p \otimes \pi_0^q$ *of* $O(p) \times O(q)$ *iff*

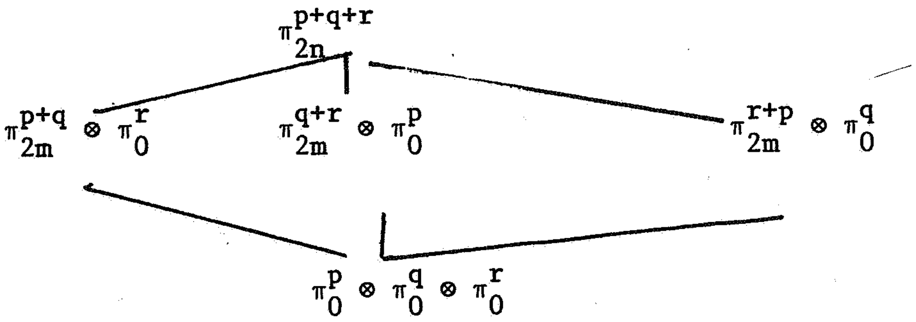
$$f(x,y) = |x|^{2m} Y(|x|^{-1}x) P_{n-m}^{(\frac{1}{2}q-1,\frac{1}{2}p-1+2k)}(1-2|y|^2), \qquad (4.1)$$

$|x|^2+|y|^2 < 1$, *for certain* $Y \in H_{2m}^p$. *Furthermore, functions of the form* (4.1) *are mutually orthogonal for different* m *and each* O(q)*-invariant* $f \in H_{2n}^{p+q}$ *can be written as a sum of functions of the form* (4.1) (m=0,1,...,n). (P_{n-m} *in* (4.1) *denotes a Jacobi polynomial.*)

Consider now the group O(p+q+r) with subgroup O(p)xO(q)xO(r) and intermediate subgroups as in the scheme:



The space of O(p)xO(q)xO(r)-invariant spherical harmonics of degree 2n on $S^{p+q+r-1}$ has in general dimension >1, but we can decompose it into subspaces of dimension 1 by using irreducible representations of one of the intermediate subgroups in the scheme:



By iteration of Lemma 4.1 we get three different orthogonal bases for the space of $O(p)\times O(q)\times O(r)$-invariant functions in H_{2n}^{p+q+r}:

$$f_{n,m}(x,y,z) := (|x|^2+|y|^2)^m P_m^{(\frac{1}{2}q-1,\frac{1}{2}p-1)}\left(1-2\frac{|y|^2}{|x|^2+|y|^2}\right) \cdot P_{n-m}^{(\frac{1}{2}r-1,\frac{1}{2}p+\frac{1}{2}q-1+2m)}(1-2|z|^2),\quad |x|^2+|y|^2+|z|^2 = 1, \tag{4.2}$$

where $m = 0,1,\ldots,n$, and two similar bases by cyclic permutation of both x,y,z and p,q,r.

We now want to express these bases in terms of each other and find the coefficients. Since $O(p)\times O(q)\times O(r)$-invariant functions on $S^{p+q+r-1}$ only depend on $|y|^2$ and $|z|^2$, we can rewrite the problem for functions in $u = |z|^2$ and $v = |y|^2$:

$$P_{n,m}^{\alpha,\beta,\gamma}(u,v) := P_{n-m}^{(\alpha,\beta+\gamma+2m+1)}(1-2u)(1-u)^m \cdot P_m^{(\beta,\gamma)}\left(1-2\frac{v}{1-u}\right), \tag{4.3}$$

where $\alpha = \frac{1}{2}r-1$, $\beta = \frac{1}{2}q-1$, $\gamma = \frac{1}{2}p-1$, and two similar families obtained by cyclic permutation of both u,v, $1-u-v$ and α,β,γ, for instance

$$Q_{n,m}^{\alpha,\beta,\gamma}(u,v) = P_{n-m}^{(\beta,\gamma+\alpha+2m+1)}(1-2v)(1-v)^m \cdot P_m^{(\gamma,\alpha)}\left(1-2\frac{1-u-v}{1-v}\right). \tag{4.4}$$

Let $\alpha,\beta,\gamma > -1$ arbitrarily. It follows from the orthogonality relations for Jacobi polynomials that both $\{P_{n,m}^{\alpha,\beta,\gamma}\}_{m=0,1,\ldots,n}$ and

$\{Q_{n,m}^{\alpha,\beta,\gamma}\}_{m=0,1,\ldots,n}$ form an orthogonal basis of the space of polynomials f on $\mathbb{R}^2$ of degree $\leq n$ for which

$$\iint_{\substack{u,v>0\\u+v<1}} f(u,v)g(u,v)u^{\alpha}v^{\beta}(1-u-v)^{\gamma}dudv = 0 \tag{4.5}$$

for all polynomials g of degree $<n$. $P_{n,m}$ respectively $Q_{n,m}$ are completely characterized up to a constant factor by the additional property

$$P_{n,m}^{\alpha,\beta,\gamma}(u,v) = \sum_{k=m}^{n}\sum_{\ell=0}^{m} a_{k,\ell}(1-u)^{k-\ell}v^{\ell}, \tag{4.6}$$

$$Q_{n,m}^{\alpha,\beta,\gamma}(u,v) = \sum_{k=m}^{n}\sum_{\ell=0}^{m} b_{k,\ell}(1-v)^{k-\ell}(1-u-v)^{\ell}, \tag{4.7}$$

for certain coefficients $a_{k,\ell}$, $b_{k,\ell}$ with $a_{n,m}$, $b_{n,m} \neq 0$. Now we have to find the coefficients in

$$Q_{n,k}^{\alpha,\beta,\gamma}(u,v) = \sum_{m=0}^{n} c_{n,m,k}^{\alpha,\beta,\gamma}\, P_{n,m}^{\alpha,\beta,\gamma}(u,v). \tag{4.8}$$

These were first obtained by Dunkl [3, Theorem 1.7] as a limit case of a similar formula for Hahn polynomials in two variables. However, there is a more direct approach by restricting (4.8) to the boundary $u = 0$ and then integrating both sides over $0 < v < 1$ with respect to the measure $P_j^{(\beta,\gamma)}(1-2v)v^{\beta}(1-v)^{\gamma}dv$. We finally arrive at

$$c_{n,m,k}^{\alpha,\beta,\gamma} = \text{elementary factor} \\ .R_n(k(k+\alpha+\gamma+1);\gamma,\beta,-n-1,\alpha+\gamma+n+1), \tag{4.9}$$

the Racah polynomial R_n being given by (2.3), which yields a new group theoretic interpretation of Racah polynomials.

It would be interesting to give an intrinsic proof that the coefficients considered here and in §2 must be the same.

5. Wilson polynomials and hyperboloid harmonics. Write elements of $\mathbb{R}^{p+q}$ as $(x,y) \in \mathbb{R}^p \times \mathbb{R}^q$. Let $H_{p,q}$ be the hyperboloid $-|x|^2+|y|^2 = 1$ in $\mathbb{R}^{p+q}$. Let $\mathcal{H}_{\lambda}^{p,q}(\lambda\in\mathbb{C})$ be the class of hyperboloid harmonics of degree $i\lambda-\frac{1}{2}(p+q)+1$, i.e. of restrictions to $H_{p,q}$ of C^{∞}-functions on

$\{(x,y)\ \mathbb{R}^{p+q}\ |-|x|^2+|y|^2>0\}$ which are even, homogeneous of degree $i\lambda-\frac{1}{2}(p+q)+1$ and are annihilated by the operator

$$\Delta_{p,x}-\Delta_{q,y} := \frac{\partial^2}{\partial x_1^2}+\ldots+\frac{\partial^2}{\partial x_p^2}-\frac{\partial^2}{\partial y_1^2}-\ldots-\frac{\partial^2}{\partial y_q^2}. \tag{5.1}$$

The (noncompact) group $O(p,q)$ of transformations which leave the form $-|x|^2+|y|^2$ invariant, acts on $\mathcal{H}_\lambda^{p,q}$ by a representation denoted by $\tau_\lambda^{p,q}$. If $\lambda > 0$ then we can associate with $\tau_\lambda^{p,q}$ an irreducible unitary transformation of $O(p,q)$ in a way which I will not make precise here. cf. Faraut [5]. Define the *Laplace-Beltrami operator* $\square_{p,q}$ on $H_{p,q}$ by the rule

$$\square_{p,q}\, f = (\Delta_{p,x}-\Delta_{q,y})F|_{H_{p,q}}, \tag{5.2}$$

where f is the restriction to $H_{p,q}$ of a C^∞-function F which is homogeneous of degree zero. Then $\mathcal{H}_\lambda^{p,q}$ consists precisely of the even C^∞-functions on $H_{p,q}$ which are eigenfunctions of $\square_{p,q}$ with eigenvalue $-\lambda^2-(\frac{1}{2}(p+q)-1)^2$.

The $O(p)\times O(q)$-invariant elements of $\mathcal{H}_\lambda^{p,q}$ are precisely the constant multiples of the Jacobi function

$$H_{p,q} \ni (x,y) \mapsto \phi_\lambda^{(\frac{1}{2}p-1,\frac{1}{2}q-1)}(\operatorname{arcsh}|x|)\ .$$

There is on $H_{p,q}$ an $O(p,q)$-invariant measure μ (unique up to a constant factor) which decomposes as

$$\int_{H_{p,q}} f(x,y)\,d\mu(x,y) = \int_{t=-\infty}^{\infty}\int_{\xi\in S^{p-1}}\int_{\eta\in S^{q-1}} f(\xi\operatorname{sh}t,\eta\operatorname{ch}t)\cdot(\operatorname{sh}t)^{p-1}(\operatorname{ch}t)^{q-1}dt\ d\xi\ d\eta,$$

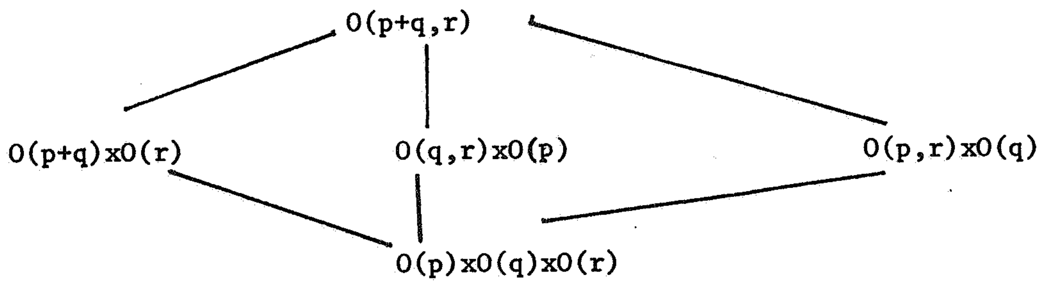
$d\xi, d\eta$ being rotation invariant measures on the spheres.

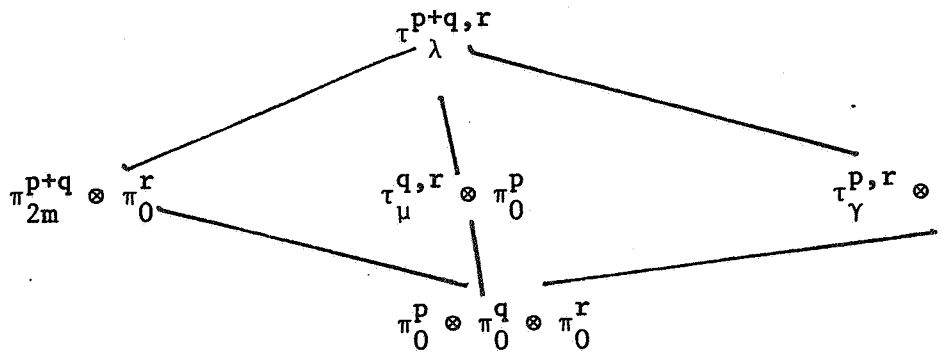
More generally, write elements of $\mathbb{R}^{p+q+r}$ as $(x,y,z) \in \mathbb{R}^p\times\mathbb{R}^q\times\mathbb{R}^r$ and consider $O(p)\times O(q)\times O(r)$-invariant elements in $\mathcal{H}_\lambda^{p+q,r}$. These are functions only depending on $|x|^2,|y|^2,|z|^2$ restricted to the hyperboloid $-|x|^2-|y|^2+|z|^2 = 1$. If we put $u := |x|^2$, $v := |y|^2$ and write the functions as functions in u,v then the property of being eigenfunctions of $\square$ translates into being eigenfunctions of

$$L := 4u(1+u)\frac{\partial^2}{\partial u^2} + 8uv\frac{\partial^2}{\partial u\partial v} + 4v(1+v)\frac{\partial^2}{\partial v^2}$$
$$+(2(p+q+r)u+2p)\frac{\partial}{\partial u} + (2(p+q+r)v+2q)\frac{\partial}{\partial v} \tag{5.3}$$

with the same eigenvalue $-\lambda^2-(\frac{1}{2}(p+q)-1)^2$. Note that the operator L is elliptic on the quarter plane $\{(u,v) \in \mathbb{R}^2 \mid u,v > 0\}$ and becomes singular at the boundary, while $\Box$ is hyperbolic and generally admits distributional eigenfunctions

The space of $O(p)\times O(q)\times O(r)$-invariant elements in $H_\lambda^{p+q,r}$ is infinite-dimensional. We can decompose it by using schemes of subgroups and corresponding representations analogous to §4:





(The representations π are as in §4.) Note that one of the intermediate subgroups is compact but the two others are noncompact. Decompositions using a noncompact subgroup will involve orthogonal systems in the generalized sense and lead to direct integral rather than direct sum decompositions. A computation shows that, corresponding to the representations $\pi_{2m}^{p+q} \otimes \pi_0^r$ and $\tau_\mu^{q,r} \otimes \pi_0^p$, respectively, we have $O(p)\times O(q)\times O(r)$-invariant elements $\phi_{\lambda,m}$ and $\psi_{\lambda,\mu}$ in $H_\lambda^{p+q,r}$ as follows. (They are written as functions of $u = |x|^2$, $v = |y|^2$.)

$$\phi_{\lambda,m}(u,v) := \phi_\lambda^{(\frac{1}{2}(p+q)+2m-1,\frac{1}{2}r-1)}(\operatorname{arcsh}(u+v)^{\frac{1}{2}}) \cdot (u+v)^m P_m^{(\frac{1}{2}p-1,\frac{1}{2}q-1)}\left(\frac{-u+v}{u+v}\right), \tag{5.4}$$

$$\psi_{\lambda,\mu}(u,v) := \phi_\lambda^{(\frac{1}{2}p-1,i\mu)}(\operatorname{arcsh}u^{\frac{1}{2}}) \cdot (u+1)^{\frac{1}{2}i\mu-\frac{1}{4}(q+r)+\frac{1}{2}}\phi_\mu^{(\frac{1}{2}q-1,\frac{1}{2}r-1)}(\operatorname{arcsh}(v^{\frac{1}{2}}(u+1)^{-\frac{1}{2}})). \tag{5.5}$$

This is all in perfect correspondence with §4. In fact, the above functions might be obtained from §4 by analytic continuation. Now we have to make precise that each representation in the last diagram occurs with multiplicity one in the representation(s) in the row above. I will make this plausible by showing that each $O(p)xO(q)xO(r)$-invariant C_c^∞-function (or L^2-function) on $H_{p+q,r}$ can be fully decomposed in terms of either the functions $\phi_{\lambda,m}$ or the functions $\psi_{\lambda,\mu}$. (For convenience, in order to avoid discrete components of the spectrum, we assume here that $q \le p+2$ and $r \le p+q+2$.)

Let f be an $O(p)xO(q)xO(r)$-invariant C^∞-function with compact support on $H_{p+q,r}$ and write it as a function of $(u,v) \in \mathbb{R}_+ \times \mathbb{R}_+$. The invariant measure on $H_{p+q,r}$ then takes the form

$$\Delta(u,v)dudv := u^{\frac{1}{2}p-1}v^{\frac{1}{2}q-1}(1+u+v)^{\frac{1}{2}r-1}dudv. \tag{5.6}$$

It follows from the orthogonality relations for Jacobi polynomials and the inversion formula for the Jacobi function transform that we have the integral transform pair

$$(F_1 f)(\lambda,m) = \int_0^\infty\int_0^\infty f(u,v)\phi_{\lambda,m}(u,v)\Delta(u,v)dudv,$$

$$f(u,v) = \sum_{m=0}^\infty \int_{\lambda=0}^\infty d\lambda (F_1 f)(\lambda,m)\phi_{\lambda,m}(u,v) \cdot \frac{m!\,\Gamma(m+\frac{1}{2}p+\frac{1}{2}q-1)(2m+\frac{1}{2}p+\frac{1}{2}q-1)}{4\pi\Gamma(m+\frac{1}{2}p)\Gamma(m+\frac{1}{2}q)} \cdot \left|\frac{\Gamma(\frac{1}{2}(i\lambda+\frac{1}{2}(p+q+r)+2m-1))\Gamma(\frac{1}{2}(i\lambda+\frac{1}{2}(p+q-r)+2m+1))}{\Gamma(\frac{1}{2}(p+q)+2m)\Gamma(i\lambda)}\right|^2. \tag{5.7}$$

It extends to an isometry of $L^2((\mathbb{R}_+)^2, \Delta(u,v)dudv)$ onto $L^2(\mathbb{R}_+\times\mathbb{Z}_+)$ with appropriate measure.

A similar transform with $\psi_{\lambda,\mu}$ involves a Jacobi function transform with unusual imaginary parameter, which still can be inverted by the methods of [7], [8,§6]:

$$G(\lambda) = \int_0^\infty F(t)\phi_\lambda^{(\alpha,i\gamma)}(t)(\mathrm{ch}t)^{i\gamma}(\mathrm{sh}t)^{2\alpha+1}\mathrm{ch}t\, dt$$

$$F(t) = (2\pi)^{-1}\int_0^\infty G(\lambda)\phi_\lambda^{(\alpha,i\gamma)}(t)(\mathrm{ch}t)^{i\gamma} \cdot \left|\frac{\Gamma(\frac{1}{2}(i\lambda+\alpha-i\gamma+1))\Gamma(\frac{1}{2}(i\lambda+\alpha+i\gamma+1))}{\Gamma(\alpha+1)\Gamma(i\lambda)}\right|^2 d\lambda \tag{5.8}$$

Note that $\phi_\lambda^{(\alpha,i\gamma)}(t)(\mathrm{ch}t)^{i\gamma}$ is real. Now combination of (1.14) and (5.8) yields the integral transform pair

$$(F_2f)(\lambda,\mu) = \int_0^\infty\int_0^\infty f(u,v)\psi_{\lambda,\mu}(u,v)\Delta(u,v)dudv,$$

$$f(u,v) = \int_0^\infty\int_0^\infty d\lambda d\mu\ (F_2f)(\lambda,\mu)\psi_{\lambda,\mu}(u,v) \cdot \left|\frac{2^{\frac{1}{2}(q+r)-2}\Gamma(\frac{1}{2}(i\lambda+\frac{1}{2}p-i\mu))\Gamma(\frac{1}{2}(i\lambda+\frac{1}{2}p+i\mu))}{\pi\Gamma(\frac{1}{2}p)\Gamma(i\lambda)\Gamma(\frac{1}{2}(i\mu+\frac{1}{2}(q+r)-1))\Gamma(\frac{1}{2}(i\mu+\frac{1}{2}(q-r)+1))}\right|^2 . \tag{5.9}$$

F_2 again extends to an isometry of L^2-spaces.

For a completely neat treatment we should relate the transforms F_1 and F_2 to the Fourier transform for general C^∞-functions with compact support on a hyperboloid, cf. Faraut [5], but we omit it here, since it is not needed for the final appearance of Wilson polynomials.

We can now state our final result

<u>Theorem 5.1</u>. *The identity*

$$\psi_{\lambda,\mu} = \sum_{m=0}^\infty \frac{(\frac{1}{2}(p+q)-1)_m}{(\frac{1}{2}p)_m(\frac{1}{2}q)_m(\frac{1}{2}(p+q)-1)_{2m}} \cdot W_m(\tfrac{1}{4}\mu^2; \tfrac{p+2i\lambda}{4}, \tfrac{p-2i\lambda}{4}, \tfrac{q+r-2}{4}, \tfrac{q-r+2}{4})\phi_{\lambda,m} \tag{5.10}$$

is valid in the weak sense that (5.10) *holds with* $\psi_{\lambda,\mu}$ *replaced by* $(\mathcal{F}_2 f)(\lambda,\mu)$ *and* $\phi_{\lambda,m}$ *by* $(\mathcal{F}_1 f)(\lambda,m)$, *for any* $f \in C^\infty([0,\infty)\times[0,\infty))$.

The theorem follows from the identity

$$(\mathrm{sh}\,t)^{-2m} \int_{-1}^{1} P_m^{(\frac{1}{2}p-1,\frac{1}{2}q-1)}(x)\psi_{\lambda,\mu}(\tfrac{1-x}{2}\,\mathrm{sh}^2 t,\ \tfrac{1+x}{2}\,\mathrm{sh}^2 t)$$
$$\cdot(1-x)^{\frac{1}{2}p-1}(1+x)^{\frac{1}{2}q-1}dx = \frac{2^{\frac{1}{2}p+\frac{1}{2}q-1}\Gamma(\frac{1}{2}p)\Gamma(\frac{1}{2}q)}{m!\,\Gamma(\frac{1}{2}(p+q)+2m)} \tag{5.11}$$
$$\cdot W_m(\tfrac{1}{4}\mu^2;\ \tfrac{p+2i\lambda}{4},\ \tfrac{p-2i\lambda}{4},\ \tfrac{q+r-2}{4},\ \tfrac{q-r+2}{4})$$
$$\cdot\phi_\lambda^{(\frac{1}{2}p+\frac{1}{2}q+2m-1,\frac{1}{2}r-1)}(t).$$

This formula, in its turn, can be proved by showing that the left hand side satisfies the differential equation (1.7) with $\alpha = \frac{1}{2}p+\frac{1}{2}q+2m-1$, $\beta = \frac{1}{2}r-1$. Hence it must be equal to

$$C_1 c_{\alpha,\beta}(\lambda)e^{(i\lambda-\alpha-\beta-1)t} + C_2 c_{\alpha,\beta}(-\lambda)e^{(-i\lambda-\alpha-\beta-1)t}$$
$$+\ o(e^{(-\alpha-\beta-1)t}) \quad \text{as} \quad t\to\infty.$$

On the other hand, by use of (5.5), estimates and asymptotics for the Jacobi functions occurring there and the dominated convergence theorem, it follows that the constants C_1 and C_2 are equal to an elementary factor times the left hand side of (1.19) with $\alpha,\beta,\delta,\lambda,\mu$ replaced by $\frac{1}{2}q-1$, $\frac{1}{2}r-1$, $\frac{1}{2}p-1$, μ, $\pm\lambda$, respectively, and that $C_1 = C_2$. Then application of (1.19) yields (5.11).

<u>Remark 5.2</u>. It would be interesting to look for the kernel which sends $O(p)\times O(q)\times O(r)$-invariant elements in $\mathcal{H}_\lambda^{p+q,r}$ labeled by $\tau_\mu^{q,r} \otimes \pi_0^p$ to ones labeled by $\tau_\nu^{p,r} \otimes \pi_0^q$. It will be a generalized orthogonal system, probably consisting of at least ${}_4F_3$-hypergeometric functions.

<u>Remark 5.3</u>. Analogues of the results of sections 4 and 5 in this paper have been obtained by Suslov [16] for Hahn polynomials respectively continuous dual Hahn polynomials, but with one degree of freedom missing in the

parameters. He did this in connection with the Schrödinger equation. It should be possible to obtain his results as limit cases of the ones given here and to relate his results to [9,(5.14)].

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