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Uniqueness of Gibbs Measures and Absorption Probabilities

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Gibbs measures are studied using a Markov chain on the non-negative integers. Uniqueness of Gibbs measures follows from absorption of the chain at {0}. To this end we derive a certain inequality. For one-dimensional systems this improves a well-known uniqueness result of Ruelle and moreover it is for models near the $1/r^2$-interaction Ising model a natural improvement of some other results.

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1. Introduction

In studying uniqueness of Gibbs measures it may be useful to make a comparison with a stochastic process. We present a method connecting absorption probabilities of a simple Markov chain at $\{0\}$ with uniqueness. Our specific result, theorem 1.1, is used for one-dimensional systems. It includes a well known result on uniqueness in RUELLE (1968) and succeeds in section 6 to find a natural improvement of other results that needed refined estimates. In our method the key role is played by a certain inequality. Comparison of processes is quite well known already in the literature: duality (see LIGGETT (1985)) is a nice technique based on an equality. We hope that our comparison method using absorption probabilities is useful in more problems.

Let us now describe our problem setting. Let S be a countable set and X a finite or countable set. For $\Lambda \subset S$ and $\sigma \in X^S$ write $\sigma_\Lambda = (\sigma_j)_{j \in \Lambda}$. We define Gibbs measures on X^S in terms of a given energy difference function $\Delta H(\sigma,\sigma')$ for the energy difference between σ and σ' in X^S. This function has to be properly defined only in case $\sigma_{\bar{\Lambda}} = \sigma'_{\bar{\Lambda}}$ for the complement $\bar{\Lambda}$ of any finite set Λ. Assume that for any finite $\Lambda \subset \mathbf{Z}$ this energy difference has the form

$$\Delta H(\sigma,\sigma') = \phi_\Lambda(\sigma) - \phi_\Lambda(\sigma') \text{ if } \sigma_{\bar{\Lambda}} = \sigma'_{\bar{\Lambda}}$$

where ϕ_Λ is a suitable real function on $X^{\mathbf{Z}}$. Let us define a probability measure $\rho_{\Lambda,\sigma}$ on X^Λ by

$$\rho_{\Lambda,\sigma}(\{\sigma'_\Lambda\}) := \frac{1}{Z_{\Lambda,\sigma}} e^{\phi_\Lambda(\sigma')}$$

where

$$Z_{\Lambda,\sigma} := \sum_{\sigma'_\Lambda \in X^\Lambda} e^{\phi_\Lambda(\sigma')} \text{ with } \sigma'_{\bar{\Lambda}} = \sigma_{\bar{\Lambda}}.$$

Clearly $\rho_{\Lambda,\sigma}$ as a function of σ does not depend on σ_Λ and moreover it is the probability measure such that

$$\frac{\rho_{\Lambda,\sigma}(\{\sigma_\Lambda\})}{\rho_{\Lambda,\sigma'}(\{\sigma'_\Lambda\})} = e^{-\Delta H(\sigma,\sigma')} \text{ if } \sigma_{\bar{\Lambda}} = \sigma'_{\bar{\Lambda}}.$$

We call a probability measure μ on X^S a *Gibbs measure* for ΔH if for any finite $\Lambda \subset S$

$$\mu(d\sigma_\Lambda, d\sigma_{\bar{\Lambda}}) = \rho_{\Lambda,\sigma}(d\sigma_\Lambda)\mu_{\bar{\Lambda}}(d\sigma_{\bar{\Lambda}}) \tag{1.1}$$

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where $\mu_{\bar\Lambda}$ is the restriction of μ to $X^{\bar\Lambda}$. Note that if (1.1) holds for Λ then it holds also for $\tilde\Lambda \subset \Lambda$.

Section 6 contains an illustration of the notation and definitions above for a one-dimensional long range interaction Ising model.

We will describe now the context of the main theorem. Suppose there are given finite sets $\emptyset = \Lambda_0 \subset \Lambda_1 \subset \cdots \uparrow S$. Write

$$\sigma_n := \sigma_{\Lambda_n \setminus \Lambda_{n-1}}, \quad n \geq 1,$$

and define

$$\phi_n(\sigma) = \phi_{\Lambda_n}(\sigma) - \phi_{\Lambda_{n-1}}(\sigma), \quad n \geq 1. \tag{1.2}$$

It is easily checked that the function ϕ_n does not depend on $\sigma_1, \cdots, \sigma_{n-1}$ and so we may write

$$\phi_n(\sigma) = \phi_n(\sigma_n, \sigma_{n+1}, \cdots).$$

Define for $k \geq 0$

$$var_k(\phi_n) := \max_{\sigma_n, \cdots, \sigma_{n+k}} [\bar\phi_n(\sigma_n, \cdots, \sigma_{n+k}) - \underline{\phi}_n(\sigma_n, \cdots, \sigma_{n+k})]$$

$$r_k := \sup_{n \geq 1} var_k(\phi_n)$$

where

$$\bar\phi_n(\sigma_n, \cdots, \sigma_{n+k}) := \sup_{(\sigma_j)_{j>n+k}} \phi_n(\sigma_n, \sigma_{n+1}, \cdots)$$

$$\underline{\phi}_n(\sigma_n, \cdots, \sigma_{n+k}) := \inf_{(\sigma_j)_{j>n+k}} \phi_n(\sigma_n, \sigma_{n+1}, \cdots).$$

Related to the result of RUELLE (1968) for uniqueness of Gibbs states is the condition

$$\sum_{k \geq 1} r_k < \infty.$$

Our result below uses the weaker condition (1.3).

THEOREM 1.1: *There exists at most one Gibbs measure for ΔH if*

$$\sum_{n \geq 1} \exp(-r_1 - \cdots - r_n) = \infty. \tag{1.3}$$

This result is temperature sensitive. Suppose e.g. that $\Delta H_\beta(\sigma,\sigma') = \beta \Delta H(\sigma,\sigma')$. One obtains a bound on the critical inverse temperature from the theorem by using the following lemma.

LEMMA 1.2:

$$\sum_{n \geq 1} \exp(-\beta(r_1 + \cdots + r_n)) = \infty \text{ for } \beta < \beta_* := \underline{\lim} \frac{\log n}{r_1 + \cdots + r_n}.$$

We leave the proof to the reader (see also section 6). The example in HOFBAUER (1977) is related to the form of our result. For chains with infinite connections BERBEE (1984) gives a uniqueness result under similar conditions but with a quite different proof. In remark 4.1 we indicate a relation using duality.

Our approach consists of an analysis of (1.1) using positive operators. This can be explained as follows. Note that $Z_{\Lambda,\sigma}$ above depends only on $\sigma_{\bar\Lambda}$. Thus the measure

$$\mu_n := \frac{1}{Z_{\Lambda_n,\cdot}} \mu_{\bar\Lambda_n} \tag{1.4}$$

on $X^{\overline{\Lambda}_n}$ is properly defined. We investigate the operators L_n for which

$$L_n\mu_n = \mu_{n-1}, n \geq 1; \mu_0 \equiv \mu .$$

In section 2 we construct a Markov matrix related to the operator L_n. Based on our approach in section 2 we obtain in section 3 a general inequality that is our key result and is formulated using absorption probabilities for a Markov chain. As a corollary we get a certain uniqueness result for Gibbs measures and an inequality for correlations. In section 4 we use this to get theorem 1.1. In section 5 we prove a result in Perron Frobenius theory and indicate how our result differs from RUELLE (1968) and section 6 discusses our Ising model application.

2. RATIO BOUNDS

We define ratio bounds of certain measures and investigate their behavior under the application of positive operators L on these measures. This leads us at the end of the section to associate a Markov operator with L.

Let $Y := \prod_{k\geq 1} Y_k$ where $Y_1, Y_2, \cdots$ are finite sets. Let $\mu,\nu \in \mathfrak{M}(Y)$ with $\mathfrak{M}(Y)$ the space of bounded measures on the space Y. We compare μ with ν using "ratio bounds" that are defined as follows, using the product structure of Y. Let $\mathcal{C}_k$ consist of the k-cylinder sets $C \subset Y$ having the form

$$C = A \times Y_{k+1} \times Y_{k+2} \times \cdots \quad \text{where } A \subset Y_1 \times \cdots \times Y_k.$$

The *ratio bounds* for the measure μ with respect to ν on Y are the coefficients

$$\rho_k := \sup_{C \in \mathcal{C}_k} \frac{\mu(C)}{\nu(C)}$$

$$\rho_{-k} := \inf_{C \in \mathcal{C}_k} \frac{\mu(C)}{\nu(C)}, \quad k \geq 0.$$

Note that $\rho_k = \rho_k(\mu,\nu)$ satisfies

$$\rho_k(\mu;\nu) = \rho_{-k}(\nu;\mu)^{-1} \tag{2.1}$$

We also have $\rho_0(\mu,\nu) = \|\mu\| / \|\nu\|$.

Let us now consider a sequence $X_1, X_2, \cdots$ of finite or countable sets and define $X_{(n)} := \prod_{k\geq 1} X_{n+k}, n \geq 0$. Fixing $n \geq 1$ we define an operator $L \equiv e^{\phi}$ such that $L : \mathfrak{M}(X_{(n)}) \rightarrow \mathfrak{M}(X_{(n-1)})$ as follows. Let $\phi \equiv \phi_n$ be a real function on $X_{(n-1)}$ and define $L \equiv L_n$ by

$$L\mu(B) := \sum_{\sigma_n \in X_n} \int_{X_{(n)}} e^{\phi(\sigma_n,x)} 1_B(\sigma_n,x)\mu(dx)$$

for $\mu \in \mathfrak{M}(X_{(n)})$. For $\mu,\nu \in \mathfrak{M}(X_{(n)})$ we write

$$\tilde{\rho}_k(\mu,\nu) := \rho_k(L\mu, L\nu).$$

Writing ϕ as $\phi_n(\sigma_n, \sigma_{n+1}, \cdots)$ with $\sigma_j \in X_j$ we have, using the definition of $var_k\phi$ in the preceding section:

PROPOSITION 2.1: *For* $1 \leq k \leq N$ *we have*

$$\tilde{\rho}_{-k} \geq \rho_{-(k-1)} e^{-var_{k-1}\phi} + \rho_{-k}(e^{-var_k\phi} - e^{-var_{k-1}\phi}) + \cdots + \rho_{-N}(e^{-var_N\phi} - e^{-var_{N-1}\phi}) \tag{2.2}$$

PROOF: Write $[\sigma_n, \cdots, \sigma_{n+k-1}]_n$ for the cylinder $C \subset X_{(n-1)}$ of the form

$$C = \{\sigma_n\} \times \{\sigma_{n+1}\} \times \cdots \times \{\sigma_{n+k-1}\} \times \prod_{j \geq n+k} X_j.$$

We want to bound $L\mu(C)/L\nu(C)$ and decompose $L\mu(C)$ as

$$\int e^{\underline{\phi}(\sigma_n, \cdots, \sigma_{n+k-1})} 1_C(\sigma_n,x)\mu(dx) + \sum_{k \leq m \leq N} \int (e^{\underline{\phi}(\sigma_n, \cdots, \sigma_{n+m})} - e^{\underline{\phi}(\sigma_n, \cdots, \sigma_{n+m-1})}) 1_C(\sigma_n,x)\mu(dx)$$

$$+ \int (e^{\phi(\sigma_n,x)} - e^{\underline{\phi}(\sigma_n,\cdots,\sigma_{n+N})}) 1_C(\sigma_n,x)\mu(dx).$$

We use after decomposing C in smaller cylinders

$$\mu([\sigma_{n+1},\cdots,\sigma_{n+m}]_{n+1}) \geq \rho_{-m}\nu([\sigma_{n+1},\cdots,\sigma_{n+m}]_{n+1}).$$

This gives

$$L\mu(C) \geq \int \{\rho_{-(k-1)}e^{\underline{\phi}(\sigma_n,\cdots,\sigma_{n+k-1})} + \sum_{k\leq m\leq N} \rho_{-m}(e^{\underline{\phi}(\sigma_n,\cdots,\sigma_{n+m})} - e^{\underline{\phi}(\sigma_n,\cdots,\sigma_{n+m-1})})$$

$$+ 0\} 1_C(\sigma_n,x)\nu(dx).$$

We rearrange the term between $\{\cdot\}$ as

$$(\rho_{-(k-1)} - \rho_{-k})e^{\underline{\phi}(\sigma_n,\cdots,\sigma_{n+k-1})} + \cdots + (\rho_{-(N-1)} - \rho_{-N})e^{\underline{\phi}(\sigma_n,\cdots,\sigma_{n+N-1})} \qquad (2.3)$$
$$+ \rho_{-N}e^{\underline{\phi}(\sigma_n,\cdots,\sigma_{n+N})}.$$

Because ρ_{-j} is nonincreasing in j the terms $(\cdot)$ are nonnegative. It is easily seen that

$$e^{\underline{\phi}(\sigma_n,\cdots,\sigma_{n+m})} \geq e^{-var_m(\phi)}e^{\phi(\sigma_n,x)}$$

where $x = (\sigma_{n+1},\sigma_{n+2},\cdots)$. We apply this to (2.3) and rearrange again. Finally we get $L\mu(C) \geq \psi L\nu(C)$ where ψ is the right hand side in (2.2). Because C is any k-cylinder set this implies the assertion on $\tilde{\rho}_{-k}$. □

Proposition 2.1 is formulated for finite N. We may let $N \to \infty$. The result thus obtained will be summarized in (2.4) using a Markov transition matrix P.

We think of P as being associated with the operator L, defined as follows. Let $\{0\}$ be an absorbing state for P by writing $P_{00} = 1$. Define a probability measure F on $\{0,1,2,\cdots\}\cup\{\infty\}$ by letting

$$F[k,\infty) = 1 - e^{-var_k(\phi)},\ k\geq 0.$$

Because in most of our applications $var_k(\phi)\downarrow 0$ as $k \to \infty$, the measure F will then be concentrated on $\{0,1,,2,\cdots\}$ and we can usually let this be the state space associated with the Markov matrix. Otherwise we add also $\{\infty\}$ as an absorbing point to the state space. Now define for $1\leq k<\infty$

$$P_{kj} := F[0,k-1] \text{ if } j = k-1\geq 0,$$
$$:= F\{j\} \qquad \text{if } j\geq k.$$

Thus the Markov matrix P, having row sums equal to 1, is finally determined. From proposition 2.1 we have the lower bound

$$\tilde{\rho}_{-k} \geq \sum_{0\leq j<\infty} P_{kj}\rho_{-j},\ k\geq 1. \qquad (2.4)$$

This nice formula, giving a lower bound for $\tilde{\rho}$ in terms of the Markov matrix P working on ρ, has a central place in the proof of key lemma 3.1.

3. An Inequality and a Markov Chain Absorbed at {0}

We take again the point of view of section 1 and connect it with section 2 in the proof of key lemma 3.1. There are given finite sets $\varnothing = \Lambda_0 \subset \Lambda_1 \subset \cdots \uparrow S$. Write $X_{(n)} := X^{\Lambda_n}$. Note that $\phi_n = \phi_n(\sigma_n,\sigma_{n+1},\cdots)$ defined by (1.2) can be identified with a function on $X_{(n-1)}$. Let $Z_n, n \geq 0$, be a Markov chain with the nonhomogeneous transition probabilities $P^{(1)},P^{(2)},\cdots$ where $P^{(n)}$ is the Markov transition matrix associated at the end of section 2 with the operator $L_n := e^{\phi_n}$.

Let $N \geq 1$ and suppose μ is a probability measure on X^S that is "right" on Λ_N for ΔH in the sense that (1.1) holds for $\Lambda = \Lambda_N$. Assume the probability measure ν is also "right" in this sense. Clearly this is valid if μ and ν are Gibbs measures for ΔH. Define for the Markov chain $Z_n, n \geq 0$, the absorption time at $\{0\}$ as

$$\tau := inf\{n \geq 0 : Z_n = 0\},$$

and let $P_k(\cdot) := P(\cdot \mid Z_0 = k)$ be the conditional probability given that the chain is started at $\{k\}$. Nice monotonicity properties of this chain are discussed in the proof of proposition 3.3. First we state our key inequality.

LEMMA 3.1: *If μ and ν are probability measures that are "right" on Λ_N for ΔH in the sense above then for any $k \geq 1$*

$$\inf_{C \in \mathfrak{B}_{\Lambda_k}} \frac{\mu(C)}{\nu(C)} \geq P_k(\tau \leq N)^2. \tag{3.1}$$

Here $\mathfrak{B}_\Lambda$ are the sets in X^S generated by the projection on X^Λ.

COROLLARY 3.2: *Suppose for all $k \geq 1$*

$$P_k(Z_n \text{ is absorbed at } \{0\}) = 1 \tag{3.2}$$

then there exists at most one Gibbs measure for ΔH.

PROOF OF COROLLARY 3.2: If μ and ν are Gibbs measures then we can apply the lemma and the right hand side of (3.1) is asymptotically 1 for any k. Hence $\mu(C) \geq \nu(C)$ for any such μ and ν and any cylinder set C. This implies uniqueness of μ. □

PROOF OF LEMMA 3.1: Note that if μ_n is defined for $0 \leq n \leq N$ by (1.4) then, defining $L_n := e^{\Phi_n}$ as in section 2 we have

$$\mu \equiv \mu_0 = L_1 \cdots L_n \mu_n, \quad 1 \leq n \leq N.$$

We prove by induction that for $n = N, N-1, \cdots, 1, 0$ holds

$$R_k(n) := \left(\frac{\rho_{-k}(\mu_n, \nu_n)}{\rho_k(\mu_n, \nu_n)}\right)^{\frac{1}{2}} \geq P(Z_N = 0 \mid Z_n = k), \; k \geq 0.$$

For $k = 0$ and also $n = N$ this is trivial. For $k \geq 1$ and any $N \geq n > 0$ we have (2.4) and similarly after using (2.1)

$$\frac{1}{\rho_k(\mu_{n-1}, \nu_{n-1})} \geq \sum_{0 \leq j < \infty} P_{kj}^{(n)} \frac{1}{\rho_j(\mu_n, \nu_n)}.$$

Using this in

$$\left(\sum_j P_{kj} x_j\right)^{\frac{1}{2}} \left(\sum_j P_{kj} \frac{1}{y_j}\right)^{\frac{1}{2}} \geq \sum_j P_{kj} \left(\frac{x_j}{y_j}\right)^{\frac{1}{2}}$$

which is a consequence of Cauchy's inequality, we find

$$R_k(n-1) \geq \sum_j P_{k,j}^{(n)} R_j(n) \geq \sum_j P_{k,j}^{(n)} P(Z_N = 0 \mid Z_n = j) = P(Z_N = 0 \mid Z_{n-1} = k)$$

using induction. This proves the induction step. Now observe

$$\left(\frac{\rho_{-k}(\mu, \nu)}{\rho_k(\mu, \nu)}\right)^{\frac{1}{2}} = R_k(0) \geq P(Z_N = 0 \mid Z_0 = k) = P_k(\tau \leq N).$$

Because

$$\rho_k(\mu, \nu) \geq \rho_0(\mu, \nu) = \frac{\|\mu\|}{\|\nu\|} = 1$$

this implies the assertion. □

Quite generally the condition of the corollary can be relaxed by the following proposition.

PROPOSITION 3.3: *Suppose* $P_1(Z_k \neq 0) > 0$ *for all* $k \geqslant 1$. *Then (3.2) holds for all* $k \geqslant 1$ *as soon as it holds for* $k = 1$.

PROOF: We analyze the Markov chain transition probabilities $P^{(n)}$. In section 2 we used a distribution function $F \equiv F^{(n)}$ to define $P^{(n)}$. Let $\epsilon_n, n \geqslant 1$, be independent random variables having distribution $F^{(n)}, n \geqslant 1$. Define

$$\begin{aligned} Z_0^{(i)} &:= i && i \geqslant 0, \\ &:= 0 && \text{if } Z_{n-1}^{(i)} = 0, \\ &:= \max(Z_{n-1}^{(i)} - 1, \epsilon_n) && \text{otherwise.} \end{aligned} \tag{3.4}$$

It is easily seen that $(Z_n^{(i)})_{n \geqslant 0}$ is distributed as the Markov chain $(Z_n)_{n \geqslant 0}$ started at $\{i\}$. All of the $Z^{(i)}$-processes are given in terms of $(\epsilon_1, \epsilon_2, \cdots)$. The event $A_k := \{Z_k^{(1)} \geqslant 1\}$ is increasing in $(\epsilon_1, \epsilon_2, \cdots)$. It has positive probability by assumption and by the *FKG*-inequality

$$P(Z_n^{(k)} \geqslant j \mid A_k) \geqslant P(Z_n^{(k)} \geqslant j). \tag{3.5}$$

On the set $A_k = \bigcap_{1 \leqslant t \leqslant k} \{Z_t^{(1)} \geqslant 1\}$ holds $Z_t^{(k)} \geqslant Z_t^{(1)} \geqslant 1$. If on A_k holds that for all $1 \leqslant t \leqslant k$ one has $Z_t^{(k)} = Z_{t-1}^{(k)} - 1$ then $Z_k^{(k)} = Z_k^{(0)} - k = 0$, contradicting $Z_k^{(k)} \geqslant 1$. Hence for some $1 \leqslant t \leqslant k$ we have $Z_t^{(k)} = \epsilon_t > Z_{t-1}^{(k)} - 1$ but then also $Z_t^{(1)} = \epsilon_t$. So on A_k there is some $1 \leqslant t \leqslant k$ such that $Z_t^{(k)} = Z_t^{(1)}$ and then $Z_n^{(k)} = Z_n^{(1)}$ for all $n \geqslant t$. Hence on A_k holds $Z_n^{(k)} = Z_n^{(1)}$ for $n \geqslant k$. Thus (3.5) becomes

$$P_1(Z_n \geqslant j \mid Z_k \neq 0) \geqslant P_k(Z_n \geqslant j).$$

Using this with $j = 1$ and assuming (3.2) for $k = 1$ we get (3.2) for any $k \geqslant 1$. □

Define the ψ-mixing coefficient $\psi(\mathcal{A}, \mathcal{B})$ between σ-fields $\mathcal{A}$ and $\mathcal{B}$ as

$$\psi(\mathcal{A}, \mathcal{B}) = sup \left| \frac{\mu(A \cap B)}{\mu(A)\mu(B)} - 1 \right|$$

where the supremum is taken over $A \in \mathcal{A}$ and $B \in \mathcal{B}$ having positive measure. This coefficient dominates the correlation between 1_A and 1_B. Our inequality gives

COROLLARY 3.4: *If* μ *is any Gibbs measure for* ΔH *then*

$$\psi(\mathcal{B}_{\Lambda_k}, \mathcal{B}_{\bar{\Lambda}_N}) \leqslant P_k(\tau \leqslant N)^{-2} - 1.$$

PROOF: The measure $\nu := \mu(\cdot \mid B)$ with $B \in \mathcal{B}_{\bar{\Lambda}_N}$ satisfies for $\Lambda = \Lambda_N$

$$\nu(d\sigma_\Lambda, d\sigma_{\bar{\Lambda}}) = \frac{1}{\mu(B)} \rho_{\Lambda,\sigma}(d\sigma_\Lambda) 1_B(\sigma_{\bar{\Lambda}}) \mu_{\bar{\Lambda}}(d\sigma_{\bar{\Lambda}}) = \rho_{\Lambda,\sigma}(d\sigma_\Lambda) \nu_{\bar{\Lambda}}(d\sigma_{\bar{\Lambda}})$$

where $\rho_{\Lambda,\sigma}$ depends only on $\sigma_{\bar{\Lambda}}$ as we noted in section 1. By lemma 3.1

$$d \leqslant \frac{\nu(A)}{\mu(A)} \leqslant d^{-1}$$

where the last inequality was obtained by interchanging μ and ν, and $d = P_k(\tau \leqslant N)^2$. Hence $\mu(A \cap B)/\mu(A)\mu(B)$ is between d and d^{-1} uniformly and this implies easily the assertion. □

REMARK 3.5: If μ is a Gibbs measure for ΔH and also (3.2) holds for all k then the relevant ψ-mixing coefficient above vanishes asymptotically as $N \to \infty$. If e.g. $S = \mathbb{Z}$ and ΔH is invariant under translation T it follows that the dynamical system $(X^{\mathbb{Z}}, \mu, T)$ is a K-system.

4. The Simple Markov Chain

The inhomogeneous chain (Z_n) of key lemma 3.1 is studied here further. A comparison with a simpler homogeneous chain (Z_n^*) gives us the proof of theorem 1.1. The use of the inhomogeneous chain directly is more complicated. In remark 6.1 we show by an example that this may give a better result. At the end of this section we give a calculation concerning the inhomogeneous chain.

Let Z_n be the Markov chain with transition matrices $P^{(1)}, P^{(2)}, \cdots$ where

$$P^{(n)} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ p_0^{(n)} & p_1^{(n)} - p_0^{(n)} & p_2^{(n)} - p_1^{(n)} & \cdots \\ 0 & p_1^{(n)} & p_2^{(n)} - p_1^{(n)} & \cdots \\ 0 & 0 & p_2^{(n)} & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix}$$

as in the preceding section. Here $\{0\}$ is an absorbing state;

$$p_k^{(n)} := e^{-var_k(\phi_n)} \tag{4.1}$$

and possibly an absorbing state $\{\infty\}$ is added also, making the row sums equal to 1. We want to investigate whether $P_k(Z_n=0)\uparrow 1$ as $n \to \infty$. By proposition 3.3 this in general only has to be done for $k = 1$ because of the special structure of the chain. We will now derive (4.7) concerning $\Delta_n := P_1(Z_n \neq 0), n \geq 0$, where Δ_n is nonincreasing.

Write for $k \geq 0$

$$G_n(k) := P_1(Z_n \leq k) \text{ and } g_n(k) := P_1(Z_n = k).$$

Then from $g_n = g_{n-1}P^{(n)}, n \geq 1$, follows

$$G_n(k) = G_{n-1}(0)(1-p_k^{(n)}) + G_{n-1}(k+1)p_k^{(n)}$$

as an easy calculation shows. Write $\Delta_n(k) = 1 - G_n(k)$. We can calculate inductively from

$$\Delta_n(k) = \Delta_{n-1}(1-p_k^{(n)}) + \Delta_{n-1}(k+1)p_k^{(n)} \tag{4.2}$$

that

$$\begin{aligned} \Delta_n(0) &= \Delta_{n-1}(1-p_0^{(n)}) + \Delta_{n-1}(1)p_0^{(n)} \\ &= \Delta_{n-1}(1-p_0^{(n)}) + \Delta_{n-2}(p_0^{(n)} - p_0^{(n)}p_1^{(n-1)}) + \Delta_{n-2}(2)p_0^{(n)}p_1^{(n-1)}. \end{aligned} \tag{4.3}$$

We continue to apply (4.2) in this way to get

$$\Delta_n = \Delta_{n-1}H^{(n)}\{1\} + \Delta_{n-2}H^{(n)}\{2\} + \cdots + \Delta_0 H^{(n)}\{n\}. \tag{4.4}$$

Here we use that $\Delta_0(0) = 1$ and $\Delta_0(1) = \Delta_0(2) = \cdots = 0$ and furthermore that $H^{(n)}$ is a probability measure such that

$$H^{(n)}\{k\} = h_{k-1}^{(n)} - h_k^{(n)} \text{ where } h_k^{(n)} := p_0^{(n)} \cdot \ldots \cdot p_{k-1}^{(n-(k-1))} \tag{4.5}$$

for $1 \leq k \leq n$, $n \geq 1$, and $h_0^{(n)} := 1$.

We now reformulate this into a description using random variables. Assume that $H^{(n)}$, $n \geq 1$, is a probability measure on $\{1,2,3,\cdots\}$ defined arbitrarily on $\{n+1, n+2, \cdots\}$. Let $Y^{(n)}$ be independent random variables distributed as $H^{(n)}$. Then (4.4) can be summarized as

$$\Delta_n = E\Delta_{n-Y^{(n)}}, \ n \geq 1, \tag{4.6}$$

if we define $\Delta_n = 0$ for $n < 0$. For $n = 0$ we have $\Delta_n = 1$ and, if we take $Y^{(n)} \equiv 0$ for all $n \leq 0$, then (4.6) is valid for all integers n. To investigate Δ_n for fixed $n \geq 0$ we study the following random walk with space inhomogeneous independent increments

$$\sigma_0 := n\,;\ \sigma_{k+1} := \sigma_k - Y^{(\sigma_k)},\ k \geq 0.$$

This random walk stops on $\{0,-1,\cdots\}$ and 0 is the only element in this set with non-vanishing Δ-value. One observes now easily that

$$\Delta_n = E\Delta_{\sigma_1} = \cdots = P_{\{n\}}(\text{ the random walk } \sigma_k \text{ hits } \{0\})\,. \tag{4.7}$$

Theorem 1.1 will follow by using renewal theory to study (4.7) and (4.4).

Proof of Theorem 1.1: We want to apply corollary 3.2. Let $P^{(n)}$ be associated to Z_n as above with $p_k^{(k)} = e^{-var_k(\phi_n)}$ by our definitions at the end of section 2 and in section 3. We compare Z_n with a time homogeneous Markov chain Z_n^* as follows. For all $n \geq 1$ replace in the definition of $P^{(n)}$ the value of $p_k^{(n)}$ by

$$p_k^* := \inf_{n \geq 1} p_k^{(n)} = e^{-r_k}$$

with r_k as in section 1. Then if $k \geq m$ we have

$$\sum_{i \geq j} P_{ki}^{(n)} \leq \sum_{i \geq j} P_{mi}^*.$$

One uses this to get $P_k(Z_n \geq j) \leq P_k(Z_n^* \geq j)$ for all $k,n \geq 1$ with Z_n^* the Markov chain with transition probability P^*. By corollary 3.2 it is sufficient to prove $P_k(Z_n \neq 0)\downarrow 0$ as $n \to \infty$ and clearly this follows from $P_k(Z_n^* \neq 0)\downarrow 0$ as $n \to \infty$. Thus by proposition 3.3 it is sufficient to prove $\Delta_n^* := P_1(Z_n^* \neq 0)\downarrow 0$ as $n \to \infty$. Now take the random walk $(\sigma_k^*)_{k \geq 0}$ as the analogue of $(\sigma_k)_{k \geq 0}$ above. It can be chosen to have increments distributed as H^* (so space homogeneous) on the positive integers and by (4.4) we have

$$\Delta_n^* = \Delta^* \otimes H^*,\ n \geq 1,$$

where $\otimes$ denotes convolution. Then $(\Delta_n^*)_{n \geq 0}$ is well-known in probability theory as a renewal sequence and by Erdös, Feller, Pollard (1949)

$$\lim_{n \to \infty} \Delta_n^* = \frac{1}{\mu}\,,$$

where μ is the mean of H^*, i.e. equals

$$\sum_{n \geq 1} nH^*(\{n\}) = \sum_{n \geq 1} n(p_0^* \cdot \ldots \cdot p_{n-2}^* - p_0^* \cdot \ldots \cdot p_{n-1}^*) = \sum_{n \geq 0} p_0^* \cdot \ldots \cdot p_{n-1}^*$$

so is infinite by assumption (1.3). So $\Delta_n^* \downarrow 0$ as $n \to \infty$ as was to be proved. □

Remark 4.1: In considering the proof above (and also L of section 3 and its associated Markov operator) the role of duality is in the background. We investigate the process (Z_n^*) constructed above. The existence of a dual Markov process $Z^\sim$ such that for $i,j \geq 0$

$$P(Z_n^* \geq j \mid Z_0^* = i) = P(Z_n^\sim \leq i \mid Z_0^\sim = j) \tag{4.8}$$

can be studied as in Siegmund (1976). It is seen that it exists because (Z_n^*) is stochastically monotone in the sense that the left hand side of (4.8) is nondecreasing in i. The transition matrix $P^\sim$ of $Z^\sim$ is seen to be

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1-p_0^* & p_0^* & 0 & 0 & \cdots \\ 0 & 1-p_1^* & 0 & p_1^* & 0 & \cdots \\ 0 & 1-p_2^* & 0 & 0 & p_2^* & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{bmatrix}$$

Now consider $\{n \geq 0 : Z^{\sim}_{n+1} = 1\}$ as "renewal epochs" (see FELLER (1969) while $Z^{\sim}_n$ is a "backward recurrence time", measuring the time lapse since the preceding renewal epoch. BERBEE (1984) contains a uniqueness theorem with similar conditions but with a rather different proof than theorem 1.1, using the process $Z^{\sim}$ instead of Z^*. Duality seems to play a role in the analogy.

The proof above used comparison with a homogeneous chain. However using the inhomogeneous chain directly may give a better bound for the critical temperature. In remark 6.1 we indicate this using the following analysis to be applied for the inhomogeneous chain. Because this chain Z_n has absorbing state $\{0\}$ we know Δ_n is nonincreasing. We may write $\Delta_n = 1-(\delta_1 + \cdots + \delta_n)$ with all $\delta_n \geq 0$.

LEMMA 4.2: *Define*

$$\psi_n(s) := \sum_{k \geq 0} h_k^{(n+k)} s^k, \quad 0 \leq s < 1. \tag{4.9}$$

Then, assuming $s^m \psi_m(s) \to 0$ *as* $m \to \infty$, $0 \leq s < 1$, *we have*

$$\psi_0(s) = 1 + \sum_{k \geq 1} s^k \psi_k(s) \delta_k.$$

Observe that $\Delta_n \downarrow 0$ is equivalent to showing $\sum_{k \geq 1} \delta_k = 1$.

PROOF: Define $\Delta(s) = \sum_{n \geq 0} \Delta_n s^n$. Using (4.4) we have

$$\Delta(s) = \Delta_0 + \sum_{n \geq 1} \sum_{1 \leq k \leq n} \Delta_{n-k}(h_{k-1}^{(n)} - h_k^{(n)}) s^n.$$

Write $s^n = s^{n-k} s^k$, exchange summation (using $s<1$) and write $m = n-k$ to get

$$\Delta(s) = 1 + \sum_{k \geq 1} \sum_{m \geq 0} \Delta_m s^m (h_{k-1}^{(m+k)} - h_k^{(m+k)}) s^k.$$

Using (4.9) a simple calculation gives

$$1 = \sum_{m \geq 0} \Delta_m (s^m \psi_m(s) - s^{m+1} \psi_{m+1}(s)).$$

Because Δ_n is increasing $\Delta_n = 1 - \delta_1 - \cdots - \delta_n$ with all $\delta_n \geq 0$. Substituting this in the equality above and using telescoping sums and exchange of summation, one finds the assertion. □

5. A PERRON FROBENIUS THEOREM

We study a positive operator and indicate at the end of the section an important difference with the Perron Frobenius theorem of RUELLE (1968). The proof of theorem 1.1 is followed closely.

Our Perron Frobenius theorem can be described as follows. Let X be a finite set and $S = \{1,2, \cdots\}$. We construct an operator $L : \mathfrak{M}(X^S) \to \mathfrak{M}(X^S)$ by defining

$$L\mu(B) := \sum_{\sigma_0 \in X} \int_{X^S} e^{\phi((\sigma_0, x))} I_B((\sigma_0, x)) \mu(dx)$$

where (σ_0, x) is seen as element of X^S and ϕ is a real function on $\prod_{n \geq 0} X$. Using the notation of section 1 we have:

THEOREM 5.1: *If*

$$\sum_{n \geq 1} \exp(-var_1(\phi) - \cdots - var_n(\phi)) = \infty \tag{5.1}$$

then there is a unique probability measure ν *with* $L\nu = \lambda\nu$ *for some* $\lambda > 0$ *and for any other bounded* *measure μ and any cylinder set $C \subset X^S$*

$$L^n\mu(C) \sim \|L^n\mu\| \nu(C) \tag{5.2}$$

$$\frac{\|L^{n+1}\mu\|}{\|L^n\mu\|} \to \lambda \quad \text{as} \quad n \to \infty.$$

PROOF: Using the argument in the proof of theorem 1.1 we find that

$$\left(\frac{\rho_{-k}(L^n\mu, L^n\nu)}{\rho_k(L^n\mu, L^n\nu)}\right)^{\frac{1}{2}} \geq P_k(\tau \geq n) \tag{5.3}$$

where τ is the absorption time into $\{0\}$ of a Markov chain $Z_n^*, n \geq 0$, with transition probabilities P associated at the end of section 2 with the operator e^{Φ}. Again as in the proof of theorem 1.1 (or using lemma 4.2 and proposition 3.3) the right hand side of (5.3) is asymptotically 1 as $n \to \infty$, and we easily get

$$\frac{L^n\mu(C)}{L^n\nu(C)} \sim \frac{\|L^n\mu\|}{\|L^n\nu\|}. \tag{5.4}$$

Existence of ν such that $L\nu = \lambda\nu$ follows by finding a fix point of the operator $\tilde{L}$ defined by $\tilde{L}\mu := L\mu/\|L\mu\|$. Such a fix point can be obtained as the limit of a convergent subsequence of

$$\frac{1}{n}\sum_{k=1}^{n}\tilde{L}^k\mu, n \geq 1$$

because X^S is compact. Uniqueness of such a ν follows from (5.4). Moreover (5.2) is also implied by (5.4). □

Above we obtained an "eigenmeasure" ν of L as any normalized limit of $L^n\mu, n \geq 1$. In RUELLE (1968) there is also constructed an eigenfunction h at eigenvalue λ for the adjoint L^*, that satisfies

$$\frac{L^n\delta_{\{x\}}}{L^n\delta_{\{y\}}} \sim \frac{h(x)}{h(y)} \quad \text{as } n \to \infty. \tag{5.5}$$

The extension we give here is interesting because our context seems more sensitive: it may be that (5.5) does not have to hold if one merely assumes (5.1).

6. A ONE-DIMENSIONAL ISING MODEL WITH LONG RANGE INTERACTION

To describe our example let $X := \{-1, 1\}$ and $S = \mathbb{Z}$. Assume for finite $\Lambda \subset \mathbb{Z}$

$$\Delta H(\sigma, \sigma') = \phi_\Lambda(\sigma') - \phi_\Lambda(\sigma) \ \text{ if } \sigma_{\bar{\Lambda}} = \sigma'_{\bar{\Lambda}}$$

with

$$\phi_\Lambda(\sigma) = -\frac{1}{2}\beta\Sigma J(|j-i|)\sigma_i\sigma_j$$

where the sum is over $i \neq j$ with $i,j \in \mathbb{Z}^2 \setminus (\bar{\Lambda} \times \bar{\Lambda})$. For simplicity we only discuss the case with $J \geq 0$, but our result could also be applied more generally.

We study β_c, the maximum of all $\beta \geq 0$ below which there is a unique Gibbs measure. If $J(n) = 1/n^\alpha$ and $\alpha > 2$ then by RUELLE (1968) there is a unique Gibbs measure at any temperature, i.e. $\beta_c = \infty$. On the other hand DYSON (1969a) showed $0 < \beta_c < \infty$ for $1 < \alpha < 2$. FRÖHLICH, SPENCER (1982) succeeded to prove this also for $\alpha = 2$. We investigate the boundary case and prove

$$\beta_c \geq \underline{\lim} \frac{\log n}{8 \sum\limits_{1 \leq k \leq n} kJ(k)}. \tag{6.1}$$

Hence $\beta_c = \infty$ if $S_n := \sum\limits_{1 \leq k \leq n} kJ(k) = o(\log n)$ and note also that the lower bound on β_c is positive if $J(n) = 1/n^2$ which makes this result seem natural. DYSON (1969b) proved earlier that $\beta_c = \infty$ if $S_n = o(\log\log n)$ and ROGERS, THOMPSON (1981) improved this to $S_n = o((\log n)^{1/2})$. FANNES, VANHEUVERZWIJN, VERBEURE (1982) obtained $\beta_c = \infty$ assuming our rate on S_n together with polynomial decay of spin correlations. Our method does not need the latter assumption. Also our corollary 3.4 gives asymptotic convergence to zero of spin correlations. Remark 6.1 discusses this further.

To derive (6.1) about uniqueness of Gibbs measures we use theorem 1.1. Take $\Lambda_0 := \varnothing$, $\Lambda_n := \{-n+1, \cdots, n\}$, $n \geq 1$, and consider ϕ_n defined by (1.2) which can be written as

$$\phi_n(\sigma) = -\beta \Sigma J(i-j)\sigma_i\sigma_j$$

where the sum is over all $i > j$ such that $(i,j) \in (\overline{\Lambda}_{n-1} \times \overline{\Lambda}_{n-1}) \setminus (\overline{\Lambda}_n \times \overline{\Lambda}_n)$. A simple calculation gives for $k \geq 0$

$$var_k(\phi_n) = 4\beta[\sum_{i>k} J(i) + \sum_{i>2n+k-1} J(i)]. \tag{6.2}$$

Hence we have

$$r_k \leq 8\beta \sum_{i \geq k} J(i).$$

By theorem 1.1 there is a unique Gibbs measure if for n large

$$\exp(-r_1 - \cdots - r_n) \geq \frac{1}{n}$$

or equivalently if

$$\beta \leq \frac{\log n}{8 \sum_{1 \leq k \leq n} \sum_{i \geq k} J(i)}.$$

Hence because

$$\sum_{1 \leq k \leq n} \sum_{i \geq k} J(i) \geq \sum_{1 \leq k \leq n} kJ(k)$$

we find for the critical inverse temperature β_c relation (6.1).

Let us now give some criticism. If $J(n) = 1/n^\alpha$ with $1<\alpha<2$ then (6.1) reduces to the trivial bound $\beta_c \geq 0$. So our result is curiously sensitive near $\alpha = 2$ but is insensitive for smaller α. FRÖHLICH and SPENCER (1982) mention a correlation inequality from which it seems our result cannot be derived. However at $\alpha = 2$ use of their inequality gives a better result than we get from corollary 3.4 for spin correlation asymptotics. As is illustrated in remark 4.1, the comparison we make in our method is related to a renewal process, which is one of the simplest 1-dimensional random processes. We hope that using other random processes may give better results. In particular it would be interesting if the bound in DOBRUSHIN (1968) could be included in an improved comparison result.

Further details are given in the following remark.

REMARK 6.1: Using Tauber theory we show for the case $J(n) = 1/n^2$ that the inhomogeneous chain of our method gives better bounds than the homogeneous chain. To know this may be of value for our method. We succeed to improve $\beta_c \geq 1/8$ to $\beta_c \geq 1/4$. A clue to this improvement is that the second term in (6.2) does not seem to play a role for large n. We should note however that use of DOBRUSHIN (1968) gives a better bound (FRÖHLICH and SPENCER (1982) give $\beta_c \geq 3/\pi^2$). By (4.5) and (4.4) we find

$$h_k^{(n+k)} = \exp(-4\beta[x_1 + \cdots + x_k + x_{2n+k+1} + \cdots + x_{2n+2k}])$$

where $x_j = \sum_{i \geq j} J(i)$. As $n \to \infty$ we have $h_k^{(n+k)} \uparrow h_k^{(\infty)}$. With the obvious notation we find from lemma 4.2

$$\frac{\psi_0(s)}{\psi_\infty(s)} = \frac{1}{\psi_\infty(s)} + \sum_{k \geq 1} s^k \frac{\psi_k(s)}{\psi_\infty(s)} \delta_k.$$

Because $\psi_k/\psi_\infty \leq 1$ and $\sum_{k \geq 1} \delta_k \leq 1$ we may apply the bounded convergence theorem. Using FELLER (1969), theorem XIII.5.5, one finds for $\beta = 1/4$ that

$$\frac{\psi_k(s)}{\psi_\infty(s)} \sim \frac{1}{2}$$

as $s \uparrow 1$. Because $\psi_\infty(1) = \infty$ we find $\sum_{k \geq 1} \delta_k = 1$ and $\Delta_n \downarrow 0$. Thus $\beta_c \geq 1/4$.

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