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Berry-Esseen Bound for Student's Statistic

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The Berry-Esseen bound for the distribution of Student's t-statistic is obtained under the sole condition that the underlying distribution has a finite 3rd moment.

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0. Introduction

We establish the Berry-Esseen bound for the distribution of Student's t-statistic under the sole assumption that the underlying distribution has a finite 3rd moment (see (1.1 - 3) below).

There are a number of results of Berry-Esseen type for various classes of statistics, e.g., for various kinds of estimators, for linear combinations of order statistics, for U-statistics etc. The case of Student's t was considered in [1] - [3]. In [1], it was assumed that the observations, X_i, have a finite moment of order $r=6$. This assumption makes it possible to apply the asymptotic theory for sums of i.i.d. random variables in a rather straightforward way because t involves the sums of X's and X^2's, these summands having then a finite 3rd moment.

In a recent paper [3] the Berry-Esseen bound for Student's t was obtained under the condition that $r>\frac{10}{3}$. (The authors of [3] require that $r>4$ but they point out that this is only needed to reduce t to a U-statistic and it appears that this reduction is valid for $r>\frac{10}{3}$ as well.)

A closely related result is that of [5] from which the Edgeworth expansion for the distribution of t can be obtained. In particular, the one-term Edgeworth expansion is valid under the present condition that $r=3$ and an additional assumption that the characteristic function of the joint distribution of (X_1, X_1^2) satisfies Cramérs condition (C).

In the present paper no conditions of the latter type are imposed. This is the main cause for differences between our proof and that of [5]. In the proof to follow we mostly omit the parts which can be carried through by specializing the corresponding parts of the proof in [5] and present in full detail those which are essentially different.

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1. THE MAIN RESULT.

Let $X_1, \ldots, X_n$ be i.i.d. random variables such that

$$EX_1 = 0, \quad EX_1^2 = 1, \quad \nu_3 = E|X_1|^3 < \infty. \tag{1.1}$$

Consider Student's statistic

$$t = \frac{n^{\frac{1}{2}}\bar{X}}{s}, \tag{1.2}$$

where

$$\bar{X} = n^{-1}\sum X_i, \quad s^2 = (n-1)^{-1}\sum(X_i-\bar{X})^2.$$

(Unless otherwise stated, the subscript i runs over $\{1, \ldots, n\}$.)

THEOREM. *Under the condition (1.1) there exists a constant $C = C(\nu_3)$ such that, for all $n \geqslant 2$,*

$$\sup_x |P\{t<x\} - \Phi(x)| \leqslant Cn^{-\frac{1}{2}}, \tag{1.3}$$

where Φ denotes the standard normal distribution function.

The results of this type were obtained in [1], [2] and [3] under more restrictive conditions.

PROOF. Let

$$Y_i = X_i^2 - 1, \quad S_n = n^{-\frac{1}{2}}\sum X_i, \quad T_n = n^{-\frac{1}{2}}\sum Y_i. \tag{1.4}$$

Then t can be rewritten as

$$t = \left(\frac{n-1}{n}\right)^{\frac{1}{2}} \frac{S_n}{(1+\frac{1}{\sqrt{n}}T_n - \frac{1}{n}S_n^2)^{\frac{1}{2}}}. \tag{1.5}$$

The arguments used in the proof of the corollary (4.1) [4] show that (1.3) will follow if we prove that for any $a \in R$ there exists $C_1 = C_1(a, \nu_3)$ such that

$$\sup_x |P\{S_n - \frac{1}{2\sqrt{n}}S_nT_n + \frac{a}{n}S_nT_n^2 < x\} - \Phi(x)| \leqslant \frac{C_1}{\sqrt{n}}. \tag{1.6}$$

Note that it is sufficient to prove (1.3) or (1.6) for all large enough n since for, say, $n \leqslant n_0$, (1.3) or (1.6) trivially hold true with C (or C_1) $= \sqrt{n_0}$.

The proof of (1.6) follows the lines of that of Theorem of [5]. The differences are due to obtaining inequalities rather than o-estimates for the remainders. Moreover we don't impose any "smoothness" conditions like Cramér's condition (C) in [5], which requires a different manner of smoothing the sums, S_n, T_n, entering into (1.6).

Let $\xi_1, \ldots, \xi_n, \eta$ be mutually independent and independent from $\{X_i\}$ r.v.'s, ξ_i being identically distributed as $N(o, \sigma^2)$ (normal with zero mean and variance σ^2) and η having the characteristic function $\theta(\frac{s}{A})$, where $\theta(s)$ is an even twice differentiable characteristic function vanishing outside $[-1, 1]$. The parameters $\sigma^2 = \sigma^2(\nu_3)$ and $A = A(\nu_3)$ will be specified in the paragraph following (1.16). We introduce the following truncated and smoothed variables

$$\tilde{X}_i = X_i 1_{\{|X_i| \leqslant \sqrt{n}\}} - EX_i 1_{\{|X_i| \leqslant \sqrt{n}\}}, \tag{1.7}$$

$$\tilde{Y}_i = Y_i 1_{\{|Y_i| \leqslant n-1\}} - EY_i 1_{\{|Y_i| \leqslant n-1\}}, \quad Y_i^* = \tilde{Y}_i + \xi_i, \tag{1.8}$$

$$\tilde{S}_n = n^{-\frac{1}{2}}\sum \tilde{X}_i, \quad S_n^* = \tilde{S}_n + \frac{\eta}{\sqrt{n}}, \quad T_n^* = n^{-\frac{1}{2}}\sum Y_i^*. \tag{1.9}$$

Denote by $p_n^*(x,u)$ and $r_n^*(x)$ the density functions of (S_n^*,T_n^*) and S_n^* and let

$$\pi_n(x,u) = \sup_{y:|y-x|\le 1} |D_y p_n^*(y,u)|. \tag{1.10}$$

Here and in what follows we write D_y and D_y^k for $\frac{\partial}{\partial y}$ and $\frac{\partial^k}{\partial y^k}$, $k=2,3,...$ Let, further,

$$B_n = \{x:|x|\le 2\ln n-1\},\quad E_x = \{u:|u|\le a_n(x)\} \tag{1.11}$$

with

$$a_n(x) = \frac{\sqrt{n}}{(1+|x|)^2}-1.$$

In what follows we denote by C constants (not necessarily the same in different places) depending on ν_3 and possibly on some other parameters. The use of this symbol will mean an assertion that such a constant exists.

By the arguments similar to those in Sections 3,4,5 and 10 of [5] the proof of (1.6) is reduced to the proof of the following relations:

There exist $A_0=A_0(\nu_3)$ and $N_0=N_0(\nu_3,A,\sigma)$ such that, for any $0<A\le A_0$, $\sigma>0$, $n\ge N_0$

$$\sup_{x\in B_n} E\{S_n^*T_n^*|S_n^*=x\}r_n^*(x)\le C(A,\sigma) \tag{1.12}$$

$$\sup_{x\in B_n} E\{S_n^*T_n^{*2}|S_n^*=x\}r_n^*(x) \le \sqrt{n}\,C(A,\sigma). \tag{1.13}$$

There exist $A_1=A_1(\nu_3,\sigma)$, $\sigma_1=\sigma_1(\nu_3)$ and $N_1=N_1(\nu_3,A,\sigma)$ such that, for any $0<A\le A_1$, $\sigma\ge\sigma_1$ and $n\ge N_1$

$$\sup_{x\in B_n} |x|\int_{E_x^c}|u|p_n^*(x,u)du\le C(\nu_3,A,\sigma), \tag{1.14}$$

$$\sup_{x\in B_n} |x|\int_{E_x^c}u^2p_n^*(x,u)du\le C(\nu_3,A,\sigma)\sqrt{n}, \tag{1.15}$$

$$\sup_{x\in B_n} x^2\int_{E_x}u^2\pi_n(x,u)du\le C(\nu_3,A,\sigma)\sqrt{n}. \tag{1.16}$$

Having obtained (1.12 - 1.16) we fix A and σ as $\sigma=\sigma_1(\nu_3)$, $A=\min\{A_0(\nu_3),A_1(\nu_3,\sigma_1(\nu_3))\}$. Then the arguments mentioned above prove (1.6) for $n\ge\max\{N_1,N_0\}$ and hence the theorem.

The proof of (1.12 - 13) follows the lines of that of theorem of [6] and will be omitted in the present paper. In what follows we prove (1.14 - 16). We shall suppress the dependence of constants on ν_3,A and σ except for A_1 and σ_1. Only the dependence on some other parameters (e.g. λ in Lemma 1) will be indicated.

Introduce the function

$$q_n(x,u) = \frac{1+x^2}{R}p_n^*(x,u), \tag{1.17}$$

where

$$R = E(1+S_n^{*2}) = 1+E\tilde{X}_1^2+\frac{1}{n}E\eta^2\le C. \tag{1.18}$$

Let

$$q_n(u) = \sup_x q_n(x,u),\quad q_n^{(1)}(u) = \sup_x |D_x q_n(x,u)| \tag{1.19}$$

and, for $z>0$,

$$Q_n(z) = \int_{|v|\ge z} q_n(v)dv,\quad Q_n^{(1)}(z) = \int_{|v|\ge z} q_n^{(1)}(v)dv. \tag{1.20}$$

The proof of (1.14 - 16) is based on the following lemma the proof of which will be given in Section 2.

LEMMA 1. *There exist positive* $A_1=A_1(\nu_3,\sigma)$, $\sigma_1=\sigma_1(\nu_3)$ *and integers* $m=m(\lambda)\geqslant 3$, $N_2=N_2(\lambda)$ *such that for all* $n\geqslant N_2$, $A\leqslant A_1$, $\sigma\geqslant\sigma_1$

$$\int q_n(v)dv\leqslant C,\quad \int q_n^{(1)}(v)dv\leqslant C \tag{1.21}$$

and for all $n\geqslant N_2$, $A\leqslant A_1$, $\sigma\geqslant\sigma_1$, $z>0$ *and* $\lambda\geqslant 1$

$$Q_n(z)\leqslant C\cdot n\cdot P\{|Y_1^*|>\frac{z\sqrt{n}}{\max(\lambda m,\sigma)}\}+C(\lambda)\frac{n^{\lambda/4}}{z^{3\lambda/2}}, \tag{1.22}$$

$$Q_n^{(1)}(z)\leqslant C\cdot\frac{n^{1/4}}{z^{3/2}}. \tag{1.23}$$

Note that (1.22) implies

$$Q_n(z)\leqslant C\cdot\frac{n^{1/4}}{z^{3/2}}. \tag{1.24}$$

PROOF OF (1.14). By (1.17) and (1.18) we have (see (1.11))

$$|x|\int_{E_n^c}|u|p_n^*(x,u)du\leqq\frac{R|x|}{1+x^2}\int_{E_n^c}|u|q_n(u)du\leqq$$

$$\leqq\frac{R|x|}{1+x^2}\int_{a_n(x)}^{\infty}Q_n(u)du.$$

(We have used here that $\lim_{z\to\infty} zQ_n(z)=0$ by (1.24).) The proof of (1.14) obtains now by a straightforward application of (1.24).

PROOF OF (1.15). Likewise, the integral in (1.15) is bounded by

$$\frac{R}{1+x^2}\int_{a_n(x)}^{\infty}uQ_n(u)du. \tag{1.25}$$

To obtain this we have yet to show that

$$\lim_{u\to+\infty}u^2Q_n(u)=0. \tag{1.26}$$

In order to prove (1.26) and to estimate (1.25) we use (1.22) with $\lambda=2$.

Note that letting $b=-EY_i1_{\{|Y_i|\leqslant n-1\}}$ (see (1.8)) we have

$$0<b=E(X_i^2-1)\cdot 1_{\{|X_i|>\sqrt{n}\}}\leqslant\frac{\nu_3}{\sqrt{n}}(\leqslant 1\ \text{ for }\ n\geqslant\nu_3^2).$$

Hence $|\tilde{Y}_i|\leqslant n$ for $n\geqslant\nu_3^2$. Moreover we have for $n\geqslant\nu_3^2$

$$E|\tilde{Y}_1|^{3/2}=E|X_1^2-1+b|^{3/2}\cdot 1_{\{|X_1|\leqslant\sqrt{n}\}}\leqslant(\nu_3^{2/3}+1-b)^{3/2}\leqslant 2^{3/2}\nu_3. \tag{1.27}$$

Now we obtain for the probability in (1.22) with $\lambda=2$

$$P\{|Y_1^*|>\frac{z\sqrt{n}}{2m}\}\leqslant P\{|\tilde{Y}_1|>\frac{z\sqrt{n}}{4m}\}+P\{|\xi_1|>\frac{z\sqrt{n}}{4m}\}, \tag{1.28}$$

$$P\{|\tilde{Y}_1|>\frac{z\sqrt{n}}{4m}\}\leqslant 2^{3/2}\nu_3\frac{(4m)^{3/2}}{z^{3/2}n^{3/4}}\ \text{ for }\ z>0, \tag{1.29}$$

$$P\{\tilde{Y}_1>\frac{z\sqrt{n}}{4m}\}=0\ \text{ for }\ z>4mn^{1/2}, \tag{1.30}$$

$$P\{|\xi|>\frac{z\sqrt{n}}{4m}\}\leqslant\frac{(4m)^4}{z^4n^2}3\sigma^4 \text{ for } z>0. \tag{1.31}$$

Making use of (1.22), (1.28 - 31) one directly verifies (1.26) and obtains a bound for (1.25) which proves (1.15).

PROOF OF (1.16). It follows from (1.17- 19) that

$$|D_yp_n^*(y,u)|\leqslant C\left[\frac{q_n^{(1)}(u)}{1+y^2}+\frac{2|y|}{(1+y^2)^2}q_n(u)\right].$$

Hence the left hand side of (1.16) is not greater than

$$C\sup_{x\in B_n}[\sup_{y:|y-x|\leqslant 1}\frac{x^2}{1+y^2}\int_{|u|\leqslant a_n(x)}u^2q_n^{(1)}(u)du+\sup_{y:|y-x|\leqslant 1}\frac{2x^2|y|}{(1+y^2)^2}\int_{|u|\leqslant a_n(x)}u^2q_n(u)du].$$

Note that $\sup_{y:|y-x|\leqslant 1}\frac{x^2}{1+y^2}\leqslant 2$ and $\sup_{y:|y-x|\leqslant 1}\frac{x^2|y|}{(1+y^2)^2}\leqslant 1$ for all $x\in R$. Now (1.16) is verified directly using (1.23) and (1.24). Thus the proof of Theorem is completed.

2. PROOF OF LEMMA 1.

Note that after we prove (1.21) it is sufficient to prove (1.22) and (1.23) for $Z\geqslant n^{1/6}$. Denote by $f_n(s,t)$ and $f_n^*(s,t)$ the characteristic functions of $(\tilde{S}_n,T_n^*)$ and (S_n^*,T_n^*) respectively. Let

$$g_n(s,u) = \frac{1}{2\pi}\int e^{-itu}f_n(s,t)dt, \quad g_n^*(s,u) = \frac{1}{2\pi}\int e^{-itu}f_n^*(s,t)dt. \tag{2.1}$$

Then

$$p_n^*(x,u) = \frac{1}{2\pi}\int e^{-isx}g_n^*(s,u)ds. \tag{2.2}$$

Hence we have (see (1.17))

$$Rq_n(x,u) = \frac{1}{2\pi}\int e^{-isx}[g_n^*(s,u)-D_s^2g_n^*(s,u)]ds \tag{2.3}$$

whence noting that $R>1$ (see (1.18)) we obtain

$$Q_n(z)\leqslant\frac{1}{2\pi}\int\int_{|u|>z}|g_n^*(s,u)-D_s^2g_n^*(s,u)|duds, \tag{2.4}$$

$$Q_n^{(1)}(z)\leqslant\frac{1}{2\pi}\int\int_{|u|>z}|s|\,|g_n^*(s,u)-D_s^2g_n^*(s,u)|\,duds. \tag{2.5}$$

Let $\theta_n(s)=\theta(\frac{s}{A\sqrt{n}})$, then (1.9) implies that

$$g_n^*(s,u) = \theta_n(s)\cdot g_n(s,u) \text{ and } g_n^*(s,u) = 0 \text{ for } |s|\geqslant A\sqrt{n}.$$

Thus (2.4) and (2.5) are rewritten as

$$Q_n^{(j)}(z)\leqslant\frac{1}{2\pi}\int_{|s|\leqslant A\sqrt{n}}\int_{|u|>z}|s|^j[|\theta_n(s)-D_s^2\theta_n(s)|\cdot|g_n(s,u)|+2|D_s\theta_n(s)|\cdot$$

$$\cdot|D_sg_n(s,u)|+|\theta_n(s)|\,|D_s^2g_n(s,u)|]dsdu, \quad j=0,1, \tag{2.6}$$

with $Q_n^{(0)}=Q_n$. We shall prove that there exist $A_1(\nu_3,\sigma)$, $\sigma_1(\nu_3)$, and integers $m>0$, $N_2>0$ such that for $n\geqq N_2$, $A\leqq A_1$, $\sigma\geqq\sigma_1$ and for $|s|\leqq A_1\sqrt{n}$

$$\int|D_s^jg_n(s,u)|du\leqslant Ce^{-cs^2}(1+|s|)^j, j=0,1,2, \tag{2.7}$$

and for any $\lambda\geqslant 1$, $z\geqslant n^{1/6}$

$$\int_{|u|>z} |D_s^j g_n(s,u)|du \leq C(\lambda,\nu_3)e^{-cs^2}(1+|s|)^j \cdot$$

$$\cdot[nP\{Y_1^* > \frac{z\sqrt{n}}{\max\{m\lambda,6\}}\} + n^{\lambda/4}z^{-3\lambda/2}], \quad j=0,1,2 \tag{2.8}$$

where $D_s^0 g = g$.

Then, by putting (2.7), (2.8), into (2.6), we obtain (1.21 - 23). Proceeding to the proof of (2.7), (2.8), introduce the functions

$$\Gamma_k(s,u) = E\{\exp[is\sum_{i=1}^{k}\tilde{X}_i] | \sum_{i=1}^{k} Y_i^* = u\}p^{k^*}(u), \tag{2.9}$$

$$k = 1,2,...$$

where $p(u)$ denotes the density of Y_1^* and the superscript k^* denotes the k-fold convolution. Note that (cf. (2.1))

$$\sqrt{n}\,\Gamma_n(\frac{s}{\sqrt{n}}, u\sqrt{n}) = g_n(s,u). \tag{2.10}$$

These functions have the following properties

$$|\Gamma_k(s,u)| \leq p^{k^*}(u), \text{ hence } \int|\Gamma_k(s,u)|du \leq 1, \tag{2.11}$$

$$\Gamma_k(s,\cdot) = \Gamma_1^{k^*}(s,\cdot) \tag{2.12}$$

(see (3.9) in [7]).

LEMMA 2. *There exists* $\sigma_0 = \sigma_0(\nu_3)$ *such that for any* $\sigma \geq \sigma_0$ *there exist constants* $C_1 > 0$, $C_2 > 0$, $A_1 > 0$, *an integer* $m > 0$ *and a function* $B(u) \geq 0$ *which depend on* ν_3 *and* σ *such that*

$$|\Gamma_m(t,u)| \leq p^{m^*}(u) - B(u)t^2 \text{ for } |t| \leq A_1 \tag{2.13}$$

and

$$\int_{|u|\leq C_2} B(u)du \geq C_1. \tag{2.14}$$

The proof will be given in Section 3.

COROLLARY. *For any* $n \geq 2m$

$$\int|\Gamma_n(t,u)|du \leq e^{-C_1 t^2 \frac{n}{2m}} \text{ for } |t| \leq A_1 \tag{2.15}$$

PROOF. Write $n = a\cdot m + k$ with $a = [n/m], 0 \leq k < m$, Then, by (2.13)

$$\int|\Gamma_n(t,u)|du = \int\Gamma_m^{a^*}(t,\cdot)^*\Gamma_k(t,u)|du \leq [\int|\Gamma_m(t,u)|du]^a \leq$$
$$\leq (1 - C_1 t^2)^a \leq \exp\{-C_1 a t^2\}.$$

It remains to note that $a \geq \frac{n}{2m}$ for $n \geq 2m$.

The constants A_1 and σ_1 which enter into the assertion concerning (2.7 - 8) can now be specified as follows. We take A_1 as given by Lemma 2 and let $\sigma_1 = \sigma_0\sqrt{2}$. The latter choice is needed to permit us the application of Lemma 2 and its corollary with $\sigma^2/2$ instead of σ^2 (see (2.16)).

For the proof of (2.7) with $j = 1,2$ we regard ξ_i (see (1.8) and the paragraph before that) as a sum of two independent r.v.'s $\xi_i^{(1)}$, $\xi_i^{(2)}$ distributed $N(0,\frac{\sigma^2}{2})$. Let $\check{Y}_i = \tilde{Y}_i + \xi_i^{(1)}$, then $Y_i^* = \check{Y}_i + \xi_i^{(2)}$. Let (cf.

(2.9))

$$\check{\Gamma}_k(s,u) = E\{\exp[is\sum_{i=1}^{k}\tilde{X}_i]|\sum_{i=1}^{k}\check{Y}_i = u\}\cdot \check{p}^{k*}(u), \quad k=1,2,... \tag{2.16}$$

where $\check{p}(u)$ denotes the density of $\check{Y}_i$. Denote the density of $N(0,k\sigma^2/2)$ by $\phi_k(u)$,

$$\phi_k(u) = \frac{1}{\sqrt{\pi k}\,\sigma}e^{-u^2/k\sigma}, \quad k>0. \tag{2.17}$$

Then we have

$$\Gamma_k(s,u) = \check{\Gamma}_k(s,\cdot)*\phi_k(u), \quad k=1,2,... \tag{2.18}$$

Now, by (2.10), (2.12) and (2.18), we obtain that the left hand side of (2.7) with $j=1$ is

$$\sqrt{n}\int|\check{\Gamma}_{n-1}(\frac{s}{\sqrt{n}},\cdot)*\check{\Gamma}_1'(\frac{s}{\sqrt{n}},\cdot)*\phi_n(u)|du\leqslant\sqrt{n}\int|\check{\Gamma}'_1(\frac{s}{\sqrt{n}},\cdot)*\phi_n(u)|du\cdot$$
$$\cdot\int|\check{\Gamma}_{n-1}(\frac{s}{\sqrt{n}},u)|du, \tag{2.19}$$

where

$$\check{\Gamma}_1'(t,\cdot) = D_t\check{\Gamma}_1(t,\cdot).$$

Applying the corollary (see (2.15)) we get a bound e^{-cs^2} for the last integral which holds for $|s|\leqslant A_1$. Thus we have to show that

$$\sqrt{n}\int|\check{\Gamma}_1'(\frac{s}{\sqrt{n}},\cdot)*\phi_n(u)|du\leqslant C(1+|s|). \tag{2.20}$$

Let

$$\mu(u) = E\{\tilde{X}_1|\check{Y}_1=u\}\check{p}(u).$$

Then

$$\check{\Gamma}_1'(t,u) = i\mu(u)+iE\{\tilde{X}_1[\exp(it\tilde{X}_1)-1]|\check{Y}_1=u\}\check{p}(u). \tag{2.21}$$

We have

$$\int\mu(u)du = E\tilde{X}_1 = 0. \tag{2.22}$$

Let

$$\overset{\circ}{\mu}(u) = \int_{-\infty}^{u}\mu(v)dv \quad (= -\int_{u}^{\infty}\mu(v)dv \quad \text{by(2.22)}).$$

Then

$$\int|\overset{\circ}{\mu}(u)|du\leqslant\int_{-\infty}^{0}\int_{-\infty}^{u}|\mu(v)|dvdu + \int_{0}^{\infty}\int_{u}^{\infty}|\mu(v)|dvdu =$$
$$= \int|u\mu(u)|du\leqslant E|\tilde{X}_1\check{Y}_1|\leqslant C. \tag{2.23}$$

Consider now the two terms arising when we put (2.21) into (2.20). The first of them is

$$\sqrt{n}\int|\mu*\phi_n(u)|du = \sqrt{n}\int|\overset{\circ}{\mu}*\phi_n'(u)|du\leqslant \tag{2.24}$$

$$\leqslant \int |\overset{\circ}{\mu}(u)|du \cdot \sqrt{n} \int |\phi_n'(u)|du \leqslant C$$

(see (2.23) and (2.17)). The second one (with $t=\frac{s}{\sqrt{n}}$) is not greater than $|s|E\tilde{X}_1^2 \leqslant C|s|$, which together with (2.24) proves (2.20), and hence (2.7) with $j=1$.

In the way similar to (2.19) we obtain that the left hand side of (2.7) with $j=2$ is not greater than

$$(n-1)\int |\check{\Gamma}_1'(\frac{s}{\sqrt{n}},\cdot)]^{2*} * \check{\Gamma}_{n-2}(\frac{s}{\sqrt{n}},\cdot)*\phi_n(u)|du + \int |\check{\Gamma}_1''(\frac{s}{\sqrt{n}},\cdot)*\Gamma_{n-2}(\frac{s}{\sqrt{n}},\cdot)*$$

$$*\phi_n(u)|du \leqslant (\sqrt{n}\int |\check{\Gamma}_1'(\frac{s}{\sqrt{n}},\cdot)*\phi_{\frac{n}{2}}(u)|\,du)^2 \int |\check{\Gamma}_{n-1}(\frac{s}{\sqrt{n}},u)|du +$$

$$+ \int |\check{\Gamma}_1''(\frac{s}{\sqrt{n}},u)|du \int |\check{\Gamma}_{n-2}(\frac{s}{\sqrt{n}},u)|du \int \phi_n(u)du. \tag{2.25}$$

Now (2.7) with $j=2$ follows from (2.11),

$$\int |\check{\Gamma}_1''(t,u)|du \leqslant E|\tilde{X}_1|^2 \tag{2.26}$$

and an analoque of (2.20) with ϕ_n replaced by $\phi_{n/2}$.

Turn to the proof of (2.8). We shall use some arguments from the proof of Theorem 1.2 in [7]. Let

$$\rho_m(s,u) = p^{m^*}(u) - B(u)s^2, \quad \rho_k(u) = p^{k^*}(u), \quad k = 1,2,... \tag{2.27}$$

and for $y>0$,

$$\rho_m(s,u,y) = \rho_m(s,u)\cdot 1_{\{u<y\}}, \quad \rho_k(u,y) = \rho_k(u)\cdot 1_{\{u\leqslant y\}}. \tag{2.28}$$

By Lemma 2 and (2.10), we have

$$\int_{|u|\geqslant z} |g_n(s,u)|du = \int_{|u|>z\sqrt{n}} |\Gamma_n(\frac{s}{\sqrt{n}},u)|du \leqslant \int_{|u|>z\sqrt{n}} \rho_m^{a*}(\frac{s}{\sqrt{n}},u)*\rho_k(u)du, \tag{2.29}$$

where a and k are as in the proof of the corollary to Lemma 2.
Hence

$$\int_{|u|>z} |g_n(s,u)|du \leqslant I_1(s,z)+I_2(s,z), \tag{2.30}$$

where

$$I_1(s,z) = \int_{|u|>z\sqrt{n}} [\rho_m^{a*}(\frac{s}{\sqrt{n}},\cdot)*\rho_k(u) - \rho_m^{a*}(\frac{s}{\sqrt{n}},\cdot,y)*\rho_k(u,y)]du, \tag{2.31}$$

$$I_2(s,z) = \int_{|u|>z\sqrt{n}} \rho_m^{a*}(\frac{s}{\sqrt{n}},\cdot,y)*\rho_k(u,y)du. \tag{2.32}$$

The integrand in (2.31) is non-negative, hence we may extend the domain of integration to the whole real line. Thus we obtain

$$I_1(s,z) \leqslant (\int \rho_m(\frac{s}{\sqrt{n}},u)du)^a - (\int \rho_m(\frac{s}{\sqrt{n}},u,y)du)^a \int \rho_k(u,y)du \leqslant$$

$$\leqslant a(1-\frac{C_1}{n}s^2)^{a-1}P\{\sum_{i=1}^{m} Y_i^* > y\} + (1-\frac{C_1}{n}s^2)^a P\{\sum_{i=1}^{k} Y_i^* > y\}.$$

Note that $a-1 \geqslant n/2m$ for $n \geqslant 4m$ and

$$P\{\sum_{i=1}^{m} Y_i^* > y\} \leqslant mP\{Y_1^* > \frac{y}{m}\}.$$

Therefore

$$I_1(s,z)\leqslant e^{-(C_1/2m)s^2}[a\cdot m\cdot P\{Y_1^*>\frac{y}{m}\} + kP\{Y_1^*>\frac{y}{k}\}]\leqslant$$

$$\leqslant e^{-(C_1/2m)s^2}n\cdot P\{Y_1^*>\frac{y}{m}\}. \tag{2.33}$$

Put $y=\frac{z\sqrt{n}}{\lambda}$. Then (2.33) gives a part of the right hand side of (2.8).

To estimate $I_2(s,z)$ we apply the arguments from [7] (see (3.25), (3.32) - (3.33) of [7]) to obtain that for an arbitrary $h>0$

$$I_2(s,z)\leqslant e^{-hz\sqrt{n}}[\int e^{hu}\rho_m(\frac{s}{\sqrt{u}},u,y)du]^a R_k(h,y) \tag{2.34}$$

with

$$R_k(h,y) = \int_{-\infty}^{y} e^{hu}p^{k*}(u)du\leqslant 1+2k\nu_{3/2}e^{hy}y^{-3/2}, \tag{2.35}$$

where

$$\nu_{3/2} = E|Y_1^*|^{3/2}\leqslant C.$$

By (2.27), (2.28) we have

$$\int e^{hu}\rho_m(\frac{s}{\sqrt{n}},u,y)du\leqslant R_m(h,y)-\frac{s^2}{n}\int_{-\infty}^{y} e^{hu}B(u)du. \tag{2.36}$$

Put

$$y = \frac{z\sqrt{n}}{\lambda},\quad h = \frac{1}{y}\log(y^{3/2}/n).$$

Then

$$e^{-hz\sqrt{n}} = n^{\lambda/4}\lambda^{3\lambda/2}z^{-3\lambda/2}$$

and, by (2.35)

$$R_k(h,y)\leqslant 1 + 2k\nu_{3/2}n^{-1},\quad R_m(h,y)\leqslant 1+2m\nu_{3/2}n^{-1}. \tag{2.37}$$

Further, since $z\geqslant n^{1/6}$ we have $y\geqslant\frac{n^{2/3}}{\lambda}$ and $y\geqslant C_2$ (see (2.14)) for $n\geqslant(\lambda C_2)^{3/2}$. Moreover, $h\leqslant\frac{3}{2}e^{-1}n^{-2/3}\leqslant\frac{1}{2}$ for $n\geqslant 2$. Therefore, by (2.14)

$$\int_{-\infty}^{y} e^{hu}B(u)du\geqslant e^{-hC_2}C_1\geqslant C_3 = C_1e^{-C_2/2}. \tag{2.38}$$

Putting (2.37), (2.38) into (2.36) and then into (2.34) we obtain

$$I_2(s,z)\leqslant\lambda^{3\lambda/2}n^{\lambda/4}z^{-3\lambda/2}\exp(2\nu_{3/2}-C_3\frac{a}{n}s^2). \tag{2.39}$$

Note that $\frac{a}{n}\geqslant\frac{1}{2m}$ for $n\geqslant 2m$. Now (2.33) and (2.39) imply (2.8) with $j=0$.

In a way similar to (2.20) we see that the left-hand side of (2.8) with $j=1$ is

$$\sqrt{n}\int_{|u|>z\sqrt{n}} |\check{\Gamma}_{n-1}(\frac{s}{\sqrt{n}},\cdot)*\check{\Gamma}_1'(\frac{s}{\sqrt{n}},\cdot)*\phi_n(u)|\,du = J_1+J_2, \tag{2.40}$$

where J_1 and J_2 are the integrals over $u>z\sqrt{n}$ and $u<-z\sqrt{n}$ respectively.

We shall use the following inequality: for two non-negative functions, f and g,

$$\int_{u>y} f*g(u)du = \int_{u+v>y} \int f(u)g(v)dudv \leqslant \int_{u>\frac{y}{2}} f(u)du \int g(v)dv + \int f(u)du \int_{v>\frac{y}{2}} g(v)dv. \tag{2.41}$$

Thus

$$J_1 \leqslant \sqrt{n} \int_{u>z\sqrt{n}/2} |\check{\Gamma}_1'(\frac{s}{\sqrt{n}},\cdot)\phi_n(u)|du \cdot \int |\check{\Gamma}_{n-1}(\frac{s}{\sqrt{n}},u)|du +$$

$$\sqrt{n}\int |\check{\Gamma}_1'(\frac{s}{\sqrt{n}},\cdot)\phi_n(u)|du \cdot \int_{u>z\sqrt{n}/2} |\check{\Gamma}_{n-1}(\frac{s}{\sqrt{n}},u)|du. \tag{2.42}$$

Note that

$$|EX_1 \cdot 1_{\{|X_1| \leqslant \sqrt{n}\}}| \leqslant E|X_1| \leqslant (EX_1^2)^{1/2} = 1.$$

Hence $|\tilde{X}_1| \leqslant \sqrt{n}+1 \leqslant 2\sqrt{n}$ (see (1.7)). Therefore

$$|\check{\Gamma}_1'(t,u)| = |E[\tilde{X}_1 e^{it\tilde{X}_1} | \check{Y}_1 = u]\check{p}(u) \leqslant 2\sqrt{n}\check{p}(u). \tag{2.43}$$

Then noting that $\check{p}*\phi_1(u) = p^*(u)$, we obtain

$$\sqrt{n} \int_{u>z\sqrt{n}/2} |\check{\Gamma}_1'(\frac{s}{\sqrt{n}},\cdot)*\phi_n(u)|du \leqslant 2n \int_{u>z\sqrt{n}/2} p^* * \phi_{n-1}(u)du \leqslant$$

$$\leqslant 2nP\{Y_1^* > \frac{z\sqrt{n}}{4}\} + 2n \int_{u>z\sqrt{n}/4} \phi_{n-1}(u)du. \tag{2.44}$$

We have for the last term

$$2n \int_{u>z\sqrt{n}/4} \phi_{n-1}(u)du \leqslant 2n \int_{u>z\ /4} \phi_1(u)du \leqslant \frac{Cn}{z^6} \leqslant Cn^{\lambda/4} z^{-3\lambda/2} \tag{2.45}$$

for $z \geqslant n^{1/6}$. Now we put (2.44) and (2.45) into (2.42). Moreover we use (2.20) and estimate the integrals of $\check{\Gamma}_{n-1}$ in the same way as those in (2.7) and (2.8) with $j=1$. The inequalities thus obtained together with similar inequalities for J_2 (see (2.40)) prove (2.8) with $j=1$.

In a similar way to (2.25) we obtain that the left hand side of (2.8) with $j=2$ is not greater than

$$n \int_{|u|>z\sqrt{n}} |[\check{\Gamma}_1'(\frac{s}{\sqrt{n}},\cdot)]^{2*} * \check{\Gamma}_{n-2}(\frac{s}{\sqrt{n}},\cdot)*\phi_n(u)|du +$$

$$+ \int_{|u|<z\sqrt{n}} |\check{\Gamma}_1''(\frac{s}{\sqrt{n}},\cdot)*\check{\Gamma}_{n-2}(\frac{s}{\sqrt{n}},\cdot)*\phi_n(u)|du = J_1 + \ldots + J_4$$

where the J's are the parts of the integrals corresponding to integration over $u>z\sqrt{n}$ and $u<-z\sqrt{n}$ respectively.

We have

$$J_1 \leqslant 2\sqrt{n} \int_{u>z\sqrt{n}/3} |\check{\Gamma}_1'(\frac{s}{\sqrt{n}},\cdot)*\phi_{\frac{n}{2}}(u)du \cdot \sqrt{n}\int |\check{\Gamma}_1'(\frac{s}{\sqrt{n}},\cdot)*\phi_{\frac{n}{2}}(u)|du \cdot$$

$$\cdot \int |\check{\Gamma}_{n-2}(\frac{s}{\sqrt{n}},u)|du + (\sqrt{n}\int |\check{\Gamma}_1'(\frac{s}{\sqrt{n}},\cdot)*\phi_{\frac{n}{2}}(u)|du)^2 \cdot$$

$$\cdot \int_{u>z\sqrt{n}/3} |\check{\Gamma}_{n-2}(\frac{s}{\sqrt{n}},u)|du, \tag{2.46}$$

$$J_3 \leqslant \int\limits_{u>z\sqrt{n}/2} |\check{\Gamma}_1''(\frac{s}{\sqrt{n}},\cdot)*\phi_n(u)|du \int |\check{\Gamma}_{n-2}(\frac{s}{\sqrt{n}},u)|du +$$

$$+\int |\check{\Gamma}_1''(\frac{s}{\sqrt{n}},\cdot)*\phi_n(u)|\,du \int\limits_{u>z\sqrt{n}/2} |\check{\Gamma}_{n-2}(\frac{s}{\sqrt{n}},u)|du \tag{2.47}$$

and similar relations for J_2, J_4 with corresponding change of the domains of integration.

In the first integral in (2.47) we have $|\check{\Gamma}_1''(t,u)| \leqslant 4n\check{p}(u)$ (cf. (2.43)). Then this integral is estimated as in (2.44), (2.45). The other integrals in (2.46), (2.47) are estimated in a similar way to (2.20), (2.44 - 45), (2.26) and (2.7), (2.8) with $j=0$. Thus the proof of (2.7 - 8) and hence the lemma is completed.

3. Proof of Lemma 2.

Let $B \subset R^2$ be defined as

$$B=\{(x,u): |x| \leqslant a, |u| \leqslant b\}, \quad a=8\nu_3, \quad b=96\nu_3^2. \tag{3.1}$$

Denote by $\tilde{P}(dx,dy)$ the distribution of $\tilde{X}_1, \tilde{Y}_1$ (see (1.7), (1.8)) and let P_B be the corresponding distribution conditional on B, i.e. for a Borel set $A \subset R^2$

$$P_B(A) = \frac{\tilde{P}(A \cap B)}{\alpha}, \text{ with } \alpha = \tilde{P}(B). \tag{3.2}$$

Consider i.i.d. random vectors (Z_{0i}, Z_i) distributed according to P_B. Denote by γ_0 and γ the expectations and by $\sigma_{0B}^2, \sigma_B^2$ the variances of Z_{01} and Z_1 respectively. Moreover, let, as before, ξ_i be i.i.d. and independent of $\{Z_{0i}, Z_i\}$ normal $N(0,\sigma^2)$ r.v.'s and $Z_i^* = Z_i + \xi_i$. Let

$$S_{nB} = n^{-1/2}\sum(Z_{0i}-\gamma_0), \quad T_{nB}^* = n^{-1/2}\sum(Z_i^*-\gamma), \sigma_B^{*2} = \mathrm{var}(Z_i^*). \tag{3.3}$$

Denote by $p_B(u)$ the density of Z_1^*.

First we shall show that α and σ_{0B} are bounded away from 0. In a similar way to (1.27) we find that, for $n \geqslant 4\nu_3^2 (\geqslant 4)$,

$$E|\tilde{X}_1| \leqslant 2\nu_3, \quad E|\tilde{X}_1\tilde{Y}_1| \leqslant 3\nu_3. \tag{3.4}$$

Then

$$\tilde{P}(B^c) \leqslant \nu_3(\frac{2}{a^3} + \frac{2^{3/2}}{b^{3/2}}),$$

$$|\int\limits_B x d\tilde{P}| = |\int\limits_{B^c} x d\tilde{P}| \leqslant \nu_3(\frac{2}{a^2}+\frac{3}{b}),$$

$$\int\limits_{B^c} x^2 d\tilde{P} \leqslant \nu_3(\frac{2}{a}+\frac{3a}{b}),$$

and with a and b as given in (3.1),

$$\alpha \geqslant \frac{127}{128},$$

$$\sigma_{0B}^2 \geqslant 1-\frac{1}{\alpha}\nu_3(\frac{2}{a}+\frac{3a}{b})-\frac{1}{\alpha^2}\nu_3^2(\frac{2}{a^2}+\frac{3}{b})^2 \geqslant \tag{3.5}$$

$$\geqslant 1-\frac{1}{2}(\frac{128}{127})^2 > \frac{1}{3}. \tag{3.6}$$

Note that $\sigma_B^{*2} = \sigma_B^2 + \sigma^2$ and $\mathrm{cov}(Z_{01}, Z_1^*) = \mathrm{cov}(Z_{01}, Z_1)$. Hence the correlation, ρ, between Z_{01} and Z_1^* satisfies the inequality

$$|\rho| \leqslant \frac{\sigma_B}{\sqrt{\sigma_B^2+\sigma^2}}. \tag{3.7}$$

Since $\sigma_B \leqslant b = 96\nu_3^2$ we have

$$|\rho| \leqslant \frac{1}{\sqrt{2}} \text{ whenever } \sigma \geqslant \sigma_0 = 96\nu_3^2. \tag{3.8}$$

Let, further

$$\Gamma_{kB}(s,u) = E\{\exp(is\sum_{i=1}^{k} Z_{0i})|\sum_{i=1}^{k} Z_i^* = u\} p_B^{k*}(u), \quad k=1,2,... \tag{3.9}$$

Moreover, for $B^c = R^2 \setminus B$ define P_{B^c} and Γ_{kB^c} in a similar way. Note that

$$\alpha p_B(u) + (1-\alpha) p_{B^c}(u) = p^*(u), \tag{3.10}$$

p^* being the density of $Y_1^* = \tilde{Y}_1 + \xi_1$ (see (1.8)),

$$\alpha \Gamma_{1B}(s,u) + (1-\alpha)\Gamma_{1B^c}(s,u) = \Gamma_1(s,u), \tag{3.11}$$

(see (2.9)) and

$$|\Gamma_{kB}(s,u)| \leqslant p_B^{k*}(u), \quad |\Gamma_{kB^c}(s,u)| \leqslant p_{B^c}^{k*}(u). \tag{3.12}$$

LEMMA 3. *For any $\sigma \geqslant \sigma_0 = 96\nu_3^2$ there exist $A_1 > 0$, an integer $m > 0$ and $d > 0$ depending on ν_3 and σ such that, for $|s| \leqslant A_1$,*

$$|\Gamma_{mB}(s,u)| \leqslant p_B^{m*}(u) - \sqrt{m}\, d s^2 1_E\left(\frac{u - m\gamma}{\sqrt{m}}\right), \tag{3.13}$$

where $E = \{u : |u| \leqslant u_0\}$ with $u_0 = \sigma_0 \sqrt{2 \ln 2}$.

We shall use this lemma to finish the proof of Lemma 2 and after that its proof will be given. We have by (2.12) and (3.11) with m as in Lemma 3

$$\Gamma_m(s,\cdot) = \Gamma_1^{m*}(s,\cdot) = \sum_{k=1}^{m} C_m^k \alpha^k (1-\alpha)^{m-k} \Gamma_{1B}^{k*}(s,\cdot) * \Gamma_{1B^c}^{(m-k)*}(s,\cdot). \tag{3.14}$$

Now we estimate the last term by (3.13) and the other terms by (3.12). Then we employ (3.10) to obtain that for $|s| \leqslant A_1$

$$|\Gamma_m(s,u)| \leqslant p^{m*}(u) - s^2 B(u), \tag{3.15}$$

where

$$B(u) = \alpha^m \sqrt{m}\, d 1_E\left(\frac{u - m\gamma}{\sqrt{m}}\right). \tag{3.16}$$

The length of the interval

$$I = \{u : \frac{u - m\gamma}{\sqrt{m}} \in E\}$$

is $2\sqrt{m} u_0$ and $I \subset [-C_2, C_2]$ for $C_2 = ma + u_0\sqrt{m}$ with a defined in (3.1). Hence

$$\int_{|u| \leqslant C_2} B(u) du = C_1 = 2u_0 m \alpha^m d \tag{3.17}$$

which completes the proof.

4. PROOF OF LEMMA 3.

Let p_{kB} be the density of T^*_{kB} (see (3.3)) and

$$g_{kB}(s,u)=E\{\exp(isS_{kB})|T^*_{kB}=u\}p_{kB}(u),\quad k=1,2,... \tag{4.1}$$

Comparing (4.1) with (3.9) we have

$$\Gamma_{kB}(s,u)=\frac{1}{\sqrt{k}}g_{kB}(s\sqrt{k},\frac{u-k\gamma}{\sqrt{k}}),\quad k=1,2... \tag{4.2}$$

Then (3.13) is equivalent to

$$|g_{mB}(s,u)|\leq P_{mB}(u)-ds^2 1_E(u) \tag{4.3}$$

for

$$|s|\leq A_2=A_1\sqrt{m}.$$

Let

$$\mu_{j,k}(u)=E\{S^j_{kB}|T^*_{kB}=u\}p_{kB}(u),\quad j,k=1,2... \tag{4.4}$$

Denote by $\phi_B(x,u)$ the density of the bivariate normal distribution with zero mean and the same covariance matrix as (Z_{01},Z^*_1). Recall that the correlation coefficient of this distribution satisfies (3.8). Let $\phi_B(u)$ be the marginal density,

$$\phi_B(u)=\int\phi_B(x,u)dx, \text{ and}$$

$$\mu_j(u)=\int x^j\phi_B(x,u)dx,\quad j=1,2,... \tag{4.5}$$

LEMMA 4. *There exist constants* $L_j=L_j(\nu_3,\sigma)$, $j=0,1,...$ *such that*

$$\sup_u|P_{kB}(u)-\phi_B(u)|\leq\frac{L_0}{\sqrt{k}},\quad k=1,2,... \tag{4.6}$$

$$\sup_u|\mu_{j,k}(u)-\mu_j(u)|\leq\frac{L_j}{\sqrt{k}},\quad k=1,2,... \tag{4.7}$$

The proof will be omitted. It uses some technique from [6] and is simplified by the fact that $Z_{0,1}$, Z_1 are bounded and Z^*_1 contains a normal component.

Expanding the exponential function in (4.1) by the Taylor formula we obtain

$$g_{kB}(s,u)=\bar{g}_{kB}(s,u)+|s|^3\zeta_k(s,u) \tag{4.8}$$

with

$$\bar{g}_{kB}(s,u)=p_{kB}(u)+is\mu_{1,k}(u)-\frac{s^2}{2}\mu_{2,k}(u), \tag{4.9}$$

$$|\zeta_k(s,u)|\leq\frac{1}{6}E[|S_{kB}|^3|T^*_{kB}=u]p_{kB}(u)\leq$$
$$\leq\zeta_k(u)=\frac{1}{6}[p_{kB}(u)+\mu_{4,k}(u)]. \tag{4.10}$$

We have

$$|\bar{g}_{kB}(s,u)|^2=p^2_{kB}(u)[1-s^2(\frac{D_k(u)}{p^2_{kB}(u)}-\frac{s^2}{4}\frac{\mu^2_{2k}(u)}{p^2_{kB}(u)})], \tag{4.11}$$

where

$$D_k(u)=p_{kB}(u)\mu_{2k}(u)-\mu^2_{1,k}(u). \tag{4.12}$$

It follows from (4.8) and (4.11) that

$$|g_{kB}(s,u)|\leqslant p_{kB}(u)-\frac{s^2}{2}\Big[\frac{D_k(u)}{p_{kB}(u)}-|s|\zeta_k(u)-\frac{s^2}{4}\frac{\mu_{2k}^2(u)}{p_{kB}(u)}\Big]. \qquad (4.13)$$

Our aim is to find m and A_2 such that the expression in brackets is bounded from below by a positive constant for $k=m, |s|\leqslant A_2, |u|\leqslant u_0$. Since we consider $\sigma\geqslant\sigma_0$ and $\sigma_B^*\geqslant\sigma$, we have

$$\phi_B(u)\geqslant\frac{1}{2}\phi_B(0) \text{ for } |u|\leqslant u_0. \qquad (4.14)$$

Moreover, since $\sigma_B^*\leqslant\sigma\sqrt{2}$, we have

$$\phi_B(0)=\frac{1}{\sqrt{2\pi}\,\sigma_B^*}\geqslant\frac{1}{2\sigma\sqrt{\pi}}. \qquad (4.15)$$

Let m_1 be the smallest integer satisfying

$$\frac{L_0}{\sqrt{m_1}}\leqslant\frac{1}{8\sqrt{\pi}\,\sigma}\Big(\leqslant\frac{1}{4}\phi_B(0)\Big) \qquad (4.16)$$

(see (4.6)). Then (4.16) and (4.14) imply that for $|u|\leqslant u_0,\ k\geqslant m_1$

$$p_{kB}(u)\geqslant\frac{1}{4}\phi_B(0), \quad \frac{\phi_B(u)}{p_{kB}(u)}\geqslant\frac{2}{3}. \qquad (4.17)$$

Let $D(u)=\phi_B(u)\mu_2(u)-\mu_1^2(u)$. Note that $D(u)/\phi_B^2(u)$ is the conditional variance of the first variable given the second one, i.e.

$$D(u)=\sigma_{0B}^2(1-\rho^2)\phi_B^2(u). \qquad (4.18)$$

Write the first term in brackets in (4.13) as

$$\frac{D_k(u)}{p_{kB}(u)}=\frac{D(u)}{p_{kB}(u)}+\frac{D_k(u)-D(u)}{p_{kB}(u)}. \qquad (4.19)$$

By (4.14), (4.15), (4.17) and (4.18) we have for $|u|\leqslant u_0,\ k\leqslant m_1$

$$\frac{D(u)}{p_{kB}(u)}=\frac{D(u)}{\phi_B(u)}\cdot\frac{\phi_B(u)}{p_{kB}(u)}\geqslant\frac{1}{3}\sigma_{0B}^2(1-\rho^2)\phi_B(0) \qquad (4.20)$$

whence, by (3.6), (3.8) and (4.15) we obtain

$$\frac{D(u)}{p_{kB}(u)}\geqslant\frac{\phi_B(0)}{18}\geqslant\frac{1}{36\sqrt{\pi}\,\sigma} \text{ for } |u|\leqslant u_0,\ k\geqslant m_1. \qquad (4.21)$$

Further, we have

$$D_k(u)-D(u)=[\mu_{2k}(u)-\mu_2(u)]p_{kB}(u)+$$
$$[p_{kB}(u)-\phi_B(u)]\mu_2(u)+[\mu_1(u)-\mu_{1k}(u)]\,[\mu_1(u)+\mu_{1k}(u)].$$

It is seen from (4.5), Lemma 4 and the fact that $\sigma_B^*\geqslant\sigma_0$ that there exists a constant $L=L(\nu_3,\sigma)$ such that, for $\sigma\geqslant\sigma_0$

$$\sup_u |D_k(u)-D(u)|\leqslant\frac{L}{\sqrt{k}}, \quad k=1,2,\ldots \qquad (4.22)$$

It follows from (4.17), (4.14) and (4.22) that we can find $m\geqslant m_1$ such that

$$\frac{|D_m(u)-D(u)|}{p_{mB}(u)}\leqslant\frac{1}{72\sqrt{\pi}\,\sigma} \text{ for } |u|\leqslant u_0 \qquad (4.23)$$

Let $d=\frac{1}{144\sqrt{\pi}\sigma}$. Then (4.19), (4.21) and (4.23) imply that

$$\frac{D_m(u)}{p_{mB}(u)}\geqslant 2d \quad \text{for } |u|\leqslant u_0 \tag{4.24}$$

By (4.17) and Lemma 4, $\frac{\mu_{2m}^2}{p_{mB}(u)}$ and $\zeta_k(u)$ are bounded for $|u|\leqslant u_0$, hence we can find $A_2>0$ such that

$$\sup_{|u|\leqslant u_0}\left[\frac{A_2^2}{4}\frac{\mu_{2,k}^2(u)}{p_{mB}(u)}+A_2\zeta_m(u)\right]\leqslant d. \tag{4.25}$$

Then (4.13), (4.24) and (4.25) imply (4.3).

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