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# Orbiting Dust under Radiation Pressure

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In this paper we consider a perturbed Keplerian system describing orbiting dust under radiation pressure. We derive an integrable second order normal form for this Hamiltonian system. Finally we analyze this integrable system by succesive reduction to a one degree of freedom system.

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## 1. Introduction

In his paper [3] Deprit considers a perturbation of bounded Keplerian motion which models the effect of radiation pressure on orbiting dust. The perturbation term can also be seen as the classical analogue of a combined Stark and Zeeman effect ( see [1] ). In the proper rotating co-ordinate system the model is given by the Hamiltonian on $(\mathbb{R}^3-\{0\})\times(\mathbb{R}^3)^*=T_0\mathbb{R}^3$

$$K(\xi,\eta)=\frac{1}{2}|\eta|^2-\frac{\mu}{|\xi|}-\epsilon m(\xi_1\eta_2-\xi_2\eta_1)+\epsilon a\xi_1=K_0(\xi,\eta)+\epsilon K_1(\xi,\eta) \quad , \tag{1.1}$$

where $m$ is the constant angular velocity of rotation of the co-ordinate frame, $a$ is the acceleration, and $\epsilon$ is a small parameter. Deprit derives and analyzes a first order normal form for $K$. In this paper we will derive and analyze a second order normal form for (1.1) using the constrained normalization algorithm described in [5].

The first step of this procedure is to write the Hamiltonian system $(T_0\mathbb{R}^3,\omega,K)$ as a perturbation of the geodesic Hamiltonian $\tilde{K}_0(q,p)=|p|$ on the punctured cotangent bundle $T^+S^3=\{(q,p)\in\mathbb{R}^8 \,|\, F_1(q,p)=|q|^2-1=0 \,,\, F_2(q,p)=<q,p>=0 \,,\, p\neq 0\}$. This is done by: (1) restricting $K$ to the negative level set $K^{-1}(-\frac{1}{2}k^2)$, (2) changing the time scale, and (3) applying Moser's regularization map. The resulting Hamiltonian system $(T^+S^3,\Omega,\tilde{K})$ is given by

$$\tilde{K}(q,p)=\tilde{K}_0(q,p)+\epsilon\tilde{K}_1(q,p) \tag{1.2}$$

where

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$$\tilde{K}_1(q,p) = -\frac{m}{k^2}(1-q_4)|p|(q_1p_2-q_2p_1) - \frac{a}{k^3}(1-q_4)|p|p_1$$
$$-\frac{a}{k^3}(1-q_4)(q_1p_4-q_4p_1) \ . \qquad (1.3)$$

Here $\Omega$ is the restriction of the standard symplectic form $\omega$ on $\mathbb{R}^8$ to $T^+S^3$. Another way of describing the system $(T^+S^3,\Omega,\tilde{K})$ is the following: on $\mathbb{R}^8$ consider the Hamiltonian $H = H_0 + \epsilon H_1$ where

$$H_0(q,p) = (|q|^2|p|^2 - <q,p>^2)^{1/2} \qquad (1.4)$$

and $H_1$ is given by the right hand side of (1.3). On $\mathbb{R}^8 - C_8$, where $C_8 = \{(q,p) \in \mathbb{R}^8 \mid H_0(q,p) = 0\}$, $H$ is a smooth function. Constraining the system $(\mathbb{R}^8 - C_8, \omega, H)$ to $T^+S^3$, gives the system $(T^+S^3,\Omega,\tilde{K})$. Note that the level set $K_0^{-1}(-\frac{1}{2}k^2)$ corresponds to the level set $H_0^{-1}(l)$ where $l = \frac{\mu}{k}$.

## 2. Computation of the second order constrained normal form

In this section we carry out the constrained normalization algorithm to find the second order normal form of $H$. The first step is to compute

$$\overline{H}_1 = \frac{1}{\pi}\int_0^{\pi} H_1 \circ \phi_t^{H_0}\, dt \ , \qquad (2.1)$$

which is the average of $H_1$ over the flow $\phi_t^{H_0}$ of $X_{H_0}$. Since

$$\phi_t^{H_0} = \begin{bmatrix} (-\frac{<q,p>}{H_0(q,p)}\sin 2t + \cos 2t)I_4 & (\frac{|q|^2}{H_0(q,p)}\sin 2t)I_4 \\ (-\frac{|p|^2}{H_0(q,p)}\sin 2t)I_4 & (\frac{<q,p>}{H_0(q,p)}\sin 2t + \cos 2t)I_4 \end{bmatrix}$$

we find that

$$\overline{q_iq_j} = \frac{1}{2}Q_iQ_j + \frac{1}{2}q_iq_j \ ,$$
$$\overline{q_ip_j} = -\frac{1}{2}Q_iP_j + \frac{1}{2}q_ip_j \ , \qquad (2.2)$$
$$\overline{p_ip_j} = \frac{1}{2}P_iP_j + \frac{1}{2}p_ip_j \ , \text{ and}$$
$$\overline{q^lp^k} = 0 \text{ if } |l| + |k| \text{ is odd (using multi index notation)} \ .$$

Here,

$$Q_i(q,p) = \frac{1}{H_0(q,p)}(<q,p>q_i - |q|^2p_i) \ ,$$
$$P_i(q,p) = \frac{1}{H_0(q,p)}(<q,p>p_i - |p|^2q_i)$$

for $1 \leq i \leq 4$; furthermore we write

$$S_{ij} = q_ip_j - q_jp_i$$

for $1 \leq i < j \leq 4$. Substituting the expression for $\phi_t^{H_0}$ into $H_1$ and using (2.2) gives

$$\overline{H}_1(q,p) = -\frac{m}{k^2}|p|S_{12} + \frac{a}{k^3}|p|(-\frac{1}{2}Q_4P_1 + \frac{1}{2}q_4p_1) - \frac{a}{k^3}|p|S_{14} \ . \qquad (2.3)$$

To simplify the above formula, we introduce the following notation: if $F,G \in C^\infty(\mathbb{R}^8 - C_8)$, then we say $F \simeq G$ if and only if $F - G$ lies in the ideal of smooth functions on $\mathbb{R}^8 - C_8$ generated by the functions $F_1$ and $F_2$. In other words, $F \simeq G$ if and only if $F \mid T^+S^3 = G \mid T^+S^3$. Consequently

$$H_0 \simeq |p| \quad , Q_i \simeq \frac{-p_i}{|p|} \quad , P_i \simeq -|p| q_i \quad ,$$
$$p_i p_j + |p|^2 q_i q_j \simeq \sum_{l=1}^{4} S_{li} S_{lj} \quad , 2|p|^2 \overline{q_i q_j} \simeq \sum_{l=1}^{4} S_{li} S_{lj} \quad ,$$
$$2|p|^2 \overline{p_i p_j} \simeq \sum_{l=1}^{4} S_{li} S_{lj} \quad , \overline{q_i p_j} \simeq \frac{1}{2} S_{ij} \quad . \tag{2.4}$$

Substituting (2.4) into (2.3) gives

$$\overline{H}_1 \simeq -\frac{m}{k^2} |p| S_{12} - \frac{3a}{2k^3} |p| S_{14} \tag{2.5}$$

which on $H_0^{-1}(l) \cap T^+S^3$ agrees with the first order normal form for $H$ found by Deprit.

The next step is to compute the generating function $R$ of the symplectic transformation $\exp L_{\epsilon R}$, which normalizes $H$ to first order. According to [2]

$$R = \frac{1}{\pi} \int_0^{\pi} t (H_1 - \overline{H}_1) \circ \phi_t^{H_0} dt \quad .$$

A straightforward calculation gives

$$R = \frac{m}{2k^2} |p| S_{12} Q_4 - \frac{a}{2k^3} S_{14} Q_4 - \frac{a}{2k^3} |p| P_1$$
$$- \frac{a}{2k^3} |p| \left[ -\frac{\pi}{4}(Q_4 P_1 - q_4 p_1) + \frac{1}{8}(Q_4 p_1 - P_1 q_4) \right]$$
$$- \frac{a}{2k^3} \frac{|p|^3 |q|^2}{H_0^2} \left[ -\frac{\pi}{4}(Q_1 p_4 - q_1 p_4) + \frac{1}{8}(Q_1 p_4 - q_1 P_4) \right] + F_2 \Re$$
$$\simeq -\frac{m}{2k^2} S_{12} p_4 - \frac{a}{2k^3} S_{14} p_4 - \frac{a}{2k^3} |p|^2 q_1 + \frac{a}{8k^3} (|p|^2 q_1 q_4 - p_1 p_4) = \tilde{R} \quad . \tag{2.6}$$

According to the constrained normalization algorithm, $R$ has to be modified to

$$R^* = R - \frac{1}{2}\{R, F_2\}(|q|^2 - 1) + \frac{1}{2}\{R, F_1\} < q,p > \tag{2.7}$$

because then the symplectic transformation $\exp L_{\epsilon R^*}$ leaves the constraint $T^+S^3$ invariant. Without changing the constrained normal form we may use $\tilde{R}^*$ instead of $R^*$. Therefore to second order the transformed Hamiltonian is

$$(\exp \epsilon L_{\tilde{R}^*}) H \simeq H_0 + \epsilon \overline{H}_1 + \epsilon^2 L_{\tilde{R}^*}(\tfrac{1}{2}(H_1 + \overline{H}_1)) + O(\epsilon^3) \tag{2.8}$$

To simplify (2.8) we may use

$$T = \frac{m}{k^2} |p| S_{12}(\tfrac{1}{2} q_4 - 1) + \frac{a}{k^3} |p| S_{14}(\tfrac{1}{2} q_4 - \tfrac{5}{4}) + \frac{a}{2k^3} |p| p_1 (q_4 - 1) \tag{2.9}$$

instead of $\frac{1}{2}(H_1 + \overline{H}_1)$, because $\frac{1}{2}(H_1 + \overline{H}_1) \simeq T$ and $T^+S^3$ is an invariant manifold of $X_{\tilde{R}^*}$. Therefore the second order term in the normal form of $H$ is

$$\overline{L_{\tilde{R}^*} T} = -\overline{\{\tilde{R}^*, T\}}$$
$$\simeq -\overline{\{\tilde{R}, T\}} + \frac{1}{2} \overline{\{\tilde{R}, F_2\}\{|q|^2, T\}} - \frac{1}{2} \overline{\{\tilde{R}, F_1\}\{<q,p>, T\}} \quad .$$

A straightforward calculation, making use of

$$\{\tilde{R},F_1\} \simeq \frac{m}{k^2}S_{12}q_4+\frac{a}{k^3}S_{14}q_4+\frac{a}{4k^3}(q_1p_4+q_4p_1) \ ,$$

$$\{\tilde{R},F_2\} \simeq \frac{m}{2k^2}S_{12}p_4+\frac{a}{2k^3}S_{14}p_4+\frac{a}{4k^3}p_1p_4+\frac{a}{2k^3}|p|^2q_1 \ ,$$

$$\{|q|^2,T\} \simeq \frac{a}{k^3}|p|(q_4-1)q_1 \ ,$$

$$\{<q,p>,T\} \simeq -\frac{m}{k^2}|p|S_{12}-\frac{5a}{4k^2}|p|S_{14}+\frac{a}{k^3}|p|(\tfrac{1}{2}q_4-1)p_1 \ ,$$

and the fact that $F \simeq G$ implies $\overline{F} \simeq \overline{G}$ ( which follows because $F_1$ and $F_2$ are integrals of $H_0$ ) gives

$$-\tfrac{1}{2}\overline{\{\tilde{R},F_2\}\{|q|^2,T\}} \simeq \frac{am}{4k^5}|p|S_{12}\overline{q_1p_4}+\frac{a^2}{4k^6}|p|S_{14}\overline{q_1p_4}$$
$$+\frac{a^2}{4k^6}|p|^3\overline{q_1^2}-\frac{a^2}{8k^6}|p|\overline{q_1q_4p_1p_4} \ .$$

$$\tfrac{1}{2}\overline{\{\tilde{R},F_1\}\{<q,p>,T\}} \simeq -\frac{5am}{8k^5}|p|S_{12}\overline{q_4p_1}-\frac{am}{8k^5}|p|S_{12}\overline{q_1p_4}$$
$$-\frac{21a^2}{32k^6}|p|S_{14}\overline{q_4p_1}-\frac{5a^2}{32k^6}|p|S_{14}\overline{q_1p_4}$$
$$+\frac{a^2}{16k^6}|p|\overline{q_1q_4p_1p_4}+\frac{a^2}{16k^6}|p|\overline{q_4^2p_1^2} \ ,$$

and

$$\overline{\{\tilde{R},T\}} \simeq \frac{a^2}{4k^6}|p|^3+\frac{m^2}{4k^4}|p|S_{12}^2+\frac{27a^2}{32k^6}|p|S_{12}^2+\frac{9am}{8k^5}|p|S_{12}S_{14}$$
$$+\frac{3am}{8k^5}|p|^2\overline{q_2q_4}-\frac{a^2}{16k^6}|p|S_{14}\overline{q_4p_1}+\frac{am}{8k^5}|p|\overline{p_2p_4}$$
$$-\frac{5a^2}{32k^6}|p|^3\overline{q_1^2}+\frac{15a^2}{32k^6}|p|^3\overline{q_4^2}+\frac{15a^2}{32k^6}|p|\overline{p_1^2}$$
$$+\frac{3a^2}{32k^6}|p|\overline{p_4^2}+\frac{a^2}{16k^6}|p|\overline{q_4^2p_1^2}-\frac{a^2}{16k^6}|p|\overline{q_1q_4p_1p_4} \ .$$

Therefore the second order term in the normal form of $H$ is

$$\overline{\{T,\tilde{R}^*\}} \simeq -\left[\frac{17a^2}{32k^6}|p|^3+(\frac{m^2}{4k^4}+\frac{am}{8k^5})|p|S_{12}^2+\frac{51a^2}{32k^6}|p|S_{14}^2 \right.$$
$$\left. -\frac{9a^2}{32k^6}|p|S_{23}^2+\frac{13am}{8k^5}|p|S_{12}S_{14}-\frac{am}{4k^5}|p|S_{23}S_{34}\right] \ , \qquad (2.10)$$

where we have used (2.4), $|p|^2 \simeq \sum_{1\leq i<j\leq 4} S_{ij}^2$, and the identity

$$\overline{q_4^2p_1^2}-\overline{q_1q_4p_1p_4}=-\overline{q_4p_1}S_{14} \ .$$

## 3. Further normalization

We can write the second order normal form of $H$ obtained in the previous section as

$$\mathcal{H}=H_0+\epsilon\mathcal{H}_1+\epsilon^2\mathcal{H}_2 \ , \tag{3.1}$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are smooth functions in $|p|$ and $S_{ij}$ which are given by (2.5) and (2.10). Hence we have two commuting integrals $H_0$ and $\mathcal{H}$ of $\mathcal{H}$. In this section we perform a further constrained normalization of $\mathcal{H}$. This further normalization introduces a third integral for the resulting normal form up to second order. More precisely, the resulting normal form

$$\hat{\mathcal{H}}=H_0+\epsilon\hat{\mathcal{H}}_1+\epsilon^2\hat{\mathcal{H}}_2$$

is Liouville integrable with integrals $\{H_0,\hat{\mathcal{H}}_1,\hat{\mathcal{H}}\}$, which Poisson commute.

To be able to perform a further constrained normalization of $\mathcal{H}$ we need a suitable Poisson algebra. The quadratic polynomials $S_{ij}$ , $1\leq i<j\leq 4$, under Poisson bracket span a Lie algebra $\mathbb{S}$ which is isomorphic to $so(4)$; moreover $|p|$ lies in the center of $\mathbb{S}$. Thus the smooth functions on $\mathbb{S}$ form a Poisson algebra $(C^\infty(\mathbb{S}), \cdot, \{\{ \ , \ \}\})$ with multiplication $\cdot$ given by pointwise multiplication of functions and Poisson bracket $\{\{ \ , \ \}\}$ defined by

$$\{\{f,g\}\}=\sum_{1\leq i,j,k,l\leq 4}\frac{\partial f}{\partial S_{ij}}\frac{\partial g}{\partial S_{kl}}\{S_{ij},S_{kl}\} \ ,$$

where $f,g\in C^\infty(\mathbb{S})$. Note that smooth functions in $|p|$ lie in the center of $(C^\infty(\mathbb{S}),\{\{ \ , \ \}\})$.

Now consider the constraint $N$ defined by

$$\mathcal{F}_1=\sum_{1\leq i,j\leq 4} S_{ij}^2-l^2=0 \ \text{ and } \ \mathcal{F}_2=S_{12}S_{34}-S_{13}S_{24}+S_{14}S_{23}=0 \ .$$

Note that $N$ is diffeomorphic to the first reduced phase space $P_l$ of section 4. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are Casimir elements of $(C^\infty(\mathbb{S}), \cdot, \{\{ \ , \ \}\})$ which span the center of this Poisson algebra, $N$ is a symplectic submanifold of $(\mathbb{S},\Omega)$, where $\Omega$ is the Kostant-Kirillov symplectic form. Since $L_S\mathcal{F}_i=0$ for every $S\in\mathbb{S}$, $N$ is invariant under the flow $t\rightarrow\exp tL_S$ for every $S\in\mathbb{S}$. Therefore when doing normalization of $\mathcal{H}$ constrained to $N$ no adjustment of the symplectic transformation needs to be made as in (2.7). Hence we need only perform an ordinary normalization of $\mathcal{H}$ on $\mathbb{S}$.

To explain this note that for any $F\in C^\infty(\mathbb{S})$, $\exp L_F$ maps a normal form of $\mathcal{H}$ into a normal form. Explicitly,

$$\begin{aligned}\tilde{\mathcal{H}}&=(\exp L_{\epsilon F})\mathcal{H}\\ &=H_0+\epsilon(\mathcal{H}_1+\{H_0,F\})+\epsilon^2(\mathcal{H}_2+\{\mathcal{H}_1,F\}+\tfrac{1}{2}\{H_0,\{H_0,F\}\})+O(\epsilon^3)\\ &=H_0+\epsilon\mathcal{H}_1+\epsilon^2(\mathcal{H}_2+\{\mathcal{H}_1,F\})+O(\epsilon^3) \ ,\end{aligned} \tag{3.2}$$

since every element of $C^\infty(\mathbb{S})$ is an integral of $H_0$. This result suggests that we try to choose $F$ so that

$$\tilde{\mathcal{H}}_2=\mathcal{H}_2+\{\mathcal{H}_1,F\}\in\ker L_{\mathcal{H}_1} \ .$$

This is possible provided that $L_{\mathcal{H}_1}$ is a smooth vector field on $\mathbb{S}$ with only periodic orbits, for then we have the splitting $C^\infty(\mathbb{S})=\ker L_{\mathcal{H}_1}\oplus \operatorname{im} L_{\mathcal{H}_1}$ [2].

To show that $L_{\mathcal{H}_1}$ has the required property we apply the linear map $\exp\lambda L_{S_{24}}$ on $\mathcal{H}_1$ to bring $\mathcal{H}_1$ into a simpler form. Because $\{S_{24},S_{12}\}=S_{14}$ and $\{S_{24},S_{14}\}=-S_{12}$ we obtain

$$\begin{aligned}\hat{\mathcal{H}}_1&=(\exp(\lambda L_{S_{24}}))\mathcal{H}_1\\ &=-|p|(\frac{m}{k^2}\cos\lambda-\frac{3a}{2k^3}\sin\lambda)S_{12}+|p|(\frac{m}{k^2}\sin\lambda+\frac{3a}{2k^3}\cos\lambda)S_{14} \ .\end{aligned}$$

Choosing $\lambda$ so that $\alpha_0 \sin\lambda = -\frac{3a}{2k^3}$ and $\alpha_0 \cos\lambda = \frac{m}{k^2}$, where $\alpha_0 = \left[\frac{m^2}{k^4} + \frac{9a^2}{4k^6}\right]^{1/2}$ gives $\hat{\mathcal{H}}_1 = -\alpha_0 \,|p|\, S_{12}$. Therefore with respect to the ordered basis $\{S_{12}, S_{13}, S_{23}, S_{34}, -S_{24}, S_{14}\}$ of $\mathbb{S}$, the vector field $L_{\hat{\mathcal{H}}_1}$ is linear and has matrix

$$-\alpha_0 \,|p|\, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} . \tag{3.3}$$

Hence $L_{\hat{\mathcal{H}}_1}$ has only periodic orbits on $\mathbb{S}$.

Applying the linear map $\exp(\lambda L_{S_{24}})$ on $\mathcal{H}_2$ with $\lambda$ chosen as above gives

$$\begin{aligned} \hat{\hat{\mathcal{H}}}_2 &= \exp(\lambda L_{S_{24}})\mathcal{H}_2 \\ &= \alpha_1 + \alpha_2 S_{12}^2 + \alpha_3 S_{12} S_{14} + \alpha_4 S_{14}^2 + \alpha_5 S_{23}^2 + \alpha_6 S_{34}^2 + \alpha_7 S_{23} S_{34} . \end{aligned} \tag{3.4}$$

Therefore we need to find $\hat{F} \in C^\infty(\mathbb{S})$ so that $\hat{\hat{\mathcal{H}}}_2 + \{\hat{\mathcal{H}}_1, \hat{F}\} \in \ker L_{\hat{\mathcal{H}}_1}$. Since the subalgebra $\ker L_{S_{12}}$ of $(C^\infty(\mathbb{S}), \cdot)$ is generated by

$$S_{12} \;,\; S_{34} \;,\; S_{13}^2 + S_{23}^2 \;,\; S_{14}^2 + S_{24}^2 \;,\; S_{13}S_{24} - S_{14}S_{23} \;,\; S_{13}S_{14} + S_{23}S_{24} . \tag{3.5}$$

and

$$L_{S_{12}}(S_{12}S_{34}) = S_{23}S_{34} \;,\; L_{S_{12}}(S_{14}S_{24}) = S_{24}^2 - S_{14}^2 \;,$$

$$L_{S_{12}}(S_{13}S_{23}) = S_{23}^2 - S_{13}^2 \;,\; L_{S_{12}}(S_{12}S_{24}) = S_{12}S_{14} \;,$$

the splitting of $S_{14}^2$ and $S_{23}^2$ into a sum of terms in $\ker L_{\hat{\mathcal{H}}_1}$ and $\operatorname{im} L_{\hat{\mathcal{H}}_1}$ is given by

$$S_{14}^2 = \tfrac{1}{2}(S_{14}^2 + S_{24}^2) - \tfrac{1}{2}(S_{24}^2 - S_{14}^2) \;,$$

$$S_{23}^2 = \tfrac{1}{2}(S_{13}^2 + S_{23}^2) - \tfrac{1}{2}(S_{13}^2 - S_{23}^2) \;.$$

Therefore the normal form of $\hat{\hat{\mathcal{H}}}_2$ with respect to $\hat{\mathcal{H}}_1$ is

$$\hat{\mathcal{H}}_2 = \alpha_1 + \alpha_2 S_{12}^2 + \tfrac{1}{2}\alpha_4 (S_{14}^2 + S_{24}^2) + \tfrac{1}{2}\alpha_5 (S_{13}^2 + S_{23}^2) + \alpha_6 S_{34}^2 \;. \tag{3.6}$$

Consequently our final second order normal form for $H = H_0 + \epsilon H_1$ is $\hat{\mathcal{H}} = H_0 + \epsilon\hat{\mathcal{H}}_1 + \epsilon^2 \hat{\mathcal{H}}_2$ where $\hat{\mathcal{H}}_1 = -\alpha_0 \,|p|\, S_{12}$ and $\hat{\mathcal{H}}_2$ is given by (3.6). Since $H_0$ and $\hat{\mathcal{H}}_1$ are integrals of $\hat{\mathcal{H}}$, which Poisson commute, $\hat{\mathcal{H}}$ is Liouville integrable.

## 4. Reduction to one degree of freedom

Since $\hat{\mathcal{H}}$ has two commuting integrals $H_0$ and $\hat{\mathcal{H}}_1$ both of which generate an $S^1$-action, we can perform reduction twice to obtain a reduced system which has only one degree of freedom. We now carry out this twofold reduction.

Recall that the quadratic polynomials $S_{ij}$ , $1 \leq i < j \leq 4$, generate the algebra of smooth functions which are invariant under the flow of $X_{H_0}$. Since this flow is periodic, the corresponding $S^1$ orbit map is

$$\rho : \mathbb{R}^8 - C_8 \to \mathbb{S} = \mathbb{R}^6 : (q,p) \to (S_{12}, S_{13}, S_{23}, S_{34}, -S_{24}, S_{14}) \tag{4.1}$$

If on $\mathbb{S}$ we apply the linear change of co-ordinates

$$A_1=S_{12}+S_{34}\quad ,\ A_2=S_{13}-S_{24}\quad ,\ A_3=S_{23}+S_{14}\quad ,$$

$$J_1=S_{12}-S_{34}\quad ,\ J_2=S_{13}+S_{24}\quad ,\ J_3=S_{23}-S_{14}\quad , \tag{4.2}$$

( which is just an isomorphism of the Lie algebras $so(4)$ and $so(3)\times so(3)$ ), we obtain another $S^1$ orbit map

$$\tilde{\rho}:\mathbb{R}^8-C_8\to\mathbb{S}=\mathbb{R}^6:(q,p)\to(A_1,A_2,A_3,J_1,J_2,J_3)\ .$$

The image of $H_0^{-1}(l)\cap T^+S^3$ under $\tilde{\rho}$ is $P_l$ which is defined by $A_1^2+A_2^2+A_3^2=l^2$ , $J_1^2+J_2^2+J_3^2=l^2$; moreover the reduced phase space of the $S^1$-action generated by the flow of $X_{H_0}\,|\,T^+S^3$ is $P_l$, which is diffeomorphic to $S^2\times S^2$. Identifying $\mathbb{R}^6$ with $(so(3)\times so(3))^*$ shows that $P_l$ is an $SO(3)\times SO(3)$ co-adjoint orbit.

Now consider the $S^1$-action on $\mathbb{S}$ generated by the flow of $L_{\hat{\mathcal{H}}_1}$. Since $\hat{\mathcal{H}}_1$ is an integral of $H_0$, the flow of $L_{\hat{\mathcal{H}}_1}$ leaves $P_l$ invariant. In fact this $S^1$-action is given by the 1-parameter group $t\to\exp tL_{\hat{\mathcal{H}}_1}$ of $SO(3)\times SO(3)$, which induces rotations on $S^2\times\{0\}$ and $\{0\}\times S^2$ that are in 1:1 resonance ( see [3]). Thus the algebra of smooth functions which are invariant under the flow of $L_{\hat{\mathcal{H}}_1}$ is generated by

$$\pi_1=A_1\quad ,\ \pi_2=A_2J_2+A_3J_3\quad ,\ \pi_3=A_3J_2-A_2J_3\quad ,$$

$$\pi_4=J_1\quad ,\ \pi_5=A_2^2+A_3^2\quad ,\ \pi_6=J_2^2+J_3^2\quad . \tag{4.3}$$

Hence the orbit map for this $S^1$-action is

$$\tau:\mathbb{S}=\mathbb{R}^6\to\mathbb{R}^6:(A,J)\to(\pi_1,\pi_2,\pi_3,\pi_4,\pi_5,\pi_6)$$

The image of $P_l\cap\hat{\mathcal{H}}_1(\tilde{c})$ under $\tau$ is the second reduced phase space $P_{l,c}$ which is defined by

$$\pi_1^2+\pi_5=l^2\quad ,\ \pi_4^2+\pi_6=l^2\quad ,$$

$$\pi_2^2+\pi_3^2=\pi_5\pi_6\quad ,\ \pi_5\geqslant 0\ \text{ and }\ \pi_6\geqslant 0\quad ,\ \pi_1+\pi_4=\frac{-2\tilde{c}}{\alpha_0 l}=2c\quad . \tag{4.4}$$

From (4.4) we find that $P_{l,c}$ is a surface of revolution in $(\pi_1,\pi_2,\pi_3)$ space defined by

$$\pi_2^2+\pi_3^2=(l^2-\pi_1^2)(l^2-(2c-\pi_1)^2)\quad ,\ -l\leqslant\pi_1\leqslant l\ \text{ and }\ -l+2c\leqslant\pi_1\leqslant l+2c\quad . \tag{4.5}$$

Consequently $P_{l,c}$ is a point if $c=\pm l$, a smooth two sphere if $0<|c|<l$, or a topological two sphere with cone-like singularities at the poles $\pm(l,0,0)$ when $c=0$. This completes the twofold reduction process.

On the second reduced phase space $P_{l,c}$ we now compute the reduced Hamiltonian $H_{l,c}$ induced by the second order normalized Hamiltonian $\hat{\mathcal{H}}$. From (4.2) and (4.3) it follows that

$$S_{12}=\frac{1}{2}(\pi_1+\pi_4)\quad ,\ S_{34}=\frac{1}{2}(\pi_1-\pi_4)\quad ,$$

$$S_{13}S_{24}-S_{14}S_{23}=\frac{1}{4}(\pi_6-\pi_5)\quad ,\ S_{13}S_{14}+S_{23}S_{24}=\frac{1}{2}\pi_3\quad , \tag{4.6}$$

$$S_{13}^2+S_{23}^2=\frac{1}{4}(\pi_5+\pi_6+2\pi_2)\quad ,S_{14}^2+S_{24}^2=\frac{1}{4}(\pi_5+\pi_6-2\pi_2)\quad .$$

Substituting $S_{12}=c$ and (4.6) into $\hat{\mathcal{H}}_2$ (3.6) yields

$$H_{l,c}^{(2)}=\tilde{\beta}_0+\tilde{\beta}_1\pi_1^2+\tilde{\beta}_2\pi_1+\tilde{\beta}_3\pi_2\quad , \tag{4.7}$$

using (4.4) and $\pi_1+\pi_4=2c$. Here

$$\tilde{\beta}_0=\alpha_1+\alpha_2c^2+\frac{1}{4}(\alpha_4+\alpha_5)l^2-\frac{1}{2}(\alpha_4+\alpha_5)c^2+\alpha_6c^2$$

$$\tilde{\beta}_1=-\frac{1}{4}(\alpha_4+\alpha_5)+\alpha_6\quad ,\ \tilde{\beta}_2=\frac{1}{2}(\alpha_4+\alpha_5)c-2\alpha_6c=-2c\tilde{\beta}_1\quad , \tag{4.8}$$

$$\tilde{\beta}_3=\frac{1}{4}(\alpha_5-\alpha_4)\quad .$$

Since $\tilde{\beta}_2=-2c\tilde{\beta}_1$ we may write $H_{l,c}^{(2)}=\beta_0+\beta_1(\pi_1-c)^2+\beta_2\pi_2$ where $\beta_0=\tilde{\beta}_0-\tilde{\beta}_1 c^2$, $\beta_1=\tilde{\beta}_1$, and $\beta_2=\tilde{\beta}_3$. Because $H_{l,c}^{(0)}=H_0=l$ and $H_{l,c}^{(1)}=\mathcal{K}_1=\tilde{c}$ on $P_{l,c}$, the second order normalized reduced Hamiltonian on $P_{l,c}$ is

$$H_{l,c}=\epsilon^2(\beta_1(\pi_1-c)^2+\beta_2\pi_2) \ , \tag{4.9}$$

after dropping inessential constants.

## 5. Qualitative analysis of $H_{l,c}$ on $P_{l,c}$.

In this section we discuss the qualitative properties of the level sets of $H_{l,c}$ on $P_{l,c}$ ( see fig.1 ). These level sets correspond to trajectories of the reduced Hamiltonian vector field $X_{H_{l,c}}$ on $P_{l,c}$.

Let $\sigma_1=\pi_1-c$ , $\sigma_2=\pi_2$ , and $\sigma_3=\pi_3$. In these variables the second reduced phase space $P_{l,c}$ is defined by

$$\sigma_2^2+\sigma_3^2=\left[(l-|c|)^2-\sigma_1^2\right]\left[(l+|c|)^2-\sigma_1^2\right]=V(\sigma_1) \ , \tag{5.1}$$

where $|\sigma_1|\leqslant l-|c|$ and $0\leqslant|c|\leqslant l$. After introducing a new time scale $s=\epsilon^2 t$, the second order normalized reduced Hamiltonian on $P_{l,c}$ is

$$H_{l,c}=\alpha\sigma_1^2+\beta\sigma_2 \ , \tag{5.2}$$

where

$$\alpha=\beta_1=-\frac{1}{4}(\alpha_4+\alpha_5)+\alpha_6$$

$$=\frac{a^2 l}{32(9a^2+4k^2m^2)k^6}(81a^2+9amk+42m^2k^2) \ , \tag{5.3}$$

$$\beta=\beta_2=\frac{1}{4}(\alpha_5-\alpha_4)$$

$$=\frac{3a^2m\ l}{32(9a^2+4k^2m^2)k^5}(3a-4mk) \ . \tag{5.4}$$

We now determine the critical points $\sigma=(\sigma_1,\sigma_2,\sigma_3)$ of $H_{l,c}$ on $P_{l,c}$. From (5.3) and (5.4) it follows that $\alpha\neq0$ but that $\beta$ can be zero. Let us first consider the special case $\alpha\neq0$ and $\beta=0$. Then by the Lagrange multiplier method we find that $\sigma$ must satisfy

$$\begin{aligned}\nu(4\sigma_1^3-4(c^2+l^2)\sigma_1)&=2\alpha\sigma_1 \ ,\\ 2\nu\sigma_2&=0 \ ,\\ 2\nu\sigma_3&=0 \ ,\\ \sigma_2^2+\sigma_3^2=V(\sigma_1) \ , \ |\sigma_1|&\leqslant l-|c| \ , \ l>0 \ .\end{aligned} \tag{5.5}$$

There are two cases to consider. (1) When $\nu=0$ the first equation in (5.5) gives $\sigma_1=0$ since $\alpha\neq0$. Hence we find that the circle $\sigma_2^2+\sigma_3^2=V(0)=(l^2-c^2)^2$ in $P_{l,c}$ lies in the critical set of $H_{l,c}$. (2) When $\nu\neq0$, the second and third equations in (5.5) imply that $\sigma_2=\sigma_3=0$ and hence $V(\sigma_1)=0$. Therefore $\sigma_1=\pm(l-|c|)$ or $\pm(l+|c|)$. But the second possibility must be disregarded since $|\sigma_1|\leqslant l-|c|$. Consequently $H_{l,c}$ has two critical points $\pm(l-|c|,0,0)$ on $P_{l,c}$, which are easily to be seen to be a maximum and a minimum. Thus when $\alpha\neq0$ but $\beta=0$ the level sets of $H_{l,c}$ on $P_{l,c}$ are given in figure 1.

After this special case we turn to the general case when $\alpha\neq0$ and $\beta\neq0$. The Lagrange multiplier equations read

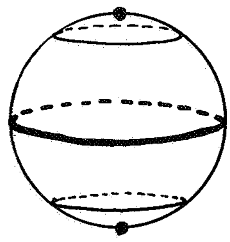

figure 1. Level sets of $H_{l,c}$ on $P_{l,c}$ when $\alpha \neq 0$ , $\beta = 0$.
The critcal set is given by the heavy curves.

$$
\begin{aligned}
\nu(4\sigma_1^3 - 4(l^2+c^2)\sigma_1) &= 2\alpha\sigma_1 \ , \\
2\nu\sigma_2 &= \beta \ , \\
2\nu\sigma_3 &= 0 \ , \\
\sigma_2^2 + \sigma_3^2 &= V(\sigma_1) \ , \ |\sigma_1| \leqslant l - |c| \ , \ l > 0 \ .
\end{aligned}
\tag{5.6}
$$

If $\nu = 0$, then the second equation in (5.6) gives $\beta = 0$ which contradicts the hypothesis. Therefore $\nu \neq 0$, which by the third equation gives $\sigma_3 = 0$. Thus every critical point of $H_{l,c}$ lies on the topological circle $S^1_{l,c} = P_{l,c} \cap \{\sigma_3 = 0\}$.

Instead of solving (5.6) with $\nu \neq 0$, we follow a different more algebraic approach. Consider the equations describing an h-level set of $H_{l,c}$ on $S^1_{l,c}$.

$$
\begin{aligned}
h &= \alpha\sigma_1^2 + \beta\sigma_2 \ , \\
\sigma_2^2 &= ((l-|c|)^2 - \sigma_1^2)((l+|c|)^2 - \sigma_1^2) \ , \ |\sigma_1| \leqslant l - |c| \ , \ l > 0 \ .
\end{aligned}
\tag{5.7}
$$

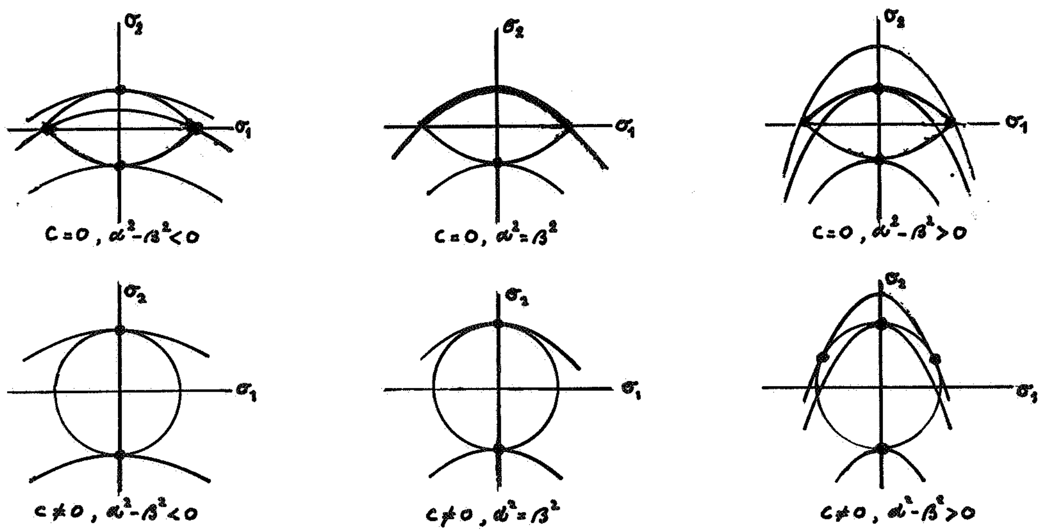


figure 2. Critical set of $H_{l,c}$ on $S^1_{l,c}$ when $\beta \neq 0$ and $(\alpha / \beta) > 0$.

Using the fact that $\beta \neq 0$, we may eliminate $\sigma_2$ from (5.7) to obtain

$$(\alpha^2-\beta^2)\sigma_1^4+2(-\alpha h+\beta^2(l^2+c^2))\sigma_1^2+(h^2-\beta^2(l^2-c^2)^2)=0 \tag{5.8}$$

together with

$$|\sigma_1|\leqslant l-|c| \quad , \quad |c|\leqslant l \quad , l>0 \ . \tag{5.9}$$

Then $(\sigma_1,\frac{1}{\beta}(h-\alpha\sigma_1^2),0)$ is a critical point of $H_{l,c}$ on $S^1_{l,c}$ if and only if $\sigma_1$ is a double root of (5.8) which satisfies (5.9) (see figure 2). Equation (5.8) has double roots precisely when its discriminant $\mathfrak{D}$ is zero. We now recall some facts about discriminants. Let $\Delta$ denote the discriminant of the biquadratic polynomial

$$x^4+ax^2+b \ . \tag{5.10}$$

Then the discriminant locus $\{\Delta=0\}$ is just the $\{c=0\}$ slice of the discriminant locus of the general quartic $x^4+ax^2+cx+b$ which in $(a,b,c)$ space is a swallowtail surface ( see [6]). We find that $\{\Delta=0\}$ in the $(a,b)$ plane is given by the line $\{b=0\}$ and the half parabola $\{a^2=4b \ , a\leqslant 0\}$.

We now begin the analysis of the discriminant locus $\{\mathfrak{D}=0\}$ of (5.8). Our analysis is divided into three parts: (1) $\alpha^2=\beta^2$, (2) $\alpha^2-\beta^2<0$, and (3) $\alpha^2-\beta^2>0$. Case (1) splits into two subcases.

1a. If

$$-\alpha h+\beta^2(l^2+c^2)=0 \ , \tag{5.11}$$

then (5.8) becomes

$$h^2=\beta^2(l^2-c^2)^2 \ . \tag{5.12}$$

Suppose that $\beta>0$. Then taking the square root of (5.12) and eliminating $h$ from (5.11) gives

$$(\beta-\alpha)l^2+(\alpha+\beta)c^2=0 \ . \tag{5.13}$$

If $\alpha+\beta=0$, then (5.13) becomes $2\beta l^2=0$, which implies that $l=0$. But this is a contradiction. Therefore $\alpha+\beta\neq 0$. But $\alpha^2=\beta^2$ by hypothesis. Hence $\alpha=\beta$ and (5.13) implies $c=0$. Hence $h=\beta l^2$. A similar argument when $\beta<0$ shows that $c=0$ and $h=-\beta l^2$.

1b. When $-\alpha h+\beta^2(l^2+c^2)\neq 0$, (5.8) has double roots if and only if

$$h^2-\beta^2(l^2-c^2)^2=0 \ . \tag{5.14}$$

Taking the above results together we see that in case (1), (5.8) has double roots if and only if

$$h^2=\beta^2(l^2-c^2)^2 \ . \tag{5.15}$$

Note that the $c=0$ slice of (5.15) is special in the sense that it corresponds to the case where $H_{l,c}=h$ and $S^1_{l,c}$ coincide along part of a parabola ( see fig.2 ). This is the only case where $H_{l,c}$ has a critical set which does not consist of isolated points.

In case (2) when $\alpha^2-\beta^2<0$ we find that the part of $\{\mathfrak{D}=0\}$ corresponding to $\{b=0\}$ piece of $\{\Delta=0\}$ is also given by (5.15). From (5.8) and (5.10) we see that

$$a=2\left[\frac{-\alpha h+\beta^2(l^2+c^2)}{\alpha^2-\beta^2}\right] \text{ and } b=\frac{h^2-\beta^2(l^2-c^2)^2}{\alpha^2-\beta^2} \ .$$

Therefore the part of $\{\mathfrak{D}=0\}$, which corresponds to the $\{a^2=4b \ , a\leqslant 0\}$ piece of $\{\Delta=0\}$, is given by

$$0=\left[\frac{-\alpha h+\beta^2(l^2+c^2)}{\alpha^2-\beta^2}\right]^2-\left[\frac{h^2-\beta^2(l^2-c^2)^2}{\alpha^2-\beta^2}\right] \ ,$$

$$\frac{-\alpha h+\beta^2(l^2+c^2)}{\alpha^2-\beta^2}\leqslant 0 \ . \tag{5.16}$$

After some simplification the equation in (5.16) reads

$$(h-\alpha(l^2+c^2))^2-4(\alpha^2-\beta^2)l^2c^2=0 \ . \tag{5.17}$$

Because $\alpha^2-\beta^2<0$, (5.17) holds if and only if $c=0$ and $h=\alpha l^2$. Consequently in each $l$ slice of $\{\mathfrak{D}=0\}$ we get just one extra point lying in the interior of the part given by (5.15) ( see fig.4).

In case (3) when $\alpha^2-\beta^2>0$ we find that a part of $\{\mathfrak{D}=0\}$ is given by (5.15). Also we obtain equations (5.16), (5.17) which describe the remaining part. In this case we may solve (5.17) to obtain

$$h=\alpha(l^2+c^2)\pm 2l\,|\,c\,|\,\sqrt{\alpha^2-\beta^2} \ . \tag{5.18}$$

For $l$=constant we find that the two parabolas in the $(c,h)$ plane given by (5.18) are tangent to $h^2=\beta^2(l^2-c^2)$ at the four points

$$Q_{1,4}=\left[\pm l\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}, \frac{2\beta^2}{\alpha-\beta}l^2\right] \ , \ Q_{2,3}=\left[\pm l\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}, \frac{2\beta^2}{\alpha-\beta}l^2\right] \ .$$

Because of the inequality in (5.16) we have to consider only the part of these curves sketched in figure 3.

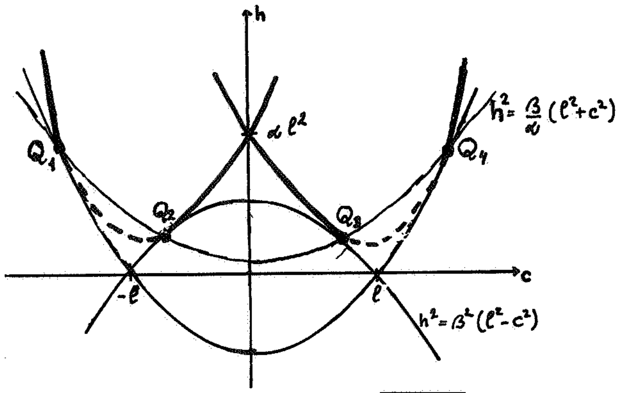


figure 3. The pieces of $h=\alpha(l^2+c^2)\pm 2l\,|\,c\,|\,\sqrt{\alpha^2-\beta^2}$ which belong to $\{\mathfrak{D}=0\}$.

It remains to investigate which points of $\{\mathfrak{D}=0\}$ are critical points of $H_{l,c}$ on $S^1_{l,c}$. Hereto we have to study the effect of the inequality (5.9). First consider the part of $\{\mathfrak{D}=0\}$ corresponding to $\{b=0\}$. Along this branch we find that $\sigma_1=0$ is the only double root of (5.8), that is, the first inequality in (5.9) is satisfied. Thus we only have the restriction $|\,c\,|\leqslant l \ , \ l>0$. Next consider the part of $\{\mathfrak{D}=0\}$ corresponding to $\{a^2=4b \ , \ a\leqslant 0\}$. When $\alpha^2-\beta^2<0$, we find that the double roots of (5.8) are given by $\sigma_1=\pm l$ when $c=0$. Again (5.9) is satisfied if we restrict to $|\,c\,|\leqslant l$. Finally consider the case $\alpha^2-\beta^2>0$. We find the double roots

$$|\,\sigma_1\,|=\left[\frac{\alpha h-\beta^2(l^2+c^2)}{\alpha^2-\beta^2}\right]^{1/2}=\left[l^2+c^2\pm\frac{2l\,|\,c\,|\,\alpha}{\sqrt{\alpha^2-\beta^2}}\right]^{1/2} \ . \tag{5.19}$$

since $h$ is given by (5.18). When $\alpha>0$, it is easy to check that the condition $|\,\sigma_1\,|\leqslant l-|\,c\,|$ is satisfied only if we take the - sign in (5.18) and (5.19). Furthermore we have to restrict to $|\,c\,|\leqslant l$. This finally gives us the set of critical values of $H_{l,c}$ on $S^1_{l,c}$ in parameter space $(c,h,l)$, which is depicted in figure 4.

In fact the curves in figure 4 describe the critical values of the energy-momentum map $T^+S^3\rightarrow\mathbb{R}^3$; $(q,p)\rightarrow(H_0,\hat{\mathcal{K}}_1,\hat{\mathcal{K}}_2)$. The total image is given by the curves and their interior. The fibers of the energy-momentum map correspond to invariant surfaces of the integrable vector field $X_{\hat{\mathcal{K}}}$. By factorization of the energy-momentum map through the orbit maps $\tilde{\rho}$ and $\tau$ the nature of the fibers can be determined in a straightforward way. We will end this section with a short description of the fibers.

Regular values correspond to one or two 3-tori. Elliptic critical values to 2-tori ( (2) indicating two of these ). Hyperbolic critical values have a fibre which includes the stable and unstable manifold, the fiber consists of two 3-tori intersecting along a hyperbolic 2-torus. An exception are those critical

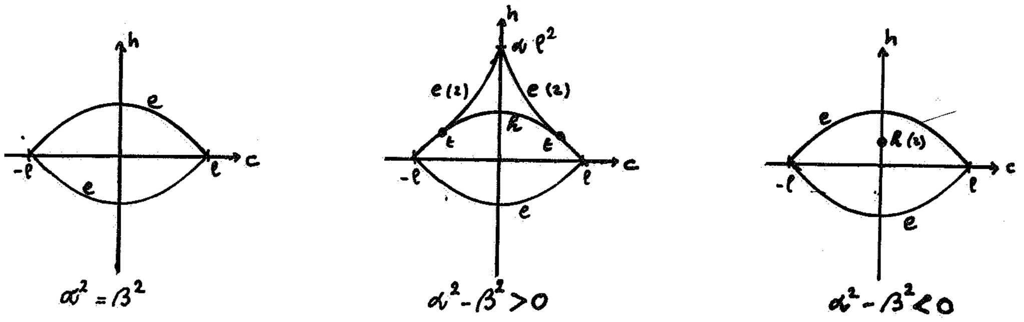


figure 4. $l>0$ slice of the set of critical points of $H_{l,c}$ on $S^1_{l,c}$,
(2) indicates two double roots,
e(lliptic),h(yperbolic),t(ransitional) indicate the stability type of the critical point.

values which correspond to the critical points on the first reduced phase space. They are given by $(c,h,l)=(0,\alpha l^2,l)$ and $(\pm l,0,l)$. For the elliptic points the fibre is just a circle. For the hyperbolic points we obtain complicated fibres containing a hyperbolic invariant circle.

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