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Transition Systems, Metric Spaces and Ready Sets in the Semantics of Uniform Concurrency

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Transition systems as proposed by Hennessy and Plotkin are defined for a series of three languages featuring concurrency. The first has shuffle and local nondeterminacy, the second synchronization merge and local nondeterminacy, and the third synchronization merge and global nondeterminacy. The languages are all uniform in the sense that the elementary actions are uninterpreted. Throughout, infinte behaviour is taken into account and modelled with infinitary languages in the sense of Nivat. A comparison with denotational semantics is provided. For the first two languages, a linear time model suffices; for the third language a branching time model with processes in the sense of De Bakker & Zucker is described. In the comparison an important role is played by an intermediate semantics in the style of Hoare & Olderog's specification oriented semantics. A variant on the notion of ready set is employed here. Precise statements are given relating the various semantics in terms of a number of abstraction operators.

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1. INTRODUCTION

Our paper aims at presenting a thorough study of the semantics of a number of concepts in concurrency. We concentrate on shuffle and synchronization merge, local and global nondeterminacy, and deadlocks. Somewhat more specifically, we provide a systematic analysis of these concepts by confronting, for three sample languages, semantic techniques inspired by earlier work due to Hennessy and Plotkin [HP, Pℓ1, Pℓ2] proposing an operational approach, De Bakker et al. [BBKM, BZ1, BZ2, BZ3] for a denotational one, and the Oxford School [BHR, OH1, OH2, RB] serving - for the purposes of our paper - an intermediate role.

Our operational semantics is based on transition systems [Ke] as employed successfully in [HP, Pℓ1, Pℓ2]; applications in the analysis of proof systems were developed by Apt [Ap1, Ap2]. Compared with previous instances, our definitions exhibit various novel features: (i) the use of a model involving languages with finite and *infinite* words (cf. Nivat [Ni]) or *streams* [Br]; (ii) the use of full recursion (based on the copy rule) rather than just iteration; (iii) an appealingly simple treatment of synchronization; (iv) a careful distinction between local and global nondeterminacy; (v) the restriction to *uniform* concurrency.

Throughout the paper we only consider uniform statements: by this we mean an approach at the *schematic* level, leaving the elementary actions uninterpreted and avoiding the introduction of notions such as assignments or states. Many interesting issues arise at this level, and we feel that it is advantageous to keep questions which arise after interpretation for a treatment at a second level (not dealt with in our paper).

We shall study three languages in increasing order of complexity:

$\mathcal{L}_0$: shuffle (arbitrary interleaving) + local nondeterminacy

$\mathcal{L}_1$: synchronization merge + local nondeterminacy

$\mathcal{L}_2$: synchronization merge + global nondeterminacy

For $\mathcal{L}_i$ with typical elements s, we shall present transition system T_i and define an induced operational semantics $\mathcal{O}_i[\![s]\!]$, $i = 0,1,2$. We shall also define three denotational semantics $\mathcal{D}_i[\![s]\!]$ based, for $i = 0,1$ on the "linear time" (LT) model which employs sets of sequences and, for $i = 2$, on the "branching time" (BT) model employing *processes* (commutative trees, with sets rather than multisets of successors for any node, and with certain closure properties) of [BBKM, BZ1, BZ2]. Throughout our paper we provide $\mathcal{D}_i$ *only* for $\mathcal{L}_i$ restricted to *guarded* recursion (each recursive call has to be preceded by some elementary action); we then have an attractive metric setting with unique fixed points for contractive functions based on Banach's fixed point theorem. (Our $\mathcal{O}_i$ do assign meaning to the unguarded case as well.)

Our main question can now be posed: Do we have that

$$\mathcal{O}_i[\![s]\!] = \mathcal{D}_i[\![s]\!] \ . \qquad (1.1)$$

We shall show that (1.1) only holds for $i = 0$. For the more sophisticated languages $\mathcal{L}_i$, $i = 1,2$, we cannot prove (1.1). In fact, we can even show that there exists no denotational $\mathcal{D}_i$ satisfying (1.1), $i = 1,2$. Rather than trying to modify $\mathcal{O}_i$ (thus spoiling its intuitive operational character) we propose to replace (1.1) by

$$\mathcal{O}_i[\![s]\!] = \alpha_i(\mathcal{D}_i[\![s]\!]) \tag{1.2}$$

where α_i, $i = 1,2$, is an *abstraction* operator which forgets some information present in $\mathcal{D}_i[\![s]\!]$. The proof of (1.2) requires an interesting technique of introducing a transition based *intermediate* semantics $\mathcal{O}_i^*[\![s]\!]$. For $i = 1$ we shall show that $\mathcal{O}_i^*[\![s]\!] = \mathcal{D}_i[\![s]\!]$. Next, we introduce our first abstraction operator α_1 (turning each failing communication into an indication of failure and deleting all subsequent actions) and prove that $\mathcal{O}_i^*[\![s]\!] = \alpha_1(\mathcal{O}_i^*[\![s]\!])$.

The case $i = 2$ is more involved, because $\mathcal{L}_1$ has *local*, and $\mathcal{L}_2$ *global* nondeterminacy. Consider a choice a or c, where a is some autonomous action and c needs a parallel $\bar{c}$ to communicate. In the case of local nondeterminacy (written as $a \cup c$) both actions may be chosen; in the global nondeterminacy case (written as $a + c$ "+" as in CCS [Mi]) c is chosen *only* when in some parallel compound $\bar{c}$ is ready to execute. Therefore, $\mathcal{L}_1$ and $\mathcal{L}_2$ exhibit different deadlock behaviours.

$\mathcal{O}_2$ is based on the transition system T_2 which is a refinement of T_1, embodying a more subtle set of rules to deal with nondeterminacy. The denotational semantics $\mathcal{D}_2$ is as in [BBKM, BZ1, BZ2]. In order to relate $\mathcal{D}_2$ and $\mathcal{O}_2$ we introduce the notion of *readies* and an associated intermediate semantics $\mathcal{O}_2^*$, inspired by ideas described in [BHR, OH1, OH2, RB].

$\mathcal{O}_2^*$ involves an extension of the LT model with some branching information (though less than the full BT model) which is amenable to a treatment in terms of transitions. Besides the operational $\mathcal{O}_2^*$ we also base an intermediate denotational semantics $\mathcal{D}_2^*$ on the domain of readies. To prove the desired result (1.2) for $\mathcal{L}_2$, we shall show that $\mathcal{O}_2^*[\![s]\!] = \mathcal{D}_2^*[\![s]\!]$ and then relate $\mathcal{O}_2$ with $\mathcal{O}_2^*$, $\mathcal{D}_2^*$ with $\mathcal{D}_2$, and thus $\mathcal{O}_2$ with $\mathcal{D}_2$ by a careful choice of suitable abstraction operators.

As main contributions of our paper we see:

1. The three transition systems T_i, in particular the refinement of T_1 into T_2.
2. The systematic treatment of the denotational semantics definitions (for the guarded case) together with the settling of the relationship $\mathcal{O}_i = \alpha_i \circ \mathcal{D}_i$. ($\alpha_0$ identity).
3. Clarification of local versus global nondeterminacy and associated deadlock behaviour.
4. The technique of intermediate semantics $\mathcal{O}_1^*$ and, in particular, $\mathcal{O}_2^*$ and $\mathcal{D}_2^*$.

The rest of our paper is organized into Sections 2 - 4 dealing with the languages $\mathcal{L}_0$ - $\mathcal{L}_2$. For each language $\mathcal{L}_i$ the corresponding section is divided into four subsections. The first three introduce the transition system T_i, the operational semantics $\mathcal{O}_i$ and the denotational semantics $\mathcal{D}_i$, respectively. Most demanding is the fourth one which settles the relationship between $\mathcal{O}_i$ and $\mathcal{D}_i$ by establishing $\mathcal{O}_i = \alpha_i \circ \mathcal{D}_i$. To avoid repetitions, we elaborate on a different aspect for each $\mathcal{L}_i$. For $\mathcal{L}_0$ we concentrate on recursion, for $\mathcal{L}_1$ on synchronization merge and for $\mathcal{L}_2$ on the intermediate ready semantics.

Finally, an appendix summarizes all results in a diagram.

2. THE LANGUAGE $\mathcal{L}_0$: SHUFFLE AND LOCAL NONDETERMINACY

Let A be a finite set of uninterpreted, elementary *actions*, with $a \in A$. Let x, y be elements of the set stmv of *statement variables* (used in fixed point constructs for recursion). The set $\mathcal{L}_0$ of (*concurrent*) *statements*, with $s, t \in \mathcal{L}_0$, is given by the following syntax:

$$s ::= a \mid s_1 ; s_2 \mid s_1 \cup s_2 \mid s_1 \| s_2 \mid x \mid \mu x[s] .$$

Thus every action $a \in A$ denotes a statement, the one which finishes (successfully terminates) after performing a. $s_1 ; s_2$ denotes (*sequential*) *composition* such that s_2 starts once s_1 has finished. $s_1 \cup s_2$ denotes *nondeterministic choice*, also known as *local nondeterminism* [FHLR]. $s_1 \| s_2$ denotes *concurrent execution* of s_1 and s_2 modelling *shuffle* (arbitrary interleaving) between the actions of s_1 and s_2. $\mu x[s]$ is a recursive statement. For example, with the definitions to be proposed presently, the intended meaning of $\mu x[(a;x) \cup b]$ is the set $a^* \cdot b \cup \{a^\omega\}$, where a^ω is the infinite sequence of a's.

In general, we will restrict attention to *syntactically closed* statements (i.e. those without free statement variables), since only such statements have a meaning under the operational semantics to be defined below. (We will not always state this explicitly.)

2.1 *The Transition System* T_0

A transition describes what a statement s can do as its next step. This concept of a transition dates back to [Ke] and to automata theoretic notions [RS]. Following Hennessy and Plotkin [HP, Pℓ1], a transition system is a syntax-directed deductive system for proving transitions (see also [Ap1, Ap2, Pℓ2]). In this section we use this idea for $\mathcal{L}_0$.

First we need some notation. Let $\perp \notin A$. Then the set A^{st} of *words* [Ni] or *streams* [Br], with $u, v, w \in A^{st}$, is defined as

$$A^{st} = A^* \cup A^\omega \cup A^* \cdot \{\perp\} .$$

A^{st} includes the set $A^\infty = A^* \cup A^\omega$ of *finite* and *infinite* words or streams over A [Ni] , and additionally the set $A^* \cdot \{\perp\}$ of *unfinished* words or streams. Let ε denote the *empty* word and $\leq$ the *prefix relation* over words. We define $\perp \cdot w = \perp$ for all w .

A *configuration* is a pair $\langle s,w \rangle$ or just a word w . A *transition relation* is a binary relation $\rightarrow$ over configurations [Ke] . A *transition* is a formula $\langle s,w \rangle \rightarrow \langle s',w' \rangle$ or $\langle s,w \rangle \rightarrow w'$ denoting an element of a transition relation. A *transition system* is a formal deductive system for proving transitions, based on *axioms* and *rules*. Using a self-explanatory notation, axioms have the format $1 \rightarrow 2$, rules have the format $\frac{1 \rightarrow 2}{3 \rightarrow 4}$. For a transition system T , $T \vdash 1 \rightarrow 2$ expresses that transition $1 \rightarrow 2$ is deducible in the system T . Then $1 \rightarrow 2$ is also called a *T-transition*. For a finite sequence $1 \rightarrow 2 \rightarrow \ldots \rightarrow n$ of T-transitions, we also write $T \vdash 1 \rightarrow^* n$.

We will present a particular transition system T_0 for $\mathcal{L}_0$. Before doing so, we introduce a notation which permits a compact representation of the transition rules.

We follow Apt [Ap1, Ap2] and explicitly allow the *empty statement* E (not present in $\mathcal{L}_0$). We then assume identifications between expressions generated by the following equalities:

$$\langle E, w\rangle = w ,$$
$$s = s;E = E;s = s\|E = E\|s .$$

Then in the notation

$$\langle s, w\rangle \rightarrow \langle s', w'\rangle , \qquad (2.1)$$

the pair $\langle s', w'\rangle$ on the r.h.s. has two possible interpretations: (i) as shown, with $s' \in \mathcal{L}_0$, and also (ii) with $s' = E$ and $\langle s', w'\rangle = w'$. Thus (2.1) represents either of the transitions

(i) $\langle s, w\rangle \rightarrow \langle s', w'\rangle$ (with $s' \in \mathcal{L}_0$),

(ii) $\langle s, w\rangle \rightarrow w'$.

We now present the system T_0. [1]

For $w \in A^\omega \cup A^* \cdot \{\perp\}$ and $s \in \mathcal{L}_0$ we put

$$\langle s,w\rangle \rightarrow w ,$$

and for $w \in A^*$ we distinguish the following cases:

(elementary action)

$$\langle a,w\rangle \rightarrow w \cdot a$$

(local nondeterminacy)

$$\langle s_1 \cup s_2,w\rangle \rightarrow \langle s_1,w\rangle$$

$$\langle s_1 \cup s_2,w\rangle \rightarrow \langle s_2,w\rangle$$

(recursion)

$$\langle \mu x[s], w\rangle \to \langle s[\mu x[s]/x], w\rangle$$

where, in general, $s[t/x]$ denotes substitution of t for x in s. Thus recursion is described here by syntactic substitution or copying.

(composition)

$$\frac{\langle s_1, w\rangle \to \langle s', w'\rangle}{\langle s_1; s_2, w\rangle \to \langle s'; s_2, w'\rangle}$$

(shuffle)

$$\frac{\langle s_1, w\rangle \to \langle s', w'\rangle}{\langle s_1 \| s_2, w\rangle \to \langle s' \| s_2, w'\rangle}$$

$$\frac{\langle s_1, w\rangle \to \langle s', w'\rangle}{\langle s_2 \| s_1, w\rangle \to \langle s_2 \| s', w'\rangle}$$

Note that our convention regarding the empty statement applies to the composition and shuffle rules given above. Thus, for example, the first shuffle rule has two interpretations: (i) as shown, with $s' \in \mathcal{L}_0$, and also (ii):

$$\frac{\langle s_1, w\rangle \to w'}{\langle s_1 \| s_2, w\rangle \to \langle s_2, w'\rangle} \quad .$$

At the beginning of this section we said that a transition describes what a statement can do as its next step. For T_0 this is made precise by the following lemma.

2.1.1 LEMMA *(Initial Step)*. $T_0 \vdash \langle s,w\rangle \to \langle s',w'\rangle$ iff there exists some $b \in A \cup \{\varepsilon\}$ with $w' = w \cdot b$ and $T_0 \vdash \langle s,\varepsilon\rangle \to \langle s',b\rangle$.

PROOF. By structural induction on s. □

2.2 *The Operational Semantics* $\mathfrak{S}_0$

By an *operational semantics* we mean here a semantics which is defined with the help of a transition system. As a first example we introduce now an operational semantics $\mathfrak{S}_0$ for $\mathcal{L}_0$. Formally, $\mathfrak{S}_0$ is a mapping

$$\mathfrak{S}_0 : \mathcal{L}_0 \to \mathbb{S}$$

with $\mathbb{S} = \wp(A^{st})$ denoting the set of *infinitary languages*, which may contain both finite and infinite words over A.

We first give some definitions.

(1) A *transition sequence* is a (finite or infinite) sequence of T_0-transitions.

(2) A *path* from s is a maximal transition sequence

$$\pi : \langle s_0,w_0\rangle \to \langle s_1,w_1\rangle \to \langle s_2,w_2\rangle \to \dots$$

where $s_0 = s$ and $w_0 = \varepsilon$.

(3) The *word* associated with a path π, $\underset{\sim}{\text{word}}\,(\pi)$, is defined according to the following three cases.

(a) π is finite, and of the form

$$\langle s_0,w_0\rangle \to \dots \to \langle s_n,w_n\rangle \to w.$$

Then word $(\pi) = w$.

(b) π is infinite:

$$\langle s_0,w_0\rangle \to \dots \to \langle s_n,w_n\rangle \to \langle s_{n+1},w_{n+1}\rangle \to \dots$$

and the sequence $(w_n)_n$ is infinitely often increasing.

Then word $(\pi) = \sup_n w_n$ (sup w.r.t. the prefix ordering), an infinite word.

(c) π is infinite as in (b), but the sequence $(w_n)_n$ is eventually constant, i.e. for some n $w_{n+k} = w_n$ for all $k \geq 0$.

Then word $(\pi) = w_n \cdot \perp$.

It is easy to see that these are the only three possibilities for a path in T_0.

We now define for $s \in \mathcal{L}_0$:

$$\mathcal{O}_0[\![s]\!] = \{\text{word } (\pi) \mid \pi \text{ is a path from } s\} .$$

EXAMPLES. $\mathcal{O}_0[\![(a_1;a_2)\|a_3]\!] = \{a_1a_2a_3,\ a_1a_3a_2,\ a_3a_1a_2\}$,

$\mathcal{O}_0[\![\mu_2[a;x) \cup b]]\!] = a^* \cdot b \cup \{a^\omega\}$,

$\mathcal{O}_0[\![\mu\, x[(x;a) \cup b]]\!] = b \cdot a^* \cup \{\perp\}$.

We conclude with two simple facts about $\mathcal{O}_0$.

2.2.1 LEMMA (*Definedness*). $\mathcal{O}_0$ is well-defined, i.e. $\mathcal{O}_0[\![s]\!] \neq \phi$ for every $s \in \mathcal{L}_0$.

PROOF. The claim follows from the fact that for each configuration $\langle s,w \rangle$ at least one transition $\langle s,w \rangle \rightarrow \langle s',w' \rangle$ exists in T_0. □

2.2.2 LEMMA (*Prolongation*). If $T_0 \vdash \langle s,\varepsilon \rangle \rightarrow^* \langle s',w \rangle$ and $w' \in \mathfrak{S}_0[\![s']\!]$, then also $w \cdot w' \in \mathfrak{S}_0[\![s]\!]$.

Proof. By the definition of $\mathfrak{S}_0$ and Lemma 2.1.1. □

We remark that corresponding lemmas will also hold for the operational semantics to be discussed subsequently.

2.3 *The Denotational Semantics* $\mathfrak{D}_0$

The operational semantics $\mathfrak{S}_0$ for $\mathfrak{L}_0$ is global in the following sense: to determine $\mathfrak{S}_0[\![s]\!]$ we first have to explore the T_0-transition sequences for all of s, and only then we can retrieve the result $\mathfrak{S}_0[\![s]\!]$. Further, in T_0, and thus in $\mathfrak{S}_0$, recursion is dealt with by syntactic copying. We now look for a *denotational semantics* $\mathfrak{D}_0$ for $\mathfrak{L}_0$. A denotational semantics should be *compositional* or *homomorphic*, i.e. for every syntactic operator $\underset{\sim}{op}$ in $\mathfrak{L}_0$ there should be a corresponding semantic operator $\underset{\sim}{op}^{\mathfrak{D}_0}$ satisfying

$$\mathfrak{D}_0[\![s_1 \underset{\sim}{op} s_2]\!] = \mathfrak{D}_0[\![s_1]\!] \underset{\sim}{op}^{\mathfrak{D}_0} \mathfrak{D}_0[\![s_2]\!],$$

and it should tackle recursion semantically with help of *fixed points*. This of course requires a suitable structure of the underlying semantic domain.

For $\mathcal{S}_0$ we shall use metric spaces (rather than the more customary cpo's) as semantic domain. Our approach is based on [BBKM], [BZ2]; for general topological notions such as *closedness*, *limits*, *continuity* and *completeness*, see [Du].

Following [BZ2], $\mathcal{S}_0$ will be defined only for *guarded* statements, a notion which we define below. We must first define the notion of an *exposed* occurrence of a substatement in a given statement.

REMARK. By "(occurrence of) a substatement of a statement s", we will always mean a statement not containing any free statement variables which are bound in s. For example, a;x is a substatement of $\mu y[a;x;y]$, but not of $\mu x[a;x;y]$.)

We now define the notion: an occurrence of a substatement t of s is *exposed* in s. The definition is by induction on the structure of s:

(a) s is *exposed* in s. (More accurately, the unique occurrence of s in s is exposed in s.)

(b) If an occurrence of t is *exposed* in s_1, then (and only then) it is also *exposed* in $s_1;s_2$, $s_1 \| s_2$, $s_2 \| s_1$, $s_1 \cup s_2$, $s_2 \cup s_1$ and $\mu x[s_1]$ (and also $s_1 + s_2$ and $s_2 + s_1$, in the case of the language $\mathcal{L}_2$ of Section 4).

EXAMPLE. In the statement $x;a \cup b;x$, the first occurrence of x is *exposed*, while the second is not.

A statement is now defined to be *guarded* (cf. [Mi] or [Ni]) if for all its recursive substatements $\mu x[t]$, t contains *no exposed occurrences* of x.

EXAMPLES. $\mu x[a;(x\|b)]$ is guarded, but $\mu x[x]$, $\mu y[y\|b]$ and $\mu x[\mu y[x]]$ (as well as statements containing these) are not.

One advantage of the guardedness restriction is that we will be able to invoke Banach's classical fixed point theorem when dealing with recursion.

Let us now introduce the metric domain for $\mathcal{S}_0$. For $u \in A^{st}$ let $u[n]$, $n \geq 0$, be the prefix of u of length n if this exists; otherwise $u[n] = u$. E.g., $abc[2] = ab$, $abc[5] = abc$. We define a natural metric d on A^{st} by putting

$$d(u,v) = 2^{-\max\{n \mid u[n] = v[n]\}}$$

with the understanding that $2^{-\infty} = 0$. For example, $d(abc,abd) = 2^{-2}$, $d(a^n,a^\omega) = 2^{-n}$. We have that (A^{st},d) is a complete metric space. For $X \subseteq A^{st}$ we put $X[n] = \{u[n] \mid u \in X\}$. A distance $\hat{d}$ on subsets X,Y of A^{st} is defined by

$$\hat{d}(X,Y) = 2^{-\max\{n \mid X[n] = Y[n]\}}.$$

Let $\mathbb{S}_c \subset \mathbb{S}$ denote the collection of all metrically *closed* subsets of A^{st}. It can be shown that $(\mathbb{S}_c, \hat{d})$ is a complete metric space (see [Ha]). A sequence $\langle x_i \rangle_{i=0}^{\infty}$ of elements of $\mathbb{S}_c$ is a *Cauchy sequence* whenever $\forall \epsilon > 0\ \exists N\ \forall n,m \geq N[\hat{d}(X_n,X_m) < \epsilon]$. For $\langle X_i \rangle_i$ a Cauchy sequence, we write $\lim_i X_i$ for its limit (which belongs to $\mathbb{S}_c$ by the completeness property).

A function $\Phi: (\mathbb{S}_c, \hat{d}) \to (\mathbb{S}_c, \hat{d})$ is called *contracting* whenever, for all X,Y, $\hat{d}(\Phi(X),\Phi(Y)) \leq \alpha$ for some real number α with $0 \leq \alpha < 1$. A classical theorem due to Banach states that in any complete metric space, a contracting function has a *unique* fixed point obtained as $\lim_i \Phi^i(X_0)$ for arbitrary starting point X_0.

We now define the semantic operators $;^{\mathcal{S}_0}$, $\cup^{\mathcal{S}_0}$ and $\|^{\mathcal{S}_0}$ on $\mathbb{S}_c$. (For ease of notation, we skip superscripts $\mathcal{S}_0$ if no confusion arises.)

a. $X,Y \subseteq A^* \cup A^* \cdot \{\bot\}$. For $X;Y =_{df} X \cdot Y$ (concatenation) and $X \cup Y$ (set-theoretic union) we adopt the usual definitions (including the clause $\bot \cdot u$ for all u). For $X\|Y$ (shuffle or merge) we introduce as auxiliary operator the so-called left-merge $⫽$ (from [BK]). It permits a particularly simple definition of $\|$ by putting

$$X\|Y = (X ⫽ Y) \cup (Y ⫽ X)$$

where $⫽$ is given recursively by $X ⫽ Y = \cup\{u ⫽ Y \mid u \in X\}$ with $\epsilon ⫽ Y = Y$, $(a \cdot u) ⫽ Y = a \cdot (\{u\}\|Y)$ and $\bot ⫽ Y = \{\bot\}$.

b. $X,Y \in \mathbb{S}_c$ where X,Y do not consist of finite words only. Then

$$X \underset{\sim}{op} Y = \lim_i (X[i] \underset{\sim}{op} Y[i]),$$

for $\underset{\sim}{op} \in \{;, \cup, \|\}$. In [BZ2] we have shown that this definition is well-formed and preserves closed sets, and the operators are *continuous* (assuming finiteness of A, as in [BBKM]).

We now turn to the definition of $\mathcal{D}_0$. We introduce the usual notion of *environment* which is used to store and retrieve meanings of statement variables. Let $\Gamma_0 = \underset{\sim}{\text{stmv}} \to \mathbb{S}_c$ be the set of environments, and let $\gamma \in \Gamma_0$. We write $\gamma' =_{df} \gamma\langle X/x\rangle$ for a *variant* of γ which is like γ but with $\gamma'(x) = X$. We define

$$\mathcal{D}_0\text{: guarded } \mathcal{L}_0 \to (\Gamma_0 \to \mathbb{S}_c)$$

as follows:

1. $\mathcal{D}_0[\![a]\!](\gamma) = \{a\}$
2. $\mathcal{D}_0[\![s_1 \underset{\sim}{op}\, s_2]\!](\gamma) = \mathcal{D}_0[\![s_1]\!](\gamma)\, \underset{\sim}{op}\, \mathcal{D}_0[\![s_2]\!](\gamma)$
3. $\mathcal{D}_0[\![x]\!](\gamma) = \gamma(x)$
4. $\mathcal{D}_0[\![\mu x[s]]\!](\gamma) = \lim_i X_i$, where $X_0 = \{\bot\}$ and $X_{i+1} = \mathcal{D}_0[\![x]\!](\gamma\langle X_i/x\rangle)$.

By the guardedness requirement, each function $\Phi = \lambda X \,.\, \mathcal{D}_0[\![\, s\,]\!]\,(\gamma\langle X/x\rangle)$ is contracting, $\langle X_i\rangle_i$ is a Cauchy sequence, and $\lim_i x_i$ equals the unique fixed point of Φ [Ni, BBKM, BZ2]. For statements s without free statement variables we write $\mathcal{D}_0[\![s]\!]$ instead of $\mathcal{D}_0[\![s]\!](\gamma)$. Since $\mathcal{D}_0[\![s]\!]$ is a set of (linear) streams, $\mathcal{D}_0$ is called a *linear time semantics* [BBKM]. (Such a semantics may constitute the basis for a linear time temporal logic for $\mathcal{L}_0$.)

REMARK. An order-theoretic approach to the denotational model is also possible ([Br, Me, BMO], see also our survey [BKMOZ]), but less convenient for our special purposes. In fact, the order-theoretic approach does not provide a *direct* treatment for the unguarded case either, it seems to require a contractivity argument for uniqueness of fixed points just as well, and, last but not least, as far as we know, it cannot be used as a basis for the branching time semantics used later in Section 4.3.

2.4 Relationship between $\mathcal{O}_0$ and $\mathcal{D}_0$

In this section we will prove:

2.4.1 THEOREM. $\mathcal{O}_0[\![s]\!] = \mathcal{D}_0[\![s]\!]$ for all (syntactically closed) guarded $s \in \mathcal{L}_0$.

The proof of Theorem 2.4.1 is by induction on the structure of s. For the induction argument we need two important facts about $\mathcal{O}_0$ which we develop first. The first fact states that $\mathcal{O}_0$ behaves compositionally over the operators $\underset{\sim}{op} \in \{;, \cup, \|\}$ of $\mathcal{L}_0$ in the sense or Section 2.3:

$$\mathcal{O}_0[\![s_1 \underset{\sim}{op}\, s_2]\!] = \mathcal{O}_0[\![s_1]\!] \underset{\sim}{op}^{\mathcal{D}_0} \mathcal{O}_0[\![s_2]\!] .$$

We shall not give a full proof here, but refer to Section 3 where this result is established in the more general setting of language $\mathcal{L}_1$.

Instead we concentrate here on the second fact dealing with recursion because its proof carries over to the languages $\mathcal{L}_1$ and $\mathcal{L}_2$ virtually without change. We wish to show that

$$\mathcal{O}_0[\![\mu x[t(x)]]\!] = \lim_n \mathcal{O}_0[\![t^{(n)}(\Omega)]\!]$$

where Ω is a certain auxiliary statement and $t^{(n)}(\cdot)$ denotes n-fold substitution (to be explained in the sequel). This proof is quite involved; it requires a number of auxiliary results on the transition system T_0 and the operational semantics $\mathcal{O}_0$.

In the following, we make the *general assumption* that all our statements are (syntactically closed and) *guarded* (unless explicitly stated otherwise). Guardedness comes into our work in two ways:

(1) in proving the technical results below on transition sequences, notably the Basic Lemma (2.4.4), and

(2) more fundamentally: $\mathcal{S}_0[\![s]\!]$ is only defined for guarded s!
(On the other hand, $\mathcal{O}_0[\![s]\!]$ is only defined for syntactically closed s.)

Let us now turn to the first fact about $\mathcal{O}_0$.

Compositionality of $\mathcal{O}_0$.

We state (more generally):

2.4.2 THEOREM.

(a) $\mathcal{O}_0[\![a]\!] = \{a\}$

(b) $\mathcal{O}_0[\![s_1 \cup s_2]\!] = \mathcal{O}_0[\![s_1]\!] \cup^{\mathcal{S}_0} \mathcal{O}_0[\![s_2]\!]$

(c) $\mathcal{O}_0[\![\mu x[s]]\!] = \mathcal{O}_0[\![s[\mu x[s]/x]]\!]$

(d) $\mathcal{O}_0[\![s_1;s_2]\!] = \mathcal{O}_0[\![s_1]\!] ;^{\mathcal{S}_0} \mathcal{O}_0[\![s_2]\!]$

(e) $\mathcal{O}_0[\![s_1 \| s_2]\!] = \mathcal{O}_0[\![s_1]\!] \|^{\mathcal{S}_0} \mathcal{O}_0[\![s_2]\!]$

PROOF. (a), (b) and (c) are clear, by considering transition sequences from $\langle a, \epsilon\rangle$, $\langle s_1 \cup s_2, \epsilon\rangle$ and $\langle \mu x[s], \epsilon\rangle$, which must start with the transition rules of elementary action, local nondeterminacy and recursion respectively. Part (d) is proved like (e), but more simply, and the proof of (e) is postponed to Section 3 (Lemma 3.4.6), in a more general context. □

We now develop a series of auxiliary results leading to the main fact about recursion (Corollary 2.4.16) used in proving Theorem 2.4.1.

Basic facts about T_0*-transitions*

NOTATION. To display all free occurrences of a variable x in a statement s, we can write $s = s(x)$. Then the result of substituting a statement t for all free occurrences of x in s is denoted formally by $s[t/x]$ and informally by $s(t)$.

We also speak of the *context* $s(\cdot)$ of the occurrence(s) of t displayed in $s(t)$.

Note that if t is a proper substatement of $s = \mu x[s_1(x)]$, then (by the remark on substatements in Section 2.3) t is a substatement of s_1, not containing x, so we can write, informally, $s = \mu x[s_1(t,x)]$.

We indicate a *specific occurrence* of a substatement t of s by *underlining* it: $s(\underline{t})$.

We also speak of the *context* $s(\cdot)$ (or $s(\underline{\cdot})$), meaning that part of the expression $s(t)$ (or $s(\underline{t})$) excluding the displayed occurrence(s) of t.

TYPES OF TRANSITIONS. We must make a closer analysis of T_0-transitions. Since every deduction rule in T_0 has only one premise, every T_0-transition

$$\langle s,w\rangle \to \langle s',w'\rangle \tag{2.2}$$

is deducible from a single axiom: *elementary action*, *nondeterminacy* or *recursion*, by a sequence of applications of the rules *composition* and *shuffle*.

There may actually be more than one deduction of (2.2). For example, the transition

$$\langle \mu x[x] \| \mu y[y],w\rangle \to \langle \mu x[x] \| \mu y[y],w\rangle$$

has two different deductions, one starting from $\mu x[x]$ and the other from $\mu y[y]$. Notice, however, that in this example the μ-substatements are unguarded. If (according to our general assumption) we restrict our attention to guarded statements, it is not hard to see that every deducible transition has a *unique deduction* (although our results do not really depend on this fact).

According to which axiom was used in its deduction (elementary action, nondeterminacy or recursion), (2.2) is called (respectively) an *a-transition*, *$\cup$-transition* or *μ-transition*.

SUBSTATEMENT INVOLVED IN A TRANSITION. Any transition

$$\langle s,w\rangle \to \langle s',w'\rangle \tag{2.3}$$

involves some (unique) *occurrence of a substatement* of s. This notion can be defined by induction on the length of the deduction of (2.3).

(i) *Basis*. If (2.3) is an axiom, then it *involves* the occurrence of s shown.

(ii) *Induction step*. If the premise of an instance of one of the rules in T_0 *involves* an occurrence of s, then the conclusion *involves* the *corresponding occurrence* of s.

For example, in the following form of the shuffle rule:

$$\frac{\langle s_1(\underline{t}),w_1\rangle \to \langle s_2,w_2\rangle}{\langle s'\|s_1(\underline{t}),w_1\rangle \to \langle s'\|s_2(\underline{t}),w_2\rangle} \quad ,$$

if the premise involves the occurrence of t shown in s_1, then the conclusion involves the *corresponding occurrence* of t shown in $s'\|s_1$.

Note that we have not defined the notion of *corresponding occurrence* precisely, but it should be clear enough.

It is clear that the substatement involved in a transition is the same as the statement on the ℓ.h.s. of the corresponding axiom.

EXAMPLES.

(1) $$\langle s_1\|(a;s_2),w\rangle \to \langle s_1\|s_2,wa\rangle$$

is an a-transition, involving the occurrence of a shown.

(2) $$\langle((s_1 \cup s_2);s_3)\|s_4,w\rangle \to \langle(s_2;s_3)\|s_4,w\rangle$$

is a U-transition, involving the occurrence of $s_1 \cup s_2$ shown.

(3) $$\langle s_1 \| \mu x\, [s_2(x)], w\rangle \rightarrow \langle s_1 \| s_2(\mu x\, [s_2(x)]), w\rangle$$

is a μ-transition, involving the occurrence of $\mu x[s_2(x)]$ shown.

PASSIVE SUBSTATEMENTS. We say that a transition

$$\langle s(\underline{t}), w\rangle \rightarrow \langle s', w'\rangle \tag{2.4}$$

affects the substatement occurrence $\underline{t}$ if it *involves* some substatement of $\underline{t}$ (perhaps $\underline{t}$ itself). Conversely, $\underline{t}$ is said to be *passive* in (2.4) if it is *not affected* by (2.4). Denote the (unique) statement occurrence involved in (2.4) by $\underline{t}_0$. Then it is easy to see that the following three statements are equivalent:

(i) t is passive in (2.4).

(ii) $\underline{t}_0$ is not contained in $\underline{t}$.

(iii) $\underline{t}$ is either disjoint from $\underline{t}_0$, or properly contained in $\underline{t}_0$.

2.4.3 LEMMA (*Substitution of Passive Substatements*). Given a T_0-transition

$$\langle s_1, w_1\rangle \rightarrow \langle s_2, w_2\rangle, \tag{2.5}$$

if s_1 has the form $s_1'(\underline{t})$, where $\underline{t}$ is *passive* in the transition, then s_2 can be written in the form $s_2'(t)$ (displaying 0, 1 or more occurrences of t), such that for any statement t', there is *corresponding* T_0-transition

$$\langle s_1'(\underline{t}'),w_1\rangle \rightarrow \langle s_2'(t'),w_2\rangle .$$

PROOF. By induction on the length of a deduction of (2.5). Briefly, the deduction of the new transition is formed simply by replacing certain occurrences of t by t' in the deduction of (2.5). The details are left to the reader. □

BASIC LEMMA ON TRANSITIONS. The following basic lemma shows the significance of the guardedness assumption. It enters three times into our working below! - (a) in the proof of Theorem 2.4.10 (via the Decreasing Exposure Lemma 2.4.7 and the Finiteness Lemma 2.4.8), (b) in the proof of Theorem 2.4.11, and (c) in the proof of Lemma 2.4.14 (via Corollary 2.4.13), which in turn is used in Theorem 2.4.15.

2.4.4 (BASIC) LEMMA. In the transition

$$\langle s_1,w_1\rangle \rightarrow \langle s_2,w_2\rangle , \qquad (2.6)$$

if a substatement occurrence $\underline{t}$ is *not exposed* in s_1, then $\underline{t}$ is *passive* (and so the lemma of the previous subsection applies).

PROOF. By induction on the length of a deduction of (2.6).

BASIS. Suppose (2.6) is an *axiom*. Then, since $\underline{t}$ is not exposed in s_1, it cannot be equal to s_1, i.e. it is a proper substatement of s_1. Hence $\underline{t}$ is passive in (2.6) (since by definition only the full statement s_1 is *involved* in an axiom (2.6).

Induction Step. Consider first the *composition* rule, and take the case

$$\frac{\langle s_1, w_1\rangle \to \langle s_2, w_2\rangle}{\langle s_1;s, w_1\rangle \to \langle s_2;s, w_2\rangle} .$$

By assumption, $\underline{t}$ is not exposed in $s_1;s$. Hence (by definition) $\underline{t}$ is either in s or (not exposed) in s_1. If $\underline{t}$ is in s, then it is certainly passive in the conclusion. Suppose $\underline{t}$ is (not exposed) in s_1. By induction hypothesis, $\underline{t}$ is passive in the premise (i.e. the substatement of s_1 involved in the premise does not occur in $\underline{t}$). Hence clearly, $\underline{t}$ is also passive in the conclusion.

The *shuffle* rule is handled similarly. □

A useful version of this lemma is given by:

2.4.5 COROLLARY. If a transition $\langle s_1, w_1\rangle \to \langle s_2, w_2\rangle$ *involves* a substatement occurrence $\underline{t}$ in s_1, then $\underline{t}$ is exposed in s_1.

PROOF. This is a trivial consequence of the Basic Lemma. (It could also easily be proved directly, by induction on the length of a deduction of the transition.) □

PASSIVE AND ACTIVE SUCCESSORS. Consider a transition $\langle s, w\rangle \to \langle s', w'\rangle$. Let $\mu_0 = \mu x[t_0(x)]$ be a μ-substatement of s, and consider a particular occurrence of μ_0 in s. Then there may be one or more *corresponding* *occurrences* of μ_0 in s', stemming from this occurrence of μ_0 in s. These are called the *successor(s)* of this occurrence of μ_0 in s.

We do not give a complete formal definition of the notion of successor; consider, as an example, the following form of the rule of *composition*:

$$\frac{\langle s_1, w\rangle \rightarrow \langle s', w\rangle}{\langle s_1; s_2(\underline{\mu_0}), w\rangle \rightarrow \langle s'; s_2(\underline{\mu_0}), w\rangle} .$$

The displayed occurrence of μ_0 on the r.h.s. is a *successor* of that on the ℓ.h.s.

Most other cases are just as trivial - call these *passive successors* - except for the case that the transition actually involves the occurrence of μ_0 considered:

$$\langle s(\underline{\mu_0}), w\rangle \rightarrow \langle s(\underline{t_0(\mu_0)}), w\rangle \tag{2.7}$$

(where, as stated above, $\mu_0 = \mu x\,[t_0(x)]$) .

In this case, each occurrence of μ_0 shown inside the occurrence of t_0 on the r.h.s. of (2.7) is an *active successor* of the occurrence of μ_0 shown on the ℓ.h.s.

The transitive relation generated by the successor relation is called *descendant*; the converse of that is called *ancestor*.

2.4.6 LEMMA *(Transitivity of exposure)*. Given a statement s_1, containing a substatement occurrence $\underline{s}_2$, containing in turn a substatement occurrence $\underline{s}_3$:

(a) If $\underline{s}_3$ is exposed in s_2, *and* $\underline{s}_2$ is exposed in s_1, then $\underline{s}_3$ is exposed in s_1. However if *either* (b) $\underline{s}_3$ is not exposed in s_2 *or* (c) $\underline{s}_2$ is not exposed in s_1, then $\underline{s}_3$ is not exposed in s_1.

PROOF. In all cases, by induction on the structure of s. □

DEGREE OF EXPOSURE OF A STATEMENT; DECREASING EXPOSURE LEMMA. The *degree of exposure* of s, $\underset{\sim}{de}(s)$, is defined to be the number of *exposed occurrences* of μ-statements of s. We have an important lemma, which uses the guardedness of statements.

2.4.7 LEMMA. (*Decreasing Exposure*). If $\langle s,w\rangle \to \langle s',w'\rangle$ is a μ-transition, then $\underset{\sim}{de}(s') < \underset{\sim}{de}(s)$.

PROOF. Suppose this transition involves an occurrence of $\mu_0 = \mu x[t_0(x)]$, and put $s = s(\underline{\mu_0})$, displaying this occurrence. Then $s' = s(\underline{t_0(\mu_0)})$. By the Basic Lemma, $\underline{\mu}_0$ is *exposed* in s. However, all its (active) successors are *not exposed* in $t_0(\mu_0)$ (since, by assumption, μ_0 is *guarded*) and hence also not exposed in s' (by the Lemma (2.4.6) on Transitivity of Exposure).

Now consider all other occurrences of μ-substatements in $s(\underline{\mu_0})$. Any occurrence which is contained in the context $s(\underline{\cdot})$ (i.e. *not* in the displayed occurrence of μ_0) has exactly one (passive) successor in $s(\underline{t_0(\mu_0)})$, which is clearly exposed if and only if the original is.

Finally, consider an occurrence of another μ-substatement, say μ_1, within μ_0, i.e. within $t_0(\cdot)$. We write $\mu_0 = \mu x[t_0(\underline{\mu}_1,x)]$, and so

$$s = s(\underline{\mu x[t_0(\underline{\mu}_1,x)]}) . \tag{2.8}$$

Now $\underline{\mu}_1$ has, in general, *many* (passive) successors in s', which we can write as

$$s' = s(\underline{t_0(\underline{\mu}_1, \mu x[t_0(\underline{\mu}_1, x)])}) . \tag{2.9}$$

The first $\underline{\mu}_1$ in (2.9) is exposed in (2.9) iff $\underline{\mu}_1$ is exposed in (2.8), that is (in both cases) iff $\underline{\mu}_1$ is exposed in $t_0(\underline{\mu}_1, x)$ (by the Lemma on Transitivity of Exposure, since $\underline{\mu}_0$ is exposed in $s(\underline{\mu}_0)$). All the other occurrences of μ_1 in (2.9) are, in any case, not exposed in s', since they are in $\mu_0 = \mu x[t_0(\underline{\mu}_1, x)]$, which is not exposed in $t_0(\mu_0)$ (again, by the assumption that μ_0 is guarded).

Putting all this together yields the result. □

The above lemma is used in the Finiteness Lemma in the following subsection.

NON-INCREASING TRANSITIONS AND TRANSITION SEQUENCES; FINITENESS LEMMA.

A transition $\langle s, w\rangle \to \langle s', w\rangle$ is said to be *non-increasing* if $w' = w$, and *increasing* otherwise (i.e. if $w' = w\cdot a$ for some $a \in A$). Similarly, a transition sequence $\langle s,w\rangle \to \ldots \to \langle s',w'\rangle$ is said to be *non-increasing* if $w' = w$.

Clearly, a transition is non-increasing iff it is a μ- or U-transition (cf. TYPES OF TRANSITIONS above), and increasing iff it is an a-transition.

We now give an important lemma, which will be used in the proof of Theorem 2.4.10 (via Corollary 2.4.9).

2.4.8 LEMMA *(Finiteness)*. Any non-increasing transition sequence is finite. In fact, for any s, there is a positive integer C, depending only on the length of s (as a string of symbols), such that any non-increasing transition sequence of the form

$$\langle s,w\rangle = \langle s_1,w\rangle \to \dots \to \langle s_n,w\rangle = \langle s',w\rangle \tag{2.10}$$

(for any s', w) has length n at most C.

Proof. Let ℓ be the length of s, and $d = \underset{\sim}{de}(s)$. Now a non-increasing transition sequence (2.10) can only contain $\cup$-transitions and μ-transitions. This can include at most d μ-transitions, by the Decreasing Exposure Lemma (2.4.7). Also, each $\cup$-transition decreases the length of the statement. Hence (by a crude estimate, since the length of a statement can be at most squared by a μ-transition) (2.10) can include at most ℓ^{2^d} $\cup$-transitions. Hence the length of (2.10) is at most $d + \ell^{2^d}$, and so (since, trivially, $d \le \ell$) we can take $C = \ell + \ell^{2^d}$. □

COUNTEREXAMPLE for an unguarded statement. Let $s = \mu x[x;\ a \cup b]$. Starting with $\langle s, \epsilon\rangle$, we can perform a μ-transition, followed by a $\cup$-transition, k times (for any k), to get:

$$\langle s,\epsilon\rangle \to \dots \to \langle s;a^k,\epsilon\rangle ,$$

a non-increasing transition sequence of length k.

2.4.9 COROLLARY. For a given s, there are only *finitely many* transition sequences of the form

$$\langle s,w\rangle \to \ldots \to \langle s',w\rangle \to \langle s'',w\cdot a\rangle \qquad (2.11)$$

(for any w, s', s'', a).

PROOF. By the Finiteness Lemma, there is a *finite upper bound* to the length of (2.11). Also, at each step there are only *finitely many* possibilities for the next transition (as is clear from an inspection of the transition rules). □

COUNTEREXAMPLE for an unguarded statement. Let (again) $s = \mu x[x;\, a \cup b]$. For any k, we construct the sequence

$$\begin{aligned}
\langle s,\epsilon\rangle &\xrightarrow{*} \langle s;a^k,\epsilon\rangle && \text{(as in counterexample after 2.4.8)}\\
&\to \langle (s;a\cup b);a^k,\epsilon\rangle && (\mu\text{-transition})\\
&\to \langle b;a^k,\epsilon\rangle && (\cup\text{-transition})\\
&\to \langle a^k,b\rangle .
\end{aligned}$$

Such sequences are distinct for different k.

Metric Closure

2.4.10 THEOREM. For any s, $\mathcal{O}_0[\![s]\!]$ is closed (in the metric on A^{st} given in Section 2.3).

PROOF. Let $(u_1, u_2, \ldots)$ be a CS (Cauchy sequence) of words in $\mathfrak{S}_0[\![s]\!]$. Let $u = \ell im_n u_n$. We must show: $u \in \mathfrak{S}_0[\![s]\!]$.

If u is finite, it is easy to see that $(u_n)_n$ is eventually constant, i.e. $u_n = u$ for n sufficiently large. Hence $u \in \mathfrak{S}_0[\![s]\!]$.

So suppose u is infinite. The idea of the proof is to find a subsequence of $(u_n)_n$ such that not only do the *words* converge, but also the *paths* producing them converge (in a suitable metric, to be discussed in 2.4.13) to a path π of s such that $u \in \underset{\sim}{\text{word}}(\pi)$, from which the result follows.

(As before, we use the notation $u[n]$ for the initial segment of a word u of length n.)

We proceed inductively.

Since $(u_n)_n$ is a CS, for n sufficiently large (say $n \geq N_1$) $u_n[1]$ is constant, i.e. u_n begins with the same letter, say a_1 (which is also the first letter of u).

For all n, let π_n be a path from s producing u_n. Consider the first part of π_n, up to the first appearance of a_1 on the r.h.s. of a configuration:

$$\pi_n: \langle s, \varepsilon \rangle \rightarrow \ldots \rightarrow \langle s_1, a_1 \rangle \rightarrow \ldots$$

By the Corollary (2.4.9) to the Finiteness Lemma, there are only *finitely many* such transition sequences possible. Hence there is a *subsequence*

$(u_{n_1}, u_{n_2}, \ldots)$ of $(u_n)_n$ such that the corresponding π_{n_k} all begin with the *same* transition sequence (up to the first appearance of a_1 on the r.h.s.).

Since $(u_{n_k})_k$ is a CS, for k sufficiently large $u_{n_k}[2]$ is constant, i.e. u_{n_k} begins with the same two letters, say a_1a_2 (which are also the first two letters of u). Again, by the Corollary to the Finiteness Lemma, we can get a *subsequence* of (u_{n_k}) such that the corresponding paths all begin in the same way, up to the first appearance of a_1a_2 on the r.h.s.:

$$\langle s,\varepsilon\rangle \to \ldots \to \langle s_1,a_1\rangle \to \ldots \to \langle s_2,a_1a_2\rangle \to \ldots$$

Continuing in this way, we get, for all k, successive subsequences of $(u_n)_n$ such that the corresponding paths all begin in the same way, up to the first appearance of k letters on the r.h.s., say $a_1a_2\ldots a_k$, which are also the first k letters of u. Finally we take the "diagonal sequence", by *piecing together* the initial segments of these paths, to obtain the path

$$\begin{aligned}\pi:\ \langle s,\varepsilon\rangle \to \ldots \to \langle s_1,\ a_1\rangle &\to \ldots \\ \ldots \to \langle s_2,\ a_1a_2\rangle &\to \ldots \\ \ldots \to \langle s_k,\ a_1a_2\ldots a_k\rangle &\to \ldots\end{aligned}$$

Clearly, $\pi \in \underset{\sim}{\text{path}}(s)$ and $u = a_1a_2\ldots a_k\ldots \in \underset{\sim}{\text{word}}(\pi)$. □

DISCUSSION (metric on the set of paths). We can define a metric $\tilde{d}$ on the set path(s) as follows: $\tilde{d}(\pi, \pi') = 2^{-n}$ if π and π' agree up to the first appearance of a word of length n on each:

$$\langle s,\varepsilon\rangle \to \ldots \to \langle s_n, a_1 \ldots a_n\rangle \to \ldots$$

(Note: this is *not* equivalent to agreeing up to the first n transitions!)

The proof of Theorem 2.4.10 produces a subsequence of $(u_n)_n$ such that the corresponding *sequence of paths also converges* (in the metric $\tilde{d}$) to a limiting path π, with $u \in$ word(π).

COUNTEREXAMPLE to Theorem 2.4.10 for an unguarded statement. Again, let $s = \mu x[x;a \cup b]$. Then $\mathcal{O}_0[\![s]\!] = b.a^* \cup \{\bot\}$. This set is not closed, since if we take $u_n = b.a^n \in \mathcal{O}_0[\![s]\!]$, then $\lim_n u_n = b.a^\omega \notin \mathcal{O}_0[\![s]\!]$.

Note that the u_n are produced by paths

$$\pi_n\colon \langle s,\varepsilon\rangle \to \ldots \to \langle a^n, b\rangle \qquad \text{(as in Counterexample after 2.4.9)}$$

$$\ldots \to b; a^n \qquad \text{(by } n \ a\text{-transitions)}.$$

But the initial parts of these paths, up to the first appearance of b on the r.h.s., are all *different*, so there is no limiting path (in the metric $\tilde{d}$)!

Linking operational and syntactic approximation.

ITERATED SUBSTITUTION; DEPTH OF A μ-STATEMENT IN A PATH. From now on, we will concentrate on a specific μ-statement, $\bar{\mu} = \mu x[\bar{t}(x)]$ (which, by our general assumption, is syntactically closed and guarded).

We define the n-*fold substitution* in $\bar{t}(x)$ by a sequence of statements $\bar{t}^n(x)$ $(n = 0,1,2,\ldots)$ where

$$\bar{t}^0(x) = n$$

$$\bar{t}^{n+1}(x) = \bar{t}(\bar{t}^n(x)) \ (= \bar{t}^n(\bar{t}(x)))$$

Since $\bar{\mu}$ is syntactically closed, $\bar{t}(x)$ contains at most x free. However, there may be many occurrences of x in $\bar{t}$ (none of the exposed!). If, for example $\bar{t}(x) = \bar{t}(\underline{x},\underline{x},\underline{x})$ (3 occurrences of x), then $\bar{t}^2(x) = \bar{t}(\underline{\bar{t}(\underline{x},\underline{x},\underline{x})}, \underline{\bar{t}(\underline{x},\underline{x},\underline{x})}, \underline{\bar{t}(\underline{x},\underline{x},\underline{x})})$.

We call a transition involving an occurrence of $\bar{\mu}$ a $\bar{\mu}$-transition.

Now consider a path from some statement s_0 containing $\bar{\mu}$:

$$\pi: \langle s_0,\epsilon\rangle \to \langle s_1,w_1\rangle \to \ldots \to \langle s_n,w_n\rangle \to \ldots$$

We define the *depth* of an occurrence of $\bar{\mu}$ in s_n (in π), by induction on n:

Basis $(n=0)$. Every occurrence of $\bar{\mu}$ in s_0 has depth 0.

Induction step $(n \to n+1)$. Given any occurrence of $\bar{\mu}$ in s_n of depth d, any passive successor (cf. PASSIVE AND ACTIVE SUCCESSORS above) of this occurrence also has depth d; all *active successors* have depth $d+1$.

In other words, the depth of an occurrence of $\bar{\mu}$ in π counts the number of $\bar{\mu}$-transitions involving ancestors of that occurrence.

SYNTACTIC BOTTOM SYMBOL; TRUNCATION OF A PATH. As a technical aid, we adjoin the symbol "Ω" to the syntax of $\mathcal{L}_0$, and the transition rules (actually axioms):

$$(\Omega_1): \quad \langle \Omega; s, w\rangle \to \langle \Omega, w\rangle$$
$$\langle \Omega \| s, w\rangle \to \langle \Omega, w\rangle$$
$$\langle s \| \Omega, w\rangle \to \langle \Omega, w\rangle$$

$$(\Omega_2): \quad \langle \Omega, w\rangle \to w.\bot$$

to T_0. We also define $\mathcal{L}_0[\![\Omega]\!](\gamma) = \{\bot\}$. This symbol will not appear in our final result (2.4.1).

We now define the n-*truncation* of a path π (w.r.t. $\bar{\mu}$), $\text{trunc}_n(\pi)$. This is the path π' formed by "truncating π at a depth of n", by (1) replacing all occurrences of $\bar{\mu}$ in π, of depth n, by Ω, and (2) replacing the first transition involving an occurrence of $\bar{\mu}$ of depth n:

$$\pi: \ldots \to \langle s(\underline{\bar{\mu}}), w\rangle \overset{①}{\to} \langle s(\underline{\bar{t}(\bar{\mu})}), w\rangle \to \ldots$$

by transitions involving Ω:

$$\pi': \ldots \to \langle s(\underline{\Omega}), w\rangle \overset{②}{\overset{*}{\to}} \langle \Omega, w\rangle \overset{\Omega_2}{\to} w\bot ,$$

thus terminating π'. The transitions in the sequence ② are deduced from instances of axiom (Ω_1) by successive applications of the composition and shuffle rules, paralleling the deduction of ① from an instance of the recursion rule.

Note that step (1) in the construction of $\underset{\sim}{\text{trunc}}_n(\pi)$ above has the effect of replacing $\bar{\mu}$-transitions, involving occurrences of $\bar{\mu}$ of depth $n-1$, by "*non-standard μ-transitions*", in which the active successor of $\bar{\mu}$ is not $\bar{t}(\bar{\mu})$ but $\bar{t}(\Omega)$.

Next we give a notation for the word associated with the n-truncation of π:

$$\underset{\sim}{\text{word}}_n(\pi) = \underset{\sim}{\text{word}}(\underset{\sim}{\text{trunc}}_n(\pi))$$

and finally define the n-*approximation* of the operational meaning of s_0:

$$\mathcal{O}_0^{(n)}[\![s_0]\!] = \{\underset{\sim}{\text{word}}_n(\pi) \mid \pi \in \underset{\sim}{\text{path}}(s_0)\} .$$

The following theorem shows that for $\mathcal{O}_0$, *operational approximation* (via n-truncation) coincides with *syntactic approximation* (via n-fold substitution). This result facilitates the subsequent considerations on metric limits.

2.4.11 THEOREM. $\mathcal{O}_0^{(n)}[\![\bar{\mu}]\!] = \mathcal{O}_0[\![\bar{t}^{(n)}(\Omega)]\!]$ for $n = 0, 1, 2, \ldots$

PROOF. We will actually prove, more generally: for any statement $s_0(x)$ (with only x free, and not containing Ω),

$$\mathcal{O}_0^{(n)}[\![s_0(\bar{\mu})]\!] = \mathcal{O}_0[\![s_0(\bar{t}^{(n)}(\Omega))]\!] .$$

(1) $\subseteq$: (This is relatively straightforward.) Let $\pi \in \underset{\sim}{\text{path}}_n(s_0(\bar{\mu}))$. We must find $\pi' \in \text{path}(s_0(\bar{t}^{(n)}(\Omega)))$ such that $\underset{\sim}{\text{word}}(\pi') = \underset{\sim}{\text{word}}(\pi)$. Note that each occurrence of $\bar{\mu}$ in π has depth

$< n$ (by definition of $\underset{\sim}{path}_n$).

Form π' from π in two steps:

(a) *Replace* each occurrence of $\bar{\mu}$ of depth $d(< n)$ by $\bar{t}^{n-d}(\Omega)$.

(b) Consider a $\bar{\mu}$-transition in π:

$$\pi: \ldots \to \langle s(\bar{\mu}), w\rangle \to \langle s(\bar{t}(\bar{\mu})), w\rangle \to \ldots$$

Actually, s may contain a number (say m) of occurrences of $\bar{\mu}$: $s(\bar{\mu}) = s(\underline{\bar{\mu}},\underline{\bar{\mu}},\ldots,\underline{\bar{\mu}})$. Suppose w.l.o.g. that the *first* of these occurrences shown is involved in the $\bar{\mu}$-transition:

$$\begin{aligned}\pi: \ldots &\to \langle s(\underline{\bar{\mu}},\bar{\mu},\ldots,\underline{\bar{\mu}}), w\rangle \\ &\to \langle s(\underline{\bar{t}(\bar{\mu})}, \underline{\bar{\mu}},\ldots,\underline{\bar{\mu}}), w\rangle \\ &\to \ldots .\end{aligned}$$

Suppose that the m occurrences of $\bar{\mu}$ shown on the l.h.s. of this transition have depths $d_1,\ldots,d_m (< n)$. Then all occurrences of $\bar{\mu}$ in $\underline{\bar{t}(\bar{\mu})}$ have depth $d_1 + 1$ (they are the *active successors* of the first $\bar{\mu}$ on the l.h.s.), and the remaining $\underline{\bar{\mu}}$'s on the r.h.s. (still) have depths $d_2,\ldots,d_m$ (they are the *passive successors* of the corresponding $\underline{\bar{\mu}}$'s on the l.h.s.). Then from step (a), π' is so far (putting $e_i = n - d_i$):

$$\begin{aligned}\pi': \ldots &\to \langle s(\underline{\bar{t}^{e_1}(\Omega)},\underline{\bar{t}^{e_2}(\Omega)},\ldots,\underline{\bar{t}^{e_m}(\Omega)}), w\rangle \\ &\to \langle s(\underline{\bar{t}(\bar{t}^{e_1-1}(\Omega))},\underline{\bar{t}^{e_2}(\Omega)},\ldots,\underline{\bar{t}^{e_m}(\Omega)}), w\rangle \\ &\to \ldots\end{aligned}$$

Now *collapse* the above "identity transition" into a single configuration

$$\pi' : \ldots \to \langle s(\ldots), w\rangle \to \ldots .$$

(2) $\supseteq$: (Trickier, here we use the Basic Lemma, and the assumption that $\bar{\mu}$ is guarded.) Let $\pi' \in \underset{\sim}{path}(s_0(\bar{t}^n(\Omega)))$. We want to find a path $\pi \in \underset{\sim}{path}_n(s_0(\bar{\mu}))$ with the same associated word. Roughly, we replace occurrences of $\bar{t}^e(\Omega)$ $(0 < e \le n)$ in π' by $\bar{\mu}$ (of depth $n-e$, as it turns out). We will construct π step by step from π'. With each configuration $\langle s, w\rangle$ in π' will be associated a finite sequence $(\underline{\bar{t}^{e_1}(\Omega)},\ldots,\underline{\bar{t}^{e_m}(\Omega)})$ $(0 < e_i < n)$ of occurrences of substatements of s. Then π is extended by adjoining a configuration $\langle s', w\rangle$, where s' is formed from s by replacing $\bar{t}^{e_i}(\Omega)$ by $\underline{\bar{\mu}}$ (of depth $n-e_i$). In detail, the construction of π from π' proceeds as follows. It starts in the obvious way (displaying the different occurrences of $\bar{t}^n(\Omega)$ in s_0):

$$\pi' : \langle s_0(\bar{t}^n(\Omega),\ldots,\bar{t}^n(\Omega)), \epsilon\rangle \to \ldots$$

$$\pi : \langle s_0(\underline{\bar{\mu}},\ldots,\underline{\bar{\mu}}), \epsilon\rangle \to \ldots$$

Now assume (inductively) that π has been constructed from π' up to a certain stage:

$$\pi' : \ldots \to \langle s(\underline{\bar{t}^{e_1}(\Omega)},\ldots,\underline{\bar{t}^{e_m}(\Omega)}), w\rangle \overset{①}{\to} \ldots$$

$$\pi : \ldots \to \langle s(\underline{\bar{\mu}},\ldots,\underline{\bar{\mu}}), w\rangle$$

where $(\underline{\bar{t}^{e_1}(\Omega)},\ldots,\underline{\bar{t}^{e_m}(\Omega)})$ is the sequence associated with the configuration in π', and (by assumption) each $\underline{\bar{t}^{e_i}(\Omega)}$ has been replaced in π by an occurrence of $\bar{\mu}$ of depth $n-e_i$ $(1 \le i \le m)$. Now consider the next transition ① in π'. There are two possibilities:

(a) Transition ① does not affect any of the $\underline{t^{e_i}(\Omega)}$ $(i = 1,\dots,m)$. Then the construction of π is extended another step in the obvious way.

(b) Transition ① affects one of the $\underline{t^{e_i}(\Omega)}$, say (w.l.o.g.) $\underline{t^{e_1}(\Omega)}$. There are two subcases:

(i) $e_1 > 1$. Now since $\bar{\mu}$ is guarded, the occurrences of x are not exposed in $\bar{t}(x)$, hence the occurrences of $\bar{t}^{e_i-1}(\Omega)$ are not exposed in $\bar{t}(\bar{t}^{e_1-1}(\Omega)) = \bar{t}^{e_1}(\Omega)$, and hence (by the Lemma (2.4.6) on ransitivity of xposure) also not in $s(\underline{\bar{t}(\bar{t}^{e_1-1}(\Omega))},\dots)$. Hence by the Basic Lemma, they are *passive* in ①, and so, by the Lemma (2.4.3) on substitution of Passive Substatements, ① has the form:

$$\begin{aligned}\pi' : \dots \to\ & \langle s(\underline{\bar{t}^{e_1}(\Omega)}, \underline{\bar{t}^{e_2}(\Omega)},\dots,\underline{\bar{t}^{e_m}(\Omega)}), w\rangle \\ =\ & \langle s(\underline{\bar{t}(\bar{t}^{e_1-1}(\Omega))}, \underline{\bar{t}^{e_2}(\Omega)},\dots,\underline{\bar{t}^{e_m}(\Omega)}), w\rangle \\ \overset{①}{\to}\ & \langle s(\underline{t'(\bar{t}^{e_1-1}(\Omega))}, \underline{\bar{t}^{e_2}(\Omega)},\dots,\underline{\bar{t}^{e_m}(\Omega)}), w\rangle \\ \to\ & \dots\end{aligned}$$

The sequence associated with this last configuration is the sequence of occurrences of $\bar{t}^{e_1-1}(\Omega)$ (shown in the context $t'(\cdot)$), followed by $\underline{\bar{t}^{e_2}(\Omega)},\dots,\underline{\bar{t}^{e_m}(\Omega)}$ as before.

Now the construction of π proceeds with a $\bar{\mu}$-transition, followed by a transition *corresponding to* ① (as given by the Lemma on the Substitution of Passive Substatements):

$$\pi: \ldots \to \langle s(\underline{\bar{\mu}}, \underline{\bar{\mu}}, \ldots, \underline{\bar{\mu}}), w\rangle$$
$$\to \langle s(\underline{\bar{t}(\bar{\mu})}, \underline{\bar{\mu}}, \ldots, \underline{\bar{\mu}}), w\rangle$$
$$\to \langle s(\underline{t'(\bar{\mu})}, \underline{\bar{\mu}}, \ldots, \underline{\bar{\mu}}), w\rangle .$$

(ii) $e_1 = 1$. Again, by the Basic Lemma, transition ① has the form:

$$\pi' : \ldots \to \langle s(\underline{\bar{t}(\Omega)}, \underline{\bar{t}^{e_2}(\Omega)}, \ldots, \underline{\bar{t}^{e_m}(\Omega)}), w\rangle$$
$$\overset{1}{\to} \langle s(\underline{t'(\Omega)}, \underline{\bar{t}^{e_2}(\Omega)}, \ldots, \underline{\bar{t}^{e_m}(\Omega)}), w\rangle$$
$$\to \ldots$$

The sequence associated with this last configuration is now $(\underline{\bar{t}^{e_2}(\Omega)}, \ldots, \underline{\bar{t}^{e_m}(\Omega)})$, and the combination of π proceeds with a non-standard $\bar{\mu}$-transition (converting $\bar{\mu}$ to $\bar{t}(\Omega)$: note that this occurrence of $\bar{\mu}$ has depth $n-1$), followed, again, by a transition corresponding to ① :

$$\pi: \ldots \to \langle s(\underline{\bar{\mu}}, \underline{\bar{\mu}}, \ldots, \underline{\bar{\mu}}), w\rangle$$
$$\to \langle s(\underline{\bar{t}(\Omega)}, \underline{\bar{\mu}}, \ldots, \underline{\bar{\mu}}), w\rangle$$
$$\to \langle s(\underline{t'(\Omega)}, \underline{\bar{\mu}}, \ldots, \underline{\bar{\mu}}), w\rangle .$$

To show that $\pi \in \underset{\sim}{path}_n(s_0(\bar{\mu}))$: notice that Ω is introduced into π (only) from non-standard $\bar{\mu}$-transitions, involving occurrences of $\bar{\mu}$ of depth n. Now we can construct a path from π, such that π is its n-truncation, by:

(1) replacing all non-standard $\bar{\mu}$-transitions by standard $\bar{\mu}$-transitions,

(2) removing all Ω_1-transitions,

(3) replacing the Ω_2-transition (assuming there is one!) by a $\bar{\mu}$-transition, and then continuing the path arbitrarily.

We leave the details to the reader. □

REMARKS. (1) We believe that the mappings between $\underset{\sim}{path}_n(s_0(\bar{\mu}))$ and $\underset{\sim}{path}(s_0(\bar{t}^n(\Omega)))$ given by the above proof are inverse bijections.

(2) Although guardedness was used in this proof (via the Basic Lemma), we cannot find a counterexample to the theorem by dropping this assumption.

Taking Limits

2.4.12 LEMMA. Consider a path from $\bar{\mu}$:

$$\langle\bar{\mu},\epsilon\rangle \to \ldots \to \langle s,w\rangle \overset{①}{\to} \langle s',w'\rangle \to \ldots$$
$$\ldots \to \langle s'',w''\rangle \overset{②}{\to} \langle s''',w'''\rangle \to \ldots$$

where transition ① involves an occurrence of $\bar{\mu}$ of depth d and transition ② involves an occurrence of a *descendant* of $\bar{\mu}$ of depth $d+1$. Then w'' is *longer* than w'.

PROOF. By the Basic Lemma, only *exposed* occurrences of $\bar{\mu}$ can be involved in a $\bar{\mu}$-transition. Since $\bar{\mu}$ is guarded, no successor of this occurrence of $\bar{\mu}$ in ① is exposed, and, in fact, no descendant of this occurrence of $\bar{\mu}$ is exposed, as long as there are only μ- and U-transitions (the proof of which is left to the reader).

Hence, before transition ② , there must be at least one a-transition, which will lengthen the word. □

Let us write $|w|$ to denote the length of the word w .

2.4.13 COROLLARY. If, in a path from $\bar{\mu}$:

$$\langle\bar{\mu},\varepsilon\rangle \to \dots \to \langle s,w\rangle \overset{①}{\to} \langle s',w'\rangle \to \dots ,$$

the transition ① involves an occurrence of $\bar{\mu}$ of depth d , then $|w| \geq d$.

COUNTEREXAMPLE for an unguarded statement. Let $s = \mu x[x;a \cup b]$. Taking the sequence described in the counterexample following 2.4.8, with transitions involving μ-statements of arbitrary depth, we remain with the empty word.

2.4.14 LEMMA. The sequence $(\mathbb{G}_0^{(n)}[\![\bar{\mu}]\!])_n$ is a Cauchy sequence in $(\mathbb{S}_c, \hat{d})$ (see Section 2.3).

PROOF. This follows from the fact that for all $\pi \in \underset{\sim}{path}(\bar{\mu})$, $\underset{\sim}{word}_n(\pi) \to \underset{\sim}{word}(\pi)$ as $n \to \infty$, *uniformly* in n (i.e. independent of π) in A^{st} . More precisely, by Corollary 2.4.13, for all $\pi \in \underset{\sim}{path}(\bar{\mu})$, n , k :

$$d(\underset{\sim}{word}_n(\pi), \underset{\sim}{word}_{n+k}(\pi)) \leq 2^{-n} .$$

Hence for all n , k:

$$\hat{d}(\mathbb{G}_0^{(n)}[\![\bar{\mu}]\!] , \mathbb{G}_0^{(n+k)}[\![\bar{\mu}]\!]) \leq 2^{-n} . \qquad \square$$

2.4.15 THEOREM. $\mathcal{O}_0[\![\bar{\mu}]\!] = \lim_n \mathcal{O}_0^{(n)}[\![\bar{\mu}]\!]$.

PROOF. By Lemma 2.4.14, the limit on the r.h.s. exists. It is equal to (see [Ha])

$$\{\lim_n w_n \mid (w_n)_n \text{ is a CS in } (A^{st},d) \text{ and } w_n \in \mathcal{O}_0^{(n)}[\![\bar{\mu}]\!]\}.$$

We will show that each side is a subset of the other.

(1) $\subseteq$: Clear, since for all $\pi \in \underset{\sim}{path}(\bar{\mu})$, $\underset{\sim}{word}(\pi) = \lim_n(\underset{\sim}{word}_n(\pi))$.

(2) $\supseteq$: Let $w = \lim_n w_n$, where $w_n \in \mathcal{O}_0^{(n)}[\![\bar{\mu}]\!]$. For all n, there exists $v_n \in \mathcal{O}_0[\![\bar{\mu}]\!]$ which extends w_n and such that $w = \lim_n v_n$ also. (Take $v_n = \underset{\sim}{word}(\pi)$ for any π such that $w_n = \underset{\sim}{word}_n(\pi)$.) Then also $w = \lim_n v_n$. Since $\mathcal{O}_0[\![\bar{\mu}]\!]$ is closed (by Theorem 2.4.10), $w \in \mathcal{O}_0[\![\bar{\mu}]\!]$. □

We can now state the main fact about recursion used in proving Theorem 2.4.1.

2.4.16 COROLLARY. $\mathcal{O}_0[\![\bar{\mu}]\!] = \lim_n \mathcal{O}_0[\![\bar{t}^n(\Omega)]\!]$.

PROOF. By Theorems 2.4.15 and 2.4.11. □

SIMPLE EXAMPLE. Let $\bar{t}(x) = a\cdot x \cup b$, $\bar{\mu} = \mu x[\bar{t}(x)]$. For all n, $\mathcal{O}_0[\![\bar{t}^n(\Omega)]\!] = \mathcal{O}_0^{(n)}[\![\bar{\mu}]\!] = \{a^i b \mid 0 \le i < n\} \cup \{a^n \bot\}$. This is a CS of sets, with limit $a^* b \cup \{a^\omega\}$, which is equal to $\mathcal{O}_0[\![\bar{\mu}]\!]$, as promised by the Theorem.

COUNTEREXAMPLE for an unguarded statement. Let $\bar{t}(x) = x \cdot a \cup b$, $\bar{\mu} = \mu x[\bar{t}(x)]$. For all n, $\mathcal{O}_0[\![\bar{t}^n(\Omega)]\!] = \mathcal{O}_0^{(n)}[\![\bar{\mu}]\!] = \{ba^i \mid 0 \leq i < n\} \cup \{\bot\}$. This is again a CS, with limit $ba^* \cup \{ba^\omega, \bot\}$. However this limit is *not* equal to

$$\mathcal{O}_0[\![\bar{\mu}]\!] = ba^* \cup \{\bot\},$$

which is not even a closed set!

Proof of Theorem 2.4.1

Finally, we are ready to prove that

$$\mathcal{O}_0[\![s]\!] = \mathcal{S}_0[\![s]\!].$$

Since we are assuming that s is syntactically closed, we do not display the environment with $\mathcal{S}_0[\![s]\!]$ above. However, in order to prove it, we must prove a more general result, in which s is not necessarily syntactically closed (but still guarded!), namely

$$\mathcal{O}_0[\![s[t_i/x_i]_{i=1}^k]\!] = \mathcal{S}_0[\![s]\!](\gamma\langle X_i/x_i\rangle_{i=1}^k) \tag{2.12}$$

where (a) $\underset{\sim}{var}(s) \subseteq \{x_1, \ldots, x_k\}$,

(b) t_i is syntactically closed for $i = 1, \ldots, k$,

(c) $\mathcal{O}_0[\![t_i]\!] = X_i$ for $i = 1, \ldots, k$.

The theorem is then (of course) a special case of (2.12) with $k = 0$.

The proof of (2.12) is by induction on the structure of s. All cases are straightforward (using Theorem 2.4.2) except for $s = \mu y[s_0]$ (assuming w.l.o.g. $y \neq x_1, \ldots, x_k$). Now

$$\mathcal{O}_0[\![\mu y\,[s_0][t_i/x_i]_{i=1}^k]\!]$$

$$= \mathcal{O}_0[\![\mu y\,[s_0[t_i/x_i]_{i=1}^k]]\!] \quad \text{(assuming w.l.o.g. no variable clashes)}$$

$$= \lim_n \mathcal{O}_0[\![r_n]\!] \quad \text{(by Corollary 2.4.16)}$$

where

$$\begin{cases} r_0 = \Omega , \\ r_{n+1} = s_0[t_i/x_i]_{i=1}^k[r_n/y] , \end{cases}$$

and

$$\mathcal{D}_0[\![\mu y\,[s_0]]\!]\,(\gamma\langle X_i/x_i\rangle_{i=1}^k) = \lim_n Y_n ,$$

where

$$\begin{cases} Y_0 = \{\bot\} , \\ Y_{n+1} = \mathcal{D}_0[\![s_0]\!](\gamma\langle X_i/x_i\rangle_{i=1}^k\langle Y_n/y\rangle) . \end{cases}$$

So it is sufficient to show

$$\mathcal{O}_0[\![r_n]\!] = Y_n \tag{2.13}$$

for all n, by induction on n.

For $n = 0$, this is clear. Assume (2.13). We must show $\mathcal{O}_0[\![r_{n+1}]\!] = Y_{n+1}$, i.e.

$$\mathcal{O}_0[\![s_0[t_i/x_i]_{i=1}^k[r_n/y]]\!] = \mathcal{D}_0[\![s_0]\!](\gamma\langle X_i/x_i\rangle_{i=1}^k\langle Y_n/y\rangle) .$$

But this follows by the main induction hypothesis on (2.12), with s_0 replacing s and $k+1$ replacing k, and using (2.13) to establish the $(k+1)$-st part of condition (c). □

3. THE LANGUAGE $\mathcal{L}_1$: SYNCHRONIZATION MERGE AND LOCAL NONDETERMINACY

For $\mathcal{L}_1$ we introduce some structure to the finite alphabet A. Let $C \subseteq A$ be a subset of so-called *communications*. From now on let c range over C and a over $A\backslash C$. Similarly to CCS [Mi] or CSP [Ho] we stipulate a bijection $\bar{-}: C \to C$ with $\bar{\bar{c}} = c$ which for every $c \in C$ yields a *matching communication* $\bar{c}$. There is a special action $\tau \in A\backslash C$ denoting the result of a synchronization of c with $\bar{c}$ [Mi].

As syntax for $s \in \mathcal{L}_1$ we give now:

$$s ::= a \mid c \mid s_1;s_2 \mid s_1 \cup s_2 \mid s_1 \| s_2 \mid x \mid \mu x[s] .$$

Apart from a distinction between communications and ordinary elementary actions, the syntax of $\mathcal{L}_1$ agrees with that of $\mathcal{L}_0$. The difference between $\mathcal{L}_1$ and $\mathcal{L}_0$ lies in a more sophisticated interpretation of $s_1 \| s_2$ to be presented in the next subsection.

3.1 The Transition System T_1

Let $\delta \notin A \cup \{\bot\}$ be an element indicating failure, with $\delta \cdot w = \delta$ for all w. The set of streams or words is extended to

$$A^{st}(\delta) = A^{st} \cup A^* \cdot \{\delta\}$$

with u,v,w now ranging over $A^{st}(\delta)$.

The transition system T_1 consists of all axioms and rules of T_0 extended with [1]

$$\langle s,w\rangle \rightarrow w \quad \text{for} \quad w \in A^{\omega} \cup A^{*}\cdot\{\delta,\bot\},$$

and for $w \in A^{*}$ with:

(*communication*)

$$\langle c,w\rangle \rightarrow w\cdot\delta$$

(an individual communication fails),

(*synchronization*)

$$\langle c\|\bar{c},\ w\rangle \rightarrow w\cdot\tau$$

(*synchronization in a context*)

$$\frac{\langle s_1\|s_2,\ w\rangle \quad \rightarrow \quad \langle s_1'\|s_2',\ w\tau\rangle}{\begin{array}{lcl} \langle (s_1;s)\|s_2,\ w\rangle & \rightarrow & \langle (s_1';s)\|s_2',\ w\tau\rangle \\ \langle (s_1\|s)\|s_2,\ w\rangle & \rightarrow & \langle (s_1'\|s)\|s_2',\ w\tau\rangle \\ \langle (s\|s_1)\|s_2,\ w\rangle & \rightarrow & \langle s'\|(s_1'\|s),\ w\tau\rangle \\ \langle s_1\|(s_2;\ s),w\rangle & \rightarrow & \langle s'\|(s_2';s),\ w\tau\rangle \\ \langle s_1\|(s_2\|s),\ w\rangle & \rightarrow & \langle s'\|(s_2'\|s),\ w\tau\rangle \\ \langle s_1\|(s\|s_2),\ w\rangle & \rightarrow & \langle s'\|(s\|s_2'),\ w\tau\rangle \end{array}}$$

where s_1' or s_2' or both may be E, and where the premise of the rule is a synchronization-transition between s_1 and s_2 such that s_1' stems from s_1 and s_2' stems from s_2.

The last rule requires some explanation. First consider a transition of the form

$$\langle s_1\|s_2,\ w\rangle \rightarrow \langle s_1',\ w'\rangle .$$

An occurrance of a substatement s of s_1' is said to *stem from* s_1 (or s_2) if whenever s_1 and s_2 were colored 'blue' and 'green' respectively, s would be exclusively colored 'blue' (or 'green'). Note that the concept of coloring is just a convenient way of tracing occurrences in configurations changed by transitions. For example, in the transition

$$\langle (c;s_1) \| (\bar{c};s), w\rangle \rightarrow \langle s_1 \| s_2, w\tau\rangle$$

s_1 stems from $c;s_1$ and s_2 stems from $\bar{c};s_2$. A transition of the form

$$\langle s_1 \| s_2, w\rangle \rightarrow \langle s_1', w\tau\rangle \qquad (3.1)$$

is called a *synchronization-transition between* s_1 *and* s_2 if a deduction of (3.1) starts with a synchronization axiom

$$\langle c \| \bar{c}, w\rangle \rightarrow w \cdot \tau$$

such that s_1 has the same color as c and s_2 has the same color as $\bar{c}$.

In contrast, a transition

$$\langle s_1 \| s_2, w\rangle \rightarrow \langle s_1' \| s_2', w'\rangle \qquad (3.2)$$

is called a *local transition* if a deduction of (3.2) starts with an axiom of the form $\langle s,w\rangle \rightarrow w'$ such that s is a substatement of either s_1 or s_2. (Note: the "$\|$" shown in (3.2) is introduced by the shuffle rule, notthe synchronization rule, and so $s_2 = s_2'$ or $s_1 = s_1'$.)

EXAMPLES. 1) $\langle (c;s_1')\|((c\|\bar{c});s_2'), w\rangle \rightarrow \langle s_1'\|(c;s_2'), w\tau\rangle$ is a synchronization-transition between $s_1 = c;s_1'$ and $s_2 = (c\|\bar{c});s_2'$.

2) $\langle (c;s_1')\|((c\|\bar{c});s_2'), w\rangle \rightarrow \langle (c;s_1')\|s_2', w\tau\rangle$ is a local transition involving only the second argument $s_2 = (c\|\bar{c});s_2'$ of the top-level "$\|$" operator.

Finally we remark that the Initial Step Lemma (2.1.1) originally stated for T_0 holds also for T_1 .

3.2 The Operational Semantics $\mathcal{O}_1$

Analogously to $\mathcal{O}_0$ we base an operational semantics $\mathcal{O}_1$ on T_1 . $\mathcal{O}_1$ is a mapping $\mathcal{O}_1 \colon \mathcal{L}_1 \rightarrow \mathbb{S}(\delta)$ with $\mathbb{S}(\delta) = \wp(A^{st}(\delta))$, and $\mathcal{O}_1[\![s]\!]$ is defined exactly the same way as $\mathcal{O}_0[\![s]\!]$ in Section 2.2.

EXAMPLES. $\mathcal{O}_1[\![c]\!] = \{\delta\}$, $\mathcal{O}_1[\![c\|\bar{c}]\!] = \{\delta, \tau\}$, $\mathcal{O}_1[\![(a;b) \cup (a;c)]\!] = \mathcal{O}_1[\![a;(b \cup c)]\!] = \{ab, a\delta\}$.

Thus under $\mathcal{O}_1$, communications c always create failures - whether or not they can synchronize with a matching communication $\bar{c}$. Also the two statements $(a;b)\cup(a;c)$ and $a;(b\cup c)$ obtain the same meaning under $\mathcal{O}_1$. This is characteristic of local nondeterminacy $s_1 \cup s_2$ where the choice of s_1 or s_2 is independent of the form of the other component s_2 or s_1 respectively. A more refined treatment will be provided in Section 4. We remark that the Definedness Lemma (2.2.1) and the Prolongation Lemma (2.2.2) of Section 2.2 hold also for $\mathcal{O}_1$. Note also that for $C = \phi$ the semantics $\mathcal{O}_1$ coincides with the previous $\mathcal{O}_0$.

REMARK 1. It is possible to do away with occurrences of δ in sets $\mathcal{O}_1[\![s]\!]$ in the case an alternative for the failure is available. Technically, this is achieved by imposing the axiom

$$\{\delta\} \cup X = X, \quad X \neq \phi . \qquad (3.3)$$

In the above example applying the axiom would turn the sets $\{\delta\}$, $\{\delta, \tau\}$ and $\{ab, a\delta\}$ into $\{\delta\}$, $\{\tau\}$ and $\{ab\}$, respectively. (For the latter case we take $\{ab, a\delta\} = a \cdot (\{b\} \cup \{\delta\}) = a \cdot \{b\} = \{ab\}$.) One might argue that imposing (3.3) throughout would be more in agreement with the intuitive understanding of communication. The reader is, of course, free to do this throughout Section 3. Our reason for not doing this is that our main result relating $\mathcal{O}_1$ and $\mathcal{D}_1$ does not depend on it. For both $\mathcal{O}_1$ and $\mathcal{D}_1$, (3.3) may or may not be imposed (simultaneously) without affecting the result of Section 3.4.

REMARK 2. Clearly, by taking $C = \phi$ the semantics $\mathcal{O}_1$ coincides with the previous $\mathcal{O}_0$.

3.3 *The Denotational Semantics* $\mathcal{D}_1$

This is as in Section 2.3, but extended/modified as shown below:

Firstly, we refine the definition of $\|: \mathbb{S}_c(\delta) \times \mathbb{S}_c(\delta) \to \mathbb{S}_c(\delta)$ as follows

1. For $X, Y \subseteq A^* \cup A^* \cdot \{\bot, \delta\}$ we define

$$X \| Y = (X \mathbin{⫫} Y) \cup (Y \mathbin{⫫} X) \cup (X | Y),$$

where

(i) $X \mathbin{\underline{\|}} Y = \bigcup\{u \mathbin{\underline{\|}} Y : u \in X\}$, $\perp \mathbin{\underline{\|}} Y = \{\perp\}$, $\delta \mathbin{\underline{\|}} Y = \{\delta\}$, $\epsilon \mathbin{\underline{\|}} Y = Y$,

$(a \cdot w) \mathbin{\underline{\|}} Y = a \cdot (\{w\} \| Y)$, and similarly with c replacing a,

(ii) $X | Y = \bigcup\{u|v : u \in X,\ v \in Y\}$, where $(c,u_1)|(\bar{c},v_1) = \tau(\{u_1\}\|\{v_1\})$

and $u|v = \phi$ for u,v not of such a form.

2. For X or Y with infinite words we define

$$X \| Y = \lim_n (X(n) \| Y(n))$$

where $X(n)$, $Y(n)$ are, as before, the sets of all n-prefixes of elements in X and Y. (This definition of $X\|Y$ is from [BK].)

The definition of $\mathcal{D}_1$ is now as follows: Let $\Gamma_1 = \underset{\sim}{stmv} \to \mathbb{S}_c(\delta)$ and let $\gamma \in \Gamma_1$. We define

$$\mathcal{D}_1 : \text{guarded } \mathcal{L}_1 \to (\Gamma_1 \to \mathbb{S}_c(\delta))$$

by the clauses

$\mathcal{D}_1[\![a]\!](\gamma) = \{a\}$ for $a \in A \backslash C$,

$\mathcal{D}_1[\![c]\!](\gamma) = \{c\}$ for $c \in C$,

$\mathcal{D}_1[\![s_1 \,\underset{\sim}{op}\, s_2]\!](\gamma) = \mathcal{D}_1[\![s_1]\!](\gamma) \,\underset{\sim}{op}^{\mathcal{D}_1}\, \mathcal{D}_1[\![s_2]\!](\gamma)$ for

$\underset{\sim}{op} \in \{\delta, \cup, \|\}$, $;^{\mathcal{D}_1} = \cdot$, $\cup^{\mathcal{D}_1} = \cup$, $\|^{\mathcal{D}_1} = \|$,

$\mathcal{D}_1[\![x]\!](\gamma) = \gamma(x)$,

$\mathcal{D}_1[\![\mu x[s]]\!](\gamma) = \lim_i X_i$, where $X_0 = \{\perp\}$ and

$X_{i+1} = \mathcal{D}_1[\![s]\!](\gamma\langle X_i/x\rangle)$.

Thus, apart from the clause for c, $\mathcal{D}_1$ is as $\mathcal{D}_0$ but for the refinement of $\|^{\mathcal{D}_1}$ with respect to $\|^{\mathcal{D}_0}$.

3.4 Relationship between $\mathcal{O}_1$ *and* $\mathcal{D}_1$

Here we do *not* simply have that

$$\mathcal{O}_1[\![s]\!] = \mathcal{D}_1[\![s]\!] \tag{3.4}$$

holds for all guarded statements $s \in \mathcal{L}_1$. As a counterexample take $s = c$. Then $\mathcal{O}_1[\![c]\!] = \{\delta\}$ but $\mathcal{D}_1[\![s]\!] = \{c\}$. Even worse, we can state:

3.4.1 THEOREM. There does not exist any denotational (implying compositional) semantics $\mathcal{D}$ satisfying (3.4).

The proof is based on:

3.4.2 LEMMA. $\mathcal{O}_1$ does not behave compositionally over $\|$, i.e. there exists no "semantic" operator

$$\|^{\mathcal{D}} : \mathbb{S}(\delta) \times \mathbb{S}(\delta) \to \mathbb{S}(\delta)$$

such that

$$\mathcal{O}_1[\![s_1 \| s_2]\!] = \mathcal{O}_1[\![s_1]\!] \|^{\mathcal{D}} \mathcal{O}_1[\![s_2]\!]$$

holds for all (guarded) $s_1, s_2 \in \mathcal{L}_1$).

PROOF. Consider $s_1 = c$ and $s_2 = \bar{c}$ in $\mathcal{L}_1$. Then $\mathcal{O}_1[\![s_1]\!] = \mathcal{O}_1[\![s_2]\!] = \{\delta\}$. Suppose now that $\|^{\mathcal{D}}$ exists. Then $\{\delta\} = \mathcal{O}_1[\![s_1 \| s_1]\!] = \mathcal{O}_1[\![s_1]\!] \|^{\mathcal{D}} \mathcal{O}_1[\![s_1]\!]$ $\mathcal{O}_1[\![s_1]\!] \|^{\mathcal{D}} \mathcal{O}_1[\![s_2]\!] = \mathcal{O}_1[\![s_1 \| s_2]\!] = \{\delta, \tau\}$. Contradiction. □

We remedy this not by redefining T_1 (which adequately captures the operational intuition for $\mathcal{L}_1$), but rather by introducing an *abstraction operator* $\alpha_1 : \mathbb{S}(\delta) \rightarrow \mathbb{S}(\delta)$ such that

$$\mathcal{O}_1[\![s]\!] = \alpha_1(\mathcal{D}_1[\![s]\!]) \qquad (3.5)$$

holds for guarded $s \in \mathcal{L}_1$. We take $\alpha_1 = \underset{\sim}{restr}_{\mathbb{S}}$ which for $W \in \mathbb{S}(\delta)$ is defined by

$$\begin{aligned} \underset{\sim}{restr}_{\mathbb{S}}(W) = \; & \{ w \mid w \in W \text{ does not contain any } c \in C\} \\ \cup & \{w \cdot \delta \mid \exists c' \in C,\ w' \in A^{st}(\delta) : w \cdot c' \cdot w' \in W \\ & \quad \text{and } w \text{ does not contain any } c \in C\} . \end{aligned}$$

Informally, $\underset{\sim}{restr}_{\mathbb{S}}$ replaces all unsuccessful synchronizations by deadlock. It thus resembles the restriction operator $\cdot \backslash C$ in CCS [Mi].

But how to prove (3.5)? Note that we cannot prove it directly by structural induction on s, because $\alpha_1 = \underset{\sim}{restr}_{\mathbb{S}}$ does not behave compositionally (over $\|$) due to Lemma 3.4.2. Our solution to this problem is to introduce a new *intermediate operational semantics* $\mathcal{O}_1^*$ such that we can show on the one hand

$$\mathcal{O}_1[\![s]\!] = \underset{\sim}{restr}_{\mathbb{S}}(\mathcal{O}_1^*[\![s]\!])$$

by purely operational, i.e. transition based arguments, and on the other hand

$$\mathcal{O}_1^*[\![s]\!] = \mathcal{D}_1[\![s]\!]$$

for guarded s, analogously to $\mathcal{O}_0[\![s]\!] = \mathcal{D}_0[\![s]\!]$ in Section 2.4. Combining these two results we will obtain the desired relationship (3.5).

For $\mathfrak{G}_1^*$ we modify the transition system T_1 into a system T_1^* which is the same as T_1 except for the communication axiom which now takes the form:

(*communication**)

$$\langle c, w\rangle \rightarrow w \cdot c .$$

We base $\mathfrak{G}_1^*$ on T_1^* just as we based $\mathfrak{G}_1$ on T_1.

EXAMPLES. $\mathfrak{G}_1^*[\![c]\!] = \{c\}$, $\mathfrak{G}_1^*[\![c \parallel \bar{c}]\!] = \{c\bar{c}, \bar{c}c, \tau\}$, $\mathfrak{G}_1^*[\![(a;b)\cup(a;c)]\!] = \mathfrak{G}_1^*[\![a;(b\cup c)]\!] = \{ab, ac\}$.

We first turn to:

3.4.3 THEOREM. $\mathfrak{G}_1[\![s]\!] = \underset{\mathbb{S}}{\underline{\text{restr}}}(\mathfrak{G}_1^*[\![s]\!])$ for every $s \in \mathcal{L}_1$.

The proof uses the following lemma which establishes the link between the underlying transition systems T_1 and T_1^*.

3.4.4 LEMMA. For all $s \in \mathcal{L}_1$, $s' \in \mathcal{L}_1 \cup \{E\}$ and $w, w' \in (A\backslash C)^*$:

(i) $T_1 \vdash \langle s, w\rangle \rightarrow \langle s', w'\rangle$

iff

$T_1^* \vdash \langle s, w\rangle \rightarrow \langle s', w'\rangle$

(ii) $T_1 \vdash \langle s, w\rangle \rightarrow \langle s', w\delta\rangle$

iff

$\exists c \in C$: $T_1^* \vdash \langle s, w\rangle \rightarrow \langle s', wc\rangle$

PROOF. Recall that $\delta \notin A$ and that T_1 and T_1^* differ only in their communication axioms:

$$\langle c, w\rangle \rightarrow w\cdot\delta \tag{3.6}$$

in T_1, and

$$\langle c, w\rangle \rightarrow w\cdot c \tag{3.6*}$$

in T_1^*. Therefore every transition in T_1 which is not a communication-transition, is also a transition in T_1^*, and vice versa. This implies (i). On the other hand, every communication-transition in T_1 corresponds to (another) communication-transition in T_1^* which is obtained by replacing axiom (3.6) by (3.6*) at the root of the proof tree, and otherwise applying exactly the same rules in T_1^* as in T_1. This argument also holds vice-versa, thus proving (ii). □

With Lemma 3.4.4 we are prepared for the

PROOF OF THEOREM 3.4.3. Observe that both

$$\mathcal{O}_1[\![s]\!], \underset{\sim}{restr}_{\mathbf{S}}(\mathcal{O}_1^*[\![s]\!]) \subseteq (A\backslash C)^* \cup (A\backslash C)^\omega \cup (A\backslash C)^* \cdot \{\bot, \delta\}.$$

Therefore we consider the following cases.

Case 1: $w \in (A\backslash C)^* \cup (A\backslash C)^\omega \cup (A\backslash C)^* \cdot \{\bot\}$.

Then as an immediate consequence of Lemma 3.4.4 (i) we have

$$w \in \mathcal{O}_1[\![s]\!] \text{ iff } w \in \mathcal{O}_1^*[\![s]\!].$$

Case 2: $w\delta \in (A\backslash C)^* \cdot \{\delta\}$.

Then

$$\begin{aligned} & w\delta \in \mathcal{O}_1[\![s]\!] \\ \text{iff } & T_1 \vdash \langle s, \epsilon\rangle \rightarrow^* w\delta \\ \text{iff } & \exists c' \in C,\ s' \in \mathcal{L}_1 \cup \{E\}:\ T_1^* \vdash \langle s, \epsilon\rangle \rightarrow \langle s', wc'\rangle \end{aligned}$$

(by Lemma 3.4.4 (ii). Note that the second alternative can arise.)

$$\text{iff} \quad (\exists c' \in C: T_1^* \vdash \langle s, \epsilon\rangle \to^* wc')$$
$$\vee\ (\exists c' \in C,\ s' \in \mathcal{L}_1,\ w' \in A^* \cup A^w \cup A^* \cdot \{\bot\}:$$
$$T_1^* \vdash \langle s, \epsilon\rangle \to^* \langle s', wc\rangle \wedge w' \in \mathcal{O}_1^*[\![s']\!])$$

(by the Definedness Lemma 2.2.1 which also holds for $\mathcal{O}_1^*$)

$$\text{iff} \quad \exists c' \in C,\ w' \in A^* \cup A^w \cup A^* \cdot \{\bot\}: wc'w' \in \mathcal{O}_1^*[\![s]\!]$$

(by the Prolongation Lemma 2.2.2 which also holds for $\mathcal{O}_1^*$)

Combining Cases 1 and 2 we find

$$\mathcal{O}_1[\![s]\!] = \underset{\sim\sim\sim}{restr}_{\mathbb{S}}(\mathcal{O}_1^*[\![s]\!]),$$

by the definition of $\underset{\sim\sim\sim}{restr}_{\mathbb{S}}$. This proves the theorem. □

Next we discuss:

3.4.5 THEOREM. $\mathcal{O}_1^*[\![s]\!] = \mathcal{D}_1[\![s]\!]$ for all (syntactically closed) guarded $s \in \mathcal{L}_1$.

Its proof has the same structure as that of "$\mathcal{O}_0[\![s]\!] = \mathcal{D}_0[\![s]\!]$" (Theorem 2.4.1). In fact, Theorems 2.4.10, 2.4.11 and 2.4.15 also hold for $\mathcal{O}_1^*$, $\mathcal{D}_1$ and $\mathcal{L}_1$ instead of $\mathcal{O}_0$, $\mathcal{D}_0$ and $\mathcal{L}_0$, with identical proofs. We therefore concentrate here only on the proof that $\mathcal{O}_1^*$ behaves compositionally over $\|$ (thereby completing the proof of Theorem 2.4.2). More precisely, we show:

3.4.6 LEMMA. $\mathcal{O}_1^*[\![s_1 \| s_2]\!] = \mathcal{O}_1^*[\![s_1]\!] \|^{\mathcal{D}_1} \mathcal{O}_1^*[\![s_2]\!]$ for all $s_1, s_2 \in \mathcal{L}_1$.

As an auxiliary tool we need a result recalling Apt's "merging lemma" in [Ap2].

3.4.7 LEMMA (*Synchronization*). $\forall s_1, s_2 \in \mathcal{L}_1\ \forall s_1', s_2' \in \mathcal{L}_1 \cup \{E\}\ \forall w, w_1, w_2 \in A^*$:

$$T_1^* \vdash \langle s_1 \| s_2, w\rangle \rightarrow \langle s_1' \| s_2', w\tau\rangle$$

where the considered transition is a synchronization-transition between s_1 and s_2 such that s_1' stems from s_1 and s_2' stems from s_2

iff

$\exists c \in C$:

$T_1^* \vdash \langle s_1, w_1\rangle \rightarrow \langle s_1', w_1 c\rangle$ and

$T_1^* \vdash \langle s_2, w_2\rangle \rightarrow \langle s_2', w_2 \bar{c}\rangle$

PROOF. By the Initial Step Lemma it suffices to prove the present lemma for $w = w_1 = w_2 = \varepsilon$ only.

"$\Rightarrow$": Suppose $T_1^* \vdash \langle s_1 \| s_2, \varepsilon\rangle \rightarrow \langle s_1' \| s_2', \tau\rangle$ as above. By the assumptions about this transition, its proof in T_1^* starts with a synchronization-axiom of the form

$$\langle c \| \bar{c}, \varepsilon\rangle \rightarrow \tau$$

where c occurs in s_1 and $\bar{c}$ in s_2. By the definition of T_1^*, s_1 and s_1' (respectively s_2 and s_2') are obtained from c and E ($\bar{c}$ and E) by successive embeddings in contexts of the form

$$\bullet;s\ ,\ \bullet\|s \ \text{ and } \ s\|\bullet \qquad (3.7)$$

for arbitrary statements $s \in \mathcal{L}_1$ (by the rule "synchronization in a context" of T_1^*).

To construct a proof of $\langle s_1, \epsilon\rangle \rightarrow \langle s_1', c\rangle$ in T_1^*, we start with the axiom

$$\langle c, \epsilon\rangle \rightarrow c$$

in T_1^* and then lift this transition to

$$\langle s_1, \epsilon\rangle \rightarrow \langle s_1', c\rangle$$

by successive applications of the rules of sequential composition and shuffle corresponding to the successive context embedding of c described in (3.7). This proves $T_1^* \vdash \langle s_1, \epsilon\rangle \rightarrow \langle s_1', c\rangle$. Analogously we prove $T_1^* \vdash \langle s_2, \epsilon\rangle \rightarrow \langle s_2', \bar{c}\rangle$.

"$\Leftarrow$": Suppose $T_1^* \vdash \langle s_1, \epsilon\rangle \rightarrow \langle s_1', c\rangle$. Let us analyze the structure of s_1 by investigating the possible proofs in T_1^* leading to a transition which produces "c". Clearly such a proof must start with the communication*-axiom

$$\langle c, \epsilon\rangle \rightarrow c ,$$

and it can proceed only applying the rules of sequential composition and shuffle. Thus s_1 has the following BNF-syntax:

$$s_1 ::= c \mid s_1;s \mid s_1 \| s \mid s \| s_1 \qquad (3.8)$$

where s is an arbitrary statement in $\mathcal{L}_1$. An analogous analysis holds for s_2 in $T_1^* \vdash \langle s_2, \epsilon\rangle \rightarrow \langle s_2', \epsilon\rangle$.

To show $T_1^* \vdash \langle s_1 \| s_2, \epsilon\rangle \to \langle s_1' \| s_2', \tau\rangle$, we start the proof with the synchronization axiom

$$\langle c \| \bar{c}, \epsilon\rangle \to \tau ,$$

and complete it by successive applications of the rule for synchronization in a context according to the structure of s_1 and s_2 as determined in (3.8). Note that we may arbitrarily "interleave" the applications concerning s_1 with those concerning s_2. This finally yields the proof of

$$\langle s_1 \| s_2, \epsilon\rangle \to \langle s_1' \| s_2', \tau\rangle$$

in T_1^*. Now by its construction this transition is a synchronization transition between s_1 and s_2 such that s_1' stems from s_1 and s_2' stems from s_2. This finishes the proof of the lemma. □

We now turn to the proof of the announced lemma.

3.4.6 LEMMA. $\mathfrak{S}_1^*[\![s_1 \| s_2]\!] = \mathfrak{S}_1^*[\![s_1]\!] \|^{\mathfrak{D}_1} \mathfrak{S}_1^*[\![s_2]\!]$ for all $s_1, s_2 \in \mathcal{L}_1$.

PROOF. "$\subseteq$": Let $w \in \mathfrak{S}_1^*[\![s_1 \| s_2]\!]$, with $w \in A^* \cup A^\omega \cup A^\omega \cup A^* \cdot \{\perp\}$. (Note that δ's are not present in $\mathfrak{S}_1^*$.) Then there exists a finite or infinite transition sequence

$$T_1^* \vdash \langle s_1 \| s_2, \epsilon\rangle = \langle s_0' \| s_0'', w_0\rangle \to \ldots \to \langle s_n' \| s_n'', w_n\rangle \to \ldots$$

such that s_n', s_n'' may be E, s_n' stems from s_1 and s_n'' from s_2, and the following holds:

(i) if $w \in A^*$ then $\exists n \geq 0$: $s'_n = s''_n = E \wedge w = w_n$

(ii) if $w \in A^\infty$ then $w = \sup_n w_n$

(iii) if $w \in A^* \cdot \{\perp\}$ then $\exists_n \geq 0 \forall_m \geq n$: $w_m = w_n \wedge w = w_n \perp$

We have to find words $u \in \mathcal{O}_1^*[\![s_1]\!]$ and $v \in \mathcal{O}_1^*[\![s_2]\!]$ with $w \in \{u\} \|_{\mathcal{S}_1} \{v\}$. To this end, we first establish the following claim.

Claim. There exist finite or infinite transition sequences

$$T_1^* \vdash \langle s_1, \epsilon \rangle = \langle t'_0, u_0 \rangle \to \ldots \to \langle t'_k, u_k \rangle \to \ldots ,$$

$$T_1^* \vdash \langle s_2, \epsilon \rangle = \langle t''_0, v_0 \rangle \to \ldots \to \langle t''_\ell, v_\ell \rangle \to \ldots$$

such that there are sequences

$$0 \leq k_0 \leq k_1 \leq k_2 \leq \ldots ,$$

$$0 \leq \ell_0 \leq \ell_1 \leq \ell_2 \leq \ldots$$

with

$$s'_n = t'_{k_n} \text{ and } s''_n = t''_{\ell_n} ,$$

$$w_n \in \{u_{k_n}\} \|_{\mathcal{S}_1} \{v_{\ell_n}\} ,$$

$$n \leq k_n + \ell_n , \quad \max\{k_n, \ell_n\} \leq n$$

for all $n \geq 0$.

Proof of the Claim. By induction on $n \geq 0$.

Basis. $n = 0$. Clear: choose $k_0 = \ell_0 = 0$.

Hypothesis. Assume the claim holds for $n \geq 0$, i.e. there are transition sequences

$$T_1^* \vdash \langle s_1, \varepsilon\rangle \to \dots \to \langle t'_{k_n}, u_{k_n}\rangle ,$$

$$T_1^* \vdash \langle s_2, \varepsilon\rangle \to \dots \to \langle t''_{\ell_n}, v_{\ell_n}\rangle$$

with $s'_n = t'_{k_n}$, $s''_n = t''_{\ell_n}$, $w_n \in \{u_{k_n}\} \|^{\mathcal{S}_1} \{v_{\ell_n}\}$, and $n \le k_n + \ell_n$.

Step $n \to n+1$: Let us analyze the final transition producing w_{n+1} in (3.9):

$$T_1^* \vdash \langle s'_n \| s''_n, w_n\rangle \to \langle s'_{n+1} \| s''_{n+1}, w_{n+1}\rangle . \tag{3.10}$$

Note that s'_{n+1} stems from s'_n and s''_{n+1} from s''_n .

Case 1: This is a local transition.

Then, say, the first component is affected, i.e.

$$T_1^* \vdash \langle s'_n, w_n\rangle \to \langle s'_{n+1}, w_{n+1}\rangle \text{ and } s''_n = s''_{n+1} .$$

(Note that we may have $w_n = w_{n+1}$.) By the Initial Step Lemma, also

$$T_1^* \vdash \langle s'_n, u_{k_n}\rangle \to \langle s'_{n+1}, u_{k_n} \cdot (w_{n+1} - w_n)\rangle .$$

Combining this transition with the hypothesis yields:

$$T_1^* \vdash \langle s_1, \varepsilon\rangle \to \dots \to \langle t'_{k_n}, u_{k_n}\rangle \to \langle s'_{n+1}, u_{k_n} \cdot (w_{n+1} - w_n)\rangle$$

(where, if w' is a word extending w , say $w' = wu$, we define $w' - w$ to be u) .

Now we define:

$$k_{n+1} = k_n + 1, \quad \ell_{n+1} = \ell_n$$
$$t_{k_{n+1}} = s'_{n+1} , \quad u_{k_{n+1}} = u_{k_n} \cdot (w_{n+1} - w_n) .$$

By the definition of $\|^{\mathcal{D}_1}$,

$$w_{n+1} = w_n \cdot (w_{n+1} - w_n)$$
$$\in \{u_{k_n} \cdot (w_{n+1} - w_n)\} \|^{\mathcal{D}_1} \{v_{\ell_n}\} = \{u_{k_{n+1}}\} \|^{\mathcal{D}_1} \{v_{\ell_{n+1}}\}$$

and of course $n+1 \leq k_{n+1} + \ell_{n+1}$. This proves the claim for $n+1$ in Case 1.

Case 2: (3.10) is a synchronization-transition between s_1 and s_2. Then $w_{n+1} = w_n\tau$ and, by the Synchronization Lemma, there exists some $c \in C$ with

$$T_1^* \vdash \langle s'_n, u_{k_n} \rangle \rightarrow \langle s'_{n+1}, u_{k_n} \cdot c \rangle ,$$
$$T_1^* \vdash \langle s''_n, v_{k_n} \rangle \rightarrow \langle s''_{n+1}, v_{k_n} \cdot \overline{c} \rangle .$$

Combining these transitions with the hypothesis yields:

$$T_1^* \vdash \langle s_1, \varepsilon \rangle \rightarrow \ldots \rightarrow \langle t'_{k_n}, u_{k_n} \rangle \rightarrow \langle t'_{n+1}, u_{k_n} \cdot c \rangle ,$$
$$T_1^* \vdash \langle s_2, \varepsilon \rangle \rightarrow \ldots \rightarrow \langle t''_{\ell_n}, v_{\ell_n} \rangle \rightarrow \langle t''_{n+1}, v_{\ell_n} \cdot \overline{c} \rangle .$$

Obviously, we define

$$k_{n+1} = k_n + 1, \quad \ell_{n+1} = \ell_n + 1,$$
$$t'_{k_{n+1}} = s'_{n+1}, \quad t''_{\ell_{n+1}} = s''_{n+1},$$
$$u_{k_{n+1}} = u_{k_n} \cdot c, \quad v_{\ell_{n+1}} = v_{\ell_n} \cdot \overline{c} .$$

By the definition of $\|^{\mathcal{D}_1}$,

$$w_{n+1} = w_n\tau \in \{u_{k_n} \cdot c\} \|^{\mathcal{D}_1} \{v_{\ell_n} \cdot \bar{c}\} = \{u_{k_{n+1}}\} \|^{\mathcal{D}_1} \{v_{\ell_{n+1}}\}$$

and of course $n + 1 \le k_{n+1} + \ell_{n+1}$. This proves the claim for $n + 1$ also in Case 2.

Hence the claim holds in general.

Using the claim, it is easy to find appropriate words u, v . The construction corresponds to the case analysis (i) - (iii) of w above. For example, we define u as follows:

- if $\exists k \ge 0$: $s'_k = E$, then $u = u_k \in A^*$,
- if $\forall k \ge 0\ \exists K > k$: $w_k < w_K$, then $u = \sup_k u_k \in A^\omega$,
- if $\exists k \ge 0\ \forall K \ge k$: $w_k = w_K$, then $u = u_k \perp \in A^* \cdot \{\perp\}$.

Analogously we proceed for v . Clearly

$$u \in \mathcal{O}_1^*[\![s_1]\!] \quad \text{and} \quad v \in \mathcal{O}_1^*[\![s_2]\!] .$$

To verify

$$w \in \{u\} \|^{\mathcal{D}_1} \{v\} \tag{3.11}$$

we examine the cases (i) - (iii) of w .

In case (i) we have a finite path

$$T_1^* \vdash \langle s_1 \| s_2, \epsilon \rangle \rightarrow \ldots \rightarrow \langle s'_n \| s''_n, w_n \rangle = \langle E \| E, w \rangle = w .$$

By the claim and the definition of u,v

$$T_1^* \vdash \langle s_1, \epsilon\rangle \to \ldots \to \langle t'_{k_n}, u_{k_n}\rangle = \langle E, u_{k_n}\rangle = u \ ,$$

$$T_1^* \vdash \langle s_2, \epsilon\rangle \to \ldots \to \langle t''_{\ell_n}, v_{\ell_n}\rangle = \langle E, v_{\ell_n}\rangle = v \ ,$$

and thus (3.11) as required.

In case (ii) we have an infinite path (3.9) producing infinitely often increasing words w_n. By the claim at least one of the paths of s_1 and s_2, say that of s_1, must also be infinite, producing infinitely often increasing words u_k, yielding an infinite $u = \sup_k u_k$. Now by definition

$$\{u\} \|^{\mathcal{D}_1} \{v\} = \lim_n(\{u[n]\} \|^{\mathcal{D}_1} \{v[n]\}) \ .$$

Consider now the approximation w_n of w. By the claim,

$$w_n \in \{u_{k_n}\} \|^{\mathcal{D}_1} \{v_{\ell_n}\} \ .$$

Since $\max\{k_n, \ell_n\} \le n$, we have

$$u_{k_n} \le u[n] \quad \text{and} \quad v_{\ell_n} \le v[n] \ .$$

Thus $\exists \widetilde{w} \in \{u[n]\} \|^{\mathcal{D}_1} \{v[n]\}$ with

$$d(w_n, \widetilde{w}) \le 2^{-|w_n|} \ .$$

This shows

$$w \in \lim_n(\{u[n]\} \|^{\mathcal{D}_1} \{v[n]\}) \ ,$$

and thus proves (3.11).

In case (iii) we have an infinite path

$$T_1^* \vdash \langle s_1 \| s_2, \varepsilon\rangle \to \dots \to \langle s'_n \| s''_n, w_n\rangle \to \langle \dots, w_{n+1}\rangle \to \dots$$

with $w_n = w_{n+1} = \dots$ and thus $w = w_n\perp$. By the claim

$$T_1^* \vdash \langle s_1, \varepsilon\rangle \to \dots \to \langle t'_{k_n}, u_{k_n}\rangle ,$$

$$T_1^* \vdash \langle s_2, \varepsilon\rangle \to \dots \to \langle t''_{\ell_n}, v_{\ell_n}\rangle ,$$

with $w_n \in \{u_{k_n}\} \|^{\mathcal{D}_1} \{v_{\ell_n}\}$. Moreover, due to the condition "$n \le k_n + \ell_n$ for all n" in the claim, at least one of the transition sequences of s_1 (or s_2) can be extended to an infinite one without expanding u_{k_n} (or v_{ℓ_n}). So $u = u_{k_n}\perp$ (or $v = v_{\ell_n}\perp$). If the other path of s_2 (or s_1) is finite, we may assume w.l.o.g. that $t''_{\ell_n} = E$ (or $t'_{k_n} = E$). So then we have $v = v_{\ell_n}$ (or $u = u_{k_n}$). Combining these facts establishes (3.11)

"$\supseteq$": Let $w \in \mathcal{O}_1^*[\![s_1]\!] \|^{\mathcal{D}_1} \mathcal{O}_1^*[\![s_2]\!]$. Then there exist words $u \in \mathcal{O}_1^*[\![s_1]\!]$, $v \in \mathcal{O}_1^*[\![s_2]\!]$ with

$$w \in \{u\} \|^{\mathcal{D}_1} \{v\} .$$

We have to prove

$$w \in \mathcal{O}_1^*[\![s_1 \| s_2]\!] .$$

By definition of $\mathcal{O}_1^*$ there are corresponding finite or infinite transition sequences in T_1^* for u and v:

$$T_1^* \vdash \langle s_1, \epsilon\rangle = \langle t_0', u_0\rangle \to \dots \to \langle t_k', u_k\rangle \to \dots , \tag{3.12}$$

$$T_1^* \vdash \langle s_2, \epsilon\rangle = \langle t_0'', v_0\rangle \to \dots \to \langle t_\ell'', v_\ell\rangle \to \dots , \tag{3.13}$$

where (in case of finite sequences) t_k' and t_ℓ'' may be E. Recall that u and v are obtained from (3.12) and (3.13) just as described for w by the cases (i) - (ii) in part "$\subseteq$". We now construct a finite or infinite path

$$T_1^* \vdash \langle s_1 \| s_2, \epsilon\rangle = \langle s_0' \| s_0'', w_0\rangle \to \dots \to \langle s_n' \| s_n'', w_n\rangle \to \dots \tag{3.14}$$

which is *maximal* w.r.t.

$$w_n \le w$$

and which moreover satisfies the following properties: there are sequences

$$0 \le k_0 \le k_1 \le \dots \quad \text{and} \quad 0 \le \ell_0 \le \ell_1 \le \dots$$

such that for each $n \ge 0$

$$s_n' = t_{k_n}', \quad s_n'' = t_{\ell_n}''$$

$$w_n \in \{u_{k_n}\} \|^{\mathcal{D}_1} \{v_{\ell_n}\} ,$$

$$\max\{k_n, \ell_n\} \le n , \quad n \le k_n + \ell_n .$$

The construction of (3.14) proceeds by induction on $n \ge 0$.

Basis: $n = 0$. Choose $k_0 = \ell_0 = 0$.

Hypothesis: Assume the construction works already up to $n \geq 0$. If the configurations

$$\langle t'_{k_n}, u_{k_n} \rangle \quad \text{and} \quad \langle t''_{\ell_n}, v_{\ell_n} \rangle \tag{3.15}$$

in (3.12) and (3.13) are both final ones, i.e. with $t'_{k_n} = t''_{\ell_n} = E$, the constructed path (3.14) is already maximal because also

$$s'_n \| s''_n = E$$

holds. In all other cases (3.14) has to be extended.

Step $n \to n+1$: We analyze the configurations (3.15).

Case 1a: Path (3.12) has a transition $\langle t'_{k_n}, u_{k_n} \rangle \to \langle t'_{k_n+1}, u_{k_n+1} \rangle$ with $u_{k_n} = u_{k_n+1}$. Then we put

$$w_{n+1} = w_n$$

and $k_{n+1} = k_n + 1$, $\ell_{n+1} = \ell_n$, $s'_{n+1} = t'_{k_n+1}$, $s''_{n+1} = s''_n$, and add the transition

$$\langle s'_n \| s''_n, w_n \rangle \to \langle s'_{n+1} \| s''_{n+1}, w_{n+1} \rangle$$

to (3.14).

Case 1b: Symmetric to Case 1a, but with regards to path (3.13).

Case 2a: Path (3.12) has a transition $\langle t'_{k_n}, u_{k_n} \rangle \to \langle t'_{k_n+1}, u_{k_n+1} \rangle$ with $u_{k_n+1} = u_{k_n} \cdot b$ where $b \in A$ and $w_n \cdot b \leq w$.

(Note: b can be an elementary action a, a communication c or τ. $w_n \cdot b \le w$ is always true for $b = a$ or $b = \tau$.) Now we put

$$w_{n+1} = w_n \cdot b$$

and $k_{n+1} = k_n + 1$, $\ell_{n+1} = \ell_n$, $s'_{n+1} = t'_{k_n+1}$, $s''_{n+1} = s''_n$, and add the transition

$$\langle s'_n \| s''_n, w_n \rangle \to \langle s'_{n+1} \| s''_{n+1}, w_{n+1} \rangle$$

to (3.14).

Case 2b: Symmetric to Case 2a, but with regards to path (3.13).

Case 3: Path (3.12) has a transition $\langle t'_{k_n}, u_{k_n} \rangle \to \langle t'_{k_n+1}, u_{k_n+1} \rangle$ with $u_{k_n+1} = u_{k_n} \cdot c$ where $c \in C$, but $w_n \cdot c \not\le w$.

Since $w \in \{u\} \|_{\mathcal{S}_1} \{v\}$, we conclude that $w_n \cdot \tau \le w$ and that path (3.13) has a transition

$$\langle t''_{\ell_n}, v_{\ell_n} \rangle \to \langle t''_{\ell_n+1}, v_{\ell_n+1} \rangle$$

with

$$v_{\ell_n+1} = v_{\ell_n} \cdot \bar{c} .$$

Then we put

$$w_{n+1} = w_n \cdot \tau$$

and

$$k_{n+1} = k_n + 1, \ \ell_{n+1} = \ell_n + 1, \ s'_{n+1} = t'_{k_n+1}, \ s''_{n+1} = t''_{\ell_n+1},$$

and add the transition

$$\langle s'_n \| s''_n, w_n \rangle \rightarrow \langle s'_{n+1} \| s''_{n+1}, w_{n+1} \rangle$$

to (3.14). This finishes the construction of path (3.14). We now claim that (3.14) yields w according to the definition of $\mathcal{O}_1^*[\![s_1 \| s_2]\!]$. This is clearly true for $w \in A^* \cup A^\omega$ due to the maximality of (3.14) and the conditions "$w_n \in \{u_{k_n}\} \|^{\mathcal{L}_1} \{v_{\ell_n}\}$ for $n \geq 0$" which link up with $w \in \{u\} \|^{\mathcal{D}_1} \{v\}$ analogously to part "$\subseteq$".

If $w \in A^* \cdot \{\perp\}$, then at least one of u or v, say u, is in $A^* \cdot \{\perp\}$ as well. Then path (3.12) is infinite. By the conditions "$\max\{k_n, \ell_n\} \leq n$ for $n \geq 0$", also the constructed path (3.14) is infinite. Thus (3.14) yields indeed w in $\mathcal{O}_1^*[\![s_1 \| s_2]\!]$. □

This also finishes our argument for Theorem 3.4.5. By combining Theorems 3.4.4 and 3.4.5 we finally obtain our desired result:

3.4.8 THEOREM. $\mathcal{O}_1[\![s]\!] = \underset{\sim\sim\sim}{restr}_{\mathcal{S}} (\mathcal{D}_1[\![s]\!])$ for every guarded $s \in \mathcal{L}_1$.

4. THE LANGUAGE $\mathcal{L}_2$: SYNCHRONIZATION MERGE AND GLOBAL NONDETERMINACY

We assume the same structure of the alphabet A as for $\mathcal{L}_1$. But the syntax for $s \in \mathcal{L}_2$ is now given by:

$$s ::= a \mid c \mid s_1;s_2 \mid s_1 + s_2 \mid s_1 \| s_2 \mid x \mid \mu x[s] .$$

The symbol " + " denoting global nondeterminacy is taken from CCS [Mi].

4.1 *The Transition System T_2*

T_2 is like T_1 but without the axioms for local nondeterminacy and for communication $(\langle c, w\rangle \rightarrow w\delta)$. Instead we have new rules for *global nondeterminacy*[1]:

(μ-unfolding)

$$\frac{\langle s_1, w\rangle \rightarrow \langle s', w\rangle}{\begin{array}{c}\langle s_1 + s_2, w\rangle \rightarrow \langle s' + s_2, w\rangle \\ \langle s_2 + s_1, w\rangle \rightarrow \langle s_2 + s', w\rangle\end{array}}$$

Here the word on the r.h.s. of the premise is equal to the word on the ℓ.h.s. (= w). This implies that the premise (and hence the conclusion) is a recursion transition.

(selection by action)

$$\frac{\langle s_1, w\rangle \rightarrow \langle s', w'\rangle}{\begin{array}{c}\langle s_1 + s_2, w\rangle \rightarrow \langle s', w'\rangle \\ \langle s_2 + s_1, w\rangle \rightarrow \langle s', w'\rangle\end{array}}$$

Here $w' = wa$ (and hence the premise is an elementary action transition) or $w' = w\tau$ (and hence the premise is a synchronization transition). Also s' may be E.

(selection by synchronization)

$$\frac{\langle s_1 \| s_2, w\rangle \rightarrow \langle s', w\tau\rangle}{\begin{array}{c}\langle (s_1 + s) \| s_2, w\rangle \rightarrow \langle s', w\tau\rangle \\ \langle (s + s_1) \| s_2, w\rangle \rightarrow \langle s', w\tau\rangle \\ \langle s_1 \| (s_2 + s), w\rangle \rightarrow \langle s', w\tau\rangle \\ \langle s_1 \| (s + s_2), w\rangle \rightarrow \langle s', w\tau\rangle\end{array}}$$

where s' may be E, and the premise of the rule is a synchronization transition between s_1 and s_2. (Note that the "$;$" and "$\|$"- context rules for $\|$ remain valid.)

REMARKS. To explain the difference between "$\cup$" and "+": note first that for $s_1, s_2 \in \mathcal{L}_1 \cap \mathcal{L}_2$

$$T_2 \vdash \langle s_1 + s_2, w\rangle \rightarrow \langle s', w'\rangle$$

implies

$$T_1 \vdash \langle s_1 \cup s_2, w\rangle \rightarrow \langle s', w'\rangle$$

but not vice versa. The essential difference between these two operators (and hence between T_1 and T_2) is how communication is treated in the presence of nondeterminacy. For example, the $\mathcal{L}_1$-statement

$$a \cup c$$

involving local nondeterminacy may choose "on its own" between a and c, i.e. in terms of T_1-transitions we have

$$\langle a \cup c,\ w\rangle \rightarrow \langle a,\ w\rangle$$

$$\langle a \cup c,\ w\rangle \rightarrow \langle c,\ w\rangle\ .$$

The first alternative yields

$$\langle a,\ w\rangle \rightarrow w \cdot a$$

whereas a communication can always deadlock in T_1 :

$$\langle c,\ w\rangle \rightarrow c \cdot \delta$$

Contrast this behavior with that of the $\mathcal{L}_2$-statement

$$a + c$$

involving global nondeterminacy. The only transition possible is

$$\langle a + c\rangle \rightarrow w \cdot a$$

(we say the first alternative of $a + c$ is selected by the action a). In particular, a communication c in isolation does not produce anything in T_2 . But in cooperation with a matching communication $\bar{c}$ in another parallel component, c may produce a synchronization transition:

$$\langle (a + c) \| \bar{c}\rangle \rightarrow w \cdot \tau$$

(we say the second alternative of $a+c$ is selected by the synchronization of c with $\bar{c}$).

This form of global noneterminacy is typical for languages like CSP [Ho], Ada [Ad] and Occam [In]. There the elementary action a corresponds to passing a true Boolean guard and the synchronization of c with $\bar{c}$ corresponds to matching communication guards in two parallel components. In the abstract setting of uniform concurrency global nondeterminacy was first discussed by Milner [Mi]. However, Milner takes from the very beginning a communication axiom corresponding (in our setting) to

$$\langle c, w\rangle \rightarrow w\cdot c \tag{4.1}$$

This enables him to state very simple transition rules for global nondeterminacy. We prefer not to adopt Milner's approach for T_2 because(4.1) does not correspond to the operational idea of CSP, Ada or Occam where a communication c proceeds only if a matching communication $\bar{c}$ is available.

Finally, note that in case of a μ-term, global nondeterminacy "+" allows us to unfold the recursion before selecting any alternative. For example,

$$\langle \mu x[a]+c, w\rangle \rightarrow \langle a+c, w\rangle \rightarrow w\cdot a$$

holds in T_2.

4.2 The Operational Semantics $\mathcal{O}_2$

$\mathcal{O}_2$ is a mapping $\mathcal{O}_2\colon \mathcal{L}_2 \rightarrow \mathbb{S}(\delta)$ with $\mathbb{S}(\delta) = \mathcal{P}(A^{st}(\delta))$ as for $\mathcal{L}_1$. The definition of $\mathcal{O}_2[\![s]\!]$ is as for $\mathcal{O}_0$ and $\mathcal{O}_1$, i.e.

$$\mathfrak{G}_2[\![s]\!] = \{\underset{\sim}{word}(\pi) \mid \pi \text{ is a path from } s\}.$$

However there is now an additional fourth clause in the definition of $\underset{\sim}{word}(\pi)$, namely:

(d) if π is finite, and of the form

$$\langle s, \epsilon\rangle = \langle s_0, w_0\rangle \to \dots \to \langle s_n, w_n\rangle$$

where no further transition $\langle s_n, w_n\rangle \to \langle s', w'\rangle$ is deducible in T_2, then $\underset{\sim}{word}(\pi) = w_n \cdot \delta$.

The pair $\langle s_n, w_n\rangle$ in (d) is called a deadlocking configuration. (Such configurations did not exist under T_0 or T_1.) Note that by (d) the Definedness Lemma 2.2.1 remains valid for $\mathfrak{G}_2$: $\mathfrak{G}_2[\![s]\!] \neq \phi$ for all $s \in \mathcal{L}_2$.

The following examples mark the differences from $\mathfrak{G}_1$.

EXAMPLES. $\mathfrak{G}_2[\![c]\!] = \{\delta\}$, $\mathfrak{G}_2[\![c \,\|\, \bar{c}]\!] = \{\tau\}$, $\mathfrak{G}_2[\![(a;b) + (a;c)]\!] = \{ab, a\delta\}$, $\mathfrak{G}_2[\![a;(b+c)]\!] = \{ab\}$. (Remember, $\mathfrak{G}_1[\![a;(b \cup c)]\!] = \mathfrak{G}_1[\![(a;b) \cup (a;c)]\!] = \{ab, a\delta\}$.)

Because it is important to see the difference between the last two examples, we shall show how they are derived:

(i) $\mathfrak{G}_2[\![(a;b) + (a;c)]\!] = \{ab, a\delta\}$.

PROOF. Note that

$$\langle a;b, \epsilon\rangle \to \langle b, a\rangle \to ab$$

and

$$\langle a;c, \epsilon\rangle \to \langle c, a\rangle$$

are deducible. So by selection by elementary action we obtain also

$$\langle (a;b) + (a;c), \epsilon \rangle \rightarrow ab$$

and

$$\langle (a;b) + (a;c), \epsilon \rangle \rightarrow \langle c, a \rangle .$$

So, since no futher deductions can be made from $\langle c, a \rangle$, we get by the definition of $\mathfrak{S}_2$: $\mathfrak{S}_2[\![(a;b) + (a;c)]\!] = \{ab, a\delta\}$.

(ii) $\mathfrak{S}_2[\![a;(b + c)]\!] = \{ab\}$.

PROOF. First note that

$$\langle a;(b + c), \epsilon \rangle \rightarrow \langle b + c, a \rangle .$$

Since we have that

$$\langle b, a \rangle \rightarrow ab ,$$

we also have

$$\langle b + c, a \rangle \rightarrow ab ,$$

and therefore

$$\langle a;(b + c), \epsilon \rangle \rightarrow ab .$$

Since we cannot deduce anything from $\langle c, a \rangle$, ab is all we can deduce from $\langle a;(b + c), \epsilon \rangle$. Consequently, $\mathfrak{S}_2[\![a;(b + c)]\!] = \{ab\}$.

Thus with global nondeterminacy "+", the statements $s_1 = (a;b) + (a;c)$ and $s_2 = a;(b + c)$ get different meanings under $\mathfrak{S}_2$. This difference can be understood as follows: If s_1 performs the elementary action a, the remaining statement is either the elementary action b or the communication c. In case of c, a deadlock occurs since no matching communication is available. However, if s_2 performs a, the remaining statement is $b + c$ which cannot deadlock because the action b is always possible. Thus communications c create deadlocks only if neither a matching communication $\bar{c}$ nor an alternative elementary action b is available.

4.3 The Denotational Semantics $\mathcal{D}_2$

We follow [BZ1, BZ2, BBKM] in introducing a *branching time* semantics for $\mathcal{L}_2$. Let, as usual, $\perp \notin A$ and let $A_\perp$ be short for $A \cup \{\perp\}$. Again, we assume a special element τ in A. Let the metric spaces $(\mathbb{P}_n, d_n)$, $n \geq 0$, be defined by

$$\mathbb{P}_0 = \wp(A_\perp),\ \mathbb{P}_{n+1} = \wp(A_\perp \cup (A \times \mathbb{P}_n))$$

where $\wp(\cdot)$ denotes all subsets of $(\cdot)$, and the metrics d_n will be defined in a moment. Let $\mathbb{P}_\omega = \bigcup_n \mathbb{P}_n$. Elements of $\mathbb{P}_\omega$ are called (finite) processes and typical elements are denoted by $p,q,\ldots$. Processes p in $\mathbb{P}_n$ are often denoted by $p_n, q_n, \ldots$. For $p \in \mathbb{P}_\omega$ we call the least n such that $p \in \mathbb{P}_n$ its *degree* . Note that each process is a *set* ; hence, a process has elements for which we use $x,y,\ldots$ (not to be confused with $x,y \in Stmv$). For each $p (\in \mathbb{P}_\omega)$ we define its n-th projection $p(n)$ as follows:

$$p(n) = \{x(n) \mid x \in p\}, \qquad n = 0,1,\ldots$$

$$x(n) = x \quad \text{if } x \in A_\perp, \qquad n = 0,1,\ldots$$

$$[a,p](n) = \begin{cases} a, & n = 0 \\ [a, p(n-1)], & n = 1,2,\ldots \end{cases}$$

We can now define d_n by

$$d_0(p_0', p_0'') = \begin{cases} 0 & \text{if } p_0' = p_0'' \\ 1 & \text{if } p_0' \neq p_0'' \end{cases}$$

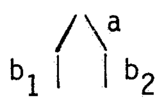
$$d_{n+1}(p_{n+1}', p_{n+1}'') = 2^{-\sup\{k \mid p_{n+1}'(k) = p_{n+1}''(k)\}}$$

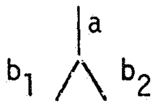
with $2^{-\infty} = 0$ as before.

On $\mathbb{P}_\omega$ we define the metric d by putting $d(p,q) = d_n(p,q)$ where $n = \max(\text{degree}(p), \text{degree}(q))$. We now define the set $\mathbb{P}$ of finite and infinite processes as the *completion* of $\mathbb{P}_\omega$ with respect to d. A fundamental result of [BZ 2] is that we have the equality (more precisely, the isometry)

$$\mathbb{P} = \mathcal{P}_{closed}(A_\perp \cup (A_\perp \times \mathbb{P})) .$$

Examples of finite elements of $\mathbb{P}$ are $\{[a,\{b_1\}], [a,\{b_2\}]$ and $\{[a,\{b_1,b_2\}]\}$. The following trees represent these:





Thus, the branching structure is preserved. An example of an infinite element of $\mathbb{P}$ is the process p which satisfies the equation $p = \{a,p], [b,p]\}$. Processes are like commutative trees which have in addition sets rather than multisets for successors of nodes and which satisfy a closedness property.

An example of a set which is not a process is $\{a,[a,\{a\}],[a,\{[a,\{a\}]\}]\ldots\}$, where this set does not include the infinite branch of a's.

REMARK. We observe that the collection of all finite and infinite trees over $A_\perp$ (where $\perp$ occurs only at the leaves), modulo Park's equivalence relation of bisimulation [Pa], is isomorphic to $\mathbb{P}$.

The empty set *is* a process and takes the role of δ. Note that in the previous *linear time* (LT) framework ϕ cannot replace δ since by the definition of concatenation (for LT) we have $a \cdot \phi = \phi$ which is undesirable for an element modelling failure. (An action which fails should not cancel all previous actions!) In the present branching time framework, $\{[a,\phi]\}$ is a process which is indeed different from (and irreducible to) ϕ.

The following operations on processes are defined. We first take the case that both processes are finite, and use induction on the degree(s) of the processes concerned:

concatenation $\circ$: $p \circ q = \bigcup\{x \circ q \mid x \in p\}$, where $\perp \circ q = \perp$, $a \circ q = [a,q]$, $[a,p'] \circ q = [a,p' \circ q]$ and similar clauses with c replacing a.

union $\cup$: $p \cup q$ is the set-theoretic union of p and q.

merge $\|$: $p \| q = (p \mathbin{\|\!\_} q) \cup (q \mathbin{\|\!\_} p) \cup (p/q)$, where $p \mathbin{\|\!\_} q = \bigcup\{x \mathbin{\|\!\_} q \mid x \in p\}$, $\perp \mathbin{\|\!\_} q = \perp$, $a \mathbin{\|\!\_} q = [a,q]$, $[a,p'] \mathbin{\|\!\_} q = [a,p' \| q]$ and similar clauses with c replacing a. Moreover, $p \mid q = \bigcup\{x \mid y : x \in p, y \in q\}$, where

$$[c, p'] \mid [\bar{c}, q'] = \{[\tau, p' \| q']\}$$
$$[c, p'] \mid q' = \{[\tau, p']\}$$
$$c \mid [\bar{c}, q'] = \{[\tau, q']\}$$
$$c \mid \bar{c} = \{\tau\}$$

and $x \mid y = \phi$ for x,y not of one of the above four forms.

For p or q infinite we have (since $\mathbb{P}$ is defined by completion of $\mathbb{P}_\omega$) that $p = \lim_n p_n$, $q = \lim_n q_n$, p_n and q_n finite, $n = 0,1,\ldots,$ and we define $p \underset{\sim}{op} q = \lim_n (p_n \underset{\sim}{op} q_n)$, where $\underset{\sim}{op} \in \{\cdot, \cup, \|\}$. It is now straightforward to define $\mathcal{D}_2$: guarded $\mathcal{L}_2 \to (\Gamma_2 \to \mathbb{P})$, where $\Gamma_2 = \underset{\sim}{Stmv} \to \mathbb{P}$, by following the clauses in the definition of $\mathcal{D}_0, \mathcal{D}_1$. Thus, we put

$$\mathcal{D}_2[\![a]\!](\gamma) = \{a\}$$

$$\mathcal{D}_2[\![c]\!](\gamma) = \{c\}$$

$$\mathcal{D}_2[\![s_1 \underset{\sim}{op} s_2]\!](\gamma) = \mathcal{D}_2[\![s_1]\!](\gamma) \underset{\sim}{op}^{\mathcal{D}_2} \mathcal{D}_2[\![s_2]\!](\gamma)$$

for $\underset{\sim}{op} \in \{;, +, \|\}$, where $;^{\mathcal{D}_2} = \circ$, $+^{\mathcal{D}_2} = \cup$, $\|^{\mathcal{D}_2} = \|$

$$\mathcal{D}_2[\![x]\!](\gamma) = \gamma(x)$$

$$\mathcal{D}_2[\![\mu x[s]]\!](\gamma) = \lim_i p_i, \quad \text{where } p_0 = \{\bot\} \text{ and}$$

$$p_{i+1} = \mathcal{D}_2[\![s]\!](\gamma \langle p_i / x \rangle).$$

Mutatis mutandis, the contractivity results for $\mathcal{D}_0$, $\mathcal{D}$, hold again.

4.4 Relationship between $\mathcal{O}_2$ and $\mathcal{D}_2$

For a suitable abstraction operator α_2 we shall show that

$$\mathcal{O}_2[\![s]\!] = \alpha_2(\mathcal{D}_2[\![s]\!]) \tag{4.2}$$

holds for all guarded $s \in \mathcal{L}_2$. We define $\alpha_2 \colon \mathbb{P} \to \mathbb{S}(\delta)$ in two steps:

1. First we define a restriction mapping $\underset{\sim}{restr}_{\mathbb{P}} : \mathbb{P} \to \mathbb{P}$. For $p \in \mathbb{P}_\omega$ we put inductively:

$$\underset{\sim}{restr}_{\mathbb{P}}(p) = \{a \mid a \in p \text{ and } a \notin C\}$$
$$\cup\{[a, \underset{\sim}{restr}_{\mathbb{P}}(q)] \mid [a,q] \in p \text{ and } a \notin C\}$$

For $p \in \mathbb{P} \setminus \mathbb{P}_\omega$ we have $p = \lim_n p_n$, with $p_n \in \mathbb{P}_n$, and we put

$$\underset{\sim}{restr}_{\mathbb{P}}(p) = \lim_n(\underset{\sim}{restr}_{\mathbb{P}}(p_n)) .$$

EXAMPLE. Let $p = \mathcal{D}_2[\![(a+c) \| (b+\bar{c})]\!] = \mathcal{D}_2[\![(a;(b+\bar{c})) + (c;(b+\bar{c})) + (b;(a+c)) + (\bar{c};(a+c)) + \tau]\!]$. Then $\underset{\sim}{restr}_{\mathbb{P}}(p) = \{[a,\{b\}],[b,\{a\}],\tau\} = \mathcal{D}_2[\![(a;b) + (b;a) + \tau]\!]$.

2. Then we define a mapping $\underset{\sim}{streams} \colon \mathbb{P} \to \mathbb{S}_c(\delta)$. For $p \in \mathbb{P}_\omega$ we put inductively:

$$\underset{\sim}{streams}(p) = \begin{cases} \{a \mid a \in p\} \cup \\ \cup\{a \cdot \underset{\sim}{streams}(q) \mid [a,q] \in p\} & \text{if } p \neq \phi \\ \{\delta\} & \text{if } p \neq \phi \end{cases}$$

Note that $a \cdot \underset{\sim}{\text{streams}}(q)$ itself is a set of streams. For $p \in \mathbb{P} \setminus \mathbb{P}_\omega$ we have $p = \lim_n p_n$, with $p_n \in \mathbb{P}_n$, and we put

$$\underset{\sim}{\text{streams}}(p) = \lim_n (\underset{\sim}{\text{streams}}(p_n)) .$$

Note that "$\lim_n$" above is taken with respect to the metric on $\mathbb{S}_c(\delta)$ [see Section 2.3].

EXAMPLE. With p as above we have $\underset{\sim}{\text{streams}}(p) = \{ab, a\bar{c}, cb, c\bar{c}, ba, bc, \bar{c}a, \bar{c}c, \tau\}$ and $\underset{\sim}{\text{streams}}(\underset{\sim}{\text{restr}}_{\mathbb{P}}(p)) = \{ab, ba, \tau\}$.

REMARK ON NOTATION. Above, and at some other places in this subsection, we are using the metavariables "a", "b" to range over all of A (instead of $A \setminus C$, according to our convention). We trust that this abuse of notation will be clear from the context and not cause confusion.

Finally we put

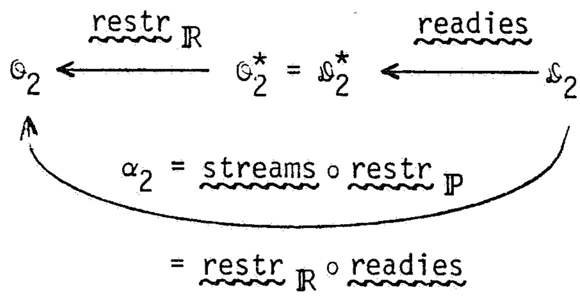
$$\alpha_2 = \underset{\sim}{\text{streams}} \circ \underset{\sim}{\text{restr}}_{\mathbb{P}}$$

in (4.2). Similarly to α_1, we cannot prove (4.2) directly by structural induction on s because α_2 does not behave compositionally. Thus again the question arises how to prove (4.2). Note that here things are rather more difficult than with $\mathcal{O}_1[\![s]\!] = \alpha_1(\mathcal{D}_1[\![s]\!])$ because the semantic domains of $\mathcal{O}_1$ and $\mathcal{D}_1$ are quite different: linear streams vs. branching processes.

Our solution to this problem is to introduce

- a new intermediate semantic domain $\mathbb{R}$,
- a new intermediate operational semantics $\mathcal{O}_2^*$ on $\mathbb{R}$,
- a new intermediate denotational semantics $\mathcal{D}_2$ on $\mathbb{R}$,

and then prove the following diagram:



where restr$_{\mathbb{R}}$ and readies are two further abstraction operators.

The Intermediate Semantic Domain $\mathbb{R}$

We start with the intermediate semantic domain. To motivate its construction, let us first demonstrate that a simple *stream-like* variant of $\mathcal{O}_2$ is not appropriate as intermediate operational semantics $\mathcal{O}_2^*$ here. Indeed, if we base $\mathcal{O}_2^*$ - similarly to $\mathcal{O}_1^*$ - on a transition system obtained by just adding the axiom

$$\langle c, w\rangle \to w \cdot c$$

to T_2, we cannot retrieve $\mathcal{O}_2$ from $\mathcal{O}_2^*$. As a counterexample consider the programs $s_1 = (a;c_1) + (a;c_2)$, $s_2 = a;(c_1 + c_2)$ and $s = \bar{c}_1$. Then $\mathcal{O}_2[\![s_1 \| s]\!] = \{a\tau,\ a\delta\} \neq \{a\tau\} = \mathcal{O}_2[\![s_2 \| s]\!]$, but $\mathcal{O}_2^*[\![s_1 \| s]\!] = \mathcal{O}_2^*[\![s_2 \| s]\!]$. Thus whatever operator α we apply to $\mathcal{O}_2^*[\![\cdot]\!]$, the results for $s_1 \| s$ and $s_2 \| s$ will turn out the same. Thus we cannot retrieve $\mathcal{O}_2$ from this $\mathcal{O}_2^*$.

To solve this problem, we introduce for $\mathcal{O}_2^*$ a new semantic domain which, besides streams $w \in A^{st}$, also includes very weak information about the *local branching structure* of a process. This information is called a *readyset* or *deadlock possibility*; it takes the form of a subset X of C, the set of communications, and may appear (locally) after every word $w \in A^*$ of successful actions. Informally, such a set X after w indicates that after w the process is ready for all communications $c \in X$ and that deadlock can be avoided only if some communication $c \in X$ can synchronize with a matching communication $\bar{c}$ in some other parallel component. Thus X can be seen as a "more informative δ". This view is confirmed by the fact that there will be no ready set X after w if the process can do an elementary action $a \in A \backslash C$ and thus avoid deadlock on its own. With some variations this notion of a ready set appears in the work of [BHR, FLP, OH1, OH2, RB].

Formally, we take $\Delta = \mathcal{P}(C)$ and define the set of *streams with ready sets* as

$$A^{rd} = A^{st} \cup A^*:\Delta$$

where $A^*:\Delta$ denotes the set of all pairs of the form $w:X$ with $w \in A^*$ and $X \in \Delta$. For $X \in \Delta$, let $\bar{X} = \{\bar{c} \mid c \in X\}$. As intermediate domain we take the *ready domain*

$$\mathbb{R} = \mathcal{P}(A^{rd}) .$$

Just as we did for A^{st} and $A^{st}(\delta)$, we can define a metric d on A^{rd} and a corresponding metric $\hat{d}$ on $\mathbb{R}$. This $\hat{d}$ turns the collection $\mathbb{R}_c \subseteq \mathbb{R}$ of closed subsets of A^{rd} into a complete metric space $(\mathbb{R}_c, \hat{d})$.

The Intermediate Operational Semantics $\mathcal{O}_2^*$

We now turn to the intermediate operational semantics $\mathcal{O}_2^*$ on $\mathbb{R}$.

It is based onthe following transition system T_2^* which consists of all axioms and rules of T_2 extended (for $w \in A^*$) by[1]:

(*communication**)

$$\langle c, w \rangle \to w \cdot c$$

(*ready sets* [or: *deadlock possibilities*])

(i) $\langle c, w \rangle \to w{:}\{c\}$

(ii) $$\frac{\langle s_1, w \rangle \to w{:}X}{\langle s_1;s_2, w \rangle \to w{:}X}$$

(iii) $$\frac{\langle s_1, w \rangle \to w{:}X,\ \langle s_2, w \rangle \to w{:}Y}{\langle s_1 + s_2, w \rangle \to w{:}(X \cup Y)}$$

(iv) $$\frac{\langle s_1, w \rangle \to w{:}X,\ \langle s_2, w \rangle \to w{:}Y}{\langle s_1 \| s_2, w \rangle \to w{:}(X \cup Y)}$$

where $X \cap \overline{Y} = \phi$.

Axiom (i) introduces ready sets or deadlock possibilities, and rules (ii)-(iv) propagate them. In particular, rule (iii) says that $s_1 + s_2$ has a deadlock possibility if s_1 and s_2 have, and rule (iv) says that $s_1 \| s_2$ has a deadlock possibility if both s_1 and s_2 have, and no synchronization is possible.

Since the rules (iii) and (iv) have two premises, deduction in T_2^* need not start any more from a single axiom. But every deduction of a transition

$$\langle s, w \rangle \to \langle s', w' \rangle$$

or

$$\langle s, w\rangle \to w'$$

or

$$\langle s, w\rangle \to w' : X$$

in T_2^* is such that all its axioms are instances of the same scheme. Thus similarly to Section 2.4 (see TYPES OF TRANSITIONS) we may talk of an (Ax)-*transition* if (Ax) is the name of the axiom. Note also that the Initial Step Lemma 2.1.1 remains valid for T_2^*.

The intermediate operational semantics

$$\mathcal{O}_2^*: \mathcal{L}_2 \to \mathbb{R}$$

is defined in terms of T_2^* just as $\mathcal{O}_2$ was defined in terms of T_2. In particular, for each finite path π of the form

$$\langle s, \varepsilon\rangle = \langle s_0, w_0\rangle \to \ldots \to \langle s_n, w_n\rangle \to w:X$$

we include $\underset{\sim}{\text{word}}(\pi) = w:X$ in $\mathcal{O}_2^*[\![s]\!]$.

EXAMPLES. (i) $\mathcal{O}_2^*[\![a;(b+c)]\!] = \{ab, ac\}$.

PROOF. We explore all transition sequences in T_2^* starting in $\langle a;(b+c), \varepsilon\rangle$:

(1)	$\langle a, \varepsilon\rangle \to a$	(elementary action)
(2)	$\langle a;(b+c), \varepsilon\rangle \to \langle b+c, a\rangle$	((1), composition)
(3)	$\langle b, a\rangle \to ab$	(elementary action)
(4)	$\langle c, a\rangle \to ac$ ↘ $a:\{c\}$	(communication)
(5)	$\langle b+c, a\rangle \to ab$ ↘ ac	((3), (4), global nondeterminacy)

No more transitions are deducible for $\langle b+c,\ a\rangle$.

(6) Thus

$$\begin{array}{lll} \langle a;(b+c),\ \epsilon\rangle \rightarrow \langle b+c,\ a\rangle & \rightarrow & ab \\ & \searrow & \\ & & ac \end{array}$$

are all transition sequences starting in $\langle a;(b+c),\ \epsilon\rangle$.

This proves the claim. □

(ii) $\mathcal{O}_2^*[\![a;b+a;c]\!] = \{ab, ac, a{:}\{c\}\}$.

PROOF. Here we only exhibit all possible transition sequences in T_2^* starting in $\langle a;b+a;c,\ \epsilon\rangle$:

$$\begin{array}{llll} \langle a;b+a;c,\ \epsilon\rangle \rightarrow & \langle b,\ a\rangle \rightarrow ab & & \\ \searrow & & & \\ & \langle c,\ a\rangle \rightarrow ac & & \\ & \searrow & & \\ & & a{:}\{c\}\ . & \square \end{array}$$

Note that we can prove $\langle a;b+a;c,\ \epsilon\rangle \rightarrow \langle c,\ a\rangle$ and $\langle c,\ a\rangle \rightarrow a{:}\{c\}$, and therefore $\langle a;b+a;c,\ \epsilon\rangle \rightarrow^* a{:}\{c\}$. However, we have $\langle a;(b+c),\ \epsilon\rangle \rightarrow \langle b+c,\ a\rangle$, but we *cannot* prove $\langle b+c,\ a\rangle \rightarrow a{:}\{c\}$. (By rule (iii) of ready sets this would only be the case if we could prove, besides $\langle c,\ a\rangle \rightarrow a{:}\{c\}$, also $\langle b,\ a\rangle \rightarrow a{:}X$ for some $X \subseteq \{c\}$. Since the only possibilities for X are ϕ and $\{c\}$, this cannot be proved.) Consequently, $\langle a;(b+c),\ \epsilon\rangle \not\rightarrow^* a{:}\{c\}$.

The Intermediate Denotational Semantics $\mathcal{D}_2^*$

We start by defining semantic operators $;^{\mathcal{D}_2^*}$, $+^{\mathcal{D}_2^*}$ and $\|^{\mathcal{D}_2^*}$ on $\mathbb{R}_C$. (Again we omit superscripts $\mathcal{D}_2^*$ whenever possible.) Let $W_1, W_2 \in \mathbb{R}_C$ and $w, w_1, w_2 \in A^{st}$.

(a) $w_1, w_2 \subseteq A^* \cup A^* \cdot \{\bot\} \cup A^*:\Delta$. Then

$$W_1;W_2 = \{w_1 \cdot w_2 \mid w_1 \in W_1 \text{ and } w_2 \in W_2\}$$
$$\cup \{w_1{:}X \mid w_1{:}X \in W_1\}$$
$$\cup \{w_1 \cdot w_2{:}X \mid w_1 \in W_1 \text{ and } w_2{:}X \in W_2\}$$

$$W_1 + W_2 = \{w \mid w \in W_1 \cup W_2\}$$
$$\cup \{\epsilon{:}(X \cup Y) \mid \epsilon{:}X \in W_1 \text{ and } \epsilon{:}Y \in W_2\}$$
$$\cup \{w{:}X \mid w \neq \epsilon \text{ and } w{:}X \in W_1 \cup W_2\}$$

$$W_1 \| W_2 = (w_1 \quad w_2) \cup (w_2 \mathbin{\underline{\|}} w_1) \cup (w_1 \mid w_2) \cup (w_1 \mathbin{\#} w_2)$$

where $w_1 \mathbin{\underline{\|}} w_2 = \bigcup\{w_1 \mathbin{\underline{\|}} W_2 \mid w_1 \in W_1\}$ with $\epsilon \mathbin{\underline{\|}} W_2 = W_2$, $(a \cdot w_1) \mathbin{\underline{\|}} W_2 = a \cdot (\{w_1\} \| W_2)$, $(a \cdot w_1{:}X) \mathbin{\underline{\|}} W_2 = a \cdot (\{w_1{:}X\} \| W_2)$, $\bot \mathbin{\underline{\|}} W_2 = \{\bot\}$, $\epsilon{:}X \mathbin{\underline{\|}} W = \emptyset$, and $w_1 \mid W_2 = \bigcup\{(w_1 \mid w_2) \mid w_1 \in W_1$ and $w_2 \in W_2\}$ with $(c \cdot u_1) \mid (\bar{c} \cdot u_2) = \tau \cdot (\{u_1\} \| \{u_2\})$ and $w_1 \mid w_2 = \phi$ for w_1, w_2 not of the above form, and

$$W_1 \mathbin{\#} W_2 = \{\epsilon{:}X \cup Y \mid \epsilon{:}X \in W_1 \text{ and } \epsilon{:}Y \in W_2 \text{ and } X \cap \bar{Y} = \phi\} .$$

(b) $W_1, W_2 \in \mathbb{R}_c$ and W_1, W_2 contain also infinite words. Then extend the previous definitions by taking limits in $\mathbb{R}_c$.

Now we define

$$\mathcal{S}_2^*\text{: guarded } \mathcal{L}_2 \to (\Gamma_2^* \to \mathbb{R}_c)$$

with $\Gamma_2^* = \underset{\sim}{Stmv} \to \mathbb{R}_c$ in the usual way:

1. $\mathcal{S}_2^*[\![a]\!](\gamma) = \{a\}$

2. $\mathcal{S}_2^*[\![c]\!](\gamma) = \{c, \epsilon{:}\{c\}\}$

3. $\mathcal{S}_2^*[\![s_1 \text{ op } s_2]\!](\gamma) = \mathcal{S}_2^*[\![s_1]\!](\gamma) \text{ op } \mathcal{S}_2^*[\![s_2]\!](\gamma)$

4. $\mathcal{S}_2^*[\![x]\!](\gamma) = \gamma(x)$

5. $\mathcal{S}_2^*[\![\mu x[s]]\!](\gamma) = \lim_i W_i$, where $W_0 = \{\perp\}$ and

$W_{i+1} = \mathcal{S}_2^*[\![s]\!](\gamma\langle W_i/x\rangle)$.

Relating $\mathcal{O}_2$ and $\mathcal{O}_2^*$

The relationship between $\mathcal{O}_2$ and $\mathcal{O}_2^*$ is similar to that between $\mathcal{O}_1$ and $\mathcal{O}_1^*$ in Section 3.4. In fact, we shall prove:

4.4.1 THEOREM. $\mathcal{O}_2[\![s]\!] = \underset{\sim}{restr}_{\mathbb{R}}(\mathcal{O}_2^*[\![s]\!])$ for every $s \in \mathcal{L}_2$.

Here $\underset{\sim}{restr}_{\mathbb{R}} : \mathbb{R} \to \mathbb{S}(\delta)$ is a restriction operator similar to $\underset{\sim}{restr}_{\mathbb{S}} : \mathbb{S}(\delta) \to \mathbb{S}(\delta)$ of Section 3.4. For $W \in \mathbb{R}$ and $w \in A^{st}$ we define

$$\underset{\sim}{restr}_{\mathbb{R}}(W) = \{w \mid w \in W \text{ does not contain any } c \in C\} \cup \{w \cdot \delta \mid \exists X \in \Delta: w{:}X \in W \text{ and } w \text{ does not contain any } c \in C\}$$

For Theorem 4.1 we need the following result concerning the transition systems T_2 and T_2^* (compare Lemma 3.4.4).

4.4.2 LEMMA. For all $s \in \mathcal{L}_2$, $s' \in \mathcal{L}_2 \cup \{E\}$ and $w, w' \in (A\backslash C)^*$:

(i) $T_2 \vdash \langle s, w\rangle \to \langle s', w'\rangle$

iff

$T_2^* \vdash \langle s, w\rangle \to \langle s', w'\rangle$

(ii) $\langle s, w\rangle$ is a deadlocking configuration for T_2

iff

$\exists X \subseteq C: T_2^* \vdash \langle s, w\rangle \to w: X$.

PROOF. *ad* (i): "$\Rightarrow$" is clear because T_2^* is an extension of T_2. For "$\Leftarrow$" note that, by the assumption $w,w' \in (A\backslash C)^*$, none of the new axioms and rules in T_2^* was used in proving the transition

$$\langle s, w\rangle \rightarrow \langle s', w'\rangle .$$

Hence it can also be proved in T_2.

ad (ii): First we analyze the structure of deadlocking configurations $\langle s, w\rangle$ in T_2: their statements s (with possible subscripts 1 and 2) have the following BNF-syntax:

$s ::= c$ for arbitrary $c \in C \mid$
$s_1;t$ for arbitrary $t \in \mathcal{L}_2 \mid s_1 + s_2 \mid$
$s_1 \| s_2$ where there is no synchronization-transition possible between s_1 and s_2.

Thus in a deadlocking configuration $\langle s, w\rangle$ all the initial actions of s are communications and in the case of a shuffle $s_1 \| s_2$ no matching initial communications (leading to a τ-action) can be found in its components s_1 and s_2. We can express this property more precisely by introducing a partial function

$$\underset{\sim}{\text{dead}}: \mathcal{L}_2 \xrightarrow{\text{part}} \Delta = \wp(C)$$

such that $\langle s, w\rangle$ is deadlocking iff $\underset{\sim}{\text{dead}}(s)$ is defined. Its definition runs as follows:

(i) $\underset{\sim}{\text{dead}}(a)$ is undefined, for $a \in A\backslash C$

(ii) $\text{dead}(c) = \{c\}$, for $c \in C$

(iii) $\text{dead}(s_1;t) = \text{dead}(s_1)$

(iv) $\text{dead}(s_1 + s_2) = \text{dead}(s_1) \cup \text{dead}(s_2)$

(v) $$\text{dead}(s_1 \| s_2) = \begin{cases} \text{dead}(s_1) \cup \text{dead}(s_2), & \text{if } \text{dead}(s_1) \cap \text{dead}(s_2) = \phi \\ \text{undefined}, & \text{otherwise}. \end{cases}$$

Now we can prove (ii):

$\langle s, w \rangle$ is a deadlocking configuration in T_2

iff $\text{dead}(s)$ is defined

(by the analysis above)

iff $\exists X \subseteq C$: $T_2^* \vdash \langle s, w \rangle \to w{:}X$ with $X = \text{dead}(s)$

(by the rules (i)-(iv) for ready sets in T_2^*). □

Intuitively, Lemma 4.4.2 (ii) says that the ready set rules (i)-(iv) of T_2^* are complete for detecting deadlocks. Using Lemma 4.4.6 we can now give the

PROOF OF THEOREM 4.4.1. Let $s \in \mathcal{L}_2$. Note that

$$\mathcal{O}_2[\![s]\!],\ \text{restr}_{\mathbb{R}}(\mathcal{O}_2^*[\![s]\!]) \subseteq (A\backslash C)^* \cup (A\backslash C)^\omega \cup (A\backslash C)^* \cdot \{\bot, \delta\}.$$

We distinguish the following cases.

Case 1: $w \in (A\backslash C)^* \cup (A\backslash C)^\omega \cup (A\backslash C)^* \cdot \{\bot\}$.

As an immediate consequence of Lemma 4.4.2 (i) and the definition of $\underset{\sim}{restr}_{\mathbb{R}}$ we have:

$$w \in \mathbb{O}_2[\![s]\!] \quad \text{iff} \quad w \in \underset{\sim}{restr}_{\mathbb{R}}(\mathbb{O}_2^*[\![s]\!]) .$$

Case 2: $w\delta \in (A\backslash C)^* \cdot \{\delta\}$.

Here we have the following chain of equivalences:

$w\delta \in \mathbb{O}_2[\![s]\!]$

iff $\langle s, w\rangle$ is a deadlocking configuration in T_2

iff $\exists X \in \Delta$: $T_2^* \vdash \langle s, w\rangle \to w{:}X$ (by Lemma 4.4.2 (ii))

iff $\exists X \in \Delta$: $w{:}X \in \mathbb{O}_2^*[\![s]\!]$

iff $w\delta \in \underset{\sim}{restr}_{\mathbb{R}}(\mathbb{O}_2^*[\![s]\!])$. □

Relating $\mathcal{S}_2$ and $\mathcal{S}_2^*$

The relationship between $\mathcal{S}_2$ and $\mathcal{S}_2^*$ is given by an abstraction operator $\underset{\sim}{readies}: \mathbb{P} \to \mathbb{R}_C$. For $p = \{a_1,\dots,a_m,[b_1,q_1],\dots,[b_n,q_n]\} \in \mathbb{P}$ we put inductively

$$\begin{aligned} \underset{\sim}{readies}(p) = \; & \{a_1,\dots,a_m\} \cup \\ & \cup \{b_j \cdot \underset{\sim}{readies}(q_j) \mid j = 1,\dots,n\} \\ & \cup \{\varepsilon{:}X \mid X = \{a_1,\dots,a_m,b_1,\dots,b_n\} \subseteq C\} \end{aligned}$$

For $p \in \mathbb{P} \backslash \mathbb{P}_\omega$ we have $p = \lim_n p_n$, with $p_n \in \mathbb{P}_n$, and put

$$\underset{\sim}{readies}(p) = \lim_n(\underset{\sim}{readies}(p_n))$$

where "$\lim_n$" is taken (as before) w.r.t the metric on $\mathbb{R}_C$.

4.4.3 THEOREM. $\mathcal{D}_2^*[\![s]\!] = \underset{\sim}{readies}(\mathcal{D}_2[\![s]\!])$ for all guarded $s \in \mathcal{L}_2$.

The proof follows from:

4.4.4 LEMMA. The operator $\underset{\sim}{readies}: \mathbb{P} \to \mathbb{R}_C$ is continuous and behaves homomorphically, i.e. for $\underset{\sim}{op} \in \{+, ;, \|\}$ and $p, p' \in \mathbb{P}$,

$$\underset{\sim}{readies}(p \; \underset{\sim}{op}^{\mathcal{D}_2} \; p') = \underset{\sim}{readies}(p) \; \underset{\sim}{op}^{\mathcal{D}_2^*} \; readies(p').$$

PROOF. Continuity is established by a variation of standard reasoning as in [BBKM], [BZ2]. For the same reason it suffices to prove the homomorphism property for $p, p' \in \mathbb{P}_\omega$ only. We proceed inductively and assume

$$p = \{a_1, \ldots, a_m, [b_1, q_1], \ldots, [b_n, q_n]\},$$

$$p' = \{a'_1, \ldots, a'_m, [b'_1, q'_1], \ldots, [b'_n, q'_{n'}]\}$$

with $m, n, m', n' \geq 0$.

Case 1: $\underset{\sim}{op} = +$

$$\underset{\sim}{readies}(p +^{\mathcal{L}_2} p') = \underset{\sim}{readies}(p \cup p')$$

$$= \{a_1, \ldots, a_m, a'_1, \ldots, a'_{m'}\} \cup$$

$$\bigcup\{b_i \cdot \underset{\sim}{readies}(q_i) \mid i = 1, \ldots, n\} \cup$$

$$\bigcup\{b'_j \cdot \underset{\sim}{readies}(g'_j) \mid j = 1, \ldots, n'\} \cup$$

$$\left\{\varepsilon : (X \cup Y) \;\middle|\; \begin{array}{l} X = \{a_1, \ldots, a_m, b_1, \ldots, b_n\} \subseteq C, \\ Y = \{a'_1, \ldots, a'_{m'}, b'_1, \ldots, b'_{n'}\} \subseteq C \end{array}\right\}$$

$= \{w \mid w \in \underset{\sim}{readies}(p) \cup \underset{\sim}{readies}(p')\}$

$\cup\{\epsilon:(X \cup Y) \mid \epsilon:X \in \underset{\sim}{readies}(p) \text{ and } \epsilon:Y \in \underset{\sim}{readies}(p')\}$

$\cup\{w:X \mid w \neq \epsilon \text{ and } w:X \in \underset{\sim}{readies}(p) \cup \underset{\sim}{readies}(p')\}$

$= \underset{\sim}{readies}(p) \overset{\mathcal{S}_2^*}{+} \underset{\sim}{readies}(p')$

Case 2: $\underset{\sim}{op} = ;$

$\underset{\sim}{readies}(p \overset{\mathcal{S}_2}{;} p') = \underset{\sim}{readies}(p \cdot p')$

$= \underset{\sim}{readies}(\{[a_1,p'],\ldots,[a_m,p'],$

$[b_1, q_1 \cdot p'],\ldots,[b_n, q_n \cdot p']\})$

$= \{\epsilon:X \mid X = \{a_1,\ldots,a_m, b_1,\ldots,b_n\} \subseteq C\} \cup$

$\bigcup\{a_i \cdot \underset{\sim}{readies}(p') \mid i = 1,\ldots,m\} \cup$

$\bigcup\{b_j \cdot \underset{\sim}{readies}(q_j \cdot p') \mid j = 1,\ldots,n\}$

$= \{\epsilon:X \mid \ldots\} \cup \bigcup\{a_i \cdot \underset{\sim}{readies}(p') \mid \ldots\} \cup$

$\bigcup\{b_j \cdot (\underset{\sim}{readies}(qj) \overset{\mathcal{S}_2^*}{;} \underset{\sim}{readies}(p')) \mid \ldots\}$ (by induction)

$= \{\epsilon:X \mid \ldots\} \cup \bigcup\{a_i \cdot \underset{\sim}{readies}(p') \mid \ldots\} \cup$

$\bigcup\{(b_j \cdot \underset{\sim}{readies}(qj)) \overset{\mathcal{S}_2^*}{;} \underset{\sim}{readies}(p') \mid \ldots\}$

$$= (\{\epsilon : X \mid X = \{a_1,\ldots,a_m,\ b_1,\ldots,b_n\} \subseteq C\} \cup$$

$$\{a_1,\ldots,a_m\} \cup$$

$$\bigcup \{bj \cdot \underset{\sim}{\text{readies}}(qj)\}) \ ;^{\wp_2^*} \ \underset{\sim}{\text{readies}}(p')\ )$$

$$= \underset{\sim}{\text{readies}}(p) \ ;^{\wp_2^*} \ \underset{\sim}{\text{readies}}(p')$$

Case 3: $\underset{\sim}{op} = \|$

By definition

$$p \| p' = (p \mathbin{\|\!\!\_} p') \cup (p' \mathbin{\|\!\!\_} p) \cup (p \mid p')$$

where

$$\begin{aligned}
p \mathbin{\|\!\!_} p' = \ & \{[a_i, p'] \mid i = 1,\ldots,m\} \\
& \cup \{[b_j, q_j \| p'] \mid j = 1,\ldots,n\}\ , \\
p' \mathbin{\|\!\!_} p = \ & \{[a'_k, p] \mid k = 1,\ldots,m'\} \\
& \cup \{[b'_\ell, q'_\ell \| p] \mid \ell = 1,\ldots,n'\}, \\
p \mid p' = \ & \left\{\tau \;\middle|\; \begin{array}{l} \exists c \in C:\ c \in \{a_1,\ldots,a_m\} \\ \quad \text{and} \ \ \bar{c} \in \{a'_1,\ldots,a'_{m'}\} \end{array}\right\} \\
& \cup \left\{[\tau, q'_\ell] \;\middle|\; \begin{array}{l} \exists c \in C:\ c \in \{a_1,\ldots,a_m\} \\ \quad \text{and} \ \ \bar{c} = b'_\ell \ \ \text{and} \ \ \ell \in \{1,\ldots,n'\} \end{array}\right\} \\
& \cup \left\{[\tau, q_j] \;\middle|\; \begin{array}{l} \exists c \in C:\ c \in \{a'_1,\ldots,a'_{m'}\} \\ \quad \text{and} \ \ \bar{c} = b_j \ \ \text{and} \ \ j \in \{1,\ldots,n\} \end{array}\right\} \\
& \cup \left\{[\tau, q_j \| q'_\ell] \;\middle|\; \begin{array}{l} \exists c \in C:\ c = b_j \ \ \text{and} \ \ \bar{c} = b'_\ell \\ \quad \text{and} \ \ j \in \{1,\ldots,n\} \\ \quad \text{and} \ \ \ell \in \{1,\ldots,n'\} \end{array}\right\} .
\end{aligned}$$

Thus

$$
\begin{aligned}
& \mathit{readies}(p \| p') \\
= \; & \left\{ \epsilon : (X \cup Y) \;\middle|\; X \cap \overline{Y} = \phi \text{ where } \begin{array}{l} X = \{a_1, \ldots, a_m, b_1, \ldots, b_n\} \subseteq C, \\ Y = \{a'_1, \ldots, a'_{m'}, b'_1, \ldots, b'_{n'}\} \subseteq C \end{array} \right\} \\
& \cup\ \mathit{readies}(p \;\|\!\lfloor\; p') \backslash \epsilon : \Delta \\
& \cup\ \mathit{readies}(p' \;\|\!\lfloor\; p) \backslash \epsilon : \Delta \\
& \cup\ \mathit{readies}(p \mid p') \backslash \epsilon : \Delta \\
= \; & \mathit{readies}(p) \;\#\; \mathit{readies}(p') \\
& \cup\ \mathit{readies}(p) \;\|\!\lfloor\; \mathit{readies}(p') \\
& \cup\ \mathit{readies}(p') \;\|\!\lfloor\; \mathit{readies}(p) \\
& \cup\ \mathit{readies}(p) \mid \mathit{readies}(p') \\
& \quad \text{(by definition of } \mathit{readies} \text{ and induction)} \\
= \; & \mathit{readies}(p) \;\|^{\&^*_2}\; \mathit{readies}(p') \; .
\end{aligned}
$$

Here we must *simultaneously* prove, by induction:

$$
\begin{aligned}
\mathit{readies}(p \;\|\!\lfloor\; p') \backslash \epsilon : \Delta &= \mathit{readies}(p) \;\|\!\lfloor\; \mathit{readies}(p') \\
\mathit{readies}(p \backslash p') \backslash \epsilon : \Delta &= \mathit{readies}(p) \mid \mathit{readies}(p') \\
\mathit{readies}(p \# p') &= \mathit{readies}(p) \;\#\; \mathit{readies}(p') \; .
\end{aligned}
$$

The details are left to the reader. □

Relating $\mathcal{O}_2^*$ *and* $\mathcal{D}_2^*$

Here we discuss

4.4.5 THEOREM. $\mathcal{O}_2^*[\![s]\!] = \mathcal{D}_2^*[\![s]\!]$ for every guarded $s \in \mathcal{L}_2$.

Again, its proof follows the structure of that for "$\mathcal{O}_0[\![s]\!] = \mathcal{D}_0[\![s]\!]$" (Theorem 2.1). In particular, Theorems 2.4.10, 2.4.11 and 2.4.15 remain valid with $\mathcal{O}_2^*$, $\mathcal{D}_2^*$ and $\mathcal{L}_2$ in place of $\mathcal{O}_0$, $\mathcal{D}_0$ and $\mathcal{L}_0$. Thus it remains to show compositionality of $\mathcal{O}_2^*$, analogously to Theorem 2.4.2, but now involving the ready domain $\mathbb{R}$ and global nondeterminacy "+" .

4.4.6 THEOREM. For $\underset{\sim}{op} \in \{+ , ; , \|\}$ and $s_1, s_2 \in \mathcal{L}$,

$$\mathcal{O}_2^*[\![s_1 \underset{\sim}{op} s_2]\!] = \mathcal{O}_2^*[\![s_1]\!] \ \underset{\sim}{op}^{\mathcal{D}_2^*} \ \mathcal{O}_2^*[\![s_2]\!] .$$

PROOF. *Case 1:* $\underset{\sim}{op} = +$

First we state some simple facts about the rule of global nondeterminacy in the transition system T_2^*:

(i) μ-unfolding:

$$T_2^* \vdash \langle s_1 + s_2, \epsilon \rangle \rightarrow \langle s', \epsilon \rangle$$

iff

$$\exists s_1' \in \mathcal{L}_2 (s' = s_1' + s_2 \wedge T_2^* \vdash \langle s_1, \epsilon \rangle \rightarrow \langle s_1', \epsilon \rangle)$$
$$\vee \ \exists s_2' \in \mathcal{L}_2 (s' = s_1 + s_2' \wedge T_2^* \vdash \langle s_2, \epsilon \rangle \rightarrow \langle s_2', \epsilon \rangle)$$

(ii) selection by an action $b \in A$:

$$T_2^* \vdash \langle s_1 + s_2, \epsilon \rangle \rightarrow \langle s', b \rangle$$

iff

$$(s' \text{ stems from } s_1 \wedge T_2^* \vdash \langle s_1, \varepsilon\rangle \rightarrow \langle s', b\rangle)$$
$$\vee\ (s' \text{ stems from } s_2 \wedge T_2^* \vdash \langle s_2, \varepsilon\rangle \rightarrow \langle s', b\rangle)$$

(iii) ready sets:

$$T_2^* \vdash \langle s_1 + s_2, \varepsilon\rangle \rightarrow \varepsilon{:}Z$$

iff

$$\exists X,Y \subseteq C\colon\ Z = X \cup Y$$
$$\wedge\ T_2^* \vdash \langle s_1, \varepsilon\rangle \rightarrow \varepsilon{:}X$$
$$\wedge\ T_2^* \vdash \langle s_2, \varepsilon\rangle \rightarrow \varepsilon{:}Y$$

Let us now analyze the possible elements of $\mathfrak{S}_2^*[\![s_1 + s_2]\!]$. These are of the form $\varepsilon{:}z$ or $b\cdot w$ with $b \in A$ and $w \in A^{rd} = A^{st} \cup A^*{:}\Delta$. (Note that $\varepsilon \notin \mathfrak{S}_2^*[\![s]\!]$ for any $s \in \mathcal{L}_2$.)

Subcase 1.1: $\varepsilon{:}Z$

$$(\varepsilon{:}Z) \in \mathfrak{S}_2^*[\![s_1 + s_2]\!]$$

iff $T_2^* \vdash \langle s_1 + s_2, \varepsilon\rangle \rightarrow^* \varepsilon{:}Z$

iff $\exists X,Y \subseteq C\colon\ Z = X \cup Y$

$\wedge\ T_2^* \vdash \langle s_1, \varepsilon\rangle \rightarrow^* \varepsilon{:}Z$

$\wedge\ T_2^* \vdash \langle s_2, \varepsilon\rangle \rightarrow^* \varepsilon{:}Y$ (by facts (i) and (iii) above)

iff $\exists X,Y \subseteq C\colon\ Z = X \cup Y \wedge (\varepsilon{:}X) \in \mathfrak{S}_2^*[\![s_1]\!]$

$\wedge\ (\varepsilon{:}Y) \in \mathfrak{S}_2^*[\![s_2]\!]$

Subcase 1.2: b·w

$$b\cdot w \in \mathfrak{S}_2^*[\![s_1 + s_2]\!]$$

iff $\exists s' \in \mathcal{L}_2 \cup \{E\}$:

$$T_2^* \vdash \langle s_1 + s_2, \epsilon\rangle \to^* \langle s', b\rangle \wedge w \in \mathfrak{S}_2^*[\![s']\!]$$

(by convention, we put here $\epsilon \in \mathfrak{S}_2^*[\![E]\!]$)

iff $\exists s' \in \mathcal{L}_2 \cup \{E\}$:

$$(T_2^* \vdash \langle s_1, \epsilon\rangle \to^* \langle s', b\rangle \wedge w \in \mathfrak{S}_2^*[\![s']\!])$$

$$\vee\ (T_2^* \vdash \langle s_2, \epsilon\rangle \to^* \langle s', b\rangle \wedge w \in \mathfrak{S}_2^*[\![s']\!])$$

(by facts (i) and (ii) above)

iff $b\cdot w \in \mathfrak{S}_2^*[\![s_1]\!] \vee b\cdot w \in \mathfrak{S}_2^*[\![s_2]\!]$

By the analysis in Subcase 1.1 and 1.2, we finally have:

$$\mathfrak{S}_2^*[\![s_1 + s_2]\!] = \{\epsilon:(X \cup Y) \mid \epsilon:X \in \mathfrak{S}_2^*[\![s_1]\!] \wedge \epsilon:Y \in \mathfrak{S}_2^*[\![s_2]\!]\}$$
$$\cup \{w \in A^{st} \mid w \in \mathfrak{S}_2^*[\![s_1]\!] \cup \mathfrak{S}_2^*[\![s_2]\!]\}$$
$$\cup \{w:X \in A^*:\Delta \mid w \neq \epsilon \wedge w:X \in \mathfrak{S}_2^*[\![s_1]\!] \cup \mathfrak{S}_2^*[\![s_2]\!]\}$$

$$= \mathfrak{S}_2^*[\![s_1]\!] \overset{\mathfrak{D}_2^*}{+} \mathfrak{S}_2^*[\![s_2]\!] .$$

Case 2: $\underset{\sim}{op}$ = ;

Straightforward.

Case 3: $\underset{\sim}{op} = \|$

First observe that the Synchronization Lemma 3.4.7 also holds for $\mathcal{L}_2$ and T_2^* instead of $\mathcal{L}_1$ and T_1^*. Note that the rules for "global nondeterminacy: selection by synchronization" in T_2^* are needed here because the contexts considered under (3.7) and (3.8) in the proof of Lemma 3.4.7 may now contain "+". E.g. in (3.8) we how have:

$$s_1 ::= c \mid s_1;s \mid s_1\|s \mid s\|s_1 \mid s_1 + s \mid s + s_1 .$$

Using the Synchronization Lemma we can prove, analogously to Lemma 3.4.6:

$$w \in \mathcal{O}_2^*[\![s_1 \| s_2]\!] \text{ iff } \exists u \in \mathcal{O}_2^*[\![s_1]\!], v \in \mathcal{O}_2^*[\![s_2]\!] : w \in \{u\} \|^{\mathcal{L}_2^*} \{v\} \tag{4.3}$$

for $w \in A^{st}$ and $s_1, s_2 \in \mathcal{L}_2$.

In the process of proving (4.3), we obtain:

$$\begin{array}{ll} & \forall s_1, s_2 \in \mathcal{L}_2 \ \forall s_1', s_2' \in \mathcal{L}_2 \cup \{E\} \ \forall w \in A^*: \\ & \quad T_2^* \vdash \langle s_1 \| s_2, \epsilon\rangle \to^* \langle s_1' \| s_2', w\rangle \\ \text{iff} & \quad \exists u, v \in A^* \ \exists X, Y \subseteq C: \\ & \quad T_2^* \vdash \langle s_1, \epsilon\rangle \to^* \langle s_1', u\rangle \\ & \wedge \ T_2^* \vdash \langle s_2, \epsilon\rangle \to^* \langle s_2', v\rangle \\ & \wedge \ w \in \{u\} \|^{\mathcal{L}_2^*} \{v\} \end{array} \tag{4.4a}$$

(compare Lemma 3.4.6). Furthermore we have

$$\forall s \in \mathcal{L}_2 \ \forall w{:}Z \in A^*{:}\Delta$$

$$w{:}Z \in \mathfrak{S}_2^*[\![s]\!] \text{ iff } \exists s' \in \mathcal{L}_2{:}\ T_2^* \vdash \langle s, \epsilon \rangle \to^* \langle s', w \rangle \tag{4.4b}$$
$$\wedge\ T_2^* \vdash \langle s', \epsilon \rangle \to \epsilon{:}Z$$

Moreover we have, as an immediate consequence of the rules for ready sets in T_2^* (4.4.2), especially rule (iv):

$$T_2^* \vdash \langle s_1 \| s_2, \epsilon \rangle \to \epsilon{:}Z$$

$$\text{iff} \quad \exists X,Y \subseteq C{:}\ Z = X \cup Y \wedge X \cap \overline{Y} = \phi \tag{4.4c}$$
$$\wedge\ T_2^* \vdash \langle s_1, \epsilon \rangle \to \epsilon{:}X$$
$$\wedge\ T_2^* \vdash \langle s_2, \epsilon \rangle \to \epsilon{:}Y$$

Combining (4.4a), (4.4b) and (4.4c) yields

$$w{:}Z \in \mathfrak{S}_2^*[\![s_1 \| s_2]\!]$$

$$\text{iff} \quad \exists u{:}X \in \mathfrak{S}_2^*[\![s_1]\!],\ v{:}Y \in \mathfrak{S}_2^*[\![s_2]\!]{:} \tag{4.5}$$
$$w \in \{u\} \|^{\mathfrak{D}_2^*} \{v\} \wedge Z = X \cup Y \wedge X \cap \overline{Y} = \phi\ .$$

With (4.3) and(4.5) we have indeed

$$\mathfrak{S}_2^*[\![s_1 \| s_2]\!] = \mathfrak{S}_2^*[\![s_1]\!] \|^{\mathfrak{D}_2^*} \mathfrak{S}_2^*[\![s_2]\!]\ .$$

This finishes the proof of Theorem 4.4.6. □

With Theorem 4.4.6 also our argument for Theorem 4.4.5 is completed.

Putting It All Together

Before we can prove the desired relationship between $\mathfrak{G}_2$ and $\mathfrak{L}_2$ (cf. (4.2)), we need one more lemma.

4.4.7 LEMMA. For every $p \in \mathbb{P}$,

$$\underset{\sim}{\text{streams}}\,(\underset{\sim}{\text{restr}}_{\mathbb{P}}(p)) = \underset{\sim}{\text{restr}}_{\mathbb{R}}\,(\underset{\sim}{\text{readies}}(p))\,.$$

PROOF. By limit considerations it suffices to prove the equation for $p \in \mathbb{P}_\omega$. We proceed inductively and assume

$$p = \{a_1,\ldots,a_m,[b_1,q_1],\ldots,[b_n,q_n]\}$$

with $X =_{df} \{a_1,\ldots,a_m,b_1,\ldots,b_n\}$. Then the ℓ.h.s. yields

$$\underset{\sim}{\text{restr}}_{\mathbb{P}}(p) = \{a_i \mid a_i \in p \text{ and } a_i \notin C\} \cup \left\{[b_j, \underset{\sim}{\text{restr}}_{\mathbb{P}}(q_j)] \;\middle|\; \begin{array}{l}[b_j, q_j] \in p\\ \text{and } b_j \notin C\end{array}\right\}$$

and thus

$$\underset{\sim}{\text{streams}}(\underset{\sim}{\text{restr}}_{\mathbb{P}}(p)) = \begin{cases} \{a_i \mid a_i \in p \text{ and } a_i \notin C\} \cup \\ \bigcup \left\{b_j \cdot \underset{\sim}{\text{streams}}(\underset{\sim}{\text{restr}}_{\mathbb{P}}(q_j)) \;\middle|\; \begin{array}{l}[b_j,q_j] \in p\\ \text{and } b_j \notin C\end{array}\right\} & \text{if } X \not\subseteq C \\ \{\delta\} & \text{if } X \subseteq C \end{cases}$$

Now the r.h.s. yields

$$\underset{\sim}{\text{readies}}(p) = \{\varepsilon : X \mid X \subseteq C\} \cup \{a_i \mid a_i \in p\} \cup \{b_j \cdot \underset{\sim}{\text{readies}}(qj) \mid [b_j, q_j] \in p\}$$

and thus

$$\underset{\sim}{\text{restr}}_{\mathbb{R}}(\underset{\sim}{\text{readies}}(p)) = \begin{cases} \{a_i \mid a \in p \text{ and } a_i \notin C\} \cup \\ \{b_j \cdot \underset{\sim}{\text{restr}}_{\mathbb{R}}(\underset{\sim}{\text{readies}}(q_j)) \mid [b_j, a_j] \in p \text{ and } b_j \notin C\} & \text{if } X \not\subseteq C \\ \{\delta\} & \text{if } X \subseteq C . \end{cases}$$

By induction, we have ℓ.h.s. = r.h.s. □

Now we are prepared for the main result on $\mathcal{L}_2$:

4.4.8 THEOREM. $\mathcal{O}_2[\![s]\!] = \alpha_2(\mathcal{D}_2[\![s]\!])$ for all guarded $s \in \mathcal{L}_2$, where $\alpha_2 = \underset{\sim}{\text{streams}} \circ \underset{\sim}{\text{restr}}_{\mathbb{P}}$.

PROOF. Theorem 4.4.1 states $\mathcal{O}_2[\![s]\!] = \underset{\sim}{\text{restr}}_{\mathbb{R}}(\mathcal{O}_2^*[\![s]\!])$ for $s \in \mathcal{L}_2$, Theorem 4.4.3 states $\mathcal{D}_2^*[\![s]\!] = \underset{\sim}{\text{readies}}(\mathcal{D}_2[\![s]\!])$ for guarded $s \in \mathcal{L}_2$, and Theorem 4.4.5 states $\mathcal{O}_2^*[\![s]\!] = \mathcal{D}_2^*[\![s]\!]$ for guarded $s \in \mathcal{L}_2$. Thus we obtain

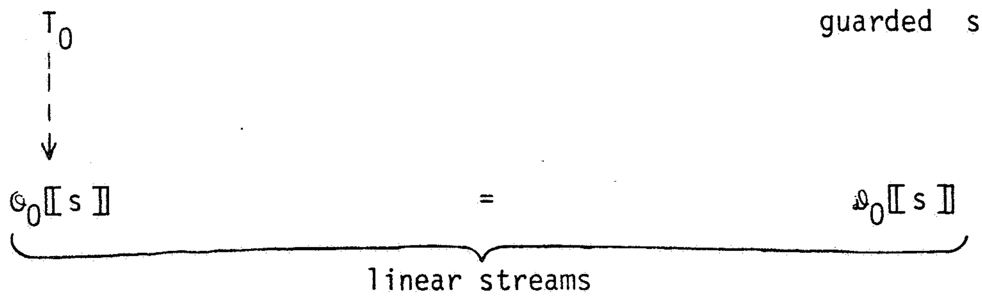
$$\mathcal{O}_2[\![s]\!] = \underset{\sim}{\text{restr}}_{\mathbb{R}}(\underset{\sim}{\text{readies}}(\mathcal{D}_2[\![s]\!])) .$$

Using Lemma 4.4.7 completes the proof of this theorem. □

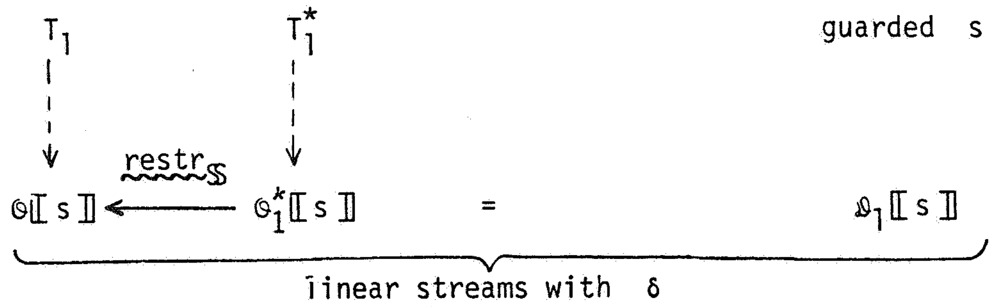
THE END

APPENDIX: DIAGRAM OF RESULTS

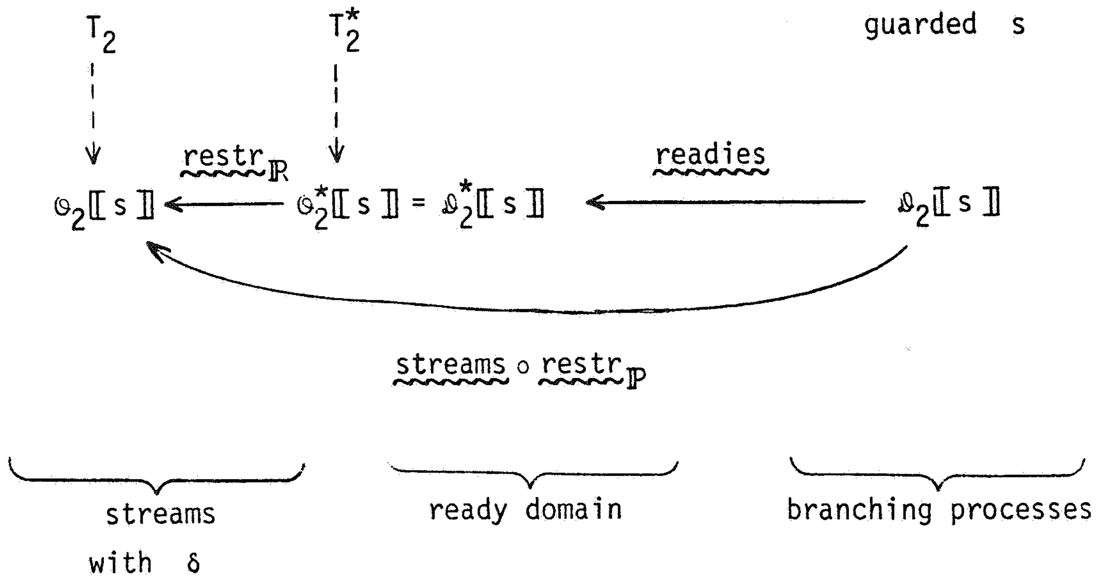
$\mathcal{L}_0$: Shuffle and Local Nondeterminacy



$\mathcal{L}_1$: Synchronization Merge and Local Nondeterminacy



$\mathcal{L}_2$: Synchronization Merge and Global Nondeterminacy



FOOTNOTES

[1]The transition rules given here are corrected versions of those given in [BMOZ].

[Ad] "Reference Manual for the Ada Programming Language".
American National Standards Institute, Inc.
ANSI MIL-STD 1815 A
LNCS 155, Springer-Verlag, 1983

[Ap1] K.R. Apt,
Recursive assertions and parallel programs.
Acta Inf. 15 (1983), 219-232.

[Ap2] K.R. Apt,
Formal justification of a proof system for
Communicating Sequential Processes,
J. Assoc. Comp. Mach. (1983), 30, 197-216.

[BBKM] J.W. de Bakker, J.A. Bergstra, J.W. Klop and J.-J.Ch. Meyer,
Linear time and branching time semantics for recursion with merge,
Theor. Comp. Science 34 (1984), 135-156.

[BKMOZ] J.W. de Bakker, J.N. Kok, J.-J.Ch. Meyer, E.-R. Olderog and J.I. Zucker
Contrasting themes in the semantics of imperative concurrency.
To appear in: Proc. Advanced School on Current Trends in Concurrency
(J.W. de Bakker, G. Rozenberg and W.P. de Roever, eds.),
LNCS, Springer-Verlag, 1986.

[BMO] J.W. de Bakker, J.-J.Ch. Meyer and E.-R. Olderog,
Infinite streams and finite observations in the semantics
of uniform concurrency.
in: Proc. 12th ICALP, Nafplion, July 1985 (W. Brauer, ed.),
LNCS 194, Springer-Verlag.

[BMOZ] J.W. de Bakker, J. J.Ch. Meyer, E.-R. Olderog and J.I. Zucker,
Transition systems, infinitary languages and the
semantics of uniform concurrency,
Proc. 17th ACM STOC, Providence, RI, May 1985, 252-262.

[BZ1] J.W. de Bakker and J.I. Zucker,
Denotational semantics of concurrency.
in: Proc. 14th ACM STOC, 1982, 153-158.

[BZ2] J.W. de Bakker and J.I. Zucker,
Processes and the denotational semantics of concurrency,
Information and Control, 54 (1982), 70-120.

[BZ3] J.W. de Bakker and J.I. Zucker,
Compactness in semantics for merge and fair merge,
in: Proc. Workshop Logics of Programs, Pittsburgh, June 1983
(E. Clarke and D. Kozen, eds.),
LNCS 164, Springer-Verlag, 18-33.

[BK] J.A. Bergstra and J.W. Klop,
Process algebra for synchronous communication,
Information and Control 60 (1984), 109-137.

[BHR] S.D. Brookes, C.A.R. Hoare and A.W. Roscoe,
A theory of communicating sequential processes,
J. Assoc. Computing Mach. 31 (1984), 560-599.

[Br] M. Broy,
Fixed point theory for communication and concurrency,
in: Proc. IFIP Working Conf.
Formal Description of Programming Concepts II, 1983 (D. Bjorner, ed.)
North-Holland, Amsterdam, 125-148.

[Du] J. Dugundji,
"Topology",
Allen and Bacon, Rockleigh, NJ, 1966.

[FHLR] N. Francez, C.A.R. Hoare, D.J. Lehmann and W.P. de Roever,
Semantics of nondeterminism, concurrency and communication,
JCSS, 19 (1979), 290-308.

[FLP] N. Francez, D.J. Lehmann and A. Pnueli,
A linear-history semantics for languages for distributed programming,
Theor. Comp. Science, 32 (1984), 25-46.

[GR] W.G. Golson and W.C. Rounds,
Connections between two theories of concurrency:
metric spaces and synchronization trees,
Information and Control, 57 (1983), 102-124.

[Ha] H. Hahn,
"Reelle Funktionen",
Chelsea, New York, 1948.

[Ho] C.A.R. Hoare,
Communicating sequential processes,
Comm. Assoc. Comp. Mach., 29 (1980), 666-677.

[HP] M. Hennessy and G.D. Plotkin,
Full abstraction for a simple programming language,
in: Proc. 8th MFCS, 1979 (J. Becvar, ed.),
LNCS 74, Springer-Verlag, 108-120.

[In] Inmos Ltd.,
"The Occam Programming Manual",
Prentice-Hall International, London, 1984.

[Ke] R. Keller,
Formal verification of parallel programs,
Comm. Assoc. Comp. Mach., 19 (1976), 371-384.

[Me] J.-J. Ch. Meyer,
Fixed points and the arbitrary and fair merge of a
fairly simple class of programs, Parts 1 and 2,
Tech. Reports IR-89 and IR-92, Dept. of Math. and Comp. Science,
Free University, Amsterdam, 1984.

[Mi] R. Milner,
"A Calculus for Communicating Systems",
LNCS 92, Springer-Verlag, 1980.

[Ni] M. Nivat,
Infinite words, infinite trees, infinite computations,
in: Foundations of Computer Science III Part 2
(J.W. de Bakker and J.van Leeuwen, eds.),
Mathematical Center Tracts 109, Amsterdam, 1979, 3-52.

[OH1] E.-R. Olderog and C.A.R. Hoare,
Specification-oriented semantics for communicating processes.
in: Proc. 10th ICALP, Barcelona, July 1983 (J. Diaz, ed.).
LNCS 154. Springer-Verlag.

[OH2] E.-R. Olderog and C.A.R. Hoare,
Specification-oriented semantics for communicating processes.
Tech. Monograph PRG-37. Oxford Univ. Programming Research Group. 1984.

[Pa] D.M.R. Park,
Concurrency and automata on infinite sequences,
in: Proc. 5th GI Conf.,
LNCS 104, Springer-Verlag.

[Pl1] G.D. Plotkin.
A structural approach to operational semantics,
Report DAIMI FN-19. Comp. Science Dept.. Aarhus Univ.. 1981.

[Pl2] G.D. Plotkin.
An operational semantics for CSP,
in: Proc. IFIP Working Conf.
Formal Description of Programming Concepts II. 1983 (D. Bjorner. ed.),
North-Holland. Amsterdam, 199-223.

[RB] W.C. Rounds and S.D. Brookes,
Possible futures, acceptances, refusals, and communicating processes.
in: Proc. 22nd IEEE Symp. FOCS, 1981, 140-149.

[RS] M.O. Rabin and D.S. Scott.
Finite Automata and their decision problem,
IBM J. Res. 3:2 (1959).