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Preservation of products by functors close to reflectors

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Preservation of Products by Functors Close to Reflectors

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It is shown that reflectors and similar functors in algebraic and topological-algebraic structures in many cases commute with products. In particular, reflectors of the category of (semi) topological semigroups into the subcategory of compact topological semigroups or groups have this property. The proofs are straightforward and avoid the use of almost periodic functions.
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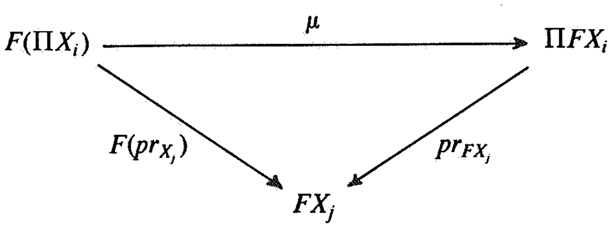
1. Introduction

In this paper we study functors close to reflectors and we consider the question in which cases they preserve products. It turns out that this is often the case when some kind of algebraic structure is involved. Our interest in this problem was stimulated by the fact that we did not understand the proof in [*Ho*] that products of topological groups are preserved by the Bohr compactification functor (it is all right if all groups involved are abelian). The proof in [*Ho*] doesn't use almost periodic functions, while all other papers dealing with this question known to us are based on the theory of those functions. Our approach is directly based on the categorical properties involved and we believe that it is essentially simpler; in addition, it applies to many other situations.

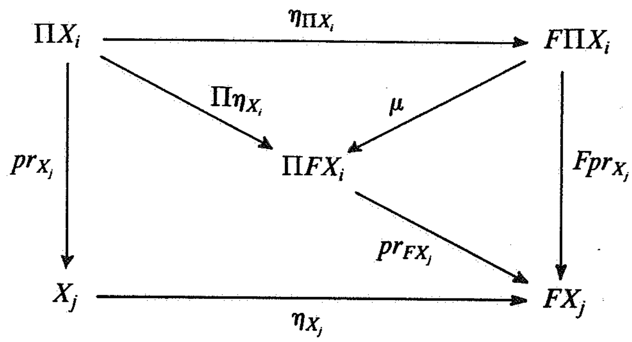
Let $F:\mathcal{K}_1 \to \mathcal{K}_2$ be a covariant functor between categories $\mathcal{K}_1$ and $\mathcal{K}_2$ (for categorical notions we refer to [*HS*]) and assume that for a set $\{X_i\}$ of objects in $\mathcal{K}_1$ both the products ΠX_i in $\mathcal{K}_1$ and ΠFX_i in $\mathcal{K}_2$ exist. Then there is a canonical morphism $\mu_{\{X_i\}}:F(\Pi X_i) \to \Pi FX_i$ (shortly: μ), defined uniquely by the condition that the following diagram commutes for every j



(here *pr* means: projection). If μ is an isomorphism in $\mathcal{K}_2$ then we shall say that "F preserves the product of $\{X_i\}$" or "F commutes with the product of $\{X_i\}$. There are many examples where F always preserves products, e.g. if F is a right adjoint, or if F is a covariant Hom-functor, or if F is a product functor. As we are more interested in reflectors, these results are of little use for us (see however the beginning of Section 2). We shall consider a situation which always occurs of F is a reflector, but which is more general: we shall assume that $\mathcal{K}_1 = \mathcal{K}_2 =: \mathcal{K}$, so that F is an endofunctor of K, and we shall assume that there is a natural transformation $\eta: 1_{\mathcal{K}} \to F$. In that case one has the equality

$$\mu_{\{X_i\}} \circ \eta_{\Pi X_i} = \Pi \eta_{X_i} \tag{1}$$

which follows from the following commutative diagram:



In the sequel we shall sometimes say that such an F is "close to a reflector". Let us now summarize several relevant known results from various structures.

EXAMPLES. 1. In the category of topological spaces one of the most studied reflections is the Čech-Stone compactification. It is known [G] that for completely regular spaces μ is a homeomorphism if and only if ΠX_i is pseudocompact (granted some non-triviality condition). A similar assertion is true for zero-dimensional spaces and the Banaschewski compactification ([Hu_2],[Br]). The problem when the Hewitt real compactification υ preserves products is still open. For partial results see e.g. [C],[Hu_1] and [Oh], where one can find other references. In any case, the property $\upsilon(X\times Y)=\upsilon X\times\upsilon Y$ is not a topological property of the space $X\times Y$ (see [Hu_1]). In the positive results for υ (and, similarly, in results for the topological completion; see [Pu]), local compactness plays an important role. This is not by accident: locally compact spaces are so-called exponential objects (i.e. $-\times X$ has a right adjoint) and in [Sch] for such objects X situations are characterized where $F(X\times Y)=FX\times FY$ (one can find in [Sch] other references to similar results, e.g. by B. Day and O. Wyler).

2. In the previous example the failure of μ to be an isomorphism (or even an injection) in general is basically due to the fact that a dense embedding of a space X into a space of the form FY is not uniquely determined by X. But this is really the case for completions of structures with uniformly continuous mappings as in **Metr, Unif, Top Vs, Norm**: completions are unique, hence the completion functor preserves all products.

3. An interesting example is Herrlich's wild reflection of **Top** into the full subcategory generated by $\{C^\kappa \mid \kappa$a cardinal number$\}$, where C is a strongly rigid Hausdorff space, e.g. Cook's continuum (see [H]). Here the reflector preserves a product ΠX_i iff every continuous mapping $\Pi X_i \to C$ depends on at most one coordinate.

4. The previous examples dealt with embeddings. Another type of reflections are those where the units are just bijections or surjections. Example 2 in Section 2 below shows that the T_0-modification in **Top** preserves finite products; as is indicated there, the T_1-, T_2- and T_3-modifications do not preserve finite products. For the $T_{3\frac{1}{2}}$-modification F of (not necessarily T_1-) topological spaces probably the strongest results are in [I]: if X is completely regular then $F(X\times Y)=X\times FY$ for every regular space Y iff X locally compact (cf. also the final remark in Example 1 above). In **Unif**, the precompact modification functor F is an example of a reflector where the units are not embeddings. It commutes with the product of $\{X_i\}$ if at most one of the spaces X_i is not precompact; moreover, for any space $X, F(X\times X)=FX\times FX$ iff X is precompact (see [Č]).

5. Let $\mathcal{K}$ be the category of partially ordered sets and monotone mappings which are either sup-preserving or inf-preserving or sup-inf-preserving. Then the reflection of $\mathcal{K}$ into the full subcategory of complete partially ordered sets preserves products (the form the of reflection depends on the type of morphisms; compare with [*HS*], p.180).

6. Let **SGrp** denote the category of semigroups; if not stated otherwise we shall assume that each semigroup has a unit and that homomorphisms of semigroups preserve the units. With **TopSGrp** (respectively, **STopSGrp**) we shall denote the category of all topological (respectively, semi-topological) semigroups; recall, that in a topological semigroup S the semigroup operation $S\times S\to S$ is simultaneously continuous, whereas in a semitopological semigroup it is only separately continuous. Apart from Holm's paper mentioned above the following papers deal with preservation of products by reflections of these categories into their full subcategories of compact objects (in obvious notation, **CompSTopSGrp**, **CompTopSGrp** and **CompTopGrp** are reflective subcategories of **STopSGrp**; the reflections of an object X in these categories are often denoted as X^{WAP} (weakly almost periodic compactification), X^{AP} (almost periodic compactification) and X^{SAP} (strongly almost periodic compactification)): generalizing work of [*LG*] and [*BM*],[*Ju*$_2$] shows that the functor $(\cdot)^{AP}$ preserves arbitrary products, and in [*BM*],[*Ju*$_1$] it is shown that $(\cdot)^{WAP}$ doesn't preserve finite products.

7. S. Dierolf proved in [*D*] that every bireflection (i.e. the unit consists of bimorphisms) in the category **TopVS** of topological vector spaces preserves products. This was generalized in [*Sy*] for endofunctors F of productive subcategories $\mathcal{K}$ of **TopVS** for which there exists a certain natural bi-transformation $\eta:1_{\mathcal{K}}\to F$. Our results in Section 2 below are even more general.

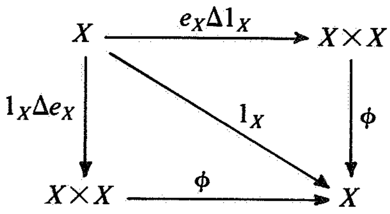
8. Let G be a topological group and let $\mathcal{K}$ be the category $\mathbf{Top}^G$ of all topological transformation groups with acting group G and continuous equivariant mappings (see e.g. [V_1]). Let for an object X of $\mathbf{Top}^G, \eta_X:X\to FX$ be the reflection of X into the subcategory of compact objects in $\mathbf{Top}^G$. Similar as in Example 1, if G is locally compact and locally connected, then $\mu:F(\Pi X_i)\to\Pi FX_i$ is an isomorphism iff ΠX_i is pseudocompact (apart from trivial cases); see [V_2],[V_3].

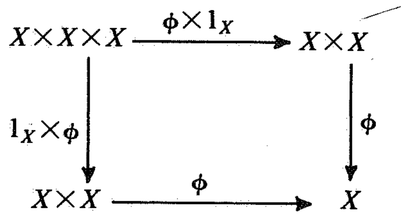
We shall present our results for the situation described in the beginning of this introduction in two parts: Section 2 deals with finite products and Section 3 with infinite products. Although in both cases the approach has a common idea, in details different procedures must be used. Also, for infinite products the results are less general.

2. Finite products

The main results of this section are stated for algebraic structures (with or without an additional topological structure). In most cases a functor close to a reflector preserves finite products. For non-algebraic structures the method gives a weaker version of preservation (e.g. μ a bijection but not necessarily an isomorphism), which is nevertheless useful.

As observed already in the Introduction, sometimes the preservation of (finite) products by reflections follows from general results. For example, let $\mathcal{K}$ be a category where finite products and coproducts exist and coincide (a so-called *semi-additive category*; see [*HS*], Section 40;) and let $F:\mathcal{K}\to\mathcal{K}_1$ be a reflector into a full subcategory $\mathcal{K}_1$ of $\mathcal{K}$. Then F preserves coproducts, hence all finite products (in $\mathcal{K}_1$, products and coproducts coincide as well). Examples of semi-additive categories are **Ab**, Rng, *R*-**Mod** (*R* any ring), their topological versions **TopAb**, **TopRng**, **TopVS** and their full subcategories (similarly, their continuity versions). Also the full subcategories of al commutative objects in **SGrp** and **TopSGrp** (*not* of **STopSGrp**) are semi-additive (together with Theorem 4 in Section 3 below this accounts e.g. for the preservation result in [*LG*]). We shall consider a slightly more general situation: an endofunctor of a semi-additive category which is close to a full reflector. Although in Theorem 1 below we shall use that finite products and coproducts have identical objects, it will nevertheless be convenient to recall the following characterization: a category $\mathcal{K}$ is semi-additive iff it has finite products, it is "pointed" (i.e., $\mathcal{K}(X,Y)$ contains a unique zero morphism $e_{X,Y}$ for any two objects X and Y in $\mathcal{K}$) and it has a "categorical" binary operation ϕ. This last condition means the following : if the diagonal product operation is denoted by Δ, then $\phi:1_{\mathcal{K}}\Delta 1_{\mathcal{K}}\to 1_{\mathcal{K}}$ is a natural transformation such that for each object X in $\mathcal{K}$ the following diagrams commute:

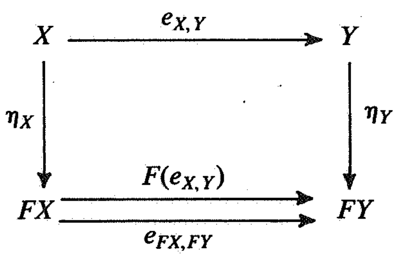




Here $e_X:=e_{X,X}$, the zero morphism of X. (That this characterization is equivalent with the definition of semi-additive category as given in [*HS*] follows easily from the observation that if $\mathcal{K}$ is semi-additive, then one can take for ϕ_X the codiagonal map; conversely, if $\mathcal{K}$ satisfies the above conditions, then "addition" of morphisms $f,g:X\to Y$ can be defined by $f+g:=\phi\circ(f\times g)\circ\delta_X$ where δ_X in the diagonal map.) In the characterization above, the condition that ϕ is a natural transformation expresses two properties, namely that each ϕ_X is a morphism in $\mathcal{K}$ and that all morphism of $\mathcal{K}$ are homomorphisms with respect to ϕ.

In the following theorem, $F(\mathcal{K})$ will denote the full subcategory of $\mathcal{K}$ generated by the F-images of objects of $\mathcal{K}$.

THEOREM 1. *Let $\mathcal{K}$ be a semi-additive category and let $F:\mathcal{K}\to\mathcal{K}$ be a covariant functor. If there is a natural transformation $\eta:1_{\mathcal{K}}\to F$ which is epi with respect to $F(\mathcal{K})$, then F preserves all finite products.*
PROOF. First observe that the functor F preserves zero-morphisms: the diagrams (for any pair X,Y of objects in $\mathcal{K}$)



commute for both $F(e_{X,Y})$ and $e_{FX,FY}$ (the latter, because compositions with zero-morphisms are again zero-morphisms). So the epi-property of η implies that $F(e_{X,Y})=e_{FX,FY}$.
Define $\nu:FX\times FY\to F(X\times Y)$ as the unique morphism in $\mathcal{K}$ such that

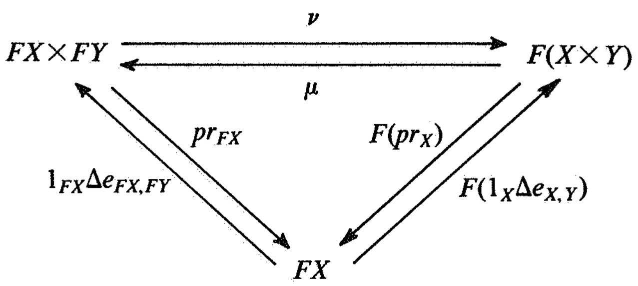
$$\nu\circ(1_{FX}\Delta e_{FX,FY}) = F(1_X\Delta e_{X,Y}), \nu\circ(e_{FY,FX}\Delta 1_{FY}) = F(e_{Y,X}\Delta 1_Y),$$

X and Y two given objects in $\mathcal{K}$ (here we use that $FX\times FY$ is the coproduct of FX and FY with canonical morphisms $1_{FX}\Delta e_{FX,FY}$ and $e_{FY,FX}\Delta 1_{FY}$). To show that μ is an isomorphism in $\mathcal{K}$ it is sufficient to prove that $\mu\circ\nu=1_{FX\times FY}$ and $\nu\circ\mu=1_{F(X\times Y)}$. Then induction on the number of factors of a products completes the proof.
The equality $\mu\circ\nu=1_{FX\times FY}$ is equivalent to four other equalities, namely,

$$pr_{FX}\circ\mu\circ\nu\circ(1_{FX}\Delta e_{FX,FY}) = pr_{FX}\circ(1_{FX}\Delta e_{FX,FY}).$$

$$pr_{FX}\circ\mu\circ\nu\circ(e_{FY,FX}\Delta 1_{FY}) = pr_{FX}\circ(e_{FY,FX}\Delta 1_{FY}).$$

and the two equalities obtained from these by replacing pr_{FX} with pr_{FY}. The proof that these equalities hold follows easily from the commutative inner and outer triangles of the following diagram:



Thus,

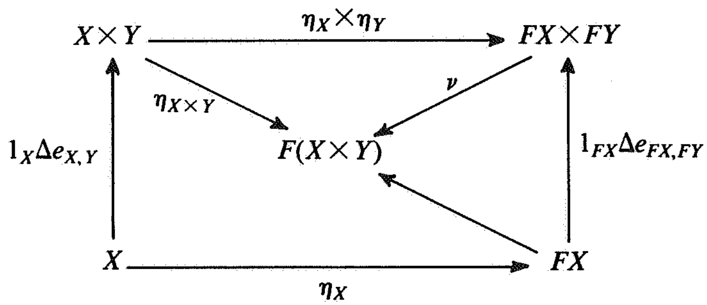
$$(pr_{FX} \circ \mu) \circ (\nu \circ (1_{FX} \Delta e_{FX,FY})) = F(pr_X) \circ F(1_X \Delta e_{X,Y}) = F(1_X) =$$
$$= 1_{FX} = pr_{FX} \circ (1_{FX} \Delta e_{FX,FY}) ,$$
$$(pr_{FX} \circ \mu) \circ (\nu \circ (e_{FY,FX} \Delta 1_{FY})) = F(pr_X) \circ F(e_{Y,X} \Delta 1_Y) = F(e_{Y,X})$$
$$= e_{FY,FX} = pr_{FX} \circ (e_{FY,FX} \Delta 1_{FY}) ,$$

and similarly for the equalities with pr_{FY} instead of pr_{FX}.

In order to prove the equality $\nu \circ \mu = 1_{F(X \times Y)}$ it is sufficient to show that $\nu \circ \mu \circ \eta_{X \times Y} = \eta_{X \times Y}$, because of the assumption on η. Note, that $\mu \circ \eta_{X \times Y} = \eta_X \times \eta_Y$, so we must show that $\nu \circ \eta_X \times \eta_Y = \eta_{X \times Y}$, which is equivalent with the following two equalities:

$$\nu \circ (\eta_X \times \eta_Y) \circ (1_X \Delta e_{X,Y}) = \eta_{X \times Y} \circ (1_X \Delta e_{X,Y})$$
$$\nu \circ (\eta_X \times \eta_Y) \circ (e_{Y,X} \Delta 1_Y) = \eta_{X \times Y} \circ (e_{Y,X} \Delta 1_Y) .$$

That these equalities hold follows without difficulties from the commutative triangles above, together with the following diagram:



□

REMARKS. 1. In the proof that $\mu \circ \nu = 1_{FX \times FY}$ the existence of η (and its epi-property) was only used in order to show that F preserves zero-morphisms. Consequently:

If F is a covariant endofunctor of a semi-additive category preserving zero-morphisms, then the canonical morphism $\mu: F(\Pi X_i) \to \Pi FX_i$ for finite products is a retraction.

The condition that F preserves zero-morphisms cannot be left out: if $\mathcal{K} = \mathbf{Ab}$, $FX:=\mathbb{Z}\times X, Ff=1_{\mathbb{Z}}\times f$, then $\mu:\mathbb{Z}\times X\times Y\to\mathbb{Z}\times X\times\mathbb{Z}\times Y$ is given by $\mu(n,x,y)=(n,x,n,y)$(X and Y objects in $\mathbf{Ab}$, $n\in\mathbb{Z},x\in X,y\in Y$), hence μ is not surjective. Note, that in general, if F preserves all finite products (as in the situation of the theorem) then in particular F preserves void products, that is, F preserves the zero object.

2. In the theorem, the epi-property for η cannot be replaced by the condition that F preserves the zero object (zero-morphisms): if $\mathcal{K}=\mathbf{Ab}$, FX the free abelian group over $|X\setminus\{0\}|$, then F preserves the zero object, but μ is not injective in general (but, by Remark 1 above, μ is a retraction). It is, in fact, easy to show that "free algebraic structure" functors do not preserve products.

3. The conditions characterizing semi-additive categories as given just before Theorem 1 (existence and properties of ϕ and e; note, that $e:X\to e_X$ is a natural transformation from $1_{\mathcal{K}}$ to $1_{\mathcal{K}}$) cannot be weakened to the assumption that $e:1_{\mathcal{K}}\to 1_{\mathcal{K}}$ is just some natural transformation. For instance, the category of left zero semigroups (i.e., with multiplication $xy:=x$ in each of its objects) satisfies these weakened conditions: for any set X, put $e_X:=1_X, \phi_X:(x,y)\mapsto x:X\times X\to X$; then ϕ and e satisfy the conditions as expressed in the commutative diagrams just before Theorem 1. If X is a topological space, then e_X and ϕ_X are continuous. Stated otherwise, also the category of topological left zero semigroups satisfies the weakened conditions. Now let for each topological left zero semigroup $X, \eta_X:X\to FX$ denote its Čech-Stone compactification (endowed with its left zero semigroup structure). Usually, $\mu:F(X\times Y)\to FX\times FY$ is not injective for Tychonov spaces X and Y.

4. The proof of Theorem 1 is of local character: it uses only $X, Y, X\times Y$ and the images of these objects under F, together with certain morphisms between these objects. It is easy to reformulate Theorem 1 so as to apply to a fixed finite product.

5. In the proof of Theorem 1 (for the case of a products $X\times Y$ of two factors) we needed only that e_X is a *right unit* of X and that e_Y is a *left unit* of Y. Remark 3 above shows, that the existence of units of some kind is necessary (see also [*Mi*],[*BM*]).

For non-commutative algebraic structures (e.g. for **Grp**) the binary operation $X\times X\to X$ and, consequently, the canonical mapping $X\times Y\to X+Y$ are not morphisms in the category under consideration, but in some auxiliary "underlying" category (e.g. **Set**). We shall modify Theorem 1 to such a situation. First a definition:

DEFINITION. A category $\mathcal{K}$ is said to be *semi-additive over a category* $\mathcal{X}$ whenever it satisfies the following conditions:

1. $\mathcal{K}$ has finite products;
2. $\mathcal{K}$ has zero-morphisms (for objects X and $Y, e_{X,Y}$ will denote the zero-morphism from X to Y, and $e_X:=e_{X,X}$);
3. There is a faithful functor $|-|:\mathcal{K}\to\mathcal{X}$ which preserves all finite products and reflects all isomorphisms;
4. There is a natural transformation $\phi:|1_{\mathcal{K}}\Delta 1_{\mathcal{K}}|\to|1_{\mathcal{K}}|$ such that

$$\phi\circ(|1_{\mathcal{K}}|\Delta|e|) = \phi\circ(|e|\Delta|1_{\mathcal{K}}|) = |1_{\mathcal{K}}| \text{ and } \phi\circ|\phi\times 1_{\mathcal{K}}| = \phi\circ(|1_{\mathcal{K}}|\times\phi)$$

(compare with the diagrams just before Theorem 1).

Note, that we do not use a characterization in terms of sums (e.g. existence of $|X+Y|$ and requiring $\mathcal{X}(|X|\times|Y|,|X+Y|)$ to have certain properties).

The categories **Grp**, **SGrp** and their full subcategories are semi-additive over **Set; TopGrp, TopSGrp** are semi-additive over **Top**. Similarly, the categories of uniform groups or convergence groups are semi-additive over the category of uniform spaces or convergence spaces, respectively. The categories of semi-topological structures (i.e. $\phi_X:XxX\to X$ is separately continuous; e.g. **STopGrp, STopSGrp**, etc.) are *not* semi-additive over some category: they are not so over **Top** (because ϕ_X is not continuous) and they are not so over any other category (**Set**, for example) because then $|-|$ doesn't reflect isomorphisms (of course, we could leave this condition out of the definition, but then we would have to include it in Theorem 2 below).

THEOREM 2. *Let the category $\mathcal{K}$ be semi-additive over the category $\mathcal{X}$, and let $F:\mathcal{K}\to\mathcal{K}$ be a covariant functor. If there is a natural transformation $\eta:1_{\mathcal{K}}\to F$ such that, for each object X of $\mathcal{K}$, $|\eta_X|$ is an epimorphism with respect to $|F(\mathcal{K})|$, then F preserves finite products.*

PROOF. Since $|-|$ is faithful, one can show in the same way as in Theorem 1 that $F(e_{X,Y})=e_{FX,FY}$ for all objects X,Y in $\mathcal{K}$. In order to show that $\mu:F(X\times Y)\to FX\times FY$ is an isomorphism in $\mathcal{K}$ it is sufficient to show that $|\mu|$ is an isomorphism in $\mathcal{X}$. We shall show that its inverse in $\mathcal{X}$ is the morphism

$$\nu:=\phi_{|F(X\times Y)|}\circ|\{F(1_X\Delta e_{X,Y})\circ pr_{FX}\}\Delta\{F(e_{Y,X}\Delta 1_Y)\circ pr_{FY}\}|$$

(X,Y objects in $\mathcal{K}$). For convenience, we shall omit in the remainder of the proof all occurrences of the functor $|-|$, understanding the intention to consider all morphisms as belonging to the category $\mathcal{X}$.

To prove $\mu\circ\nu=1_{FX\times FY}$ is equivalent with showing that $pr_{FX}\circ\mu\circ\nu=pr_{FX}$ and $pr_{FY}\circ\mu\circ\nu=pr_{FY}$. We shall prove the first of these equalities:

$$pr_{FX}\circ\mu\circ\nu\overset{(1)}{=}\phi_{FX}\circ[F(pr_X)\times F(pr_X)]\circ\{[F(1_X\Delta e_{X,Y})\circ pr_{FX}]\Delta[F(e_{Y,X}\Delta 1_Y)\circ pr_{FY}]\}$$

$$=\phi_{FX}\circ\{[F(pr_X)\circ F(1_X\Delta e_{X,Y})\circ pr_{FX}]\Delta[F(pr_X)\circ F(e_{Y,X}\Delta 1_Y)\circ pr_{FY}]\}$$

$$\overset{(2)}{=}\phi_{FX}\circ\{pr_{FX}\Delta e_{FX\times FY,FX}\}$$

$$=\phi_{FX}\circ(pr_{FX}\times pr_{FX})\circ\{1_{FX\times FY}\Delta e_{FX\times FY}\}$$

$$\overset{(3)}{=}pr_{FX}\circ\phi_{FX\times FY}\circ(1_{FX\times FY}\Delta e_{FX\times FY})\overset{(4)}{=}pr_{FX}\,.$$

Here equality (1) is based on the fact that $F(pr_X)\circ\phi_{F(X\times Y)}=\phi_{FX}\circ(F(pr_X)\times F(pr_X))$ which follows from ϕ being a natural transformation; note also that $pr_{FX}\circ\mu=F(pr_X)$. Equality (3) follows similarly from ϕ being a natural transformation. In (2) it is used that $F(pr_X\circ(1_X\Delta e_{X,Y}))=F(1_X)=1_{FX}$ and $F(pr_X\circ(e_{Y,X}\Delta 1_Y))=F(e_{Y,X})=e_{FY,FX}$. Finally, (4) uses one of the axioms of ϕ.

Next we show that $\nu\circ\mu=1_{F(X\times Y)}$ or equivalently (by the assumption on η), $\nu\circ\mu\circ\eta_{X\times Y}=\eta_{X\times Y}$. Since $\mu\circ\eta_{X\times Y}=\eta_X\times\eta_Y$ we must prove $\nu\circ(\eta_X\times\eta_Y)=\eta_{X\times Y}$; as follows:

$$\nu\circ(\eta_X\times\eta_Y)\overset{(5)}{=}\phi_{F(X\times Y)}\circ\{[F(1_X\Delta e_{X,Y})\circ\eta_X\circ pr_X]\Delta[F(e_{Y,X}\Delta 1_Y)\circ\eta_Y\circ pr_Y]\}$$

$$=\phi_{F(X\times Y)}\circ\{[\eta_{X\times Y}\circ(1_X\Delta e_{X,Y})\circ pr_X]\Delta[\eta_{X\times Y}\circ(e_{Y,X}\Delta 1_Y)\circ pr_Y]\}$$

$$=\phi_{F(X\times Y)}\circ(\eta_{X\times Y}\times\eta_{X\times Y})\circ\{(1_X\times e_Y)\Delta(e_X\times 1_Y)\}$$

$$=\eta_{X\times Y}\circ\phi_{X\times Y}\circ\{(1_X\times e_Y)\Delta(e_X\times 1_Y)\}\overset{(6)}{=}\eta_{X\times Y}\,.$$

Here properties of ϕ and η as natural transformations are used. Also, (5) requires the definition of ν and the equality $pr_{FX}\circ(\eta_X\times\eta_Y)=\eta_X\circ pr_X$ (similarly for pr_{FY}), and (6) follows from the equality

$$\phi_{X\times Y}\circ\{(1_X\times e_Y)\Delta(e_X\times 1_Y)\}=1_{X\times Y}\,,$$

which can be proved by composing both sides with pr_X and pr_Y:

$$pr_X\circ\phi_{X\times Y}\circ\{(1_X\times e_Y)\Delta(e_X\times 1_Y)\}=$$

$$=\phi_X\circ(pr_X\times pr_X)\circ\{(1_X\times e_Y)\Delta(e_X\times 1_Y)\}$$

$$=\phi_X\circ\{(pr_X\circ(1_X\times e_Y))\Delta[pr_X\circ(e_X\times 1_Y)]\}$$

$$=\phi_X\circ\{(1_X\circ pr_X)\Delta(e_X\circ pr_X)\}$$

$$= \phi_X \circ (1_X \Delta e_X) \circ pr_X = pr_X \,,$$

and similarly for the composition with pr_Y. □

Remarks. The epi-property of η (cf. Theorem 1) is now needed in the category $\mathcal{X}$. If X is the category **Set**, then this requirement for η means that it is a surtransformation (all η_X's are surjections); we need this even if we consider the epi-property of η in $\mathcal{X}$ only with respect to the morphisms of $\mathcal{K}$ (i.e. morphisms in $\mathcal{X}$ which are homomorphisms in $\mathcal{K}$). This implies that for discrete algebraic structures Theorem 2 gives no better results than Theorem 3 below. But in categories of algebraic structures endowed with a continuity, like topological semigroups, convergence groups, etc., with the underlying category $\mathcal{X}$ equal to **Top**, **Conv**, **Unif**, etc., there are many examples of functors F (even reflectors) satisfying the conditions of Theorem 2 with η not a surtransformation. For concrete examples, see after Theorem 3.

Most of the remarks following Theorem 1 can be repeated here (but concerning Remark 1, one has only that $|\mu|$ is a retraction in $\mathcal{X}$). Note, that the reflection of isomorphisms is only needed with respect to $|F(\mathcal{K})|$ (in particular: $|\mu|$ an isomorphism in $\mathcal{X}$ must imply μ an isomorphism in $\mathcal{K}$); so if one considers a specific example, then this condition concerning $|-|$ can be relaxed slightly. As to the conditions for ϕ, it is essential in the proof that for each object X in $\mathcal{K}, \phi_{|X|}$ is a morphism in $\mathcal{X}$. If this holds only for objects in $F(\mathcal{K})$, then the first part of the proof is still valid, so $|\mu|$ is a retraction in $\mathcal{X}$. Also, most of the second part is valid, but the proof of equality (6) falls through. In concrete categories this equality can be proved, however, by applying the forgetful functor $\mathcal{X} \to$ **Set** (provided it preserves finite products). For example, if $\mathcal{K} :=$ **STopSGrp** (not semi-additive over **Top**, nor over **Set**) and $\mathcal{X} =$ **Top** (note, that the forgetful functor $|-|:\mathcal{K} \to \mathcal{X}$ reflects isomorphisms), then any functor F admitting a natural transformation $\eta: 1_{\mathcal{K}} \to F$ satisfying the conditions of Theorem 2 and which has $F(\mathcal{K}) \subseteq$ **TopSGrp** preserves finite products. See also Example 4 below.

The most general situation (and weakest result) is the following theorem. We shall say that a functor $|-|:\mathcal{K} \to \mathcal{X}$ *lifts constants* whenever for all objects X, Y in $\mathcal{K}$ the image of the set $\mathcal{K}(X, Y)$ under $|-|$ contains all constant morphisms of $\mathcal{X}(|X|, |Y|)$.

Theorem 3. *Let $\mathcal{K}$ be a category having finite products, let $\mathcal{X}$ denote either the category* **Set** *or the category* **SGrp**, *and let $|-|:\mathcal{K} \to \mathcal{X}$ be a faithful functor preserving finite products and lifting constants. If $F:\mathcal{K} \to \mathcal{K}$ is a covariant functor and $\eta: 1_{\mathcal{K}} \to F$ is a natural transformation then $|\mu|$ is injective on the image of $|\eta_{\Pi X_i}|$ for each finite family $\{X_i\}$ of objects in $\mathcal{K}$.*

Proof. It is sufficient to prove the assertion for products of two factors X and Y. In view of formula (1) in the Introduction the following must be shown: if $(x,y),(x',y') \in |X| \times |Y|$ and $|\eta_X \times \eta_Y|(x,y) = |\eta_Y \times \eta_Y|(x',y')$, then $|\eta_{X\times Y}|(x,y) = |\eta_{X\times Y}|(x',y')$. First, suppose that $\mathcal{X} =$ **Set**. Then for every $b \in |Y|$ we have the following commutative diagram in $\mathcal{X}$ (here c_b denotes the lifted constant morphism $X \to Y$ that has the value b in $|Y|$:

$$\begin{array}{ccc} X\times Y & \xrightarrow{\eta_{X\times Y}} & F(X\times Y) \\ \uparrow{\scriptstyle 1_X \Delta c_b} & & \uparrow{\scriptstyle F(1_X \Delta c_b)} \\ X & \xrightarrow{\eta_X} & FX \end{array}$$

Together with the equality $|\eta_X|(x)=|\eta_X|(x')$ this implies

$$|\eta_{X\times Y}|(x,b) = \eta_{X\times Y}(x',b)\,. \qquad (2)$$

Similarly, the assumption $|\eta_Y|(y)=|\eta_Y|(y')$ implies for every $a\in|X|$:

$$|\eta_{X\times Y}|(a,y) = |\eta_{X\times Y}|(a,y')\,. \qquad (3)$$

Substituting $b:=y$ in (2) and $a:=x'$ in (3) one gets the desired result.

In the case that $\mathfrak{X}$ = **SGrp** one obtains in a similar way the equalities (2) and (3), but now with $a:=e_X, b:=e_Y$, i.e., only for the unit elements. But as

$$(x,y) = (x,e_Y).(e_X,y),\ (x',y') = (x',e_Y).(e_X,y')$$

and $|\eta_{X\times Y}|$ preserves the multiplication in the semigroups, it follows easily that $|\eta_{X\times Y}|(x,y)=|\eta_{X\times Y}|(x',y')$, as desired. □

REMARK. As in the proof of Theorem 1, in the above proof for a product of two factors X and Y only the existence of a right unit in X and a left unit in Y is needed.

For the following corollaries, recall that if $|-|:\mathfrak{K}\to\mathfrak{X}$ is a faithful functor, then a morphism $f:X\to Y$ in $\mathfrak{K}$ is said to be a *quotient* (w.r.t. $|\text{-}|:\mathfrak{K}\to\mathfrak{X}$) if $|f|$is an epimorphism in $\mathfrak{X}$ and if, in addition, $g\circ|f|\in|\mathfrak{K}(X,Z)|$ for some morphism g in $\mathfrak{X}$ and object Z in $\mathfrak{K}$, implies $g\in|\mathfrak{K}(Y,Z)|$. In our case, where $\mathfrak{X}$ is either **Set** or **SGrp**, quotients are always surjective (or rather, their "underlying" mappings in $\mathfrak{X}$ are surjective; but we prefer to use adjectives like surjective, injective, etc. also for morphisms in $\mathfrak{K}$).

COROLLARY 1. *Under the assumptions of Theorem 3, if η is a surtransformation, then μ is bijective for finite products. If, moreover, the faithful functor $|-|:\mathfrak{K}\to\mathfrak{X}$ reflects isomorphisms of $|F(\mathfrak{K})|$, then F preserves all finite products.* □

COROLLARY 2. *Under the assumptions of Theorem 3, if both η_X and η_Y are quotient and also $\eta_X\times\eta_Y$ is quotient, then $\mu:F(X\times Y)\to FX\times FY$ is an isomorphism.*

PROOF. If η_X and η_Y are surjective then μ is a surjection (use formula (1)), hence a bijection. If $\eta_X\times\eta_Y$ is quotient, then (1) implies that μ is quotient. Now observe, that every bijective quotient is an isomorphism. □

EXAMPLES. 1. For purely algebraic categories like **Grp, Ab, R-Mod, SGrp, BoolAlg, Rng**, and their full subcategories, Theorem 3 implies that each endofunctor F which admits a surtransformation (in particular: each sur-reflection) preserves finite products. As observed earlier, in this situation one cannot obtain stronger results via Theorems 1 and 2.

2. For categories of purely continuity structures like **Top, Conv, Unif**, and their full subcategories the forgetful functor into **Set** almost never reflects isomorphisms. In these cases Theorem 3 gives only that if there is a surtransformation $\eta:1_{\mathfrak{K}}\to F$ then $\mu:F(\Pi X_i)\to\Pi FX_i$ in a bijection for finite products, that is, $F(\Pi X_i)$ may be regarded as ΠFX_i but endowed with a finer structure. This is the case for the regular and completely regular modification functors in **Top** and the precompact modification functor in **Unif** (cf. also Example 4 in the Introduction). The T_0-modification in **Top** is a quotient reflection (i.e. each η_X is quotient: it is even an open mapping) and Corollary 2 shows that it preserves all finite products. The T_1- and T_2- modifications are also quotient, but since in general $\eta_X\times\eta_Y$ needs not be quotient, Corollary 2 cannot be applied, and actually, the T_1- and T_2- modifications do not preserve all finite products (consider $1\times\eta:Q\times X\to Q\times Y$, where $X=\omega\times(\omega+1)$ with the topology in which all points of $\omega\times\omega$ are discrete, while points of the form $(n,\omega)\in\omega\times(\omega+1)$ have a nbd base consisting of sets of the form $\{(i,\omega)|i\leqslant n\}$; the space Y is the quotient of X obtained by identification of the subset $\{(n,\omega)|n\in\omega\}$ to one point; the quotient map $\eta:X\to Y$ is the T_2- (hence T_1)-modification of X).

Nevertheless, Corollary 2 can be applied in some cases, e.g. the T_1-modification of symmetric spaces (i.e., spaces in which $x \in \overline{\{y\}}$ implies $y \in \overline{\{x\}}$, or the T_2- modification of completely regular (not necessarily T_1-) spaces preserve finite products (which is also easy to prove directly). There are categories of continuous structures where quotients are productive: **Unif, Prox** (cf. [*HR*]), and the categories of merotopic spaces or of convergence spaces (not in **Near**, [*R*]). In such categories, all quotient reflections preserve finite and (by Corollary 2 of Theorem 4 below) infinite products.

3. An exception of the general statement with which 2 above begins is the category **Comp** of compact Hausdorff spaces: here the forgetful functor to **Set** reflects isomorphisms (also, surjective morphisms are quotients and quotients are productive). So if F: **Top** $\rightarrow$ **Top** is a covariant functor with F (**Top**) $\subseteq$ **Comp** and $\eta: 1 \rightarrow F$ is a surtransformation, then F preserves all finite (and, by Theorem 4 below) all infinite products. In particular, every epireflector F: **Comp** $\rightarrow$ **Comp** preserves products. Example 3 from the Introduction shows that one cannot remove the epi-condition. Similar remarks can be made for the category $\mathbf{Ban}_1$ of all Banach spaces and bounded linear transformations: by the Open Mapping Theorem, bijective morphisms are isomorphisms. Thus, for example, every epireflection F: $\mathbf{Ban}_1$ $\rightarrow$ $\mathbf{Ban}_1$ preserves all products. Also, in the category of standard Borel spaces and Borel mappings, every bijective morphism is an isomorphism (see [*Ma*] for references); we leave the conclusions to the reader.

4. Applications of Theorems 1 and 2 are mainly in categories of structures that are both of algebraic and of continuous character. Thus, if $\mathcal{K}$:= **STopSGrp** and $\eta_X: X \rightarrow FX$ denotes the reflection of an object X from **STopSGrp** into **CompTopSGrp** or into **CompTopGrp** (the almost periodic, respectively strongly almost periodic compactification of X), then F preserves all finite products (for infinite products, see Theorem 5 below). For comments and references, see Example 6 in the Introduction. Here we stress the fact that our proof uses only the categorical properties of these compactifications (the proof is "intrinsic") and make no use of (weakly) almost periodic functions. More generally, every surreflection from **STopSGrp** into **TopSGrp** preserves finite products, and every dense-reflection (i.e. each $\eta_X: X \rightarrow FX$ has a dense range) from **STopSGrp** into $\mathbf{TopSGrp}_{Haus}$ preserves finite products (cf. the Remark following Theorem 2). A completely different application is the one, mentioned in Example 7 of the Introduction: by Theorem 2, every covariant functor F: **TopVS** $\rightarrow$ **TopVS** (or $\mathcal{K} \rightarrow \mathcal{K}$, where $\mathcal{K}$ is a productive full subcategory of **TopVS**) with values in the full subcategory of Hausdorff spaces and for which there is a dense-transformation $\eta: 1_{\mathcal{K}} \rightarrow F$ preserves finite products. See also the Remark after Theorem 4 below.

5. The condition that $|-|$ lifts constants cannot be omitted from Theorem 3. Let $\mathcal{K}$ be the category $\mathbf{Top}^G$ (cf. Example 8 in the Introduction). For each object $<X,\pi>$ in $\mathbf{Top}^G$ (i.e. π is the action of G on X) let X/C_π be the orbit space of $<X,\pi>$ and τ the trivial action of G on X/C_π. Then the quotient mapping $\eta_X: <X,\pi> \rightarrow <X/C_\pi, \tau>$ is a morphism in $\mathbf{Top}^G$ and it is a quotient (in fact, $\eta_X: X \rightarrow X/C_\pi$ is an open mapping). Although η is a surtransformation, μ is not injective in general: take $X=Y=G$ with $\pi(t,k):=tx$ for $t,x \in G$. Then X/C_π is a singleton, hence $(X/C_\pi)\times(Y/C_\pi)$ is a singleton. On the other hand, the orbit space of $X \times Y$ is (the underlying topological space of) G.

6. It is known that the category of complete convergence groups is a full reflective subcategory of the category of all convergence groups (see e.g. [*N*], also for earlier references to convergence groups and their completions, and [*K*] for later results). The natural map from a convergence group to its universal completion needs not be injective, and completions need not be unique. It follows from Theorem 2 that the reflector from the category of convergence groups into the category of complete convergence groups commutes with finite products. (This result was known to R. Frič and V. Koutník, but not published).

7. By the facts, mentioned in Example 3 above, if a functor F into a category of compact structures preserves a product ΠX_i and$G$ is an epireflection into a smaller category of compact structures, then $G \circ F$ preserves that product as well. For instance, if G_1 is an epireflection from $\mathbf{Top}_{3\frac{1}{2}}$ into a subcategory of **Comp** and ΠX_i is pseudocompact, then $G_1(\Pi X_i) = \Pi G_1 X_i$, because G_1 factorizes as $G_1 = G \circ F$ with F the Čech-Stone reflector. It is interesting to compare this result with [*CH*] where in some sense a converse is obtained: if G: $\mathbf{Top}_2 \rightarrow \mathcal{A}$ and F: $\mathbf{Top}_2 \rightarrow \mathcal{B}$ are epireflections, $\mathcal{A} \subseteq \mathcal{B} \subseteq Reg\mathcal{A}$,

then for all objects X, Y in $Reg\mathcal{C}$, the equality $G(X\times Y)=GX\times GY$ implies $F(X\times Y)=FX\times FY$.

3. Infinite Products

Easy examples show that in categories of discrete algebraic structures reflections do not preserve infinite products even if they preserve finite ones (according to Example 1 in Section 2). For example, take $\mathcal{K}:=$ **Grp** and for $\eta_X: X\to FX$ the quotient map of the group X onto X/X_0, where X_0 is the torsion subgroup of X. Then for $\mathbb{Z}_n$, the cyclic group of n elements, $F\mathbb{Z}_n=\{0\}$, hence $\Pi_{n=1}^{\infty}F\mathbb{Z}_n=\{0\}$, but $\Pi_{n=1}^{\infty}\mathbb{Z}_n$ is not a torsion group, hence $F(\Pi_{n=1}^{\infty}\mathbb{Z}_n)\neq\{0\}$. A similar example can be given for the reflection F: **Grp** $\to$ **Ab**.

In algebraic structures, the finite products are directly determined by their factors (e.g. in **SGrp**, $(x,y)=(x,e_Y)\cdot(e_X,y)$ for $(x,y)\in X\times Y$): to determine infinite products by finite ones, one needs some kind of convergence. This is done in the following theorem, which is the infinite counterpart of Theorem 3:

Theorem 4. *Let $\mathcal{K}$ be a category which admits a faithful functor $|-|:\mathcal{K}\to$ **Top** and assume that $|-|$ preserves products. Moreover, let $F:\mathcal{K}\to\mathcal{K}$ be a covariant functor and $\eta: 1_{\mathcal{K}}\to F$ a natural transformation. If $|\mu|$ is injective on the image of $|\eta_{\Pi X_i}|$ for all finite products, then $|\mu|$ is injective on the image of $|\eta_{\Pi X_i}|$ for all infinite products with $|F(\Pi X_i)|$ a Hausdorff space.*

Proof. Suppose that κ is an infinite ordinal number, that $\{X_\alpha\}_{\alpha\in\kappa}$ is a family of objects of $\mathcal{K}$ for which both ΠX_α and ΠFX_α exist, and that $|F(\Pi X_\alpha)|$ is a Hausdorff space. Take $x,y\in|\Pi X_\alpha|$ such that $|\Pi\eta_\alpha|(x)=|\Pi\eta_\alpha|(y)$; as in the proof of Theorem 3 we have to show that $|\eta|(x)=|\eta|(y)$ (for simplicity we write η_α instead of η_{X_α} and η instead of $\eta_{\Pi X_\alpha}$). For $\beta\leqslant\kappa$ denote by z_β the point of $|\Pi X_\alpha|$ such that

$$pr_\alpha z_\beta=\begin{cases}pr_\alpha y & \text{for } \alpha<\beta\\ pr_\alpha x & \text{for } \alpha\geqslant\beta\end{cases}$$

(here pr_α is the projection of $|\Pi X_\alpha|$ onto $|X_\alpha|$). Observe, that $z_0=x$ and $z_\kappa=y$. We shall prove by transfinite induction that $|\eta|(z_\beta)=|\eta|(z_0)=|\eta|(x)$ for all $\beta\leqslant\kappa$. Obviously, this is true for $\beta=0$. Suppose our claim is true for all $\beta<\gamma$, where $0<\gamma\leqslant\kappa$. If γ is isolated (i.e. $\gamma-1$ exists) then, taking into account that the two-factor product $X_{\gamma-1}\times(\prod_{\alpha\neq\gamma-1}X_\alpha)$ is preserved by F, one easily sees that the equality of the images of the points $z_{\gamma-1}$ and z_γ under $|\eta_{X_{\gamma-1}}\times\eta_{\prod_{\alpha\neq\gamma-1}X_\alpha}|$ implies the equality of their images under $|\eta|$. Thus, one has $|\eta|(z_{\gamma-1})=|\eta|(z_\gamma)$. Together with the induction hypothesis it follows that $|\eta|(z_\gamma)=|\eta|(x)$. If γ is a limit then $z_\gamma=\lim_{\beta<\gamma}z_\beta$ hence $|\eta|(z_\gamma)=\lim_{\beta<\gamma}|\eta|(z_\beta)$; so by the induction hypothesis, $|\eta|(z_\gamma)=|\eta|(x)$ (observe, that $F(\Pi X_\alpha)$ has unique limits). This completes the proof, because for $\beta=\kappa$ the equality $|\eta|(z_\beta)=|\eta|(x)$ gives the desired result. □

Remarks. In the above proof, continuity of $|\eta|$ is needed, but only for special nets indexed over chains of length not larger than the cardinality of the index set of the product. Also, the functor $|-|:\mathcal{K}\to$ **Top** needs not preserve products in the full sense of the word: it suffices that $|\Pi X_i|$ is a cartesian product endowed with a topology which is coarser than the product topology obtained when all factors are given the discrete topology (or even coarser than the chain-net coreflection of that product). So instead of **Top** one may take **Conv**, and in sequential structures countable product are preserved. This is formulated in the following Corollaries; here we mean by a *continuity structure* a structure where convergence of chains is defined such that constant nets have its value as limit and such that subnet of a net having a limit has the same limit. The morphisms are required to preserve the convergence.

Corollary 1. *Let $\mathcal{K}$ be a category of continuous structures having a faithful functor into* **Set** *or* **SGrp** *which preserves products and lifts constants. If $F:\mathcal{K}\to\mathcal{K}$ is a covariant functor, has values in structures with unique limits and admits a surtransformation $\eta:1_{\mathcal{K}}\to F$, then μ is bijective on all products.*

PROOF. Combine Theorem 4 (together with the Remarks above) with Theorem 3 in order to see that μ is injective. Surjectivity follows easily from equation (1) in the Introduction. □

COROLLARY 2. *Under the assumptions of Corollary 1 for $\mathcal{K}$ and F, if η is a quotient-transformation then $\mu:F(\Pi X_i)\to\Pi FX_i$ is an isomorphism in $\mathcal{K}$ for a product ΠX_i in $\mathcal{K}$ iff $\Pi\eta_{X_i}$ is quotient.* □

REMARKS. We leave the formulation of similar Corollaries for sequential structures and countable products to the reader. Note, that in Corollary 1, if the faithful functor from $\mathcal{K}$ into **Set** or **SGrp** reflects isomorphisms in $|F(\mathcal{K})|$, then F preserves products (cf. one of the remarks following Theorem 2). Also, observe that no compatibility of algebraic and continuity structures was required: we needed only continuity of $|\eta|$.

The main application of Theorem 4 and Corollary 1 lies in balanced categories (i.e. bijective morphisms are isomorphisms), and most categories with an algebraic and a continuity structure are *not* balanced. Thus, in such categories, if F is an endofunctor admitting a surtransformation $\eta:1\to F$ and F has values in Hausdorff structures, then $F(\Pi X_i)$ and ΠFX_i have the same underlying set, but in general the continuity structure on the former in finer than that on the latter. In categories like **Unif**, **Conv**, **STopGrp**, **TopGrp**, **TopVS**, where quotients are productive, Corollary 2 can be used. For example, the reflector from **STopGrp** into **TopGrp** $_{Haus}$ preserves all products. Another example is a modification of the example at the beginning of this Section: in the category **TopGrp** denote for an object X the torsion subgroup by X_t; then $FX:=X/\overline{X}_t$ defines a reflection of **TopGrp** into **TopGrp** $_{Haus}$ (in fact, the torsion free Hausdorff groups), and F preserves all products. A similar example is obtained if one replaces X_t by X_c, the commutator subgroup of X (then one obtains the reflection into abelian Hausdorff groups). Notice, that Theorem 4 and Corollary 1 are also of interest in categories of compact structures and of Banach spaces (cf. Example 3 in Section 2).

One cannot hope to obtain more than the conclusion of Theorem 4, namely, that $|\mu|$ is injective on the image of $|\eta|$. To this end, consider Example 3 of the Introduction. It is interesting (namely, in connection with the first case in Theorem 5 below) that this example can also be given within the category **TopGrp** $_{Haus}$: by [vM] there exists a topological Hausdorff group S admitting no continuous endomorphisms but the obvious ones (the constant mapping with value the identity, and the identity mapping); with this object S, the procedure outlined in [H] can be performed in **TopGrp** $_{Haus}$.

Our final result is formulated in a local form in order to keep the presentation as general as possible and at the same time understandable.

THEOREM 5. *Let $|-|:\mathcal{K}\to$ **STopSGrp** be a faithful functor which preserves products, lifts constants and reflects isomorphisms; moreover, let $F:\mathcal{K}\to\mathcal{K}$ be a covariant functor and $\eta:1_{\mathcal{K}}\to F$ a natural transformation. If $\{X_i\}_{i\in I}$ is a set of objects in $\mathcal{K}$ then in the following cases $\mu:F(\Pi X_i)\to\Pi FX_i$ is an isomorphism:*

1. *$|F(\Pi X_i)|$ is a Hausdorff topological semigroup and $|\eta_{\Pi X_i}|$ is surjective;*
2. *$|F(\Pi X_i)|$ is a compact Hausdorff topological semigroup and $|\eta_{\Pi X_i}|$ maps $|\Pi X_i|$ onto a dense subset of $|F(\Pi X_i)|$.*

PROOF. We have to prove that $|\mu|$ is an isomorphism in the category **STopSGrp**. First, we shall show that $|\mu|$ is surjective. To this end, observe that for each $j\in I$ the canonical projection $p_j:\Pi X_i\to X_j$ is a retraction, the diagonal product $q_j:X_j\to\Pi X_i$ of 1_{X_j} and the zero-morphism $X_j\to\Pi_{i\neq j}X_i$ being a section (note, that $\mathcal{K}$ has zero-morphisms, obtained as liftings of the constant morphisms in **STopSGrp** that have unit elements as values). It follows that $F(p_j)$ is a retraction, so that $|F(p_j)|$ is surjective for each $j\in I$. However, $|\eta_{X_j}|\circ|p_j|=|F(p_j)|\circ|\eta_{\Pi X_i}|$, and this implies that $|\eta_{X_j}|$ is surjective (has dense range, respectively) if $|\eta_{\Pi X_i}|$ is surjective (has dense range, respectively). Now equation (1) in the Introduction implies that in case 1, $|\mu|$ is surjective and that in case 2, $|\mu|$ has a dense range. But in case 2, each $|FX_j|$ has a compact Hausdorff topology (being retract of $|F(\Pi X_i)|$ under $|F(p_j)|$), so that $|\Pi FX_i|$ has a Hausdorff topology, and therefore $|\mu|$ is a surjection in this case as well.

Next, we show that $|\mu|$ is injective. This will be sufficient for the second case, since we know already that $|\mu|$ is a surjection of compact Hausdorff structures. To prove injectivity of $|\mu|$ in case 1, we need only refer to Corollary 1 of Theorem 4 (or rather, a version of this Corollary for the given product ΠX_i, requiring only that each $|\eta_{X_i}|$ is surjective; cf. the proof of the Corollary). In case 2, proceed as follows. For any subset J of I, consider the following diagram in $\mathcal{K}$:

$$\begin{array}{ccccc}
\Pi_I X_i & \xrightarrow{\ \eta\ } & F(\Pi_I X_i) & \xrightarrow{\ \mu\ } & \Pi_I FX_i \\
\alpha_J \uparrow \downarrow p_J & & F(\alpha_J) \uparrow \downarrow F(p_J) & & \downarrow q_J \\
\Pi_J X_i & \xrightarrow{\ \eta_J\ } & F(\Pi_J X_i) & \xrightarrow{\ \mu_J\ } & \Pi_J FX_i
\end{array}$$

Here $\eta_J := \eta_{\Pi_J X_i}, \mu_J := \mu_{\{X_i|i\in J\}}, \eta := \eta_I, \mu := \mu_I$, the p_J and q_J are projections, and α_J is the diagonal product of $1_{\Pi_J X_i}$ with the zero-morphism $\Pi_J X_i \to \Pi_{I\setminus J} X_i$. Note, that for finite J the morphism $|\mu_J|$ is an isomorphism (cf. Example 4 in Section 2), so that in order to prove injectivity of $|\mu|$ it suffices to show that for $x,y \in |F(\Pi X_i)|, x \neq y$ implies that there exists a finite subset J of I with $|F(p_J)|(x) \neq |F(p_J)|(y)$. For the proof it will be convenient to introduce the following notation:

$$|\alpha_J \circ p_J| \ = \ :w_J \text{ and } \rho_J := |F(w_J)| \ = \ |F\alpha_J \circ Fp_J| \ = \ |F\alpha_J| \circ |Fp_J| \,.$$

Consider any point x in $|F(\Pi X_i)|$. We claim that the net $\{\rho_J x | J \in [I]^{<\omega}\}$ converges to x in $|F(\Pi X_i)|$. Assume the contrary: there is an open nbd U of x such that the set $\mathcal{F} = \{J | J \in [I]^{<\omega}, \rho_J x \notin U\}$ is cofinal in $[I]^{<\omega}$. By compactness, the set $\{\rho_J x | J \in \mathcal{F}\}$ has an accumulation point p in $|F(\Pi X_i)|$. Then $p \notin U$, so p has a nbd V such that $x \notin V$. Since $p = pe$(e the unit element in $|F(\Pi X_i)|$) and the binary operation in the semigroup $|F(\Pi X_i)|$ is continuous, there are nbds V^1 of p and V_e of e such that $V^1.\overline{V_e} \subseteq V$. Continuity of $|\eta|$ implies that there is a nbd W of $\{e_i\}_{i\in I}$ in $|\Pi_I X_i|$ such that $|\eta|(W) \subseteq V_e$. There is a finite subset J of I such that $w_{I\setminus J}(y) \in W$ for all $y \in \Pi_I X_i$: any finite subset of I determining a basic nbd of $\{e_i\}_{i\in I}$ in $\Pi_I X_i$, included in W, suffices; also, J can be taken large enough to guarantee that $J \in \mathcal{F}$ and $\rho_J x \in V^1$. Since $\rho_{I\setminus J}(|\eta| y) = |\eta| \circ w_{I\setminus J}(y) \in |\eta|(W) \subseteq V_e$ for all $y \in \Pi X_i$ and $|\eta|$ has a dense range, it follows that $\rho_{I\setminus J}(z) \in \overline{V_e}$ for all $z \in |F(\Pi X_i)|$. Next, notice that $y = w_J(y).w_{I\setminus J}(y)$ for all $y \in \Pi X_i$, hence $z = \rho_J(z).\rho_{I\setminus J}(z)$ for all z in the (dense) range of η. By a continuity argument, this equality holds for all $z \in |F(\Pi X_i)|$, which gives

$$x \ = \ \rho_J(x).\rho_{I\setminus J}(x) \in V^1.\overline{V_e} \subseteq V \,,$$

contradicting the choice of V. This proves our claim.

Clearly, this implies immediately that if $x,y \in |F(\Pi X_i)|$, and $x \neq y$, there is $J \in [I]^{<\omega}$ with $\rho_J(x) \neq \rho_J(y)$, hence $|Fp_J|(x) \neq |Fp_J|(y)$, as desired. This completes the proof that $|\mu|$ is injective in case 2. It remains to show that $|\mu|$ is an isomorphism in case 1. We know already that it is a continuous bimorphism. That $|\mu|$ is a homeomorphism can be proved as follows.

First, notice that in $|\Pi X_i|$ for each point y the net $\{w_J(y) | J \in [I]^{<\omega}\}$ converges to y. Since $|\eta|$ is a continuous surjection, it follows that in $|F(\Pi X_i)|$ for each point x the net $\{\rho_J x | J \in [I]^{<\omega}\}$ converges to x. Now consider a point x and an open nbd U of x in $|F(\Pi X_i)|$, and let V and V_e be nbds of x and e, respectively, such that $V.V_e \subseteq U$. As in the proof above one shows that there is $J \in [I]^{<\omega}$ such that $\rho_J x \in V$ and $\rho_{I\setminus J}(z) \in V_e$ for all $z \in |F(\Pi X_i)|$. Continuity of $|F\alpha_J|$ implies the existence of a nbd W of $|Fp_J|(x)$ in $|F(\Pi_J X_i)|$ with $|F\alpha_J|(W) \subseteq V$. Since $|\mu_J|$ is a homeomorphism (Example 4 in Section 2), $W^1 := q_J^{-1}(|\mu_J|(W))$ is a nbd of $|\mu|(x)$ in $\Pi_I |FX_i|$. Now for every point $y \in |\mu|^{-1}(W^1)$ one has $q_J |\mu|(y) \in |\mu_J|(W)$, that is, $|\mu_J|(|Fp_J|(y)) \in |\mu_J|(W)$, hence $|Fp_J|(y) \in W$ and consequently

$\rho_J(y) \in |F\alpha_J|(W) \subseteq V$. As before, $y = \rho_J(y).\rho_{I \setminus J}(y)$; since by the choice of J we have $\rho_{I \setminus J}(y) \in V_e$, it follows that $y \in V.V_e \subseteq U$. This shows that $W^1 \subseteq |\mu|(U)$ and $|\mu|$ is a homeomorphism. □

REMARKS. In the last paragraph of the above proof it was observed that if $|\eta_{\Pi X_i}|$ is surjective, then $\{\rho_J x | J \in [I]^{<\omega}\}$ converges to x in $|F(\Pi X_i)|$. If $|F(\Pi X_i)|$ has Hausdorff topology, then this can be used to give another proof of Theorem 4.

The above proof (also for case 2) can be so modified as to use only chains (then chain-compactness for chains of a certain length would be sufficient in case 2).

Finally, as in previous results, the functor $|-|$ needs only to reflect isomorphisms from $|F(\mathfrak{X})|$, which is in both cases a subcategory of $\mathbf{TopSGrp}_{Haus}$.

The most important applications of Theorem 5 are formulated in the following Corollaries:

COROLLARY 1. *The strongly almost periodic compactification* **STopSGrp** $\to$ **CompTopGrp** *and the almost periodic compactification* **STopSGrp** $\to$ **CompTopSGrp** *preserve all products.*

PROOF. Both functors are reflectors, satisfying the conditions of case 2 of Theorem 5. □

COROLLARY 2. *Every surreflector of* **STopSGrp** *into a full subcategory of* $\mathbf{TopSGrp}_{Haus}$ *preserves all products.* □

REMARKS. 1. The conditions on η in Theorem 5 cannot be omitted. To this end, modify Example 3 of the Introduction to one in the category $\mathbf{TopGrp}_{Haus}$: let F be the reflector of this category into the subcategory $\{G^\kappa | \kappa \text{ a cardinal }\}$, where G is a strongly rigid topological Hausdorff group (cf. [vM]).

2. Case 2 of Theorem 5 can also be adapted to the situation of a countable product of convergence groups and its reflection into the category of complete convergence groups. Indeed, by Example 6 of Section 2, finite products are preserved, and the proof that $|\mu|$ is injective can be given in a similar way as above: the only modification is that one has to show that the net $\{\rho_J x | J \in [I]^{<\omega}\}$ has a convergent subnet not by using a compactness argument but by observing that, for a countable product, $F(\Pi X_i)$ is complete and $\{\rho_J x | J \in [I]^{<\omega}\}$ is a Cauchy-sequence (which is easy to prove). Now also the last paragraph of the proof of Theorem 5 can be modified so as to work in the present situation (one has to choose V and V_e such that $V.\overline{V}_e \subseteq U$). Thus, completion of convergence groups preserves all countable products.

3. In the category **TopGrp**, Corollary 2 above can be improved so as to hold for dense-reflections into $\mathbf{TopGrp}_{Haus}$. We shall indicate a proof of the following statement: if F: **TopGrp** $\to$ $\mathbf{TopGrp}_{Haus}$ *is a covariant functor and* $\eta: 1 \to F$ *is a dense-transformation, then for all products* $\mu: F(\Pi X_i) \to \Pi F X_i$ *is an embedding.* To prove this, notice that η can be factorized as $1 \xrightarrow{\eta'} F' \xrightarrow{\eta''} F''$ where η' is a surtransformation and η'' is an embedding-transformation. By case 1 of Theorem 5, F' preserves all products. Thus, we need only to prove that our statement holds for the case that η is a dense-embedding transformation. With notation as in the proof of Theorem 5, let $x \in F(\Pi X_i), x \neq e$ and $\mu(x) = e$ (all identities are denoted e). There are disjoint nbds U_x of x, U_e of e in $F(\Pi X_i)$ and a canonical nbd V_e in ΠX_i depending on some $J \in [I]^{<\omega}$ such that $\eta(V_e) \subseteq U_e$. Then $\eta^{-1}(U_x) \cap V_e = \phi$, hence $pr_J \eta^{-1}(U_x) \cap pr_J V_e = \phi$. But $x \in \overline{\eta(\eta^{-1}(U_x))}$ because η has a dense range, hence $e = F(pr_J)(x) \in F(pr_J)\overline{\eta(\eta^{-1} U_x)} = \overline{\eta_J(pr_J \eta^{-1}(U_x))}$. As this set is disjoint from $pr_J V_e (\eta_J$ is injective) this is a contradiction, so μ is injective. From this it follows from a straightforward argument (taking into account that $\Pi F X_i$ as a product of Hausdorff groups is a regular space into which $F(\Pi X_i)$ is continuously injected by μ in such a way that the dense subspace $\eta(F(\Pi X_i))$ is topologically embedded) that μ is an embedding.

The following example shows that in this result μ needs not be surjective: consider a sequence of topological groups $\{G_n\}_{n \in \mathbb{N}}$ such that the only continuous homomorphism from G_n to G_m for $m \neq n$ is the constant map with value the identity of G_m; note, that the only continuous endomorphisms of G_m

are the constant map and the identity mapping. Also, taking none of the G_n compact, the image $\tilde{G}_n$ of G_n in G_n^{SAP} is a proper subgroup of G_n^{SAP}. Now let $G := \{x \in \Pi G_n^{SAP} | \# \{n | pr_n x \notin \tilde{G}_n\} < \omega\}$, and let F be the reflector of **TopGrp** into the epireflective hull in **TopGrp** $_{Haus}$ of $\{G\}$. Taking into account that $G_n^{SAP} \subseteq G$ for each n and that G_n admids no other continuous homomorphism into G than the obvious one (coming from the canonical morphism $G_n \to G_n^{SAP}$) it follows that $FG_n = G_n^{SAP}$ for each n. On the other hand, $G = F(\Pi G_n)$ which is a proper subset of ΠFG_n.

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