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The sieve method in multi-stage sampling

by

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ABSTRACT

It is shown that the sieve method (a technique for probability proportional to size sampling) may, under certain conditions, be validly used for two- or three-stage sampling schemes, even though the statistical evaluation is based on true random sampling.

NOTE: This note is the product of work of the author (assisted by Dr. R.D. Gill of the C.W.I.) as part of his study at the Catholic University of Nijmegen.

The subject of the note comes from a consultation project at C.W.I. with Klynveld Kraayenhof & Co, Accountants.

The note must be seen as the continuation of a report, written by Dr. R.D. Gill, dealing with the sieve method in single-stage sampling.

KEY WORDS & PHRASES: *Sieve sampling, Two-stage sampling, Hoeffding inequalities.*

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0. INTRODUCTION

In his note [1], Dr. R.D. Gill describes both true random sampling and sieve sampling as applied to an (accounting) population, and proves that the statistical evaluation for both can be taken as the same, namely the evaluation using tables based on the Poisson distribution.

Chapter 1 is an introduction for those readers who are not familiar with the subject of sieve sampling or of sampling in accountancy.

Chapter 2 deals with the testing problem accountants mainly have to do with, under two-stage sampling. Both the dollar-unit method (true random sampling) and the sieve method are considered, and results are deduced and tables are constructed.

Chapter 3 looks at the testing problem under three stage sampling and one example is elaborated on.

1. ONE-STAGE SAMPLING.

1.1. True random sampling

An accounting population is generally made up of a number of items of various sizes and these items are often present in some physical sequence. Each item consists of monetary units, say dollars, and the size of an item is the amount of dollars in it. One can investigate every item and so determine the amount of errors. By convention we say that the lowest numbered dollars in each item are the bad ones, see figure 1.

figure 1: accounting population arranged in sequence of items each consisting of monetary units.

The true random dollar-unit sample is obtained by selecting, completely at random and independently of one another, a number of dollars from the population. These dollars are investigated and are given the qualification "good" or "bad". One can now make a statement or come to a decision concerning the fraction of bad dollars in the population, using the tables based on the Poisson-distribution.

This is based on the fact that, if the dollars are chosen at random and independently of one another, the (random) number of bad dollars found in the sample has the hypergeometric distribution with parameters N (population size-number of dollars), n (sample size) and K (number of bad dollars in the population).

This distribution is very close to the binomial distribution with parameters n and $p = \frac{K}{N}$ (error rate).

In turn the binomial distribution is very close to the Poisson distribution with parameter $\lambda = np$ (at least, for small values of p).

The Poisson-distribution has only one parameter and is therefore easy to work with, as illustrated in the two following well-known problems in statistics.

The estimation problem

Let β be the chosen risk and x the number of errors found in the sample. We say that the upper confidence limit λ_u is that λ such that:

$$\sum_{y=0}^{x} \frac{(\lambda)^y}{y!} e^{-\lambda} = \beta$$

or in words: the chance of finding x or less errors equals β when λ equals λ_u. We then make the statement: $\lambda = np < \lambda_u$. The risk we run that λ_u will actually be lower than the true λ is less than β whatever λ may be. The same can be done for lower confidence limits and two-sided confidence limits.

The testing problem

Let $1-\beta$ be the chosen confidence level for the following testing problem:

H_0: $\lambda \geq \lambda_1$

H_1: $\lambda \leq \lambda_0$ with $\lambda_0 < \lambda_1$, two given values.

The standard Poisson-evaluation is that we will reject the null-hypothesis (and thus accept the population) if in the sample less than or equal to x_1 errors are found with x_1 such that:

$$\sum_{y=0}^{x_1} \frac{(\lambda_1)^y}{y!} e^{-\lambda_1} = \beta.$$

Then the chance of rejecting the null-hypothesis while it is true is less than or equal to β.

1.2. Sieve sampling

One can also look at the problem of the unknown error rate from a completely different point of view. In stead of considering the accounting population as a collection of dollar-units, each of which has an equal chance of being selected, we now consider the population as a collection of items each of which has a different chance (proportional to its value) of being selected. So one exploits the physical composition of the population and this has techniqually a lot of advantages. The sampling method that is based on this idea was discovered in 1959 by C. Rietveld of Klynveld Kraayenhof & Co and further developed by him later (see Rietveld 1978, 1979, 1984).

Suppose the population consists of N dollars and m items. Suppose we take sample size n with n such that all item sizes are less than or equal to N/n (if items are larger they either will be split into smaller items or investigated completely). Now consider m cells of N/n dollar-units, and place every item in a cell (the cell need not be filled up). See figure 2.

figure 2: population in sieve sampling

Suppose for every $i \in \{1,\ldots,m\}$ the following:

- item i has value a_i (consists of a_i dollar-units)
- item i contains e_i bad dollars (errors)

For every cell, independently of the other cells we draw a random number from 1 to N/n, say z_i. If $z_i \leq a_i$, we select item i; this happens with probability $\frac{a_i}{(N/n)} = \frac{n.a_i}{N}$. Next, if an item i is selected we investigate the selected dollar and see if it is bad or good. It is bad with probability $\frac{e_i}{a_i}$. So in item i a bad dollar (an error) is found with probability

$$\frac{n\ a_i}{N} \cdot \frac{e_i}{a_i} = \frac{e_i}{N/n} .$$

Another way to look at this is to imagine that every item is laid on a sieve with random mesh size, uniformly and continuously distributed between zero and N/n.

For every item a number is generated from this distribution and is said to be the item sieve. With probability $\frac{a_i}{N/n}$ item i remains on the sieve. We investigate this selected item and establish the total amount of error. Next we imagine this total amount of error of item i lying on the same sieve. Now with probability $\frac{e_i}{N/n}$ it remains on the sieve, hence is discovered, and 1 must be added to the total amount of errors found.

In contrast with dollar-unit sampling, in sieve sampling the (random) number of errors found in the sample generally does not have hypergeometric, binomial or Poisson distribution; not even by approximation. The distribution strongly depends on the concentration of errors in the population or rather in the items. We give two extreme examples:

1 - the bad dollars are evenly spread over the items

2 - all bad dollars are concentrated in one or more items.

In situation 1, if the error rate is p, we have pN bad dollars and so in each item $\frac{pN}{m}$. In every item independently of the others with probability

$$\frac{\frac{pN}{m}}{N/n} = \frac{n.p}{m}$$

an error is found. There are m items and therefore the total number of errors found in the sample is binomial distributed with parameters m and $\frac{np}{m}$ so very close to the Poisson distribution (if m is large enough) with parameter np.

In situation 2 if the error rate is p and if we have precisely np items of size $\frac{N}{n}$ (the cell is filled) that contain only bad dollars, then the sample gives us exactly n.p bad dollars (errors).

Clearly for both situations, the ordinary Poisson-based evaluation as used for dollar-unit sampling, is valid too, at least in a conservative way: in the estimation problem one runs a risk less or equal to β of making an incorrect statement and in the testing problem the chance of rejecting the null-hypothesis while it is true is less than or equal to β, if $\beta \le e^{-1}$, see [1].

In situations between these two extremes, the corresponding distribution of the number of errors found lies in a certain way also between the distribution in the extreme situations. (Hoeffding's theorem [2]). In the note [1] of Dr. R.D. Gill this, together with some results of Anderson & Samuels [3], is used to demonstrate that the ordinary Poisson-based evaluation applied to a sieve sample is conservative, in both estimation problems and testing problems.

We will demonstrate this in more detail and further refer to the note of R.D. Gill [1].

If for every $i \in \{1,\dots,m\}$, p_i is the probability of finding an error in item i, $\underline{x}$ is the random total number of errors found in the sample, and for a fixed number x: $P_r[\underline{x} \le x \mid p,\dots,p_m]$ is the notation for the chance of finding x or less errors in the sample when $p_1,\dots,p_m$ are the p_i's earlier discribed, then the theorem of Hoeffding states that:

for any $p_1,\dots,p_m$ with $\sum_{i=1}^{m} p_i = \lambda$ and $x < [\lambda]$ we have:

$$0 \le P_r[\underline{x} \le x \mid p_1,\dots,p_m] \le \sum_{y=0}^{x} \binom{m}{y} \left(\frac{\lambda}{m}\right)^y \left(1-\frac{\lambda}{m}\right)^{m-y}.$$

Note that the left value (0) corresponds with situation 2 and the right with situation 1 . Anderson & Samuels [3] have proved that the last term is smaller then

$$\sum_{y=0}^{x} \frac{\lambda^y}{y!} e^{-\lambda}.$$

The estimation problem

For the formulation of the problem see §1.1.

To show that the evaluation for sieve sampling is conservative, it must be proved that $p_r[\lambda_u \le \lambda \mid p_1,\dots,p_m] \le \beta$ whatever the values of $p_1,\dots,p_m$ may be. We have: $\lambda_u \le \lambda$ if and only if $\underline{x} \le x_0$ with x_0 such that

$$\sum_{y=0}^{x_0} \frac{\lambda^y}{y!} e^{-\lambda} \le \beta \quad \text{but} \quad \sum_{y=0}^{x_0+1} \frac{\lambda^y}{y!} e^{-\lambda} > \beta.$$

If $\beta \le e^{-1}$ then $x_0 \le \lambda-1$ (see the Poisson table in the appendix).

So

$$P_r[\lambda_u \le \lambda \mid p_1,\ldots,p_m] = P_r[\underline{x} \le x_0 \mid p_1,\ldots,p_m]$$
$$< \sum_{y=0}^{x_0} \frac{\lambda^y}{y!} e^{-\lambda} \quad \text{(Hoeffding)}$$
$$\le \beta \quad \text{(by definition of } x_0\text{).}$$

A similar result for lower confidence limits and two-sided confidence limits is obtained by using the second part of the theorem of Hoeffding, see [1].

The testing problem

For the formulation, see 1.1.

To show that the evaluation for sieve sampling is conservative, we must prove that if in fact $\lambda \ge \lambda_1$, the chance of rejecting the null hypothesis is less than or equal to β whatever $p_1,\ldots,p_m$ may be.

If $\beta \le e^{-1}$, then $x_0 \le \lambda_1 - 1 \le \lambda - 1$ (see the Poisson table in the appendix).

and hence:

$$P_r[\underline{x} \le x_0 \mid p_1,\ldots,p_m] < \sum_{y=0}^{x_0} \frac{\lambda^y}{y!} e^{-\lambda} \quad \text{(Hoeffding)}$$
$$\le \sum_{y=0}^{x_0} \frac{\lambda_1^y}{y!} e^{-\lambda_1} \quad \text{(because } \lambda \ge \lambda_1\text{)}$$
$$= \beta \quad \text{(by definition of } x_0\text{).}$$

2. TWO-STAGE SAMPLING.

2.1. True random sampling

2.1.1. Introduction

Two-stage sampling is a form of multi-stage sampling, a device by which we may sometimes cut inspection short based on the results one has obtained early in the inspection process. For suppose one wants to take a sample from an (accounting) population, using the dollar-unit method (true random sampling). One selects at random and independently dollars from the population and at the same time qualifies the selected dollars as "good" or "bad".

Halfway one takes a rest and looks at the results so far. If now the amount of errors is very low or high, one could already draw a conclusion about the population.
If not, one goes on selecting dollars and makes a conclusion when one is finished. If the sampling plan reckons with the possibility of stopping selecting early on, one can with the same reliability as in one-stage sampling say something about the population, with on the average less inspection. We will elaborate on this in an example later on.
On account of the possibility of accepting or rejecting the population in an early stage of the sample, this method of sampling is very useful in testing problems.

2.1.2. The testing problem

In this note we will from now on only consider testing problems and in fact only a particular one. This is done for the reason that accountants in their statistical work mainly have to deal with the problem:
"Is the error rate p (i.e. the amount of errors divided by the total amount of elements) in an accounting population, less than or equal to a certain acceptable percentage?"
For the reason that accepting a population with a too high percentage of errors is a far more serious error than rejecting a population with a very low percentage of errors, we can draw up the following testing problem, which will be a starting-point:

null hypothesis, H_0 : $p \geq p_1$
alternative hypothesis, H_1: $p \leq p_0$ with p_0, p_1 given values and $p_0 < p_1$
with confidence level $1-\beta$.

One can conceive of p_1 as the unacceptable error rate of the population and p_0 as the expected error rate of the population.

The test we will use is given in the form of an acceptance-criterium: accept the population if and only if
"in the first sample with size n_1 less than or equal to r_1 errors are found or in the first sample more than r_1, but in the extended sample with size (n_1+n_2) less than or equal to r_2 errors are found". $(r_2>r_1)$.

Rejection of H_0 when it is true is called a Type I error and in this situation it corresponds to accepting the population although the error rate p is unacceptable ($p \geq p_1$). The size of a test is defined as the supremum of the probabilities that a Type I error is made. For this test we will give this value the name:
"*chance of acceptance wrongly*". The confidence level of the test is $1-\beta$, which means that the chance of acceptance wrongly must be less or equal to β. This imposes restrictions on the sample sizes n_1 and n_2, but still there are many combinations of (n_1,n_2) possible.

Acceptance of H_0, when it is false, is called a Type II error and in this situation it corresponds to rejecting the population although the error rate p is acceptable. ($p \leq p_0$).
We define "*the chance of rejection wrongly*" as the supremum of probabilities, that a Type II error is made and indicate this chance with $\alpha(n_1,n_2)$
Remark that to every pair (n_1,n_2) another value of α is attached. To prefer two-stage sampling to one-stage-sampling is to prefer, on the average less inspection to enlargement of the chance of rejection wrongly.

EXAMPLE

The testingproblem is

$$H_0: p \geq p_1 = 0.05$$
$$H_1: p \leq p_0 = 0.005 \qquad \text{at confidence level } 1-\beta = 1-0.01$$

In one-stage sampling we use the acceptance-criterion:
"total number of errors found must be zero or one"
We calculate the sample size n, by using the Poisson-distribution (usual evaluation): The chance of accepting the population is (approximately) $(1+np)e^{-np}$. Then "the chance of acceptance wrongly" is (approximately) $(1+np_1)e^{-np_1}$ (because $p \geq p_1$). Equalising this chance to β gives us: n = 133.
The chance of rejecting the population is (approximately) $1-(1+np)e^{-np}$.
Then "the chance of rejection wrongly" (α) is (approximately) $1-(1+np_0)e^{-np_0} = 0.1434$ (because $p \leq p_0$).
In two-stage sampling we use the acceptance criterion:
"in the first sample zero errors are found or in the first sample one error and in the extended sample one error is found".

We calculate the sample sizes n_1 and n_2 using the Poisson distribution. The chance of accepting the population is (approximately) $e^{-n_1 p} + n_1 p e^{-(n_1+n_2)p}$

Thus "the chance of acceptance wrongly" is (approximately)

$e^{-n_1 p_1} + n_1 p_1 e^{-(n_1+n_2)p_1}$ (because $p \geq p_1$).

(2.1.2.a) The sample sizes n_1, n_2 must satisfy the equation:

$$\beta = e^{-n_1 p_1} + n_1 p_1 e^{-(n_1+n_2)p_1}.$$

The chance of rejecting the population is (approximately):

$1-\{e^{-n_1 p} + n_1 p e^{-(n_1+n_2)p}\}$.

(2.1.2.b) Then "the chance of rejection wrongly" is (approximately):

$1-\{e^{-n_1 p_0} + n_1 p_0 e^{-(n_1+n_2)p_0}\}$ (because $p \leq p_0$)

which (by definition) equals $\alpha(n_1, n_2)$.

Two combinations of (n_1, n_2) that satisfy (2.1.2a) are:

$$n_1 = 93, \quad n_2 = 93 \quad \text{and} \quad n_1 = 100, \; n_2 = 47.$$

The "chance of rejection wrongly" $\alpha(n_1, n_2)$ is 0.1884 and 0.1537 respectively, which are both larger than this chance in one-stage sampling.

If we base ourselves on the assumption, that the error rate is the one we expect (p_0), then an error is found in the first sample with probability $n_1 p_0 e^{-n_1 p_0}$. With this probability the second sample is taken and therefore we can say that the expected size of the sample is: $n_1 + (n_1 p_0 e^{-n_1 p_0})\, n_2$.

For the two mentioned combinations this results in 121 and 115 respectively. Both are smaller than the sample size in the one-stage sample.

2.2. Sieve sampling

2.2.1. Introduction

Our aim is to use the sieve method in two-stage sampling and to obtain the same results as in chapter I. This means that we want to show that the evaluation of two-stage sampling with the dollar-unit method using the Poisson distribution can also be applied to two-stage sampling with the sieve method. For this we first adjust the sieve method to two-stage sampling. Then we consider the chance of acceptation wrongly and the chance of rejection wrongly in the testing problem when one samples with the sieve-method, and deduce some results and construct tables at the end of this chapter.

2.2.2. The Sieve method

The definition of two-stage sampling using the dollar-unit method was obvious. One first selects n_1 dollars and if necessary, one selects another n_2 dollars. These selections are of course all at random and independently of one another. Because of the very different structure of the sieve-method such a definition doesn't work for two-stage sampling with the sieve method. To define two-stage sampling with the sieve method in a useful way the definition must imply that even if an error is not found in a particular item in the first sample, there is a chance of finding it in the extended sample. This can be achieved by reducing, for each item i, the item-sieve that was generated for the first sample in proportion to the cell size reduction in the extension of the sample. In formula's: if z_i is the item-sieve of item i in the first sample, then

$$\left(\frac{n_1}{n_1+n_2}\right) * z_i$$

will be taken as the item-sieve of item i in the extended sample (with size n_1+n_2). See also figure 3.
In this manner we define the sieve method in two-stage sampling, and thus determine the probability mechanism of the sampling scheme.

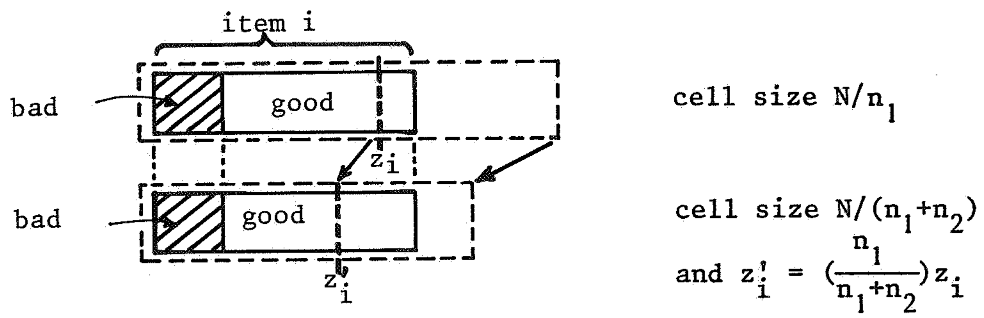
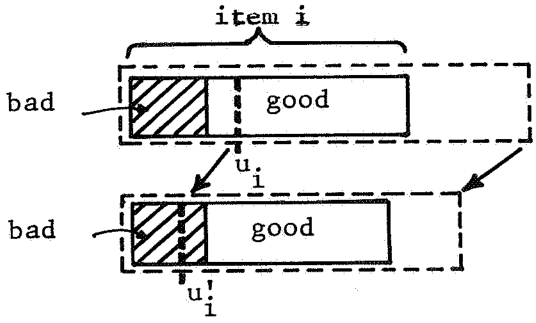


figure 3. the sieve method in two-stage sampling (1)

From figure 3 we see that there are three possible situations, according to whether or not an error is found, and if so, according to whether it is

found in the first or second stage of sampling. The first one is shown in figure 3; no error is discovered, neither in the first sample with sample size n_1 nor in the extended sample with size (n_1+n_2). Figure 4 shows the situation in which no error is discovered in the first sample but is discovered in the extended sample.
Figure 5 finally shows the situation in which both in the first sample and in the extended sample an error is found.

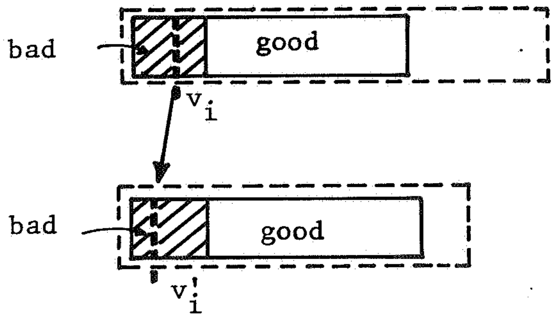


size of the cell: N/n_1

size of the cell: $N/(n_1+n_2)$

$u_i' = (\frac{n_1}{n_1+n_2})u_i$

figure 4. sieve method in two-stage sampling (2)



size of the cell: $N/(n_1)$

size of the cell: $N/(n_1+n_2)$

$v_i' = (\frac{n_1}{n_1+n_2})v_i$

figure 5. sieve method in two-stage sampling (3)

For the reason that the sieve method, in contrast to the dollar-unit method, doesn't proceed from individual elements (dollars), and so the extended sample doesn't consist of just selecting a few elements more, the remarks in the previous paragraph about expected average sample sizes don't hold in the sieve method.
But it is clear that if one takes a smaller first sample size n_1, less items have to be investigated on average. On the other side, if one takes a smaller first sample size n_1, the sample size of the extended sample (n_1+n_2) will be greater and so more items have to be investigated if one

has to extend the sample.

2.3. The testing problem

We again consider the testing problem:

$$H_0: p \geq p_1$$

$$H_1: p \leq p_0 \text{ with } p_0, p_1 \text{ given values and } p_0 < p_1.$$

The test we will use is the same as in 2.1.2; accept the population if and only if "in the first sample with size n_1 less than or equal to r_1 errors are found or more than r_1 in the first sample but less than or equal to r_2 in the extended sample". with size (n_1+n_2).
We have seen how the evaluation was done when sampling with the dollar-unit method. When we have given fixed values of β, p_0, p_1, r_1 and r_2 we can calculate the sample sizes n_1 and n_2, carry out the sample and based on the acceptance criterion, reject or accept the population, with a risk less than or equal to β. We will use these calculated n_1 and n_2 to calculate "the chance of acceptance wrongly" if the sample is carried out by using the sieve method. We would like to see that this chance underestimates β.
Then one can conclude that the evaluation based on the Poisson-distribution of numbers of errors found is valid when using the sieve method.

2.2.4. The chance of acceptance wrongly

We introduce some notation:

m	number of items
a_i	size of item i, i = 1,...,m
e_i	amount of error in item i, i = 1,...,m
$N = \sum_{i=1}^{m} a_i$	population size
$K = \sum_{i=1}^{m} e_i$	total error amount
$p = \frac{K}{N}$	error rate
n_1	nominal first sample size
n_2	nominal second sample size (thus (n_1+n_2) size of the extended sample)

$C_1 = N/n_1$ effective cell size in first sample with size n_1

$C_2 = N/(n_1+n_2)$ effective cell size in extended sample with size (n_1+n_2)

$\rho = n_1/(n_1+n_2)$

$$q_i = \frac{(n_1+n_2)e_i}{N}$$

$$\lambda = \sum_{i=1}^{m} q_i = (n_1+n_2)p$$ expected number of errors found in the extended sample

$\lambda_1 = (n_1+n_2)p_1, \quad \lambda_0 = (n_1+n_2)p_0.$

We suppose now that $\rho \in (0,1)$ and $0 \le e_i \le a_i \le C_2 \le C_1$ for $i = 1,\ldots,m$ so all items are smaller than the effective cell size on the extended sample with size (n_1+n_2).

From figure 3.4.5 we see that for each item i independently of the other itmes, in the sieve method:

- with probability $\frac{e_i}{N/n_1} = \rho . q_i$ an error is found in the first sample therefore also in the extended sample (figure 5)
- with probability $\frac{e_i}{N/\ n_2} = (1-\rho)q_i$ an error is found in the extended sample but <u>not</u> in the first sample (figure 4)
- with probability $1- \frac{(n_1+n_2)e_i}{N} = 1-q_i$ no error is found in either the first sample or in the extended sample (figure 3)

We define some random variables.

Let $\underline{X}_{1i}$ be the random variable which takes the value 1 if an error is found in item i in the first sample with size n_1, zero otherwise.

Let $\underline{X}_{2i}$ be the random variable which takes the value 1 if an error is found in item i, in the extended sample with size n_1+n_2, zero otherwise.

So, $\underline{X}_{11}, \underline{X}_{12},\ldots,\underline{X}_{1m}$ are independent Bernoulli variables with

$P_R[\underline{X}_{1i} = 1] = 1-P_R[\underline{X}_{1i} = 0] = \rho . q_i \quad i = 1,\ldots,m.$

Also $\underline{X}_{21},\underline{X}_{22},\ldots,X_{2m}$ are independent Bernoulli variables with

$P_R[\underline{X}_{2i} = 1] = 1-P_R[\underline{X}_{2i} = 0] = q_i . \quad i = 1,\ldots,m$

Notice that if $\underline{X}_{1i} = 1$ then $\underline{X}_{2i} = 1$ for $i = 1,\dots,m$.

Let $\underline{X}_1 = \sum_{i=1}^{m} \underline{X}_{1i}$ denote the total number of errors found in the first sample.

Let $\underline{X}_2 = \sum_{i=1}^{m} \underline{X}_{2i}$ denote the total number of errors found in the extended sample.

Then in terms of these random variables the acceptance criterion becomes:

$$"\underline{X}_1 \le r_1 \text{ or } \{\underline{X}_1 > r_1 \text{ and } \underline{X}_2 \le r_2\}".$$

For the chance of accepting the population when using the sieve method, we write:

$$P_R[\underline{X}_1 \le r_1 \text{ or } (\underline{X}_1 \ge r_1 \text{ and } \underline{X}_2 \le r_2) \mid q_1,\dots,q_m] \text{ or } {}^{m}f_{r_1,r_2}(q_1,\dots,q_m)$$

(since it depends on $q_1,\dots,q_m$).

<u>THEOREM 1</u>. *Let* $r_1, r_2 \in \mathbb{N}_0$ *with* $r_1 < r_2$ *and let* $\lambda = \Sigma q_i$ *be fixed. Then*

$$P_R[\underline{X}_1 \le r_1 \text{ or } (\underline{X}_1 > r_1 \text{ and } \underline{X}_2 \le r_2) \mid q_1,\dots,q_m]$$

attains its maximum (minimum) value if for every

$i = 1,\dots,m$: $q_i = 0$
or $q_i = 1$
or $q_i = a$ with $0 < a < 1$.

<u>PROOF</u>. See end of the paragraph.

So the chance of acceptation attains its maximum (minimum) if the amount of errors in each item is one of the next three values:

- zero, i.e. no error in the item (q_i=0)
- maximum, i.e. the amount of error is the effective cell size in the extended sample (q_i=1)
- a value between zero and the maximum (q_i=0, with $0 < a < 1$).

For general r_1 and r_2 it seems very hard indeed to give useful conditions under which one can prove that this chance of acceptation doesn't exceed β. Therefore we will consider only a few cases and treat one of them more extensively. Other cases can be tackled in the same way.
The investigated cases are:

$r_1 = 0$ and $r_2 = 1$
$r_1 = 0$ and $r_2 = 2$
$r_1 = 1$ and $r_2 = 2$.

As a guide we will use the case $r_1 = 0$ and $r_2 = 1$ and we will compare this with the example in 2.1.2.

An elaborated example of the sieve method in two-stage sampling:

Our testingproblem is $H_0: p \geq p_1$
$H_1: p \leq p_0$ p_0, p_1 give values with $p_0 < p_1$

at confidence level 1-β (see 2.2.3).
The acceptation criterion is:

"zero errors in the first sample
or one error in the first sample but only one error in the extended sample"

The chance of accepting the population is: $P_R[\underline{X}_1 = 0$ or $(\underline{X}_1 = 1$ and $\underline{X}_2 = 1) \mid q_1,\ldots,q_m]$.
If $\tilde{q} = (q_1,\ldots,q_m)\frac{\text{maximizes}}{(\text{minimizes})}$ this chance, then it follows from Theorem 1 that we can say that $\tilde{q}$ consists of the following components

s one-components
r zero-components
and t=m-r-s a-components with $a = \frac{\lambda-s}{t}$ for $\sum_{i=1}^{m} q_i = \lambda$.

The numbers s,r and t must therefore satisfy $0 < \frac{\lambda-s}{t} < 1$.
(The fact that $t > 0$ will be proved in lemma 1).

We can draw up the following scheme for every item i, independently of all the other items:

	1-comp.	0-comp.	$\frac{\lambda-s}{t}$ -comp.
number	s	r	t
chance of an error in first sample	ρ	0	$\rho(\frac{\lambda-s}{t})$
chance of an error in extended but not in first sample	$1-\rho$	0	$(1-\rho)(\frac{\lambda-s}{t})$
chance of no error in either first or extended sample	0	1	$1-(\frac{\lambda-s}{t})$

The chance of acceptation is the chance of zero errors in the first sample plus the chance of one error in the first and in the extended sample.
Now, the chance of zero errors in the first sample is clearly:
$(1-\rho)^s(1-\rho(\frac{\lambda-s}{t}))^t$.
The chance of one error in the first and in the extended sample can we split up according to the possible values of s.
If $s \geq 2$ then surely two errors will occur in the extended sample and so this chance is zero.
If $s = 1$, then surely one error occurs in the extended sample and this error therefore must also occur in the first sample and this with probability ρ.
In the other items, no error may be found, neither in the first nor in the extended sample;
this with probability $(1-\frac{\lambda-1}{t})^t$
Thus the chance is $\rho(1-\frac{\lambda-1}{t})^t$.
If $s = 0$, in one of the items, that are partially filled with errors, an error must be found in the first sample: this with probability $\rho(\frac{\lambda}{t})$.
In the other items, no error may be found in either the first or in the extended sample: this with probability $(1-\frac{\lambda}{t})^{t-1}$.
Thus the chance is: $t\,(\frac{\rho\lambda}{t})\;(1-\frac{\lambda}{t})^{t-1} = \rho\lambda(1-\frac{\lambda}{t})^{t-1}$.

This gives us the chance of acceptance, split up according to the values of s:

$s = 0 \quad (1-\frac{\rho\lambda}{t})^t \quad + \rho\lambda(1-\frac{\lambda}{t})^{t-1}$

$s = 1 \quad (1-\frac{\rho(\lambda-1)}{t})^t(1-\rho) \quad + \rho(1-\frac{\lambda-1}{t})^t$

$\lambda > s \geq 2 \; (1-\frac{\rho(\lambda-s)}{t})^t \; (1-\rho)^s.$

In one-stage sampling we have seen that if λ is large enough, the case when the errors are evenly spread over the items leads to a maximum chance of acceptance. The corresponding case in two-stage sampling is when $s = 0$ and $r = 0$ and $t = m$ and then the chance of accepting the population is:

$(1-\frac{\rho\lambda}{m})^m + \rho\lambda(1-\frac{\lambda}{m})^{m-1}$ which is approximately: $e^{-\rho\lambda} + \rho\lambda e^{-\lambda}$.

For all $\lambda \geq \lambda_1$ this is less than or equal to :

$e^{-\rho\lambda_1} + \rho\lambda_1 e^{-\lambda_1}$.

This equals $e^{-n_1p_1} + n_1p_1e^{-(n_1+n_2)p_1}$ which is exactly "the chance of acceptance wrongly" in the dollar-unit method and thus equals β (see 2.1.2.a). If now "the evenly spread" case corresponds to the largest value of the chance of accepting the population (as in one-stage sampling) and thus "the chance of acceptance wrongly" does not exceed β, then we can conclude that evaluation of the sample according to the dollar-unit method is also valid when sampling is done with the sieve method. In this case, ($r_1 = 0$, $r_2 = 1$), this condition is satisfied when $\lambda \geq e$, which is stated and proved in Lemma 1 at the end of this chapter.

For if $\lambda_1 \geq e$, then for all $\lambda \geq \lambda_1 \geq e$:

the chance of accepting the population $\leq (1-\frac{\rho\lambda}{m})^m + \rho\lambda(1-\frac{\lambda}{m})^{m-1}$. (Lemma 1)

Then "the chance of acceptance wrongly" $\leq (1-\frac{\rho\lambda_1}{m})^m + \rho\lambda_1(1-\frac{\lambda_1}{m})^{m-1}$ (because $\lambda \geq \lambda_1$)

$$< e^{-\rho\lambda_1} + \rho\lambda_1 e^{-\lambda_1} \qquad \text{(Lemma 1)}$$

$$= e^{-n_1p_1} + n_1p_1e^{-(n_1+n_2)p_1}$$

$$= \beta \qquad \text{(definition)}$$

Indeed we can evaluate two-stage sampling with the sieve method as two-stage sampling with the dollar-unit method.

To prove this, we can also use a more direct method namely by showing directly that all possible chances of acceptance (in the sieve method) underestimate: $e^{-\rho\lambda} + \rho\lambda e^{-\lambda}$.
Because if this can be done, then;

the chance of accepting the population $\le e^{-\rho\lambda} + \rho\lambda e^{-\lambda}$

Then "the chance of acceptance wrongly" $\le e^{-\rho\lambda_1} + \rho\lambda_1 e^{-\lambda_1}$ (because $\lambda \ge \lambda_1$)

$= \beta$ (definition).

Lemma 1.a. states a condition under which this holds.
Lemma 1.a. is stated and proved at the end of this paragraph.

So there are two ways of showing that "the chance of acceptance wrongly" on the sieve method is less than or equal to β, i.e. "the chance of acceptance wrongly" in the dollar-unit method.
For general r_1 and r_2, we can make the same remarks as in the previous example. If we say that $\tilde{q} = (q_1,\ldots,q_m)$ maximizes the chance of accepting the population and $\tilde{q}$ consists of s one-component, r zero-components and t components with value $\frac{\lambda-s}{t}$, then the "evenly spread" case corresponds to the situation in which $s = 0$, $r = 0$, $t = m$. Then the first way of argumentation that the evaluation is valid is:
I: the chance of accepting the population

$$(2.2.4.a) \quad \le \sum_{k=0}^{r_1} \binom{m}{k} \left(\frac{\rho\lambda}{m}\right)^k \left(1-\frac{\rho\lambda}{m}\right)^{m-k}$$

$$+ \sum_{k=r_1+1}^{r_2} \sum_{l=k}^{r_2} \binom{l}{k} \rho^k (1-\rho)^{l-k} \binom{m}{l} \left(\frac{\lambda}{m}\right)^l \left(1-\frac{\lambda}{m}\right)^{m-l} \quad \text{(lemma *)}$$

Then "the chance of acceptance wrongly"

$$(2.2.4.b) \quad \le \sum_{k=0}^{r_1} \binom{m}{k} \left(\frac{\rho\lambda_1}{m}\right)^k \left(1-\frac{\rho\lambda_1}{m}\right)^{m-k} +$$

$$+ \sum_{k=r_1+1}^{r_2} \sum_{l=k}^{r_2} \binom{l}{k} \rho^k (1-\rho)^{l-k} \binom{m}{l} \left(\frac{\lambda_1}{m}\right)^l \left(1-\frac{\lambda_1}{m}\right)^{m-l} \quad \text{(because } \lambda \ge \lambda_1\text{)}$$

$$(2.2.4.c) \quad < \sum_{k=0}^{r_1} \frac{(\rho\lambda_1)^k}{k!} e^{-\rho\lambda_1} + \sum_{k=r_1+1}^{r_2} \sum_{l=k}^{r_2} \binom{l}{k} \rho^k (1-\rho)^{l-k} \frac{\lambda_1^l}{l!} e^{-\lambda_1} \qquad \text{(lemma *)}$$

and if we substitute $\rho = \frac{n_1}{n_1+n_2}$ and $\lambda_1 = (n_1+n_2)p_1$ in (2.2.4.c) and rewrite a little, we get:

$$(2.2.4.d) \quad \sum_{k=0}^{r_1} \frac{(n_1p_1)^k}{k!} e^{-n_1p_1} + \sum_{k=r_1+1}^{r_2} \sum_{l=0}^{r_2-k} \frac{(n_2p_1)^l}{l!} e^{-n_2p_1} \frac{(n_1p_1)^k}{k!} e^{-n_1p_1}$$

which is the approximated (by using the Poisson distribution) "chance of acceptance wrongly" in the dollar-unit method, and therefore (2.2.4.c) equals β.

Thus if all the steps can be proven, then the evaluation is valid.

Therefore one has to find conditions under which (2.2.4.a) is satisfied (lemma *).

In the second step we used the fact that $\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$ decreases in p.

If $\rho\lambda_1 \geq r_1+1$ and $\lambda_1 \geq r_2+1$, then (2.2.4.c) is satisfied. This follows from the theorem of Hoeffding (see also Anderson & Samuels, Theorem 2.1).

For the case $r_1 = 0$, $r_2 = 1$ this method of proof is worked out in lemma 1.

For the case $r_1 = 0$, $r_2 = 2$ it is contained worked out in lemma 2, which can also be found at the end of this chapter.

The second way of argumentation is.

II: The chance of accepting the population

$$(2.2.4.e) \quad \leq \sum_{k=0}^{r_1} \frac{(\rho\lambda)^k}{k!} e^{-\rho\lambda} + \sum_{k=r_1+1}^{r_2} \sum_{l=k}^{r_2} \binom{l}{k} \rho^k (1-\rho)^{l-k} \frac{\lambda^l}{l!} e^{-\lambda} \qquad \text{(lemma (**))}$$

Then "the chance of acceptance wrongly"

$$\leq \sum_{k=0}^{r_1} \frac{(\rho\lambda_1)^k}{k!} e^{-\rho\lambda_1} + \sum_{k=r_1+1}^{r_2} \sum_{l=k}^{r_2} \binom{l}{k} \rho^k (1-\rho)^{l-k} \frac{\lambda_1^l}{l!} e^{-\lambda_1} \qquad (\text{because } \lambda \geq \lambda_1)$$

$$= \beta.$$

Thus if one can establish lemma (**) then the validity of the evaluation is proven.
In the second step we used the fact that $\sum_{k=0}^{n} \frac{\lambda^k}{k!} e^{-\lambda}$ decreases in λ.
In the case $r_1 = 0$, $r_2 = 1$ this method of proof is carried out in lemma 1b.
In the case $r_1 = 1$, $r_2 = 2$ we can also use this method, and deduce lemma 3.
Thus for general (r_1,r_2) we have two possible strategies for proving that under certain conditions the evaluation of two-stage sampling with the sieve-method can be taken the same as that of two-stage sampling with the dollar-unit-method. It should be remarked that the second way seems easier.

2.2.5. The chance of rejection wrongly

We still consider the testing problem and test in (2.2.3). In the previous paragraph we have taken a close look at "the chance of acceptance wrongly", i.e. the size of the test.
We have seen, that in several different acceptation-criterions (r_1,r_2), under some conditions this chance does not exceed "the chance of acceptance wrongly" in the dollar-unit method (β).

In this paragraph we will examine "the chance of rejection wrongly", which is the supremum of the probabilities of accepting H_0 when it is false or in other words the probabilities of rejecting the population although the error rate p is acceptable ($p \le p_0$).

First we consider the example $r_1 = 0$, $r_2 = 1$.
We now calculate this chance in the sieve method.
First remark that chance of rejecting the population is equal to one minus the chance of accepting the population and this last chance we know.
For the reason, that we look at the Type II error, which means accepting

H_0 when it is false, we have to do with the case $\lambda \le \lambda_0$. And in practice $\lambda_0 \le 1$. Therefore we assume in this paragraph further $\lambda_0 \le 1$. We know from theorem 1, that ${}^m f_{0,1}(\tilde{q})$ attains its minimum value by a q, whose nonone components and nonzero-components are all equal. This minimum value , the chance of acceptance, is calculated in 2.2.4 and because $\lambda \le \lambda_0 \le 1$, the only case that is possible is the one in which s = 0, which gives the chance of acceptance (if $\lambda \le 1$):

$$(2.2.5.d) \quad \left(1-\frac{\rho\lambda}{t}\right)^t + \rho\lambda\left(1-\frac{\lambda}{t}\right)^{t-1} \qquad \text{with } m = r+t \text{ and } 0 < \lambda < t.$$

If we look at the "evenly spread" case, that is when t = m, we get: the chance of acceptance is:

$$\left(1-\frac{\rho\lambda}{m}\right)^m + \rho\lambda\left(1-\frac{\lambda}{m}\right)^{m-1}$$

and then the chance of rejection is:

$$1-\left\{\left(1-\frac{\rho\lambda}{m}\right)^m + \rho\lambda\left(1-\frac{\lambda}{m}\right)^{m-1}\right\} .$$

For all $\lambda \le \lambda_0$ this is less than or equal to

$$1-\left\{\left(1-\frac{\rho\lambda_0}{m}\right)^m + \rho\lambda_0\left(1-\frac{\lambda_0}{m}\right)^{m-1}\right\}$$

and this is approximately:

$$1-\{e^{-\rho\lambda_0} + \rho\lambda_0 e^{-\lambda_0}\}$$

which is exactly "the chance of acceptance wrongly" in the dollar-unit method

So if now "the evely spread" case attaches the smallest value to the chance of acceptance and thus the largest value to the chance of rejection and therefore "the chance of rejection wrongly" does not exceed $\alpha(n_1,n_2)$ then we can conclude that for this pair (n_1,n_2) the sieve method is "as good as" the dollar-unit method with respect to the "chance of rejection wrongly" (the first way of argumentation). In this case ($r_1 = 0$, $r_2 = 1$), this condition is satisfied when $\lambda \le 1$ and

$\rho\lambda \leq 2-\sqrt{2}$, which is stated in lemma 4.
For if $\lambda_0 \leq 1$ and $\rho\lambda_0 \leq 2-\sqrt{2}$ then for all $\lambda \leq \lambda_0$ we have $\lambda \leq 1$ and $\rho\lambda \leq 2-\sqrt{2}$ and thus:

the chance of accepting the population

$$\geq (1-\frac{\rho\lambda}{m})^m + \rho\lambda(1-\frac{\lambda}{m})^{m-1} \qquad \text{(lemma 4 and theorem 1).}$$

Then the chance of rejecting the population

$$\leq 1-\{(1-\frac{\rho\lambda}{m})^m + \rho\lambda(1-\frac{\lambda}{m})^{m-1}\}.$$

Then "the chance of rejection wrongly"

$$\leq 1-\{(1-\frac{\rho\lambda_0}{m})^m + \rho\lambda_0(1-\frac{\lambda_0}{m})^{m-1}\} \qquad \text{(because } \lambda \leq \lambda_0\text{)}$$

$$< 1-\{e^{-\rho\lambda_0} + \rho\lambda_0 e^{-\lambda_0}\} \qquad \text{(lemma 4)}$$

$$= \alpha(n_1,n_2) \qquad (2.2.5.a)$$

This example of the case $r_1 = 0$, $r_2 = 1$ can easily be extended to other cases, if we assume that: $\lambda_0 \leq 1$, which is in practice a very reasonable assumption. We can of course also use the second way of argumentation applied to "the chance of rejection wrongly" which leads for the case $r_1 = 0$, $r_2 = 2$ to lemma 5 and for the case $r_1 = 1$, $r_2 = 2$ to lemma 6. The lemmas are placed at the end of this chapter, but the results of them together with the results of the lemmas in paragraph 2.2.4 are summarized in the next paragraph.

2.2.6. The conditions for sieve sampling

In this paragraph we exhibit a scheme summarizing conditions for applying the sieve method in two-stage sampling. In the scheme only the three cases we have treated are inserted.

We choose first β, p_0 and p_1.
With these three values we can, in each case (r_1,r_2), calculate a series of possible sample sizes n_1 and n_2 and corresponding α.

Then we calculate:

$$\rho = \frac{n_1}{n_1+n_2}, \quad \lambda_1 = (n_1+n_2)p_1, \quad \lambda_0 = (n_1+n_2)p_0.$$

Then the conditions are for:

	$r_1 = 0$, $r_2 = 1$	$r_1 = 0$, $r_2 = 2$	$r_1 = 1$, $r_2 = 2$
The chance of acceptance wrongly <= BETA	$\lambda_1 \geq e$ or $\begin{cases} \lambda_1 \in (2,e) \\ \rho \in (0.3,1) \end{cases}$	$\lambda_1 \geq e+\sqrt{e(e-2)}$	$\lambda_1 \geq e+\sqrt{e(e-2)}$ $\lambda_1 \geq 1 + 1/\rho$ $\lambda_1 \geq \frac{1}{\rho} - \frac{\rho}{(1-\rho)(\rho+\log(1-\rho))}$
The chance of rejection wrongly <= ALPHA	$\begin{cases} \lambda_0 \leq 1 \\ \rho \cdot \lambda_0 \leq 2-\sqrt{2} \end{cases}$	$\lambda_0 \leq 1$	$\lambda_0 \leq 1$

2.2.7. Tables

It seems hard to construct a table, by means of which, having chosen the parameter β, p_0 and p_1, one can see what sampling plan (r_1,r_2) is preferable. The tables in the appendix give the user a whole range of possibilities for a particular choice of β, p_0 and p_1. Such tables are easy to construct for any other β, p_0 and p_1. In this paragraph we first show how the tables in the appendix are drawn up and thus how to draw up similar tables.

We elaborate on the table with $r_1 = 0$, $r_2 = 1$, $\beta = 0.01$, $p_0 = 0.005$, $p_1 = 0.05$. The sample sizes n_1 and n_2 are calculated by solving equation (2.1.2.b). The chance of rejection wrongly in the dollar-unit method (α) can be calculated for every pair (n_1,n_2) by using the expression in (2.1.2.a). The columns denoted "chance of acceptation wrongly" and "chance of rejection wrongly" refer to possibilities when using the *sieve method*. The scheme in paragraph 2.2.6 gives conditions under which these chances, for each pair (n_1,n_2), are less than or equal to β and less than or equal to α respectively when using the sieve method. To verify the conditions are satisfied for each pair one has to calculate

$$\rho = \frac{n_1}{n_1+n_2}, \quad \lambda_1 = (n_1+n_2)p_1 \quad \text{and} \quad \lambda_0 = (n_1+n_2)p_0.$$

If the condition is satisfied under which the chance of acceptance wrongly in the sieve method is less than or equal to β, we write <= BETA in that column, similarly for the other. If the condition is not satisfied, we write "?": this does not mean that the sieve method gives a larger chance, but only that the lemmas that we have now do not cover the corresponding pair (n_1,n_2). As an example we take $n_1 = 95$ and $n_2 = 69$.
We get

$$\rho = 0.5793, \quad \lambda_1 = 8.2000, \quad \lambda_0 = 0.8200, \quad \text{thus } \rho.\lambda_0 = 0.4750$$

All conditions are satisfied and thus in the columns we write

"<= BETA" and "<= ALPHA".

If one wants to make a table with other β, p_0 and p_1, the procedure is the same. We give it for general r_1 and r_2.
If the risk β, the expected error rate (p_0) and the unacceptable error rate (p_1) are chosen one calculates a range of possible sample sizes n_1 and n_2 by equalising (2.2.4.a) to β.
The chance of rejection wrongly (α) for these n_1 and n_2 is given by (2.2.5.b).
The conditions that have to be satisfied to be able to write "<= BETA" or "<= ALPHA" are given in the scheme in paragraph 2.2.6.

The choice of a sampling plan, in particular the choice of r_1 and r_2, depends strongly on the β, p_0, p_1, α one wants to work with.
For instance if we choose $\beta = 0.01$, $p_0 = 0.005$, $p_1 = 0.05$ and we want to keep the α as low as possible, from the three cases that are tabled, the plans $r_1 = 0$, $r_2 = 2$ and $r_1 = 1$, $r_2 = 2$ are reasonable. The difference between these two plans is that in the case $r_1 = 0$, $r_2 = 2$ the pair (n_1,n_2) that can be used, when sampling with the sieve method, have a smaller first sample size n_1 than in the case $r_1 = 1$, $r_2 = 2$.

But the second sample size n_2 is larger (all this is with the same value of α). A final choice is hard to make without knowing the priorities of the user.

2.2.8. Proofs

THEOREM 1. *Let*

$$r_1, r_2 \in \mathbb{N}_0, \quad \text{with} \quad r_1 < r_2,$$

then for fixed

$$\lambda = \sum_{i=1}^{m} q_i$$

$$p_R[\underline{X}_1 \le r_1 \quad \text{or} \quad (\underline{X}_1 > r_1 \text{ and } \underline{X}_2 \le r_2) \mid q_1,\ldots,q_m]$$

attains its maximum (minimum) value if for every $i = 1,\ldots,m$:

$$q_i = 0 \quad \text{or} \quad q_i = 1 \quad \text{or} \quad q_i = a \quad \text{with} \quad 0 < a < 1.$$

PROOF. Write $\tilde{q} = (q_1,\ldots,q_m)$ for the vector of m components chose i'th component is q_i and write $\tilde{q}_{ij}$ for the vector obtained from $\tilde{q}$ by deleting the i'th and j'th components.
Define:

$$^m f_{r_1,r_2}(\tilde{q}) := p_R[\underline{X}_1 \le r_1 \quad \text{or} \quad (\underline{X}_1 > r_1 \text{ and } \underline{X}_2 \le r_2) \mid q_1,\ldots,q_m]$$

We consider a fixed value of m and a fixed value of $\lambda = \sum_{i=1}^{m} q_i$.
Take two different items i and j and split up the event "$\underline{X}_1 \le r_1$ or $(\underline{X}_1 > r_1$ and $\underline{X}_2 \le r_2)$" according to the events for both i and j, that:
- an error was found in the first sample (with probability ρq_i and ρq_j)
- an error in the extended but not in the first sample (with probability $(1-\rho)q_i$ and $(1-\rho)q_j$)
- no error was found (with probability $(1-q_i)$ and $(1-q_j)$)

(see also paragraph 2.2.4.).

This gives us:

$$
\begin{aligned}
{}^{m}f_{r_1,r_2}(\tilde{q}) &= \rho q_i.\rho q_j.{}^{m-2}f_{r_1-2,r_2-2}(\tilde{q}_{ij}) \\
&+ \rho q_i.(1-\rho)q_j.{}^{m-2}f_{r_1-1,r_2-2}(\tilde{q}_{ij}) + \rho q_i.(1-q_j).{}^{m-2}f_{r_1-1,r_2-2}(\tilde{q}_{ij}) \\
&+ (1-\rho)q_i.\rho q_j.{}^{m-2}f_{r_1-1,r_2-2}(\tilde{q}_{ij}) + \\
&+ (1-\rho)q_i.(1-\rho)q_j.{}^{m-2}f_{r_1,r_2-2}(\tilde{q}_{ij}) + (1-\rho)q_i.(1-q_j).{}^{m-2}f_{r_1,r_2-2}(\tilde{q}_{ij}) \\
&+ (1-q_i).\rho q_j.{}^{m-2}f_{r_1-1,r_2-1}(\tilde{q}_{ij}) \\
&+ (1-q_i).(1-\rho)q_j.{}^{m-2}f_{r_1,r_2-1}(\tilde{q}_{ij}) + (1-q_i).(1-q_j).{}^{m-2}f_{r_1,r_2}(\tilde{q}_{ij}) \\
&= q_iq_j.A + B \qquad \text{(A and B both depend on the choices of i and j)}
\end{aligned}
$$

where

$$
\begin{aligned}
A = \rho^2.{}^{m-2}f_{r_1-2,r_2-2}(\tilde{q}_{ij}) &+ 2\rho(1-\rho).{}^{m-2}f_{r_1-1,r_2-2}(\tilde{q}_{ij}) \\
&- 2\rho.{}^{m-2}f_{r_1-1,r_2-1}(\tilde{q}_{ij}) \\
+ (1-\rho)^2.{}^{m-2}f_{r_1,r_2-2}(\tilde{q}_{ij}) &- 2(1-\rho).{}^{m-2}f_{r_1,r_2-1}(\tilde{q}_{ij}) + {}^{m-2}f_{r_1,r_2}(\tilde{q}_{ij})
\end{aligned}
$$

and

$$
\begin{aligned}
B = (q_i+q_j).(\rho.{}^{m-2}f_{r_1-1,r_2-1}(\tilde{q}_{ij}) + (1-\rho).{}^{m-2}f_{r_1,r_2-1}(\tilde{q}_{ij}) - {}^{m-2}f_{r_1,r_2}(\tilde{q}_{ij})) \\
+ {}^{m-2}f_{r_1,r_2}(\tilde{q}_{ij}).
\end{aligned}
$$

Note that if $\tilde{q}_{ij}$ is held fixed, and we vary q_i and q_j but keep their sum fixed, then in the equation:

$$^{m}f_{r_1,r_2}(\tilde{q}) = q_i.q_j.A + B$$

only $q_i.q_j$ varies while A and B remain the same.
Suppose we can replace q_i by $q_i+\varepsilon$ and q_j by $q_j-\varepsilon$ for some small quantity ε.
Then $^{m}f_{r_1,r_2}(\tilde{q})$ changes only in the factor $q_i.q_j$, which becomes
$q_iq_j + \varepsilon.((q_j-q_i)-\varepsilon)$.
Suppose now, that $q_i < q_j$.
If A is positive, you can take ε positive and small, so that $^{m}f_{r_1,r_2}(\tilde{q})$ increases.
If A is negative, you can take ε negative and small, so that $^{m}f_{r_1,r_2}(\tilde{q})$ increases.
If A is zero, then $^{m}f_{r_1,r_2}(\tilde{q})$ remains the same.

Suppose now, that $\tilde{q} = (q_1,\ldots,q_m)$ maximizes $^{m}f_{r_1,r_2}(\tilde{q})$.
(At least one maximizing value does exist, but there may be more).
If any two components of $\tilde{q}$ exist not equal to one another and not equal to zero or one, say $0 < q_i < q_j < 1$, then the corresponding term A must be zero, otherwise we could increase $^{m}f_{r_1,r_2}(\tilde{q})$ still further.
So $^{m}f_{r_1,r_2}(\tilde{q}) = B$ and therefore we can replace both q_i and q_j by their average $(q_i+q_j)/2$ and still keep $^{m}f_{r_1,r_2}(\tilde{q})$ the same.
We can repeat this procedure infinitely often at each step choosing for q_i and q_j the smallest nonzero respectively and the largest nonone component in the vector $\tilde{q}$ while $^{m}f_{r_1,r_2}(\tilde{q})$ stays the same.
So we get a vector $\tilde{q}$ in which the components that are nonzero and nonone get closer and closer to one another.
In the limit all these components are equal and $^{m}f_{r_1,r_2}(\tilde{q})$ still has that same maximum value.
Thus $^{m}f_{r_1,r_2}(\tilde{q}) = P_r[\underline{X}_1 \leq r_1$ or $(\underline{X}_1 > r_1$ and $\underline{X}_2 \leq r_2)$ $(q_1,\ldots,q_m)]$ attains its maximum by a $\tilde{q}$ whose nonzero and nonone components are all equal.
When we replace q_i by $q_i-\varepsilon$ and q_j by $q_j+\varepsilon$ you can prove analogously that $^{m}f_{r_1,r_2}(\tilde{q})$ attains its minimum value by a $\tilde{q}$ whose nonzero and nonone components are all equal.
Now a theorem of Anderson and Samuels will be given that will be used in several lemmas. (Theorem 3.1 of Anderson and Samuels [3]).

<u>Lemma of Anderson and Samuels:</u>

i) If $\lambda \in \mathbb{R}^{+}\backslash\{0\}$, then

$$(1-\frac{\lambda}{t})^{t} > (1-\frac{\lambda}{t-1})^{t-1} \quad \text{for } t \in \{2,3,\ldots\} \text{ and } t-1 > \lambda.$$

ii)a) if $\lambda \geq 2$, then

$$\binom{t}{1}(\frac{\lambda}{t})^{1}(1-\frac{\lambda}{t})^{t-1} > \binom{t-1}{1}(\frac{\lambda}{t-1})^{1}(1-\frac{\lambda}{t-1})^{t-2} \quad \text{for } t \in \{2,3,\ldots\} \text{ and } t-1 > \lambda$$

ii)b) if $0 < \lambda < 2$, then there is a $t'_{\lambda} \in \mathbb{N}_0$ so that:

$$\binom{t}{1}(\frac{\lambda}{t})^{1}(1-\frac{\lambda}{t})^{t-1} < \binom{t-1}{1}(\frac{\lambda}{t-1})^{1}(1-\frac{\lambda}{t-1})^{t-2} \quad \text{for } t \in \{t'_{\lambda}, t'_{\lambda}+1,\ldots\} \text{ and } t-1 > \lambda$$

iii)a) if $0 < \lambda \leq 2-\sqrt{2}$ or $\lambda \geq 2+\sqrt{2}$, then

$$\binom{t}{2}(\frac{\lambda}{t})^{2}(1-\frac{\lambda}{t})^{t-2} > \binom{t-1}{2}(\frac{\lambda}{t-1})^{2}(1-\frac{\lambda}{t-1})^{t-3} \quad \text{for } t \in \{3,4,\ldots\} \text{ and } t-1 > \lambda$$

iii)b) if $2-\sqrt{2} < \lambda < 2+\sqrt{2}$, then there is a $t''_{\lambda} \in \mathbb{N}_0$ so that:

$$\binom{t}{2}(\frac{\lambda}{t})^{2}(1-\frac{\lambda}{t})^{t-2} < \binom{t-1}{2}(\frac{\lambda}{t-1})^{2}(1-\frac{\lambda}{t-1})^{t-3} \quad \text{for } t \in \{t''_{\lambda}, t''_{\lambda}+1,\ldots\} \text{ and } t-1 > \lambda.$$

<u>PROOF</u>. i) $(1-\frac{\lambda}{t})^{t}$ is an increasing function of t for all $\lambda \in \mathbb{R}^{+}$, with $t > \lambda$.

ii)

$$\log\binom{t}{1}(\frac{\lambda}{t})^{1}(1-\frac{\lambda}{t})^{t-1} = \log\lambda + (t-1)\log(1-\frac{\lambda}{t})$$

$$= \log\lambda - \lambda + \sum_{r=1}^{\infty}(\frac{1}{t})^{r}\{\frac{\lambda^{r}}{r} - \frac{\lambda^{r+1}}{r+1}\} \quad \text{with } t > \lambda.$$

a) if $\lambda \geq 2$, then $\{\frac{\lambda^r}{r} - \frac{\lambda^{r+1}}{r+1}\} \leq 0$ for all $r \in \{1,2,...\}$ and thus

$\binom{t}{1}(\frac{\lambda}{t})(1-\frac{\lambda}{t})^{t-1}$ increases as function of t if $t > \lambda$.

b) if $0 < \lambda < 2$, then the coefficient of $(\frac{1}{t})^1$ - i.e. $\lambda - \frac{1}{2}\lambda^2$, is positive and therefore when you have a t, say t'_λ, such that

$$\binom{t}{1}(\frac{\lambda}{t})^1(1-\frac{\lambda}{t})^{t-1} < \binom{t-1}{1}(\frac{\lambda}{t-1})^1(1-\frac{\lambda}{t-1})^{t-2}$$

then it is true for all $t \geq t_\lambda^2$

iii) $$r(2;t) := \frac{\binom{t}{2}(\frac{\lambda}{t})^2(1-\frac{\lambda}{t})^{t-2}}{\binom{t-1}{2}(\frac{\lambda}{t-1})^2(1-\frac{\lambda}{t-1})^{t-3}}$$ for $t \in \{3,4,...\}$ and $t-1 > \lambda$

decreases for $0 < \lambda < 2$ and increases for $\lambda > 2$.

a) we are ready when we prove that $\log(\binom{t}{2}(\frac{\lambda}{t})^2(1-\frac{\lambda}{t})^{t-2})$ is an increasing function of t at the values: $\lambda = 2 - \sqrt{2}$ and $\lambda = 2 + \sqrt{2}$.

well $\log\binom{t}{2}(\frac{\lambda}{t})^2(1-\frac{\lambda}{t})^{t-2} = \log \lambda^2 - \log 2 + \log(1-\frac{\lambda}{t}) + (t-2)\log(1-\frac{\lambda}{t})$

$$= -\sum_{r=1}^{\infty}(\frac{1}{t})^r\{\frac{1}{r} + \frac{\lambda^{r+1}}{r+1} - \frac{2\lambda^r}{r}\} - \lambda + \log \lambda^2 - \log 2$$

we show it for $\lambda = 2 + \sqrt{2}$: It is enough to proof that for all $r \in \{1,2,...\}$: $\{\frac{1}{r} + \frac{\lambda^{r+1}}{r+1} - \frac{2\lambda^r}{r}\} \geq 0$.

Substitute $\lambda = 2 + \sqrt{2}$, we get:

$$\frac{1}{r(r+1)} \cdot \{(r+1) + r(2+\sqrt{2})^{r+1} - 2(r+1)(2+\sqrt{2})^r\}$$

$$= \frac{1}{r(r+1)}(2+\sqrt{2})^r\{(r+1)(1-\frac{1}{\sqrt{2}})^r + \sqrt{2}.r - 2\}$$

$$\geq \frac{1}{r(r+1)} 2.(2+\sqrt{2})^r\{(1-\frac{1}{\sqrt{2}})^r(1+\tfrac{1}{4}(r-1)) + (\frac{r}{\sqrt{2}}-1)\}$$ because $\frac{1}{2}(r+1) \geq 1+\frac{1}{4}(r-1)$

$$\geq \frac{1}{r(r+1)} 2.(2+\sqrt{2})^r\{(1-\frac{1}{\sqrt{2}})^r + \tfrac{1}{4}(r-1).(1-\frac{1}{\sqrt{2}})^r + \frac{r}{\sqrt{2}} - 1\}$$

$$\geq \frac{1}{r(r+1)}.2.(2+\sqrt{2})^r\{\tfrac{1}{4}(r-1)(1-\frac{1}{\sqrt{2}})^r\}$$ because $(1-\frac{1}{\sqrt{2}})^r \geq 1 - \frac{r}{\sqrt{2}}$

≥ 0

b) if $2-\sqrt{2} < \lambda < 2+\sqrt{2}$ then the coefficient of $(\frac{1}{t})^1$, i.e. $-1 - \frac{1}{2}\lambda^2+2\lambda$ is positive and with this we have the last part of the lemma.

Consequence:

COROLLARY 1. If $\lambda_0 > 0.65$ and $\lambda_0 < 1$, then

$$\binom{t}{2}\left(\frac{\lambda_0}{t}\right)^2\left(1-\frac{\lambda_0}{t}\right)^{t-2} \geq \tfrac{1}{2}\lambda_0^2\, e^{-\lambda_0} \quad \text{for all } t \geq 3.$$

PROOF. From (iii)b) from Lemma A.S. we can calculate the collection:

$$\{\lambda_0 \in (2-\sqrt{2},\ 2+\sqrt{2}) \mid t''_{\lambda_0} = 4\} = (0.69,\ 2.56).$$

Thus if

$$\lambda_0 \in (0.69, 2.56) \quad \text{then} \quad \binom{t}{2}\left(\frac{\lambda_0}{t}\right)^2\left(1-\frac{\lambda_0}{t}\right)^{t-1}$$

is a decreasing series in t with $t \geq 3$, with limit: $\frac{1}{2}\lambda_0^2\, e^{-\lambda_0}$.

If $\lambda_0 \in (0.65, 0.69)$, then

$$\binom{t}{2}\left(\frac{\lambda_0}{t}\right)\left(1-\frac{\lambda_0}{t}\right)^{t-2}$$

increases as function of t on $[3,t''_\lambda)$ and decreases on $[t''_\lambda,\infty)$ and so for $t \in [3,t''_{\lambda_0})$:

$$\binom{t}{2}\left(\frac{\lambda_0}{t}\right)\left(1-\frac{\lambda_0}{t}\right)^{t-2} \geq 3.\ \frac{\lambda_0^2}{y}(1-\tfrac{1}{2}\lambda_0) \geq \tfrac{1}{2}\lambda_0^2\, e^{-\lambda_0}$$

and for $t \in [t''_{\lambda_0},\infty)$:

$$\binom{t}{2}\left(\frac{\lambda_0}{t}\right)^2\left(1-\frac{\lambda_0}{t}\right)^{t-2} \geq \tfrac{1}{2}\lambda_0^2 e^{-\lambda_0}.$$

LEMMA 1. *If* $\lambda \geq e$, *then* ${}^m f_{0,1}(\tilde{q})$ *attains its maximum value, when* $\tilde{q} = (q_1,\ldots,q_m)$ *consists of all equal components. i.e. when* $q_i = \lambda/m$, $i = 1,\ldots,m$.

PROOF. Suppose $\tilde{q} = (q_1,\dots,q_m)$ maximizes ${}^{m}f_{0,1}(\tilde{q})$ and therefore (Theorem 1) consists of r zero-components, s one-components and $t = m-r-s$ components with value $a = \frac{\lambda-s}{t}$, with $0 < a < 1$, and thus $0 < \lambda-s < t$.

We first show that $t > 0$.

Suppose $t=0$. Then it must be true that $\lambda = [\lambda] = s$ and $r = m-\lambda$, thus $s \geq 2$ and ${}^{m}f_{0,1}(\tilde{q}) = (1-\rho)^{\lambda}$.

But this is not the maximum value, for if we take $r = s = 0$, $t = m$ then ${}^{m}f_{0,1}(\tilde{q}) = (1-\frac{\rho\lambda}{m})^m + \rho\lambda(1-\frac{\lambda}{m})^{m-1} > (1-\frac{\rho\lambda}{m})^m > (1-\rho)^{\lambda}$.

So $t > 0$.

${}^{m}f_{0,1}(\tilde{q})$ is (see 2.2.4):

$$s = 0: \quad (1-\frac{\rho\lambda}{t})^{t} + \rho\lambda(1-\frac{\lambda}{t})^{t-1}$$

$$s = 1: \quad (1-\frac{\rho(\lambda-1)}{t})^{t}(1-\rho) + \rho(1-\frac{\lambda-1}{t})^{t}$$

$$\lambda > s \geq 2: \quad (1-\frac{\rho(\lambda-s)}{t})^{t}(1-\rho)^{s}$$

with

$$r+s+t = m \quad \text{and} \quad 0 < \lambda-s < t.$$

First we prove that $r=0$ by showing that for every fixed $s \in [0,\lambda) \cap \mathbb{N}_0$, the series, written down above, are increasing in t.

By applying lemma A.5 we see, that if $\lambda \geq 2$ all series increase in t. Thus $r=0$, and $m = s+t$, which gives us the following formulas:

i) $$s = 0: \quad (1-\frac{\rho\lambda}{m})^{m} + \rho\lambda(1-\frac{\lambda}{m})^{m-1}$$

ii) $$s = 1: \quad (1-\frac{\rho(\lambda-1)}{m-1})^{m-1}(1-\rho) + \rho(1-\frac{\lambda-1}{m-1})^{m-1}$$

iii) $$\lambda > s \geq 2: \quad (1-\frac{\rho(\lambda-s)}{m-s})^{m-s}(1-\rho)^{s} \quad \text{with} \quad 0 \leq s < \lambda < m.$$

To prove that $s = 0$, it is sufficient to prove (iii) $\leq$ (i) and (ii) $\leq$ (i).

If $\lambda \geq e$ then

$$\rho(1-\frac{\lambda-1}{m-1})^{m-1} < \rho\lambda(1-\frac{\lambda}{m})^{m-1}.$$

Furthermore

$$(1-\frac{\rho(\lambda-s)}{m-s})^{m-s}(1-\rho)^s$$

decreases as function of s on $[0,\lambda)$.
Thus $s = 0$, and $t = m$, and ${}^m f_{0,1}(\tilde{q})$ attains its maximum value when $\tilde{q} = (\lambda/m,\ldots,\lambda/m)$, and

$${}^m f_{0,1}(\lambda/m,\ldots,\lambda/m) = (1-\frac{\rho\lambda}{m})^m + \rho\lambda(1-\frac{\lambda}{m})^{m-1} .$$

<u>LEMMA 2</u>. *If* $\lambda \geq e + \sqrt{e(e-2)}$, *then* ${}^m f_{0,2}(\tilde{q})$ *attains its maximum value when* $\tilde{q} = (q_1,\ldots,q_m)$ *consists of all equal components.*
i.e. when $q_i = \lambda/m \quad i = 1,\ldots,m$.

<u>PROOF</u>. This proof is analogous to the previous one. Again we consider a $\tilde{q}$ that maximizes ${}^m f_{0,2}(\tilde{q})$, consisting of r zero-components. s one-components and $t = m-r-s$ components with value $a = \frac{\lambda-s}{t}$ $(0<a<1)$. From the fact that $\lambda \geq 3$, we see that $t > 0$. We now give the probabilities under each possible value of r,s and t, referring to (2.2.4) for the way in which they are found. ${}^m f_{0,2}(\tilde{q})$ is

$$s = 0:\ (1-\frac{\rho\lambda}{t})^t + \rho\lambda(1-\frac{\lambda}{t})^{t-1} + \rho(2-\rho)\binom{t}{2}(\frac{\lambda}{t})^2(1-\frac{\lambda}{t})^{t-2}$$

$$s = 1:\ (1-\frac{\rho(\lambda-1)}{t})^t(1-\rho) + \rho(1-\frac{\lambda-1}{t})^t + \rho(2-\rho)(\lambda-1)(1-\frac{\lambda-1}{t})^{t-1}$$

$$s = 2:\ (1-\frac{\rho(\lambda-2)}{t})^t(1-\rho)^2 + \rho(2-\rho)\ (1-\frac{\lambda-2}{t})^t$$

$$\lambda > s \geq 3:\ (1-\frac{\rho(\lambda-s)}{t})^t(1-\rho)^s$$

with

$$m = r+s+t \quad \text{and} \quad 0 < \lambda-s < t.$$

By applying lemma A.S. we see that the series increase in t if $\lambda \geq 2+\sqrt{2}$. Thus if $\lambda \geq 2+\sqrt{2}$, then $r = 0$ and $m = s+t$, which gives us for ${}^m f_{0,2}(\tilde{q})$:

(i) $s = 0$: $(1-\frac{\rho\lambda}{m})^{m} + \rho\lambda(1-\frac{\lambda}{m})^{m-1} + \rho(2-\rho)\ \binom{m}{2}(\frac{\lambda}{m})^{2}(1-\frac{\lambda}{m})^{m-2}$

(ii) $s = 1$: $(1-\frac{\rho(\lambda-1)}{m-1})^{m-1}(1-\rho) + \rho(1-\frac{\lambda-1}{m-1})^{m-1} + \rho(2-\rho)\ (\lambda-1)(1-\frac{\lambda-1}{m-1})^{m-2}$

(iii) $s = 2$: $(1-\frac{\rho(\lambda-2)}{m-2})^{m-2}(1-\rho)^{2} + \rho(2-\rho)\ (1-\frac{\lambda-2}{m-2})^{m-2}$

(iv) $\lambda > s \geq 3$: $(1-\frac{\rho(\lambda-s)}{m-s})^{m-s}(1-\rho)^{s}$ (with $0 \leq s < \lambda < m$).

To prove that $s = 0$, it suffices to prove that (iv) $\leq$ (i), (iii) $\leq$ (i), and (ii) $\leq$ (i). The first and the second term in the cases $s = 0$ and $s = 1$ have already been treated in the previous lemma with the result : $\lambda \geq e$.
Furthermore if

$$\lambda \geq e + \sqrt{e(e-2)} \text{ then } (\lambda-1)(1-\frac{\lambda-1}{m-1})^{m-2} \geq \binom{m}{2}(\frac{\lambda}{m})^{2}(1-\frac{\lambda}{m})^{m-2}$$

for all $m \geq 5$.
Also if

$$\lambda \geq \sqrt{2}.e \text{ then } (1-\frac{\lambda-2}{m-2})^{m-2} \leq \binom{m}{2}(\frac{\lambda}{m})^{2}(1-\frac{\lambda}{m})^{m-2}$$

for all $m \geq 4$.
Thus if $\lambda \geq e + \sqrt{e(e-2)}$ we have $s = 0$, and thus $m = t$, which means that ${}^{m}f_{0,2}(\tilde{q})$ attains its maximum value when $\tilde{q} = (\lambda/m,\dots,\lambda/m)$ and

$$^{m}f_{0,2}(\lambda/m,\dots,\lambda/m) =$$

$$(1-\frac{\rho\lambda}{m})^{m} + \rho\lambda(1-\frac{\lambda}{m})^{m-1} + \rho(2-\rho)\tfrac{1}{2}\lambda^{2}(\frac{m-1}{m})(1-\frac{\lambda}{m})^{m-2} .$$

<u>LEMMA 3</u>. *Let* $\rho \in (0,1)$ *if* $\lambda \geq e+\sqrt{e(e-2)}$ *and* $\lambda \geq 1 + 1/\rho$ *and*

$$\lambda \geq -1/\rho - \frac{\rho}{(1-\rho)(\rho+\log(1-\rho))} ,$$

then

(L.3.1) $\quad {}^{m}f_{1,2}(\tilde{q}) \le (1+\rho\lambda)e^{-\rho\lambda} + \frac{1}{2}(\rho\lambda)^2 e^{-\lambda}$

for all

$$\tilde{q} = (q_1,\dots,q_m) \text{ with } \sum_{i=1}^{m} q_i = \lambda.$$

PROOF. Let $\lambda \in \mathbb{R}$ be such that the conditions in the lemma are satisfied. Let $\tilde{q} = (q_1,\dots,q_m)$ be the vector that maximizes ${}^{m}f_{0,2}(\tilde{q})$ and therefore consists of r zero-components, s one-components and $t = m-r-s$ components with value $a = \frac{\lambda-s}{t}$, $(0<a<1)$.
From the fact $\lambda-1 \ge 2$, we have that $t > 0$.
${}^{m}f_{1,2}(\tilde{q})$ is:

$$s = 0:\ (1-\frac{\rho\lambda}{t})^{t} + \rho\lambda(1-\frac{\rho\lambda}{t})^{t-1} + \rho^2\binom{t}{2}(\frac{\lambda}{t})^2(1-\frac{\lambda}{t})^{t-2}$$

$$s = 1:\ (1-\frac{\rho(\lambda-1)}{t})^{t}(1-\rho) + \rho.(1-\frac{\rho(\lambda-1)}{t})^{t} +$$
$$+ \rho(1-\rho)(\lambda-1)(1-\frac{\rho(\lambda-1)}{t})^{t-1} + \rho^2(\lambda-1)(1-\frac{\lambda-1}{t})^{t-1}$$

$$s = 2:\ (1-\frac{\rho(\lambda-2)}{t})^{t}(1-\rho)^2 + 2\rho(1-\rho)(1-\frac{\rho(\lambda-2)}{t})^{t} +$$
$$+ \rho(1-\rho)^2(\lambda-2)(1-\frac{\rho(\lambda-2)}{t})^{t-1} + \rho^2(1-\frac{\lambda-2}{t})^{t}$$

$$\lambda > s \ge 3:\ (1-\frac{\rho(\lambda-s)}{t})^{t}(1-\rho)^{s} + s\rho(1-\rho)^{s-1}(1-\frac{\rho(\lambda-s)}{t})^{t}$$
$$+ \rho(1-\rho)^{s}(\lambda-s)(1-\frac{\rho(\lambda-s)}{t})^{t-1}$$

with

$$m = r+s+t,\quad 0 < \lambda-s < t.$$

We will prove (L.3.1) by showing that for every s, the series increase as functions of t. Next we calculate for every s the limit of the series for $t \to \infty$ and then we show that these limits are less than or equal to $(1+\rho\lambda)e^{-\rho\lambda} + \frac{1}{2}(\rho\lambda)^2 e^{-\lambda}$.

With lemma A.S.we see that the series increase in t, when we only look at the last terms in the cases s = 0,1,2 if $\lambda \geq 2 + \sqrt{2}$. For the other terms, with lemma 3.a., we see that, if $\lambda \geq 1 + 1/\rho$, the series increase in t.

Thus for every s, the series increase in t. Their limits are:

$$s = 0 \quad e^{-\rho\lambda} + \rho\lambda e^{-\rho\lambda} + \tfrac{1}{2}(\rho\lambda)^2 e^{-\lambda}$$

$$s = 1 \quad e^{-\rho(\lambda-1)}(1-\rho) + \rho e^{-\rho(\lambda-1)} + \rho(1-\rho)(\lambda-1)e^{-\rho(\lambda-1)} + \\ + \rho^2(\lambda-1)e^{-(\lambda-1)}$$

$$s = 2 \quad e^{-\rho(\lambda-2)}(1-\rho)^2 + 2\rho(1-\rho)e^{-\rho(\lambda-2)} + \rho(1-\rho)^2(\lambda-2)e^{-\rho(\lambda-2)} + \\ + \rho^2 e^{-(\lambda-2)}$$

$$\lambda > s \geq 3 \quad e^{-\rho(\lambda-s)}(1-\rho)^s + s\rho(1-\rho)^{s-1}e^{-\rho(\lambda-s)} + \\ + (1-\rho)^s\rho(\lambda-s)e^{-\rho(\lambda-s)}.$$

First, we consider the last terms in the cases s = 0,1,2 and notice that if $\lambda \geq e + \sqrt{e(e-2)}$ these terms are less than or equal to $\tfrac{1}{2}(\rho\lambda)^2 e^{-\lambda}$.
Lemma 3.b shows us that the last terms treated as "one group" are less than or equal to the "group" for s = 0, if

$$\lambda \geq -\ 1/\rho - \frac{\rho}{(1-\rho)(\rho+\log(1-\rho))}\ .$$

Thus the limits are all less than or equal to the limit in the case s=0, i.e.

$$(1+\rho\lambda)e^{-\rho\lambda} + \tfrac{1}{2}(\rho\lambda)^2 e^{-\lambda}.$$

LEMMA 3.a. *Let* $\rho \in (0,1)$, *if* $\lambda \geq 1 + 1/\rho$ *then for every* $s \in [0,\lambda) \cap \mathbb{N}_0$ *we have*:

$$\text{(L.3a.1)} \qquad (1-\rho)^s\left(1-\frac{\rho(\lambda-s)}{t}\right)^t + s\rho(1-\rho)^{s-1}\left(1-\frac{\rho(\lambda-s)}{t}\right)^t +$$

$$+ (1-\rho)^s \rho(\lambda-s)(1-\frac{\rho(\lambda-s)}{t})^{t-1}$$

increases in t, *with* $t > \lambda-s$.

PROOF. Let $\lambda \in \mathbb{R}$, with $\lambda \geq 1 + 1/\rho$ and $s \in \mathbb{N}_0$ with $0 \leq s < \lambda$. The derivative of (L.3a.1) with respect to t is

$$\text{(L.3a.2)} \qquad (1-\frac{\rho(\lambda-s)}{t})^{t-1}.[((1-\rho)^s+s\rho(1-\rho)^{s-1})$$

$$(1-\frac{\rho(\lambda-s)}{t}).\{\log(1-\frac{\rho(\lambda-s)}{t}) + \frac{\rho(\lambda-s)}{t-\rho(\lambda-s)}\}$$

$$+ (1-\rho)^s\rho(\lambda-s).\{\log(1-\frac{\rho(\lambda-s)}{t}) + (\frac{t-1}{t})(\frac{\rho(\lambda-s)}{t-\rho(\lambda-s)})\}].$$

We will prove that for every $\alpha \in (0,1)$, $\frac{\lambda-s}{\alpha}$ substituted in (L.3a.2) for t gives a positive result. It is clear that we are ready then. Let $a \in (0,1)$ we then get the inequality

$$(1-\rho\alpha)^{(\frac{\lambda-s}{\alpha})-1}.(1-\rho)^{s-1}[(1-\rho+s\rho)(1-\rho\alpha)\{\log(1-\rho\alpha) + \frac{\rho\alpha}{1-\rho\alpha}\} +$$

$$+ (1-\rho)\rho(\lambda-s)\{\log(1-\rho\alpha) + (\frac{\lambda-s-\alpha}{\lambda-s})(\frac{\rho\alpha}{1-\rho\alpha})\}] \geq 0$$

If we isolate λ, we get the inequality:

$$\lambda\rho(1-\rho)\{\log(1-\rho\alpha) + \frac{\rho\alpha}{1-\rho\alpha}\} \geq s.\rho(1-\rho)\{\log(1-\rho\alpha) + \frac{\rho\alpha}{1-\rho\alpha}\} +$$

$$+ \alpha\rho(1-\rho)(\frac{\rho\alpha}{1-\rho\alpha}) +$$

$$+ (\rho-1-s\rho)(1-\rho\alpha)\{\log(1-\rho\alpha) + \frac{\rho\alpha}{1-\rho\alpha}\}.$$

This gives us:

$$\text{(L.3a.3)} \qquad \lambda \geq s + \frac{\rho\alpha^2}{\rho\alpha+(1-\rho\alpha)\log(1-\rho\alpha)} + (\frac{\rho-1-s\rho}{\rho(1-\rho)}).(1-\rho\alpha).$$

Notice that the last term is an increasing function of α on $(0,1)$, so:

$$\left(\frac{\rho-1-s\rho}{\rho(1-\rho)}\right)(1-\rho\alpha) < \frac{\rho-1-s\rho}{\rho} .$$

The second term is a decreasing function of α on $(0,1)$ so:

$$\frac{\rho\alpha^2}{\rho\alpha+(1-\rho\alpha)\log(1-\rho\alpha)} < \frac{2}{\rho} \quad \text{(apply the theorem of l'Hôpital twice).}$$

Thus if $\lambda \geq s + \frac{2}{\rho} + \frac{\rho-1-s\rho}{\rho} = 1 + \frac{1}{\rho}$, then equation (L.3a.3) is satisfied and we are ready.

<u>LEMMA 3.b</u>. *Let* $\rho \in (0,1)$. *If*

$$\lambda \geq -1/\rho - \frac{\rho}{(1-\rho)(\rho+\log(1-\rho))} ,$$

then

(L.3b.1) $$((1-\rho)^s + s\rho(1-\rho)^{s-1} + (1-\rho)^s\rho(\lambda-s))e^{-\rho(\lambda-s)}$$

decreases as function of s *on* $[0,\lambda)$.

LEMMA . *Let* $\lambda \in \mathbb{R}$, *with*

$$\lambda \geq -1/\rho - \frac{\rho}{(1-\rho)\{\rho+\log(1-\rho)\}} .$$

The derivative of (L3b.1) with respect to s is:

(L.3b.2) $$[(1-\rho)^{s-1}e^{-\rho(\lambda-s)}]\,[((1-\rho)(1+\rho\lambda)+\rho^2 s).\{\rho+\log(1-\rho)\}+\rho^2].$$

It is enough to prove that for every $\alpha \in (0,1)$, if we substitute $\lambda\alpha$ for s in (L.3b.2), the result is non-positive.

Let $\alpha \in (0,1)$, we then get the equation:

$$[(1-\rho)^{\lambda\alpha-1}e^{\rho(1-\alpha)\lambda}]\,[((1-\rho)(1+\rho\lambda)+\rho^2\alpha\lambda).\{\rho+\log(1-\rho)\}+\rho^2] \leq 0$$

This gives:

$$\text{(L.3b.3)} \qquad \lambda \geq \frac{-\rho^2 - (1-\rho)\{\rho+\log(1-\rho)\}}{(\rho+(\alpha-1)\rho^2)\{\rho+\log(1-\rho)\}}$$

The last expression is a decreasing function of α on $[0,1)$ so:

$$\frac{-\rho^2-(1-\rho)\{\rho+\log(1-\rho)\}}{(\rho+(\alpha-1)\rho^2)\{\rho+\log(1-\rho)\}} \leq -1/\rho - \frac{\rho}{(1-\rho)\{\rho+\log(1-\rho)\}} .$$

Thus if

$$\lambda \geq -1/\rho - \frac{\rho}{(1-\rho)\{\rho+\log(1-\rho)\}} ,$$

then equation (L.3b.3) is satisfied and we are ready.

LEMMA 1.a. *If* $2 < \lambda < e$ *and* $\rho \in (0,1)$ *and* $\rho > \hat{\rho}$ *(with* $\hat{\rho}$ *the solution of the equation:* $(1-\rho)e^{\rho} + (e-2)e^{-2}\rho e^{2\rho} = 1$*) then*

$${}^{m}f_{0,1}(\tilde{q}) \leq e^{-\rho\lambda} + \rho\lambda e^{-\lambda}$$

for all $\tilde{q} = (q_1,\ldots,q_m)$ *with* $\sum_{i=1}^{m} q_i = \lambda$.

PROOF. Let $\lambda \in (2,e)$ and $\rho > \hat{\rho}$ and $\rho \in (0,1)$. ($\hat{\rho} \approx 0.3$).From the proof of Lemma 1, we know that, because $\lambda > 2$, for every $s \in [0,\lambda) \cap \mathbb{N}_0$ the series of ${}^{m}f_{0,1}(\tilde{q})$ increases in t. We now calculate their limits and show that these are greater than or equal to $e^{-\rho\lambda} + \rho\lambda e^{-\lambda}$. These limits are:

$$s = 0: \; e^{-\rho\lambda} + \rho\lambda e^{-\lambda}$$

$$s = 1: \; (1-\rho)e^{-\rho(\lambda-1)} + \rho e.e^{-\lambda}$$

$$s = 2: \; (1-\rho)^2 e^{-\rho(\lambda-2)}.$$

Now

$$(1-\rho)^2 e^{-\rho(\lambda-2)} = ((1-\rho)e^{\rho})^2 e^{-\rho\lambda} < e^{-\rho\lambda} \text{ for } (1-\rho)e^{\rho} < 1.$$

$$(1-\rho)e^{-\rho(\lambda-1)} + \rho e e^{-\lambda} < e^{-\rho\lambda} + \rho\lambda e^{-\lambda}$$

can be written in the form:

$$1-(1-\rho)e^{\rho}-(e-\lambda)\rho e^{(\rho-1)\lambda} > 0.$$

The expression on the left hand side of the inequality-sign is an increasing function of λ on $(2,e)$. Furthermore $1-(1-\rho)e^{\rho}-(e-2)\rho e^{(\rho-1)2}$ increases on $(0.25,1]$ and

$$1-(1-\hat{\rho})e^{\hat{\rho}}-(e-2)\hat{\rho}e^{(\hat{\rho}-1)2} = 0.$$

Thus

$$1-(1-\rho)e^{\rho}-(e-\lambda)\rho e^{(\rho-1)\lambda} \geq 1-(1-\rho)e^{\rho}-(e-2)\rho e^{(\rho-1)2} > 0 \text{ if } \rho > \hat{\rho}.$$

Thus all limits are less than or equal to $e^{-\rho\lambda} + \rho\lambda e^{-\lambda}$ and

$$^{m}f_{0,1}(\tilde{q}) \leq e^{-\rho\lambda} + \rho\lambda e^{-\lambda}$$

for all

$$\tilde{q} = (q_1,\ldots,q_m) \text{ with } \sum_{i=1}^{m} q_i = \lambda.$$

LEMMA 4. *Let* $\lambda \in (0,1)$. *Let* $m \in \mathbb{N}$, *with* $\lambda \leq m$.
$(r_1=0,r_2=1)$ *If* $\rho \in (0,1)$ *and* $0 < \rho.\lambda \leq 2-\sqrt{2}$ *then for all* $t \in \{1,\ldots,m-1\}$.

$$\left(1-\frac{\rho\lambda}{t}\right)^{t} + \rho\lambda\left(1-\frac{\lambda}{t}\right)^{t-1} > \left(1-\frac{\rho\lambda}{m}\right)^{m} + \rho\lambda\left(1-\frac{\lambda}{m}\right)^{m-1}.$$

PROOF. We distinguish two cases:

1) $0 < \lambda \leq 2-\sqrt{2}$

2) $2-\sqrt{2} < \lambda < 1$.

First we treat case 1):

Let $t \in \{1,\dots,m-1\}$ and $0 < \lambda \le 2-\sqrt{2}$.

Define $f_t\colon [0,1] \to \mathbb{R}$ by

$$(L.4.1) \qquad f_t(\rho) = \left(1-\frac{\rho\lambda}{t}\right)^{t} + \rho\lambda\left(1-\frac{\lambda}{t}\right)^{t-1} - \left(1-\frac{\rho\lambda}{m}\right)^{m} - \rho\lambda\left(1-\frac{\lambda}{m}\right)^{m-1}.$$

Then derivative of $f_t(\rho)$ with respect to ρ is:

$$(L.4.2) \qquad \lambda\left[\left(1-\frac{\rho\lambda}{m}\right)^{m-1} - \left(1-\frac{\rho\lambda}{t}\right)^{t-1} - \left\{\left(1-\frac{\lambda}{m}\right)^{m-1} - \left(1-\frac{\lambda}{t}\right)^{t-1}\right\}\right].$$

Also (L.4.2) = 0 if and only if $\rho = 1$.

This can be proved by looking at:

$$(L.4.3) \qquad \left(1-\frac{\rho\lambda}{m}\right)^{m-1} - \left(1-\frac{\rho\lambda}{t}\right)^{t-1}$$

and showing that if $\rho\lambda < 2-\sqrt{2}$ this is a strictly decreasing function on $[0,1]$ of ρ and thus takes the value

$$\left(1-\frac{\lambda}{m}\right)^{m-1} - \left(1-\frac{\lambda}{t}\right)^{t-1}$$

if and only if $\rho = 1$.

Substituting $\rho = 0$ in (L.4.2) gives a result that is positive and thus $f_t(\rho)$ strictly increases on $[0,1]$ and $f_t(\rho) > 0$ on $[0,1)$.

Now case 2).

Let $t \in \{1,\dots,m-1\}$ and let $2-\sqrt{2} < \lambda < 1$.

Look at the function in (L.4.1) and (L.4.3) on the domain $[0, \frac{2-\sqrt{2}}{\lambda}]$.

Because $\rho\lambda \le 2-\sqrt{2}$ we have that (L.4.3) strictly decreases on $[0, \frac{2-\sqrt{2}}{\lambda}]$ and because

$$(L.4.4) \qquad \left(1-\frac{2-\sqrt{2}}{m}\right)^{m-1} - \left(1-\frac{2-\sqrt{2}}{t}\right)^{t-1} > \left(1-\frac{\lambda}{m}\right)^{m-1} - \left(1-\frac{\lambda}{t}\right)^{t-1}$$

we have that (L.4.2) is positive on $[0, \frac{2-\sqrt{2}}{\lambda}]$ and thus

$$f_t(\rho) > 0 \quad \text{on} \quad [0, \frac{2-\sqrt{2}}{\lambda}].$$

The proof of (L.4.4) can be given by showing that

$$\left(1 - \frac{2-\sqrt{2}}{t}\right)^{t-1} - \left(1 - \frac{\lambda}{t}\right)^{t-1}$$

is an increasing function of t.

LEMMA 5. *Let* $\lambda \in (0,1)$ *and* $\rho \in (0,1)$, *then for all* $t \in \{1,2,...\}$:

$$\left(1 - \frac{\rho\lambda}{t}\right)^{t} + \rho\lambda\left(1 - \frac{\lambda}{t}\right)^{t-1} + \rho(2-\rho)\binom{t}{2}\left(\frac{\lambda}{t}\right)^{2}\left(1 - \frac{\lambda}{t}\right)^{t-2}$$
$$\geq e^{-\rho\lambda} + \rho\lambda e^{-\lambda} + \rho(2-\rho).\tfrac{1}{2}\lambda^{2}e^{-\lambda}.$$

PROOF. The cases $t = 1$ and $t = 2$ are clear, noting that

$$e^{-\rho\lambda} + \rho\lambda e^{-\lambda} + \rho(2-\rho).\tfrac{1}{2}\lambda^{2}e^{-\lambda} \text{ is a decreasing function of } \rho.$$

Suppose now $t \geq 3$. Define the function $g : [0,1] \to \mathbb{R}$ by:

$$g(\rho) = \left(1 - \frac{\rho\lambda}{t}\right)^{t} + \rho\lambda\left(1 - \frac{\lambda}{t}\right)^{t-1} + \rho(2-\rho)\binom{t}{2}\left(\frac{\lambda}{t}\right)^{2}\left(1 - \frac{\lambda}{t}\right)^{t-2} - e^{-\rho\lambda} - \rho\lambda e^{-\lambda} - \rho(2-\rho).\tfrac{1}{2}\lambda^{2}e^{-\lambda}$$

Notice that $g(0) = 0$. We calculate the first, second and third derivative of g with respect to ρ and see that $g'(1) = 0$, $g''(1) = 0$, $g'''(0) > 0$ and

$$g'''(\rho) = \frac{6}{\rho^{3}}\left[\frac{(\rho\lambda)^{3}}{3!}e^{-\rho\lambda} - \binom{t}{3}\left(\frac{\rho\lambda}{t}\right)\left(1 - \frac{\rho\lambda}{t}\right)^{t-3}\right] \text{ with } \rho \in (0,1].$$

From lemma A.S. we see that $g'''(\rho) > 0$ for all $\rho \in [0,1]$, (because $\rho\lambda \leq 1 \leq 3-\sqrt{3}$). With this we can conclude that $g(\rho) > 0$ for all $\rho \in (0,1)$.

LEMMA 6. *Let* $\lambda \in (0,1)$ *and* $\rho \in (0,1)$, *then for all* $t \in \{1,2,3,...\}$:

$$(1-\frac{\rho\lambda}{t})^{t} + \rho\lambda(1-\frac{\rho\lambda}{t})^{t-1} + \rho^2\binom{t}{2}(\frac{\lambda}{t})^2(1-\frac{\lambda}{t})^{t-2}$$

$$\geq e^{-\rho\lambda} + \rho\lambda . e^{-\rho\lambda} + \tfrac{1}{2}\rho^2\lambda^2 e^{-\lambda}.$$

PROOF. The cases $t = 1$ and $t = 2$ are clear. Suppose now $t \geq 3$.

Define the function $f : [0,1] \to \mathbb{R}$ by

$$f(\rho) = (1-\frac{\rho\lambda}{t})^{t} + \rho\lambda(1-\frac{\rho\lambda}{t})^{t-1} + \rho^2\binom{t}{2}(\frac{\lambda}{t})^2(1-\frac{\lambda}{t})^{t-2} - e^{-\rho\lambda} - \rho\lambda e^{-\rho\lambda} - \tfrac{1}{2}\rho^2\lambda^2 e^{-\lambda}.$$

Notice $f(0) = 0$. We calculate the first derivative of f with respect to ρ and see that: $f'(\rho) = 2\rho[h(1)-h(\rho)]$ with the function $h : [0,1] \to \mathbb{R}$ defined by

$$h(\rho) = \frac{3}{3}[\frac{(\rho\lambda)^3}{3!}e^{-\rho\lambda} - \binom{t}{3}(\frac{\rho\lambda}{t})^3(1-\frac{\rho\lambda}{t})^{t-3}]$$

From lemma A.S. we see that (because $\rho\lambda \leq 1 \leq 3-\sqrt{3}$)

$h'(\rho) > 0$ for all $\rho \in [0,1]$. So $f(\rho) > 0$ for all $\rho \in (0,1)$.

3. THREE STAGE SAMPLING

3.1. Introduction

In chapter 2 we have discussed in detail two-stage sampling. The obtained results are that under some conditions one can evaluate the sample obtained by using the sieve method, in the same way as a sample that was obtained using the dollar-unit method.

Of course one wonders if this could be extended to multi-stage sampling and even to sequential sampling.

For the reason that in two-stage sampling already no general conditions, that is conditions in terms of r_1 and r_2, can be obtained for the valid application of the sieve method, we will not try to do so in multi-stage sampling in general. However, to form a picture of the possibilities in this field, we will consider a form of multi-stage sampling namely three-stage sampling.

We will treat three-stage sampling much the same as we treated two-stage sampling but with less comments. Furthermore we will elaborate on one example of three-stage sampling with the dollar-unit method and the sieve method.

3.2. The testing problem

We will consider for both sampling methods the same testing problem as in chapter 2.

H_0: $p \geq p_1$

H_1: $p \leq p_0$ (with $p_0 < p_1$) with confidence level $1-\beta$.

We will use for both sampling methods the following test: accept the population if and only if
"in the first sample with size n_1 less than or equal to r_1 errors are found *or* in the first sample more than r_1 but in the once extended sample with size (n_1+n_2) less than or equal to r_2 errors are found *or* in the first sample more than r_1 and in the once extended sample more than r_2 but in the twice extended sample with size $(n_1+n_2+n_3)$ less than or equal to r_3 errors are found".
(n_i is the nominal i'th sample size).

3.3. True random sampling

When we sample, using the dollar-unit method the chances of acceptance and rejection are easy to calculate.
We will give these chances only for one case of three-stage sampling, namely the case in which $r_1 = 0$, $r_2 = 1$, $r_3 = 2$.
The acceptance-criterion becomes (see paragraph 3.2):
"in the first sample (n_1) 0 errors *or* in the first sample (n_1) 1 error and in the second (n_2) 0 errors *or* in the first sample (n_1) 1 error and in the second (n_2) 1 error and in the third (n_3) 0 errors *or* in the first sample (n_1) 2 errors and in the second (n_2) 0 errors and in the third (n_3) 0 errors".
The chance of acceptance wrongly is (approximately):

$$(3.3.a) \quad e^{-n_1p_1} + n_1p_1e^{-(n_1+n_2)p_1} + (n_1n_2p_1^2 + \tfrac{1}{2}(n_1p_1)^2)e^{-(n_1+n_2+n_3)p_1}.$$

Thus the sample sizes must be such that (3.3.a) equalizes β. To diminish the number of possible combinations of sample sizes we assume:

(3.3.b) $$n_2-n_1 = n_3-n_2.$$

The resulting simple scheme can be considered as a prerunner of sequential sampling.
The chance of rejection wrongly is:

(3.3.c) $$1-\{e^{-n_1p_0} + n_1p_0e^{-(n_1+n_2)p_0} + (\tfrac{1}{2}(n_1p_0)^2 + n_1n_2p_0{}^2)e^{-(n_1+n_2+n_3)p_0}\}.$$

We define $\alpha = \alpha(n_1,n_2,n_3)$ as the result of (3.3.c). Thus for each combination (n_1,n_2,n_3) we have a different α.

3.4. Sieve sampling

3.4.1. The sieve method

To achieve that if an error is not found in the first sample, a chance is present of finding it in the once extended sample and that if an error is not found in the once extended sample, a chance is present of finding it in the twice extended sample, we adapt the sieve method to three-stage sampling in the same way as we did for two-stage sampling. Thus if z_i is the item-sieve in the first sample of item i, then the item-sieve in the once extended sample is

$$\left(\frac{n_1}{n_1+n_2}\right) \cdot z_i$$

and the item-sieve in the twice extended sample is

$$\left(\frac{n_1}{n_1+n_2+n_3}\right) \cdot z_i.$$

Then there are four different possibilities of finding errors or not, corresponding with four different item-sieves. We will illustrate one of them but for each calculate the corresponding probabilities. We need some new notations (for the rest see 2.2.4):

C_3 effective cell size in the twice extended sample

$q_i = \frac{(n_1+n_2+n_3)e_i}{N} \quad i = 1,\ldots,m$

$\rho_1 = \frac{n_1}{n_1+n_2+n_3}$

$\rho_2 = \frac{n_2}{n_1+n_2+n_3}$

$\lambda = \sum_{i=1}^{m} q_i = (n_1+n_2+n_3)p$ expected number of errors found in the twice extended sample

$\lambda_1 = (n_1+n_2+n_3)p_1$

$\lambda_0 = (n_1+n_2+n_3)p_0$

We suppose: $0 < \rho_1 < 1$, $0 < \rho_2 < 1$, and $0 \leq e_i \leq a_i \leq C_3 \leq C_2 \leq C_1$ for $i = 1,\ldots,m$.

So all items are smaller than the effective cell size in the twice extended sample.

In the sieve method, in each item i, independently of one another, we have the following four possibilities:

1- an error is found in the first sample with size n_1

with probability $\frac{e_i}{N/(n_1)} = \rho_1 \cdot q_i$

2- an error is found in the once extended sample with size (n_1+n_2) but <u>not</u> in the first sample;

with probability $\frac{e_i}{N/(n_1+n_2)} - \frac{e_i}{N/n_1} = \rho_2 \cdot q_i$

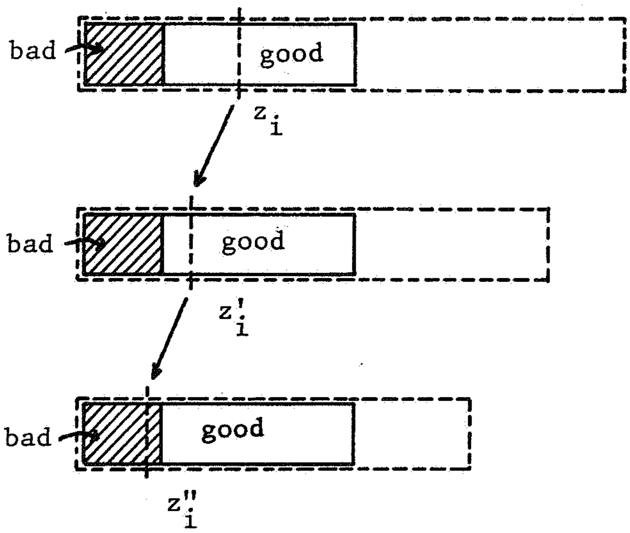
3- an error is found in the twice extended sample with size $(n_1+n_2+n_3)$ but not in the once extended sample;

with probability $\frac{e_i}{N/(n_1+n_2+n_3)} - \frac{e_i}{N/(n_1+n_2)} = (1-\rho_1-\rho_2)q_i$

4- no error is found in the twice extended sample;

with probability $1 - \frac{e_i}{N/(n_1+n_2+n_3)} = 1-q_i$

As an example we will illustrate the third possibility (see figure 6).



size of the cell: $N/(n_1)$

size of the cell: $N/(n_1+n_2)$

$$z_i' = \left(\frac{n_1}{n_1+n_2}\right) \cdot z_i$$

size of the cell: $N/(n_1+n_2+n_3)$

$$z_i'' = \left(\frac{n_1}{n_1+n_2+n_3}\right) \cdot z_i$$

figure 6: an error is found in the twice extended sample with size $(n_1+n_2+n_3)$ but not in the once extended sample with size (n_1+n_2).

3.4.2. The chance of acceptance wrongly

We consider the testing problem and in particular the test of paragraph 3.2. We make the following definitions:

Let $\underline{X}_{1i}$ be the random variable which takes the value 1, if in item i an error is found in the first sample with size n_1, zero otherwise.

Let $\underline{X}_{2i}$ be the random variable which takes the value 1, if in item i an error is found in the once extended sample with size n_1+n_2, zero otherwise.

Let $\underline{X}_{3i}$ be the random variable which takes the value 1, if in item i an error is found in the twice extended sample with size $n_1+n_2+n_3$, zero otherwise. Notice, that if $\underline{X}_{1i} = 1$, then $\underline{X}_{2i} = 1$ and if $\underline{X}_{2i} = 1$, then $\underline{X}_{3i} = 1$ for, $i = 1,\dots,m$.

Thus $\underline{X}_{11},\dots,\underline{X}_{1m}$ are independent Bernoulli variables with

$$P[\underline{X}_{1i} = 1] = 1-P[\underline{X}_{1i} = 0] = \rho_1 q_i, \quad i = 1,\dots,m.$$

$\underline{X}_{21},\dots,\underline{X}_{2m}$ are independent Bernoulli variables with

$$P[\underline{X}_{2i} = 1] = 1-P[\underline{X}_{2i} = 0] = (\rho_1+\rho_2) q_i, \quad i = 1,\dots,m.$$

$\underline{X}_{31},\ldots,\underline{X}_{3m}$ are independent Bernoulli variables with

$$P[\underline{X}_{3i} = 1] = 1-P[\underline{X}_{3i} = 0] = q_i, \quad i = 1,\ldots,m.$$

Let

$$\underline{X}_1 = \sum_{i=1}^{m} \underline{X}_{1i}, \quad \underline{X}_2 = \sum_{i=1}^{m} \underline{X}_{2i}, \quad \underline{X}_3 = \sum_{i=1}^{m} \underline{X}_{3i}.$$

Then we can write the chance of acceptance in the sieve method as:

$$P_r[\underline{X}_1 \le r_1 \text{ or } (\underline{X} > r_1 \text{ and } \underline{X}_2 \le r_2) \text{ or } (\underline{X}_1 > r_1 \text{ and } \underline{X}_2 > r_2 \text{ and }$$
$$\underline{X}_3 \le r_3) \mid q_1,\ldots,q_m] \text{ or } {}^m f_{r_1,r_2,r_3}(q_1,\ldots,q_m).$$

We deduce a theorem that states, corresponding to theorem 1, that the chance of acceptance attains its maximum (minimum) value, if the amount of errors in each item is one of the following three values:

- zero
- maximum, i.e. the effective cell size in the twice extended sample
- a value between zero and the maximum

The statement and the proof are placed at the end of this chapter.
We will, concerning three stage sampling, only treat the case $r_1 = 0$, $r_2 = 1$, $r_3 = 2$. For this case we can, using the theorem mentioned above, derive a condition under which the chance of acceptance wrongly in the sieve method does not exceed β.
This condition is given by lemma 7 which states that if this condition is satisfied, the chance of acceptance in the sieve method is less than or equal to

$$(2.4.2.a) \quad e^{-\rho_1\lambda} + \rho_1\lambda e^{-(\rho_1+\rho_2)\lambda} + (\tfrac{1}{2}\rho_1^2 + \rho_1\rho_2)\lambda^2 e^{-\lambda} \text{ for all } \tilde{q} = (q_1,\ldots,q_m) \text{ with } \sum_{i=1}^{m} q_i = \lambda.$$

Then the chance of acceptance wrongly in the sieve method is less than or equal to:

$$e^{-\rho_1\lambda_1} + \rho_1\lambda_1 e^{-(\rho_1+\rho_2)\lambda_1} + (\tfrac{1}{2}\rho_1^2+\rho_1\rho_2)\lambda_1^2 e^{-\lambda_1} \quad \text{(because } \lambda \geq \lambda_1\text{)}$$

and this is exactly (3.3.a), the chance of acceptance wrongly in the dollar-unit method, and thus equals β.

We made the assumption that $n_2-n_1 = n_3-n_2$, in paragraph 3.3. Translated to paragraph 3.4. this means:

$$\rho_1 = \frac{n_1}{3n_2} \quad \text{and} \quad \rho_2 = 1/3.$$

3.4.3. <u>The chances of rejection wrongly</u>

We only treat the case $r_1 = 0$, $r_2 = 1$, $r_3 = 2$. For this case if we assume that $\lambda_0 \leq 1$, we can calculate the chance of acceptance and thus the chance of rejection.

Lemma 8 gives a condition under which all possible chances of acceptance in the sieve method (if $\lambda \leq 1$) overestimate:

$$e^{-\rho_1\lambda} + \rho_1\lambda e^{-(\rho_1+\rho_2)\lambda} + (\rho_1^2+2\rho_1\rho_2)\tfrac{1}{2}\lambda^2 e^{-\lambda}$$

then:

the chance of rejecting the population

= 1- the chance of accepting the population

$$\leq 1-\{e^{-\rho_1\lambda} + \rho_1\lambda e^{-(\rho_1+\rho_2)\lambda} + (\rho_1^2+2\rho_1\rho_2)\frac{\lambda^2}{2}e^{-\lambda}\}.$$

Then "the chance of rejection wrongly"

$$\leq 1-\{e^{-\rho_1\lambda_0} + \rho_1\lambda_0 e^{-(\rho_1+\rho_2)\lambda_0} + (\rho_1^2+\rho_1\rho_2)\frac{\lambda_0^2}{2}e^{-\lambda_0}\} \quad \text{(because } \lambda \leq \lambda_0\text{)}$$

$$= \alpha(n_1,n_2) \qquad \text{(see 3.3.c.)}$$

Together with the condition of Lemma 7 we can check whether or not the sieve method can be applied and can be evaluated as the dollar-unit method. We will give the procedure again, as a guide for making the tables.

First we choose β, ρ_0 and ρ_1.
Then, (using 3.3.b), we calculate the sample sizes n_1, n_2, n_3 by equalizing (3.3.a) to β. Then for every combination (n_1, n_2, n_3) we calculate the corresponding α, using (3.3.c). Then calculate $\rho_1, \rho_2, \lambda_1$ and λ_0.
Now we can state:

If $\lambda_1 \geq e + \sqrt{e(e-2)}$ and $\lambda_1 \geq \dfrac{(\rho_1+\rho_2)e^{\rho_1+\rho_2}}{1-(1-\rho_1-\rho_2)e^{\rho_1+\rho_2}}$

then the chance of acceptance wrongly in the sieve method is $<= \beta$.

If $\lambda_0 \leq 1$

then the chance of rejection wrongly in the sieve method is $<= \alpha$.

3.4.4. Proofs

THEOREM 2. *For* $r_1, r_2, r_3 \leq \mathbb{N}_0$ *with* $0 \leq r_1 < r_2 < r_3$:

$$ {}^m f_{r_1,r_2,r_3}(\tilde{q}) = {}^m f_{r_1,r_2,r_3}(q_1,\ldots,q_m) $$

attains its maximum (minimum) value if for every $i \in \{1,\ldots,m\}$:

$$ q_i = 0 $$

$$ \text{or } q_i = 1 $$

$$ \text{or } q_i = a \text{ with } 0 < a < 1. $$

PROOF. Consider a fixed value of m and of $\sum_{i=1}^{m} q_i = \lambda$. Take two different items i and j. Split the chance up into, for i and j, the four possibilities of finding errors as written down in paragraph 3.4.1. This gives us:

$$ {}^m f_{r_1,r_2,r_3}(\tilde{q}) = A.q_i q_j + B $$

with

$$A = \rho_1^2 . {}^{m-2}f_{r_1-2,r_2-2,r_3-2}(\tilde{q}_{ij}) + 2\rho_1\rho_2 . {}^{m-2}f_{r_1-1,r_2-2,r_3-2}(\tilde{q}_{ij})$$

$$+ 2\rho_1(1-\rho_1-\rho_2) . {}^{m-2}f_{r_1-1,r_2-1,r_3-2}(\tilde{q}_{ij})$$

$$- 2\rho_1 . {}^{m-2}f_{r_1-1,r_2-1,r_3-1}(\tilde{q}_{ij})$$

$$+ \rho_2^2 . {}^{m-2}f_{r_1,r_2-2,r_3-2}(\tilde{q}_{ij}) + 2\rho_2(1-\rho_1-\rho_2) . {}^{m-2}f_{r_1,r_2-1,r_3-2}(\tilde{q}_{ij})$$

$$-2\rho_2 . {}^{m-2}f_{r_1,r_2-1,r_3-1}(\tilde{q}_{ij}) + (1-\rho_1-\rho_2)^2 . {}^{m-2}f_{r_1,r_2,r_3-2}(\tilde{q}_{ij})$$

$$-2(1-\rho_1-\rho_2) . {}^{m-2}f_{r_1,r_2,r_3-1}(\tilde{q}_{ij}) + {}^{m-2}f_{r_1,r_2,r_3}(\tilde{q}_{ij})$$

and

$$B = (q_i+q_j).\{\rho_1 . {}^{m-2}f_{r_1-1,r_2-1,r_3-1}(\tilde{q}_{ij}) + \rho_2 . {}^{m-2}f_{r_1,r_2-1,r_3-1}(\tilde{q}_{ij})$$

$$+ (1-\rho_1-\rho_2) . {}^{m-2}f_{r_1,r_2,r_3-1}(\tilde{q}_{ij}) - {}^{m-2}f_{r_1,r_2,r_3}(\tilde{q}_{ij})\}$$

$$+ {}^{m-2}f_{r_1,r_2,r_3}(\tilde{q}_{ij}).$$

Following the same argumentation as in the proof of theorem 1, we can conclude that ${}^{m}f_{r_1,r_2,r_3}(\tilde{q})$ attains its maximum (minimum) value by a $\tilde{q}$ whose nonzero and nonone components are all equal.

<u>LEMMA 7</u>. *Let* $\rho_1,\rho_2 \in (0,1)$. *If* $\lambda \geq e + \sqrt{e(e-2)}$ *and*

$$\lambda \geq \frac{(\rho_1+\rho_2)e^{\rho_1+\rho_2}}{1-(1-\rho_1-\rho_2)e^{\rho_1+\rho_2}}$$

then

$${}^{m}f_{0,1,2}(\tilde{q}) \leq e^{-\rho_1\lambda} + \rho_1\lambda e^{-(\rho_1+\rho_2)\lambda} + (\tfrac{1}{2}\rho_1^2+\rho_1\rho_2)\lambda^2 e^{-\lambda}$$

for all $\tilde{q} = (q_1,\ldots,q_m)$ *with* $\sum_{i=1}^{m} q_i = \lambda$

PROOF. Let $\lambda \in \mathbb{N}$ satisfying the conditions stated above. Suppose $\tilde{q}$, that maximizes ${}^{m}f_{0,1,2}(\tilde{q})$, consists of r zero components, s one components and $t = m-r-s$ components with value $a = \frac{\lambda s}{t}$ $(0<a<1)$.

First we show that $t > 0$. If $t = 0$, then $\tilde{q}$ consists of λ one components and $m-\lambda$ zero components. Then ${}^{m}f_{0,1,2}(q) = (1-\rho_1)^{\lambda} + \rho_1\lambda(1-\rho_1-\rho_2)^{\lambda-1}$. But this is not the maximum value for if we take a vector $\tilde{q}$ with 0 one components and m components with value $\frac{\lambda}{m}$ then

$$
\begin{aligned}
{}^{m}f_{0,1,2}(\tilde{q}) &= (1-\rho_1\cdot\tfrac{\lambda}{m})^{m} + \rho_1\lambda(1-(\rho_1+\rho_2)\tfrac{\lambda}{m})^{m-1} + \ldots \\
&> (1-\rho_1)^{\lambda} + \rho_1\lambda(1-(\rho_1+\rho_2))^{\lambda-1} \quad \text{if} \quad (\rho_1+\rho_2)\lambda \geq 2.
\end{aligned}
$$

This last condition is satisfied by λ (see below). Thus $t > 0$.
We split up ${}^{m}f_{0,1,2}(\tilde{q})$ according to the possible values of s. Then ${}^{m}f_{0,1,2}(\tilde{q})$ is:

$$s = 0: \; (1-\rho_1\tfrac{\lambda}{t})^{t} + \rho_1\lambda(1-(\rho_1+\rho_2)\tfrac{\lambda}{t})^{t-1} + (\rho_1^2+2\rho_1\rho_2)\tbinom{t}{2}(\tfrac{\lambda}{t})^{2}(1-\tfrac{\lambda}{t})^{t-2}$$

$$
\begin{aligned}
s = 1: \; & (1-\rho_1(\tfrac{\lambda-1}{t}))^{t}(1-\rho_1) + (1-\rho_1-\rho_2)\rho_1(\lambda-1)(1-(\rho_1+\rho_2)(\tfrac{\lambda-1}{t}))^{t-1} \\
& + \rho_1(1-(\rho_1+\rho_2)(\tfrac{\lambda-1}{t}))^{t} + (\rho_1^2+2\rho_1\rho_2)(\lambda-1)(1-\tfrac{\lambda-1}{t})^{t-1}
\end{aligned}
$$

$$
\begin{aligned}
s = 2: \; & (1-\rho_1(\tfrac{\lambda-2}{t}))^{t}(1-\rho_1)^2 + (1-\rho_1-\rho_2)^2\rho_1(\lambda-2)(1-(\rho_1+\rho_2)(\tfrac{\lambda-2}{t}))^{t-1} \\
& + 2\rho_1(1-\rho_1-\rho_2)(1-(\rho_1+\rho_2)(\tfrac{\lambda-2}{t}))^{t} + (\rho_1^2+2\rho_1\rho_2)(1-\tfrac{\lambda-2}{t})^{t}
\end{aligned}
$$

$$
\begin{aligned}
\lambda > s \geq 3: \; & (1-\rho_1(\tfrac{\lambda-s}{t}))^{t}(1-\rho_1)^{s} + (1-\rho_1-\rho_2)^{s}\rho_1(\lambda-s)(1-(\rho_1+\rho_2)(\tfrac{\lambda-s}{t}))^{t-1} \\
& + s\rho_1(1-\rho_1-\rho_2)^{s-1}(1-(\rho_1+\rho_2)(\tfrac{\lambda-s}{t}))^{t}
\end{aligned}
$$

with $m = r+s+t$, $0 < \lambda-s < t$.
We will prove Lemma 7 by first showing that for every s the series increase in t. Next we calculate for every s the limit of the series for $t \to \infty$ and

then show that all these limits are less than or equal to:

$$e^{-\rho_1\lambda} + \rho_1\lambda\, e^{-(\rho_1+\rho_2)\lambda} + (\tfrac{1}{2}\rho_1^2+\rho_1\rho_2)\lambda^2\, e^{-\lambda}.$$

With Lemma A.S. and Lemma 7.a we see that the series increase in t, if $\lambda \geq 2+\sqrt{2}$ and $(\rho_1+\rho_2)\lambda \geq 2$. These conditions are satisfied, for if $\lambda \geq e+\sqrt{e(e-2)}$ then $\lambda \geq 2+\sqrt{2}$, and if

$$\lambda \geq \frac{(\rho_1+\rho_2)e^{\rho_1+\rho_2}}{1-(1-\rho_1-\rho_2)e^{\rho_1+\rho_2}} \quad \text{then} \quad \lambda \geq \frac{2}{\rho_1+\rho_2}.$$

We now give the limits of the series:

(i) $s = 0$: $e^{-\rho_1\lambda} + \rho_1\lambda\, e^{-(\rho_1+\rho_2)\lambda} + (\rho_1^2+2\rho_1\rho_2)\tfrac{1}{2}\lambda^2 e^{-\lambda}$

(ii) $s = 1$: $(1-\rho_1)e^{-\rho_1(\lambda-1)} + \rho_1((1-\rho_1-\rho_2)(\lambda-1)+1)e^{-(\rho_1+\rho_2)\lambda-1} +$

$$+ (\rho_1^2+2\rho_1\rho_2)(\lambda-1)e^{-(\lambda-1)}$$

(iii) $s = 2$: $(1-\rho_1)^2 e^{-\rho_1(\lambda-2)} + \rho_1((1-\rho_1-\rho_2)^2(\lambda-2)+2(1-\rho_1-\rho_2))e^{-(\rho_1+\rho_2)(\lambda-2)} +$

$$+ (\rho_1^2+2\rho_1\rho_2)e^{-(\lambda-2)}$$

(iv) $\lambda > s \geq 3$: $(1-\rho_1)^s e^{-\rho_1(\lambda-s)} + \rho_1[(1-\rho_1-\rho_2)^s(\lambda-s) +$

$$+ s(1-\rho_1-\rho_2)^{s-1}]e^{-(\rho_1+\rho_2)(\lambda-s)} \quad .$$

To show, that (ii) $\leq$ (i) and (iii) $\leq$ (i) and (iv) $\leq$ (i) we start with the last term of (ii) and (iii) which gives us the condition $\lambda \geq e+\sqrt{e(e-2)}$ and $\lambda \geq \sqrt{2}.e$. These are satisfied. The first terms clearly get smaller when s gets larger. From Lemma 7.b. it follows that the second term decreases when s increases if

$$\lambda \geq \frac{(\rho_1+\rho_2)e^{\rho_1+\rho_2}}{1-(1-\rho_1-\rho_2)e^{\rho_1+\rho_2}} .$$

Thus all limits are smaller then or equal to the one in the case $s = 0$.

LEMMA 7a. *Let* $\rho_1,\rho_2 \in (0,1)$. *If*

$$\lambda \geq \frac{2}{(\rho_1+\rho_2)}\text{, then for every } s \in [0,\lambda) \cap \mathbb{N}_0 .$$

(L.7.a.1) $$\rho_1(1-\rho_1-\rho_2)^s(\lambda-s)(1-(\rho_1+\rho_2)(\frac{\lambda-s}{t}))^{t-1} + $$
$$+ \rho_1 s(1-\rho_1-\rho_2)^{s-1}(1-(\rho_1+\rho_2)(\frac{\lambda-s}{t}))^t$$

increases as function of t, *with* $t > \lambda-s$.

PROOF. Write $\rho = \rho_1+\rho_2$. Let $\lambda \in \mathbb{N}^+$ with $\lambda \geq 2/\rho$. Let $s \in [0,\lambda) \cap \mathbb{N}_0$.
The derivative of (L.7.a.1) to t is

(L.7.a.2) $$\rho_1(1-\rho)^{s-1}(1-\rho(\frac{\lambda-s}{t}))^{t-1}.[\log(1-\rho(\frac{\lambda-s}{t})).\{(1-\rho)(\lambda-s)+s.(1-\frac{\rho(\lambda-s)}{t})\}$$
$$+ (1-\rho)(\lambda-s)(\frac{t-1}{t})(\frac{\rho(\lambda-s)}{t-\rho(\lambda-s)}) + \frac{\rho s(\lambda-s)}{t}].$$

We are ready if we prove that for every $\alpha \in (0,1)$, if we substitute $t = \frac{\lambda-s}{\alpha}$ in (L.7.a.2) the result is non-negative.
Let $\alpha \in (0,1)$, we get the inequality:

$$\log(1-\rho\alpha).\{(1-\rho)(\lambda-s)+s(1-\rho\alpha)\}+ \frac{\rho(1-\rho)(\lambda-s-\alpha)\alpha}{1-\rho\alpha} + s\rho\alpha \geq 0.$$

Isolate λ, we get the inequality:

$$\lambda\{(1-\rho)(1-\rho\alpha)\log(1-\rho\alpha) + (1-\rho)\rho\alpha\} \geq s\{(1-\rho)(1-\rho\alpha)\log(1-\rho\alpha)+(1-\rho)\rho\alpha\}$$
$$-s(1-\rho\alpha)\{(1-\rho\alpha)\log(1-\rho\alpha)+\rho\alpha\} + \rho(1-\rho)\alpha^2$$

this gives us:

$$\lambda \geq s - \frac{s(1-\rho\alpha)}{1-\rho} + \frac{\rho\alpha^2}{\rho\alpha+(1-\rho\alpha)\log(1-\rho\alpha)} .$$

This inequality is satisfied if $\lambda \geq 2/\rho = \frac{2}{(\rho_1+\rho_2)}$.

LEMMA 7.b. *Let* $\rho_1, \rho_2 \in (0,1)$. *If*

$$\lambda \geq \frac{(\rho_1+\rho_2)e^{\rho_1+\rho_2}}{1-(1-\rho_1-\rho_2)e^{\rho_1+\rho_2}}$$

then for every $s \in [0,\lambda) \cap \mathbb{N}_0$:

$$\text{(L.7.b.1)} \quad ((1-\rho_1-\rho_2)^s(\lambda-s) + s(1-\rho_1-\rho_2)^{s-1})e^{(\rho_1+\rho_2)s} \leq \lambda.$$

PROOF. Write $\rho = \rho_1+\rho_2$, let $\lambda \in \mathbb{N}^+$, that satisfies the condition stated above. The case $s = 0$ is clear. Herefore we assume now $s \geq 1$ and rewrite (L.7.b.1) into:

$$\text{(L.7.b.2)} \quad \lambda \geq \frac{\rho s(1-\rho)^{s-1}e^{\rho s}}{1-(1-\rho)^s e^{\rho s}} \quad .$$

We will prove Lemma 7.b by showing that the right hand term of the inequality in (L.7.b.2) decreases when s increases, which means that it is a decreasing series of s with $s \geq 1$. Therefore we calculate the quotient of two following elements on the sieve.

$$\frac{\dfrac{\rho(s+1)(1-\rho)^s e^{\rho(s+1)}}{1-(1-\rho)^{s+1}e^{\rho(s+1)}}}{\dfrac{s\rho(1-\rho)^{s-1}e^{\rho s}}{1-(1-\rho)^s e^{\rho s}}} = \left(\frac{s+1}{s}\right)(1-\rho)e^{\rho}.\left(\frac{1-((1-\rho)e^{\rho})^s}{1-((1-\rho)e^{\rho})^{s+1}}\right)$$

$$= \left(\frac{s+1}{s}\right)\beta.\left(\frac{1-\beta^s}{1-\beta^{s+1}}\right)$$

with $\beta = (1-\rho)e^{\rho}$ and thus $\beta \in (0,1)$.

$$= \left(\frac{s+1}{s}\right).\beta.\frac{(1-\beta)(1+\beta+\ldots+\beta^{s-1})}{(1-\beta)(1+\beta+\ldots+\beta^s)}$$

$$= \left(\frac{s+1}{s}\right)\left(1-\frac{1}{1+\beta+\ldots+\beta^s}\right) \leq \left(\frac{s+1}{s}\right)\left(1-\frac{1}{1+s}\right) = 1.$$

<u>LEMMA 8</u>. *Let* $\lambda \in (0,1)$ *and let* $\rho_1,\rho_2 \in (0,1)$ so that $\rho_1+\rho_2 < 1$.

Then for all $t \in \{1,2,3,\ldots\}$:

$$\left(1-\frac{\rho_1\lambda}{t}\right)^t + \rho_1\lambda\left(1-\frac{(\rho_1+\rho_2)\lambda}{t}\right)^{t-1} + (\rho_1^2+2\rho_1\rho_2)\left(\frac{\lambda}{t}\right)^2\left(1-\frac{\lambda}{t}\right)^{t-2}$$
$$\geq e^{-\rho_1\lambda} + \rho_1\lambda e^{-(\rho_1+\rho_2)\lambda} + (\rho_1^2+2\rho_1\rho_2).\tfrac{1}{2}\lambda^2 e^{-\lambda}.$$

<u>PROOF</u>. The cases $t = 1$ and $t = 2$ are clear, noticing that the right hand term of the inequality decreases as function of ρ_2.

Suppose now $t \geq 3$. We define the function $h : [0,1-\rho_1] \to \mathbb{R}$ by

$$h(\rho_2) = \left(1-\frac{\rho_1\lambda}{t}\right) + \rho_1\lambda\left(1-\frac{(\rho_1+\rho_2)\lambda}{t}\right)^{t-1} + (\rho_1^2+2\rho_1\rho_2)\binom{t}{2}\left(\frac{\lambda}{t}\right)^2\left(1-\frac{\lambda}{t}\right)^{t-2}$$
$$- e^{-\rho_1\lambda} - \rho_1\lambda e^{-(\rho_1+\rho_2)\lambda} - (\rho_1^2+2\rho_1\rho_2)\tfrac{1}{2}\lambda^2 e^{-\lambda}.$$

Notice that $h(0) > 0$ (see lemma 6) and $h'(1-\rho_1) = 0$.

For all $\rho_2 \in [0,1-\rho_1]$:

$$h''(\rho_2) = \frac{3!\rho_1}{(\rho_1+\rho_2)^3}\left[\binom{t}{3}\left(\frac{(\rho_1+\rho_2)\lambda}{t}\right)^3\left(1-\frac{(\rho_1+\rho_2)\lambda}{t}\right)^{t-3} - \frac{((\rho_1+\rho_2)\lambda)^3}{3!}e^{-(\rho_1+\rho_2)\lambda}\right]$$

for all $\rho_2 \in [0,1-\rho_1]$.

Thus from lemma A.S. we see that (because $(\rho_1+\rho_2)\lambda \leq 1 \leq 3-\sqrt{3}$) for all $\rho_2 \in [0,1-\rho_2]$: $h''(\rho_2) \leq 0$.

So $h(\rho_2) \geq 0$ for all $\rho_2 \in [0,1-\rho_1]$.

4. REFERENCES

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5. APPENDIX

5.1. Poisson Tables

Upper limits for $\lambda = np$

Products of sample size n and error rate p when:

- x errors are found
- β equals the accepted risk of making an incorrect statement

x \ β	0.001	0.01	0.05	0.37(*)	0.50
0	6.91	4.60	3.00	1.00	0.69
1	9.23	6.64	4.74	2.15	1.68
2	11.23	8.41	6.30	3.26	2.67
3	13.06	10.05	7.75	4.35	3.67
4	14.79	11.60	9.15	5.43	4.67
5	16.45	13.11	10.51	6.51	5.67
6	18.06	14.57	11.84	7.58	6.67
7	19.63	16.00	13.15	8.64	7.67
8	21.16	17.40	14.43	9.70	8.67
9	22.66	18.78	15.71	10.75	9.67
10	24.13	20.14	16.96	11.81	10.67
11	25.59	21.49	18.21	12.86	11.67
12	27.03	22.82	19.44	13.90	12.67
13	28.45	24.14	20.67	14.95	13.67
14	29.85	25.45	21.89	16.00	14.67
15	31.24	26.74	23.10	17.04	15.67
16	32.62	28.03	24.30	18.08	16.67
17	33.99	29.31	25.50	19.12	17.67
18	35.35	30.58	26.69	20.16	18.67
19	36.80	31.85	27.88	21.20	19.67
20	38.04	33.10	29.06	22.24	20.67

The value of λ given in the table is the solution to the equation

$$\sum_{y=0}^{x} \lambda^{y} e^{-\lambda}/y! = \beta.$$

Boxed values violate the requirement $x \leq \lambda - 1$, or satisfy $x = \lambda - 1$.

(*) $e^{-1} \doteq 0.3679$

5.2. Tables for sieve sampling.

TWO-STAGE SAMPLING WITH R1=0 AND R2=1

BETA: 0.01000
PO: 0.00500
P1: 0.05000

N1	N2	ALPHA	PROBABILITY OF INCORRECTLY ACCEPTING	PROBABILITY OF INCORRECTLY REJECTING
93	93	0.1884	<= BETA	<= ALPHA
94	78	0.1761	<= BETA	<= ALPHA
95	69	0.1689	<= BETA	<= ALPHA
96	63	0.1645	<= BETA	<= ALPHA
97	58	0.1609	<= BETA	<= ALPHA
98	54	0.1582	<= BETA	<= ALPHA
99	50	0.1554	<= BETA	<= ALPHA
100	47	0.1537	<= BETA	<= ALPHA
101	44	0.1519	<= BETA	<= ALPHA
102	42	0.1513	<= BETA	<= ALPHA
103	40	0.1506	<= BETA	<= ALPHA
104	38	0.1498	<= BETA	<= ALPHA
105	36	0.1490	<= BETA	<= ALPHA
106	34	0.1482	<= BETA	<= ALPHA
107	32	0.1473	<= BETA	<= ALPHA
108	30	0.1464	<= BETA	<= ALPHA
109	29	0.1468	<= BETA	<= ALPHA
110	27	0.1458	<= BETA	<= ALPHA
111	26	0.1462	<= BETA	<= ALPHA
112	24	0.1451	<= BETA	<= ALPHA
113	23	0.1454	<= BETA	<= ALPHA
114	22	0.1457	<= BETA	<= ALPHA
115	20	0.1445	<= BETA	<= ALPHA
116	19	0.1448	<= BETA	<= ALPHA
117	18	0.1450	<= BETA	<= ALPHA
118	17	0.1453	<= BETA	?
119	15	0.1440	<= BETA	?
120	14	0.1442	<= BETA	?
121	13	0.1443	<= BETA	?
122	12	0.1445	<= BETA	?
123	11	0.1447	<= BETA	?
124	10	0.1448	<= BETA	?
125	9	0.1449	<= BETA	?
126	7	0.1434	<= BETA	?
127	6	0.1435	<= BETA	?
128	5	0.1436	<= BETA	?
129	4	0.1436	<= BETA	?
130	3	0.1437	<= BETA	?
131	2	0.1437	<= BETA	?
132	1	0.1437	<= BETA	?

TWO-STAGE SAMPLING WITH R1=0 AND R2=2

BETA: 0.01000
P0: 0.00500
P1: 0.05000

N1	N2	ALPHA	PROBABILITY OF INCORRECTLY ACCEPTING	PROBABILITY OF INCORRECTLY REJECTING
93	139	0.0909	<= BETA	?
94	123	0.0812	<= BETA	?
95	113	0.0755	<= BETA	?
96	106	0.0718	<= BETA	?
97	100	0.0687	<= BETA	<= ALPHA
98	96	0.0670	<= BETA	<= ALPHA
99	92	0.0652	<= BETA	<= ALPHA
100	89	0.0641	<= BETA	<= ALPHA
101	85	0.0623	<= BETA	<= ALPHA
102	83	0.0618	<= BETA	<= ALPHA
103	80	0.0606	<= BETA	<= ALPHA
104	78	0.0601	<= BETA	<= ALPHA
105	76	0.0596	<= BETA	<= ALPHA
106	74	0.0591	<= BETA	<= ALPHA
107	72	0.0586	<= BETA	<= ALPHA
108	70	0.0580	<= BETA	<= ALPHA
109	68	0.0575	<= BETA	<= ALPHA
110	66	0.0569	<= BETA	<= ALPHA
111	65	0.0570	<= BETA	<= ALPHA
112	63	0.0564	<= BETA	<= ALPHA
113	61	0.0559	<= BETA	<= ALPHA
114	60	0.0560	<= BETA	<= ALPHA
115	59	0.0561	<= BETA	<= ALPHA
116	57	0.0555	<= BETA	<= ALPHA
117	56	0.0556	<= BETA	<= ALPHA
118	54	0.0549	<= BETA	<= ALPHA
119	53	0.0550	<= BETA	<= ALPHA
120	52	0.0551	<= BETA	<= ALPHA
121	50	0.0545	<= BETA	<= ALPHA
122	49	0.0545	<= BETA	<= ALPHA
123	48	0.0546	<= BETA	<= ALPHA
124	47	0.0547	<= BETA	<= ALPHA
125	46	0.0547	<= BETA	<= ALPHA
126	44	0.0541	<= BETA	<= ALPHA
127	43	0.0541	<= BETA	<= ALPHA
128	42	0.0542	<= BETA	<= ALPHA
129	41	0.0542	<= BETA	<= ALPHA
130	40	0.0543	<= BETA	<= ALPHA
131	39	0.0543	<= BETA	<= ALPHA
132	38	0.0544	<= BETA	<= ALPHA
133	37	0.0544	<= BETA	<= ALPHA
134	35	0.0537	<= BETA	<= ALPHA
135	34	0.0537	<= BETA	<= ALPHA
136	33	0.0538	<= BETA	<= ALPHA
137	32	0.0538	<= BETA	<= ALPHA

138	31	0.0538	<= BETA	<= ALPHA
139	30	0.0539	<= BETA	<= ALPHA
140	29	0.0539	<= BETA	<= ALPHA
141	28	0.0539	<= BETA	<= ALPHA
142	27	0.0539	<= BETA	<= ALPHA
143	26	0.0539	<= BETA	<= ALPHA
144	25	0.0540	<= BETA	<= ALPHA
145	24	0.0540	<= BETA	<= ALPHA
146	23	0.0540	<= BETA	<= ALPHA
147	22	0.0540	<= BETA	<= ALPHA
148	21	0.0540	<= BETA	<= ALPHA
149	20	0.0540	<= BETA	<= ALPHA
150	19	0.0540	<= BETA	<= ALPHA
151	18	0.0541	<= BETA	<= ALPHA
152	17	0.0541	<= BETA	<= ALPHA
153	16	0.0541	<= BETA	<= ALPHA
154	15	0.0541	<= BETA	<= ALPHA
155	14	0.0541	<= BETA	<= ALPHA
156	13	0.0541	<= BETA	<= ALPHA
157	12	0.0541	<= BETA	<= ALPHA
158	11	0.0541	<= BETA	<= ALPHA
159	10	0.0541	<= BETA	<= ALPHA
160	9	0.0541	<= BETA	<= ALPHA
161	8	0.0541	<= BETA	<= ALPHA
162	7	0.0541	<= BETA	<= ALPHA
163	6	0.0541	<= BETA	<= ALPHA
164	5	0.0541	<= BETA	<= ALPHA
165	4	0.0541	<= BETA	<= ALPHA
166	3	0.0541	<= BETA	<= ALPHA
167	2	0.0541	<= BETA	<= ALPHA
168	1	0.0541	<= BETA	<= ALPHA

TWO-STAGE SAMPLING WITH R1=1 AND R2=2

BETA: 0.01000
P0: 0.00500
P1: 0.05000

N1	N2	ALPHA	PROBABILITY OF INCORRECTLY ACCEPTING	PROBABILITY OF INCORRECTLY REJECTING
133	113	0.0791	<= BETA	?
134	80	0.0685	<= BETA	?
135	68	0.0646	<= BETA	?
136	60	0.0621	<= BETA	<= ALPHA
137	54	0.0603	<= BETA	<= ALPHA
138	50	0.0593	<= BETA	<= ALPHA
139	46	0.0583	<= BETA	<= ALPHA
140	43	0.0577	<= BETA	<= ALPHA
141	40	0.0570	<= BETA	<= ALPHA
142	37	0.0563	<= BETA	<= ALPHA
143	35	0.0561	<= BETA	<= ALPHA
144	33	0.0558	<= BETA	<= ALPHA
145	31	0.0555	<= BETA	<= ALPHA
146	29	0.0552	<= BETA	<= ALPHA
147	27	0.0549	<= BETA	<= ALPHA
148	25	0.0545	<= BETA	<= ALPHA
149	24	0.0547	<= BETA	<= ALPHA
150	22	0.0543	<= BETA	<= ALPHA
151	21	0.0545	<= BETA	<= ALPHA
152	19	0.0541	<= BETA	<= ALPHA
153	18	0.0542	<= BETA	<= ALPHA
154	16	0.0538	<= BETA	<= ALPHA
155	15	0.0539	<= BETA	<= ALPHA
156	14	0.0540	<= BETA	<= ALPHA
157	13	0.0541	<= BETA	<= ALPHA
158	11	0.0536	<= BETA	<= ALPHA
159	10	0.0537	<= BETA	<= ALPHA
160	9	0.0537	<= BETA	<= ALPHA
161	8	0.0538	?	<= ALPHA
162	7	0.0539	?	<= ALPHA
163	6	0.0539	?	<= ALPHA
164	5	0.0540	?	<= ALPHA
165	4	0.0540	?	<= ALPHA
166	3	0.0541	?	<= ALPHA
167	2	0.0541	?	<= ALPHA
168	1	0.0541	?	<= ALPHA

THREE-STAGE SAMPLING WITH R1=0, R2=1 AND R3=2

BETA: 0.01000
P0: 0.00500
P1: 0.05000

N1	N2	N3	ALPHA	PROBABILITY OF INCORRECTLY ACCEPTING	PROBABILITY C INCORRECTLY REJEC
93	94	95	0.1360	<= BETA	<= ALPHA
94	80	66	0.1215	<= BETA	<= ALPHA
95	74	53	0.1162	<= BETA	<= ALPHA
96	70	44	0.1131	<= BETA	<= ALPHA
97	67	37	0.1112	<= BETA	<= ALPHA
98	64	30	0.1094	<= BETA	<= ALPHA
99	63	27	0.1096	<= BETA	<= ALPHA
100	61	22	0.1088	<= BETA	<= ALPHA
101	60	19	0.1091	<= BETA	<= ALPHA
102	59	16	0.1094	<= BETA	<= ALPHA
103	58	13	0.1097	<= BETA	<= ALPHA
104	58	12	0.1110	<= BETA	<= ALPHA
105	57	9	0.1113	<= BETA	<= ALPHA
106	56	6	0.1116	<= BETA	<= ALPHA
107	56	5	0.1128	<= BETA	<= ALPHA
108	55	2	0.1131	<= BETA	<= ALPHA
109	55	1	0.1144	<= BETA	<= ALPHA

TWO-STAGE SAMPLING WITH R1=0 AND R2=1

BETA: 0.05000
P0: 0.00500
P1: 0.05000

N1	N2	ALPHA	PROBABILITY OF INCORRECTLY ACCEPTING	PROBABILITY OF INCORRECTLY REJECTING
60	132	0.1443	<= BETA	<= ALPHA
61	81	0.1129	<= BETA	<= ALPHA
62	67	0.1039	<= BETA	<= ALPHA
63	59	0.0991	<= BETA	<= ALPHA
64	53	0.0956	<= BETA	<= ALPHA
65	49	0.0937	<= BETA	<= ALPHA
66	45	0.0916	<= BETA	<= ALPHA
67	42	0.0904	<= BETA	<= ALPHA
68	39	0.0891	<= BETA	<= ALPHA
69	36	0.0877	<= BETA	<= ALPHA
70	34	0.0872	<= BETA	<= ALPHA
71	32	0.0867	<= BETA	<= ALPHA
72	30	0.0861	<= BETA	<= ALPHA
73	28	0.0855	<= BETA	<= ALPHA
74	26	0.0848	<= BETA	<= ALPHA
75	25	0.0853	<= BETA	<= ALPHA
76	23	0.0845	<= BETA	<= ALPHA
77	21	0.0837	<= BETA	<= ALPHA
78	20	0.0840	<= BETA	<= ALPHA
79	19	0.0843	<= BETA	<= ALPHA
80	17	0.0834	<= BETA	<= ALPHA
81	16	0.0837	<= BETA	<= ALPHA
82	15	0.0839	<= BETA	<= ALPHA
83	13	0.0829	<= BETA	<= ALPHA
84	12	0.0831	<= BETA	<= ALPHA
85	11	0.0832	<= BETA	<= ALPHA
86	10	0.0834	<= BETA	<= ALPHA
87	9	0.0836	<= BETA	<= ALPHA
88	8	0.0837	<= BETA	<= ALPHA
89	7	0.0838	<= BETA	<= ALPHA
90	6	0.0839	<= BETA	<= ALPHA
91	4	0.0826	<= BETA	<= ALPHA
92	3	0.0826	<= BETA	<= ALPHA
93	2	0.0827	<= BETA	<= ALPHA
94	1	0.0827	<= BETA	<= ALPHA

TWO-STAGE SAMPLING WITH R1=0 AND R2=2

BETA: 0.05000
P0: 0.00500
P1: 0.05000

N1	N2	ALPHA	PROBABILITY OF INCORRECTLY ACCEPTING	PROBABILITY OF INCORRECTLY REJECTING
60	180	0.0739	<= BETA	?
61	124	0.0485	<= BETA	<= ALPHA
62	109	0.0424	<= BETA	<= ALPHA
63	100	0.0391	<= BETA	<= ALPHA
64	93	0.0367	<= BETA	<= ALPHA
65	88	0.0351	<= BETA	<= ALPHA
66	83	0.0335	<= BETA	<= ALPHA
67	80	0.0329	<= BETA	<= ALPHA
68	76	0.0317	<= BETA	<= ALPHA
69	73	0.0310	<= BETA	<= ALPHA
70	71	0.0307	<= BETA	<= ALPHA
71	68	0.0300	<= BETA	<= ALPHA
72	66	0.0297	<= BETA	<= ALPHA
73	64	0.0294	<= BETA	<= ALPHA
74	62	0.0290	<= BETA	<= ALPHA
75	60	0.0287	<= BETA	<= ALPHA
76	58	0.0284	<= BETA	<= ALPHA
77	56	0.0280	<= BETA	<= ALPHA
78	55	0.0281	<= BETA	<= ALPHA
79	53	0.0277	<= BETA	<= ALPHA
80	52	0.0278	<= BETA	<= ALPHA
81	50	0.0275	<= BETA	<= ALPHA
82	49	0.0275	<= BETA	<= ALPHA
83	47	0.0271	<= BETA	<= ALPHA
84	46	0.0272	<= BETA	<= ALPHA
85	45	0.0273	<= BETA	<= ALPHA
86	43	0.0269	<= BETA	<= ALPHA
87	42	0.0269	<= BETA	<= ALPHA
88	41	0.0270	<= BETA	<= ALPHA
89	39	0.0266	<= BETA	<= ALPHA
90	38	0.0266	<= BETA	<= ALPHA
91	37	0.0267	<= BETA	<= ALPHA
92	36	0.0267	<= BETA	<= ALPHA
93	35	0.0268	<= BETA	<= ALPHA
94	34	0.0268	<= BETA	<= ALPHA
95	32	0.0263	<= BETA	<= ALPHA
96	31	0.0264	<= BETA	<= ALPHA
97	30	0.0264	<= BETA	<= ALPHA
98	29	0.0264	<= BETA	<= ALPHA
99	28	0.0265	<= BETA	<= ALPHA

TWO-STAGE SAMPLING WITH R1=1 AND R2=2

BETA: 0.05000
P0: 0.00500
P1: 0.05000

N1	N2	ALPHA	PROBABILITY OF INCORRECTLY ACCEPTING	PROBABILITY OF INCORRECTLY REJECTING
95	120	0.0442	<= BETA	?
96	75	0.0352	<= BETA	<= ALPHA
97	62	0.0326	<= BETA	<= ALPHA
98	54	0.0310	<= BETA	<= ALPHA
99	49	0.0302	<= BETA	<= ALPHA
100	44	0.0294	<= BETA	<= ALPHA
101	40	0.0287	<= BETA	<= ALPHA
102	37	0.0283	<= BETA	<= ALPHA
103	34	0.0279	<= BETA	<= ALPHA
104	32	0.0278	<= BETA	<= ALPHA
105	29	0.0274	<= BETA	<= ALPHA
106	27	0.0272	<= BETA	<= ALPHA
107	25	0.0270	<= BETA	<= ALPHA
108	23	0.0268	<= BETA	<= ALPHA
109	22	0.0270	<= BETA	<= ALPHA
110	20	0.0268	<= BETA	<= ALPHA
111	18	0.0265	<= BETA	<= ALPHA
112	17	0.0266	<= BETA	<= ALPHA
113	15	0.0264	<= BETA	<= ALPHA
114	14	0.0265	<= BETA	<= ALPHA
115	13	0.0266	<= BETA	<= ALPHA
116	11	0.0262	<= BETA	<= ALPHA
117	10	0.0263	<= BETA	<= ALPHA
118	9	0.0264	?	<= ALPHA
119	8	0.0265	?	<= ALPHA
120	7	0.0265	?	<= ALPHA
121	6	0.0266	?	<= ALPHA
122	5	0.0266	?	<= ALPHA
123	3	0.0261	?	<= ALPHA
124	2	0.0262	?	<= ALPHA
125	1	0.0262	?	<= ALPHA

100	27	0.0265	<= BETA	<= ALPHA
101	26	0.0265	<= BETA	<= ALPHA
102	25	0.0265	<= BETA	<= ALPHA
103	24	0.0266	<= BETA	<= ALPHA
104	23	0.0266	<= BETA	<= ALPHA
105	22	0.0266	<= BETA	<= ALPHA
106	21	0.0266	<= BETA	<= ALPHA
107	20	0.0266	<= BETA	<= ALPHA
108	19	0.0266	<= BETA	<= ALPHA
109	18	0.0266	<= BETA	<= ALPHA
110	17	0.0267	<= BETA	<= ALPHA
111	16	0.0267	<= BETA	<= ALPHA
112	14	0.0262	<= BETA	<= ALPHA
113	13	0.0262	<= BETA	<= ALPHA
114	12	0.0262	<= BETA	<= ALPHA
115	11	0.0262	<= BETA	<= ALPHA
116	10	0.0262	<= BETA	<= ALPHA
117	9	0.0262	<= BETA	<= ALPHA
118	8	0.0262	<= BETA	<= ALPHA
119	7	0.0262	<= BETA	<= ALPHA
120	6	0.0262	<= BETA	<= ALPHA
121	5	0.0262	<= BETA	<= ALPHA
122	4	0.0262	<= BETA	<= ALPHA
123	3	0.0262	<= BETA	<= ALPHA
124	2	0.0262	<= BETA	<= ALPHA
125	1	0.0262	<= BETA	<= ALPHA

THREE-STAGE SAMPLING WITH R1=0, R2=1 AND R3=2

BETA: 0.05000
P0: 0.00500
P1: 0.05000

N1	N2	N3	ALPHA	PROBABILITY OF INCORRECTLY ACCEPTING	PROBABILITY OF INCORRECTLY REJECT
60	132	204	0.1170	<= BETA	?
61	81	101	0.0763	<= BETA	?
62	69	76	0.0675	<= BETA	?
63	62	61	0.0631	<= BETA	?
64	58	52	0.0611	<= BETA	?
65	54	43	0.0592	<= BETA	?
66	52	38	0.0588	<= BETA	?
67	50	33	0.0585	<= BETA	<= ALPHA
68	48	28	0.0581	<= BETA	<= ALPHA
69	47	25	0.0586	<= BETA	<= ALPHA
70	46	22	0.0590	<= BETA	<= ALPHA
71	45	19	0.0594	<= BETA	<= ALPHA
72	44	16	0.0598	<= BETA	<= ALPHA
73	43	13	0.0603	<= BETA	<= ALPHA
74	43	12	0.0614	<= BETA	<= ALPHA
75	42	9	0.0619	<= BETA	<= ALPHA
76	41	6	0.0623	<= BETA	<= ALPHA
77	41	5	0.0635	<= BETA	<= ALPHA
78	41	4	0.0646	<= BETA	<= ALPHA
79	40	1	0.0651	<= BETA	<= ALPHA

5.3. Computerprogram

This computerprogram is written in Pascal and gives us the tables for sieve sampling in the case $r_1 = 0$, $r_2 = 1$, $p_0 = 0.005$, $p_1 = 0.05$ and for several choices of beta.

```
PROGRAM KK01 (OUTPUT);
TYPE RIJ =ARRAY [1..12] OF REAL;
VAR Q,P0,P1,R,L,RL,A :REAL;
    M,N,N1,N2,I,J :INTEGER;
    B              :RIJ;
FUNCTION F01 (N:INTEGER;P:REAL):REAL;
 BEGIN
   F01:= EXP(-N1*P)+N1*P*EXP(-(N1+N)*P)
 END;
FUNCTION FM (N:INTEGER;P:REAL):REAL;
 BEGIN
  FM := (1+N*P)*EXP(-N*P)
 END;
FUNCTION ZOEK (B:REAL;START:INTEGER;PE:REAL;
                       FUNCTION F (N:INTEGER;P:REAL):REAL):INTEGER;
 BEGIN
  WHILE (F(START,PE) > B)
  DO     START := START + 100;
  START := START - 100;
  WHILE (F(START,PE) > B)
  DO     START := START + 10;
  START := START - 10;
  WHILE (F(START,PE) > B)
  DO     START := START + 1;
  ZOEK := START
 END;
FUNCTION ZOEKM (B:REAL;START:INTEGER;PE:REAL;
                       FUNCTION F (N:INTEGER;P:REAL):REAL):INTEGER;
 BEGIN
  WHILE (F(START,PE) > B)
  DO     START := START + 100;
  START := START - 100;
  WHILE (F(START,PE) > B)
  DO     START := START + 10;
  START := START - 10;
  WHILE (F(START,PE) > B)
  DO     START := START + 1;
  ZOEKM := START
 END;
BEGIN
PAGE;
 P0:= 0.005;
 P1 := 0.05;
 Q := P1/P0;
 FOR I:= 1 TO 10 DO
  B[I]:= I/100;
 B[11]:= 0.15; B[12]:= 0.2;
 FOR J:= 1 TO 12 DO
```

```
BEGIN
 PAGE;
 WRITELN;
 WRITELN('    TWO-STAGE SAMPLING WITH R1=0 AND R2=1');
 WRITELN;
 WRITELN('    BETA:  ',B[J]:1:5);
 WRITELN('      P0:  ',P0:1:5);
 WRITELN('      P1:  ',P1:1:5);
 WRITELN;
 WRITELN('                                             PROBABILITY OF   ',
         '      PROBABILITY OF ');
 WRITELN('     N1      N2          ALPHA      INCORRECTLY ACCEPTING',
         '   INCORRECTLY REJECTING');
 WRITELN;
 N:= TRUNC(-LN(B[J])/P1) + 1;
 M:= ZOEKM ( B[J],0,P1,FM);
 FOR N1 := N TO M DO
 BEGIN
  IF   N1 = N
  THEN N2 := ZOEK(B[J],0,P1,F01);
  IF   N2 > 1
  THEN BEGIN
         WHILE (F01(N2,P1) <= B[J])
         DO    N2 := N2 - 1;
         N2 := N2 + 1;
         R := N1/(N1 + N2);
         L := (N1 + N2)*P1;
         RL := N1*P1;
         A:= 1 - F01(N2,P0);
         WRITE(N1:7,N2:7,'     ',A:10:4);
         IF   L >= 2
         THEN IF   L < EXP(1)
              THEN IF   R > 0.3
                   THEN WRITE('               <= BETA      ')
                   ELSE WRITE('                  ?         ')
              ELSE WRITE('               <= BETA      ')
         ELSE WRITE('               ?         ');
         IF   (L < Q) AND (RL <= Q*(2 - SQRT(2)))
         THEN WRITELN('          <= ALPHA')
         ELSE WRITELN('             ?');
         END;
   END;WRITELN;WRITELN;
   END;
END.
```

```
*********************************************************************************************
*********************************************************************************************
**
**                                       COMMENT
**
**      LINE          DESCRIPTION
**
**       6  -   9     THE PROBABILITY OF ACCEPTING
**      10  -  13     THE PROBABILITY OF ACCEPTING WHEN N2 IS ZERO
**      14  -  26     FUNCTION THAT FOR GIVEN VALUES BETA, P1 AND N1
**                    THE SAMPLE SIZE N2 CALCULATES
**      27  -  39     FUNCTION THAT FOR GIVEN VALUES BETA AND P1
**                    THE MAXIMUM POSSIBLE SAMPLE SIZE N1 CALCULATES
**      42  -  47     INITIALISATION OF THE PARAMETERS BETA, P0 AND P1
**      63  -  64     DETERMINATION OF THE BOUNDS OF N1
**      67  -  73     CALCULATION OF N2 FOR EVERY N1
**      77            CALCULATION OF ALPHA FOR EVERY PAIR (N1,N2)
**      79  -  89     CHECK ON THE CONDITIONS, WHETHER OR NOT
**                    THE SIEVE METHOD CAN BE APPLIED
**
**
*********************************************************************************************
*********************************************************************************************
```