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Necessary Conditions for Resonance in Turning Point Problems for Ordinary Differential Equations

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In the theory of turning point problems for ordinary linear differential equations of second order necessary conditions for ACKERBERG-O'MALLEY resonance are studied by earlier writers. The present paper gives a sequence of necessary conditions for resonance, which is derived in an iterative way. Special cases are considered as illustrative examples.

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1. INTRODUCTION

Since 1970, there has been published a large number of papers that consider the singularly perturbed turning point problem of the form

$$\epsilon y'' + f(x,\epsilon)y' + g(x,\epsilon)y = 0, \ (0<\epsilon\ll 1, \ -a<x<b) \tag{1.1}$$

$$y(-a) = \alpha, \ y(b) = \beta, \tag{1.2}$$

where a and b are positive numbers and $f(0, 0)=0$. This problem was studied first by ACKERBERG and O'MALLEY [1]. They gave the condition under which the boundary value problem

$$\epsilon y'' - p(x)y' + g(x)y = 0 \tag{1.3}$$

$$y(-a) = \alpha, \ y(b) = \beta, \tag{1.4}$$

with $p(0)=0$, $p'(0)>0$ exhibits "resonance". That is, under which condition the limit of its solution for $\epsilon\to 0$ is non-trivial. In 1971, WATT [2] showed by an example that the condition given in [1] :

$$\frac{g(0)}{p'(0)} = N, \ (N\text{:non-negative integer }) \tag{1.5}$$

is not sufficient for exhibiting resonance. In 1973, COOK and ECKHAUS [3] gave an improved condition for resonance of the boundary value problem (1.1) - (1.2), which is

$$-\frac{g(0,\epsilon)}{f_x(0,\epsilon)} = N+\mu_1\epsilon \ (N\text{:non-negative integer}),$$

where $\mu_1 = -[\, g_x^2 + g_{xx}(N+\frac{1}{2})]$. In 1975-1976, MATKOWSKY [4] examined several examples and proposed to analyse the related eigenvalue problem to test the resonance. In 1978, OLVER [5] formulated

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sufficient conditions for testing the resonance. These studies have stimulated the development of a theory for this kind of turning point problems. We shall point out in this paper that all of these conditions given by former authors, except [5], are only necessary conditions for resonance, we shall show by examples that they are not sufficient. Moreover, we find that there is a sequence of necessary conditions for resonance which can be derived in an iterative way. As special cases we consider $g(x,\epsilon)\equiv 0$, and $f(x,\epsilon)\equiv -Ax$, $g(x,\epsilon)\equiv B$, where A, B are constants. It turns out that the first necessary condition given by ACKERBERG and O'MALLEY [1]

$$\frac{g(0,0)}{f'(0,0)} = N, \ (N\text{:non-negative integer})$$

implies the whole sequence of necessary conditions for resonance, so it is also sufficient for these cases.

II. EXAMPLE

Consider the turning point problem for the differential equation of the form

$$\epsilon y''-x(1+x^2)y'+(2+B(\epsilon))y = 0, \tag{2.1}$$

where

$$B(\epsilon) \sim p_1\epsilon+p_2\epsilon^2+\cdots. \tag{2.2}$$

Suppose its outer solution has the expansion of the form

$$y \sim y_0(x)+\epsilon y_1(x)+\epsilon^2 y_2(x)+\epsilon^3 y_3(x)+\cdots. \tag{2.3}$$

Substituting (2.3) into (2.1) and equating the terms with identical powers of ϵ, we obtain the recurrent system of differential equations for y_n, $(n=0,1,2,\cdots)$

$$-x(1+x^2)y'_n+2y_n = -y''_{n-1}-\sum_{i=1}^{n}p_i y_{n-i}\,, \tag{2.4}$$

with $y_{-1}(x)\equiv 0$.

From (2.4) with $n=0$ we have

$$y_0(x) = C_0\frac{x^2}{1+x^2}, \tag{2.5}$$

where C_0 is an undetermined constant. Substituting (2.5) into (2.4) with $n=1$, we have

$$-x(1+x^2)y'_1+2y_1 = -y''_0-p_1y_0 \equiv G_0(x)\,. \tag{2.6}$$

Its solution is

$$y_1 = C_1I_0(x) + I_0(x)\int^x \frac{G_0(s)}{-s(1+s^2)I_0(s)}ds \tag{2.7}$$

where $I_0(x)=\dfrac{x^2}{1+x^2}$. Since

$$G_0(x) = \frac{-2+6x^2}{x^2(1+x^2)^2}C_0I_0(x) - p_1C_0I_0(x)\,, \tag{2.8}$$

we have from (2.7)

$$\begin{aligned} y_1 &= C_1I_0(x) + C_0I_0(x)\int^x\left[\frac{2-6s^2}{s^3(1+s^2)^3} + \frac{p_1}{s(1+s^2)}\right]ds \\ &= C_1I_0(x) + \left[\frac{-1-9x^2-6x^4}{x^2(1+x^2)^2} - (6-\frac{p_1}{2})\ln\frac{x^2}{1+x^2}\right]C_0I_0(x)\,. \end{aligned} \tag{2.9}$$

Because $y_1(x)$ should be analytic at $x=0$, we must have

$$p_1 = 12 . \tag{2.10}$$

Otherwise, it is non-resonant. We should take then $C_0=0$.

From (2.4) with $n=1$ we obtain the equation for y_2:

$$-x(1+x)y'_2 + 2y_2 = -y''_1 - p_1y_1 - p_2y_0 \equiv G_1 \tag{2.11}$$

where $p_1=12$. Its solution is

$$y_2 = C_2I_0(x) + I_0(x)\int^x \frac{G_1(s)}{-s(1+s^2)I_0(s)}ds . \tag{2.12}$$

If we only want to test whether C_0 is equal to zero, the process can be simplified. Let $y_n^{(0)}$ denote the particular solution of (2.4) with only C_0 as factor. Then from (2.9) we know that

$$y_1^{(0)} = C_0I_0(x)\frac{-1-9x^2-6x^4}{x^2(1+x^2)^2} . \tag{2.13}$$

Since

$$\begin{aligned} G_1^{(0)}(x) &= -(y_1^{(0)})''-12y_1^{(0)} - p_2y_0 \\ &= \left[12\frac{2+x^2+25x^4+24x^6+6x^8}{x^2(1+x^2)^4} - p_2\right]C_0I_0(x), \end{aligned} \tag{2.14}$$

we have from (2.12)

$$\begin{aligned} y_2^{(0)} = \; & 12C_0I_0(x)\left[\frac{1}{x^2(1+x^2)^4} + \frac{44}{8}\frac{1}{(1+x^2)^4} \right. \\ & + \frac{3}{2}\frac{1}{(1+x^2)^3} + \frac{9}{4}\frac{1}{1+x^2} + 5\frac{x^2}{(1+x^2)^4} + \frac{3}{2}\frac{x^4}{(1+x^2)^4} \\ & \left. + \frac{9}{2}\ln\frac{x^2}{1+x^2}\right] + \frac{p_2}{2}C_0I_0(x)\ln\frac{x^2}{1+x^2} . \end{aligned} \tag{2.15}$$

It is only when

$$p_2 = -108 \tag{2.16}$$

that $y_2^{(0)}$ is analytic at $x=0$. Otherwise, we should take $C_0=0$.

EQ. (2.16) is the second condition for the boundary value problem of differential equation (2.1) to be resonant. Evidently, in the present approach, a sequence of numbers p_i, $(i=3, 4, \cdots)$ should be determined, and it is not sufficient to solve a single related eigenvalue problem as proposed by MATKOWSKY in [4].

III. GENERAL CASE

We return to the boundary value problem (1.1)-(1.2), and write it as

$$\epsilon y'' - xA(x,\epsilon)y' + B(x,\epsilon)y = 0 \quad (0<\epsilon\ll 1, \; -a<x<b) \tag{3.1}$$

where

$$A(x, \epsilon) \sim A_0(x) + \epsilon A_1(x) + \epsilon^2 A_2(x) + \cdots , \tag{3.2}$$

$$B(x, \epsilon) \sim B_0(x) + \epsilon B_1(x) + \epsilon^2 B_2(x) + \cdots , \tag{3.3}$$

with $A_0(0)>0$. Suppose that $A_i(x)$, $B_i(x)$, $(i=0,1,2, \cdots)$ are analytic in $[-a,b]$.

Let the outer expansion of its solution be

$$y \sim y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \cdots. \tag{3.4}$$

Substituting (3.4) into (2.1), and equating the terms with equal powers of ϵ, we obtain the recursive equations governing y_n, $(n=0,1,2,\cdots)$

$$-xA_0(x)y'_n + B_0(x)y_n = -y''_{n-1} + x\sum_{i=1}^{n} A_i y'_{n-i} - \sum_{i=1}^{n} B_i y_{n-i} \tag{3.5}$$

with $y_{-1} \equiv 0$.

From (3.5) (with $n=0$) we obtain the equation for y_0 :

$$-xA_0(x)y'_0 + B_0(x)y_0 = 0\,, \tag{3.6}$$

its solution is

$$y_0 = C_0 \exp\left[\int^x \frac{B_0(s)}{sA_0(s)} ds\right] \equiv C_0 I_0(x)\,. \tag{3.7}$$

Suppose the expansions of $A_i(x)$, $B_i(x)$, $(i=0,1,2,\cdots)$ near $x=0$ are

$$A_i(x) = A_{i,0} + A_{i,1}x + A_{i,2}x^2 + \cdots,$$

$$B_i(x) = B_{i,0} + B_{i,1}x + B_{i,2}x^2 + \cdots,$$

and the expansions of $A_i^{-1}(x)$, $A_i^{-3}(x)$, $(i=0,1,2,\cdots)$ near $x=0$ are

$$A_i^{-1}(x) = \tilde{A}_{i,0} + \tilde{A}_{i,1}x + \tilde{A}_{i,2}x^2 + \cdots,$$

$$A_i^{-3}(x) = \tilde{\tilde{A}}_{i,0} + \tilde{\tilde{A}}_{i,1}x + \tilde{\tilde{A}}_{i,2}x^2 + \cdots,$$

then $I_0(x)$ has the expansion

$$I_0(x) = x^{B_{0,0}\tilde{A}_{0,0}} \exp\left[\sum_{n=1}^{\infty} \left(\sum_{i=0}^{n} B_{0,i}\tilde{A}_{0,n-i}\right)x^n\right]. \tag{3.8}$$

In order that $y_0(x)$ is analytic at $x=0$, we must have

$$B_{0,0}\tilde{A}_{0,0} \equiv \frac{B(0,0)}{A(0,0)} = N \text{ (N:non-negative integer)}, \tag{3.9}$$

which is the condition for resonance given by ACKERBERG and O'MALLEY [1]. We shall show later on that it is only the first necessary condition in the sequence of necessary conditions for resonance.

From (3.5) with $n=1$ we obtain the equation for y_1:

$$-xA_0(x)y'_1 + B_0(x)y_1 = -y''_0 + xA_1y'_0 - B_1y_0 \equiv L_0[y_0],$$

where L_0 is the differential operator of the form:

$$L_0 \equiv -\frac{d^2}{dx^2} + xA_1\frac{d}{dx} - B_1\,. \tag{3.10}$$

The solution of (3.10) is

$$y_1 = C_1I_0(x) + I_0(x)\int^x \frac{L_0[y_0(s)]}{-sA_0(s)I_0(s)} ds\,. \tag{3.11}$$

Since

$$y'_0 = \frac{B_0(x)}{xA_0(x)} C_0I_0(x),$$

$$y''_0 = \frac{xA_0B'_0 - xA'_0B_0 - A_0B_0 + B_0^2}{x^2A_0^2}\, C_0I_0(x)\,,$$

we have that

$$L_0\,[\,y_0(x)] = \left[\frac{-xA_0B'_0 + xA'_0B_0 + A_0B_0 - B_0^2 + x^2A_0B_0A_1}{x^2A_0^2} - B_1\right] C_0I_0(x)\,.$$

From (3.11) we derive

$$\begin{aligned} y_1 &= C_1I_0(x) + C_0I_0(x)\int^x \left[\frac{sA_0B'_0 - sA'_0B_0 - A_0B_0 + B_0^2 - s^2A_0B_0A_1}{s^3A_0^3} + \frac{B_1}{sA_0}\right] ds \\ &= C_1I_0(x) + C_0I_0(x)\left[\frac{a_{0,-2}}{x^2} + \frac{a_{0,-1}}{x} + a_{0,0}\ln x + a_{0,1}x + a_{0,2}x^2 + \cdots\right. \\ &\quad + B_{1,0}\tilde{A}_{0,0}\ln x + (B_{1,0}\tilde{A}_{0,1} + B_{1,1}\tilde{A}_{0,0})x \\ &\quad \left. + \tfrac{1}{2}\Big(\sum_{i=0}^{2} B_{1,i}\tilde{A}_{0,2-i}\Big)x^2 + \cdots\right], \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} a_{0,-2} &= -\tfrac{1}{2}(-A_{0,0}B_{0,0} + B_{0,0}^2)\tilde{\tilde{A}}_{0,0}\,, \\ a_{0,-1} &= -\,[(-\,A_{0,0}B_{0,0} + B_{0,0}^2)\tilde{\tilde{A}}_{0,1} + (-A_{0,0}B_{0,1} - A_{0,1}B_{0,0} + B_{0,0}B_{0,1} + B_{0,1}B_{0,0} \\ &\qquad + A_{0,0}B_{0,1} - A_{0,1}B_{0,0})\tilde{\tilde{A}}_{0,0}], \\ a_{0,0} &= (-\,A_{0,0}B_{0,0} + B_{0,0}^2)\tilde{\tilde{A}}_{0,2} + (-A_{0,0}B_{0,1} - A_{0,1}B_{0,0} + B_{0,0}B_{0,1} + B_{0,1}B_{0,0} \\ &\quad + A_{0,0}B_{0,1} - A_{0,1}B_{0,0})\tilde{\tilde{A}}_{0,1} + \Big(-\sum_{i=0}^{2}A_{0,i}B_{0,2-i} + \sum_{i=0}^{2}B_{0,i}B_{0,2-i} \\ &\quad + \sum_{i=1}^{2} iB_{0,i}A_{0,2-i} - \sum_{i=1}^{2} iA_{0,i}B_{0,2-i} - A_{0,0}B_{0,0}A_{1,0}\Big)\tilde{\tilde{A}}_{0,0}, \end{aligned}$$

....

$$\begin{aligned} a_{0,n} &= \frac{1}{n}\Bigg\{(-\,A_{0,0}B_{0,0} + B_{0,0}^2)\tilde{\tilde{A}}_{0,n+2} + (-A_{0,0}B_{0,1} - A_{0,1}B_{0,0} + B_{0,0}B_{0,1} \\ &\quad + B_{0,1}B_{0,0} + A_{0,0}B_{0,1} - A_{0,1}B_{0,0})\tilde{\tilde{A}}_{0,n+1} + \sum_{p=2}^{n+2}\Bigg[-\sum_{i=0}^{p}A_{0,i}B_{0,p-i} \\ &\quad + \sum_{i=0}^{p}B_{0,i}B_{0,p-i} + \sum_{i=1}^{p} iB_{0,i}A_{0,p-i} - \sum_{i=1}^{p} iA_{0,i}B_{0,p-i} \\ &\quad - \sum_{j=0}^{p-2}\Big(\sum_{i=0}^{j}A_{0,i}B_{0,j-i}\Big)A_{1,p-2-j}\Bigg]\tilde{\tilde{A}}_{0,n+2-p}\Bigg\}. \end{aligned}$$

From (3.12) we see that if we wish to have $y_1(x)$ analytic at $x=0$, it is required that

$$a_{0,0} + B_{1,0}\tilde{A}_{0,0} = 0\,. \tag{3.13}$$

This is the second necessary condition for resonance after (3.9).

Especially, if $A(x,\epsilon) \equiv 1$, $B(x,\epsilon) \equiv g(x,\epsilon)$, condition (3.13) reduces to

$$g_\epsilon(0,0) = -\left[g_x^2(0,0) + (N+\tfrac{1}{2})g_{xx}(0,0)\right] \tag{3.14}$$

N is the non-negative integer that appeared in the first necessary condition (3.9). For the differential equation

$$\epsilon y'' - x(\alpha + \bar{\alpha}\epsilon)y' + (\rho + \bar{\rho}\epsilon + \gamma x + \delta x^2)y = 0 \tag{3.15}$$

the second necessary condition (3.13) reduces to

$$\alpha\delta + 2\rho\delta + \gamma^2 - \rho\alpha\bar{\alpha} + \bar{\rho}\alpha^2 = 0 \tag{3.16}$$

since

$$a_{0,0} = \frac{\alpha\delta + 2\rho\delta + \gamma^2 - \rho\alpha\bar{\alpha}}{\alpha^3}, \quad B_{1,0} = \bar{\rho}.$$

They are in agreement with those derived by COOK and ECKHAUS [3]. We remark that the above conditions are not sufficient, which can be shown by the following example:

$$\epsilon y'' - x(1 + 4\epsilon)y' + (1 + 2x)y = 0.$$

Condition (3.16) is satisfied, and its outer solution is

$$y = y_0(x) + \epsilon y_1(x) + \epsilon^2 y_2(x) + \cdots,$$

where

$$y_0(x) = C_0 x e^{2x}, \quad y_1(x) = C_1 x e^{2x} + C_0 x e^{2x}\left(\frac{-4}{x} - 8x\right),$$

$$y_2(x) = C_2 x e^{2x} - 4C_1 x e^{2x}\left(\frac{1}{x} + 2\right) + 32C_0(1 - x\ln x + x^2 + x^3)e^{2x}$$

but $y_2(x)$ is not analytic at $x=0$, unless $C_0=0$.

By the same process we can obtain the third necessary condition. From (3.5) with $n=2$, we obtain the equation for $y_2(x)$:

$$-xA_0(x)y'_2 + B_0(x)y_2 = -y''_1 + x(A_1y'_1 + A_2y'_0) - B_1y_1 - B_2y_0. \tag{3.17}$$

Owing that we only want to find the necessary condition for resonance, we can just consider the particular solution corresponding to the terms with C_0 as factor. Let $y_1^{(0)}$ be these terms in y_1 then

$$y_1^{(0)} = C_0 I_0(x)\left[\frac{\tilde{a}_{0,-2}}{x^2} + \frac{\tilde{a}_{0,-1}}{x} + \tilde{a}_{0,1}x + \tilde{a}_{0,2}x^2 + \cdots + \tilde{a}_{0,n}x^n + \cdots\right] \tag{3.18}$$

where $\tilde{a}_{0,-2} = a_{0,-2}$, $\tilde{a}_{0,-1} = a_{0,-1}$, $\tilde{a}_{0,n} = a_{0,n} + \frac{1}{n}\sum_{i=0}^{n} B_{1,i}A_{0,n-i}$, $(i \geq 1)$. Consider the solution of the following equation

$$\begin{aligned} -xA_0(x)y'_2 + B_0(x)y_2 &= -(y_1^{(0)})'' + x(A_1 y_1^{(0)\prime} + A_2y'_0) - B_1y_1^{(0)} - B_2y_0 \\ &\equiv L_0[y_1^{(0)}] + L_1[y_0], \end{aligned} \tag{3.19}$$

where L_1 is the differential operator of the form

$$L_1 \equiv xA_2\frac{d}{dx} - B_2. \tag{3.20}$$

Since

$$y_1^{(0)\prime} = C_0 I_0(x)\left[\frac{B_0}{xA_0}\sum_{n=-2}^{\infty}\tilde{a}_{0,n}x^n + \frac{1}{x}\sum_{n=-2}^{\infty} n\tilde{a}_{0,n}x^n\right]$$

$$(y_1^{(0)})'' = C_0 I_0(x)\left[\frac{xA_0B'_0 - xA'_0B_0 - A_0B_0 + B_0^2}{x^2A_0^2}\sum_{n=-2}^{\infty}\tilde{a}_{0,n}x^n\right.$$

$$\left. + \frac{2B_0}{x^2A_0}\sum_{n=-2}^{\infty} n\tilde{a}_{0,n}x^n + \frac{1}{x^2}\sum_{n=-2}^{\infty} n(n-1)\tilde{a}_{0,n}x^n\right],$$

with $\tilde{a}_{0,0}=0$, we have that

$$\frac{L_0[y_1^{(0)}]}{-xA_0(x)I_0(x)} = C_0\left\{\frac{1}{x^3A_0^3}\left[(-A_{0,0}B_{0,0} + B_{0,0}^2) + \sum_{n=1}^{\infty}\sum_{i=0}^{n}(-A_{0,i}B_{0,n-i}\right.\right.$$

$$\left. + B_{0,i}B_{0,n-i} + iB_{0,i}A_{0,n-i} - iA_{0,i}B_{0,n-i})x^n\right]\times$$

$$\times \sum_{n=-2}^{\infty}\tilde{a}_{0,n}x^n + \frac{2x}{x^3A_0^3}\left[\sum_{n=0}^{\infty}\sum_{i=0}^{n}A_{0,i}B_{0,n-i}\,x^n\right]\times$$

$$\times \sum_{n=-2}^{\infty} n\tilde{a}_{0,n}x^{n-1} - \frac{x^2}{x^3A_0^3}\sum_{n=0}^{\infty}\left[\sum_{i+j+k=n} A_{1,i}A_{0,j}B_{0,k}x^n\right]\times$$

$$\times \sum_{n=-2}^{\infty}\tilde{a}_{0,n}x^n - \frac{x}{xA_0}\sum_{n=0}^{\infty}A_{1,n}x^n \times \sum_{n=-2}^{\infty} n\tilde{a}_{0,n}x^{n-1}$$

$$\left. + \frac{1}{xA_0}\sum_{n=0}^{\infty}B_{1,n}x^n \times \sum_{n=-2}^{\infty}\tilde{a}_{0,n}x^n + \frac{1}{xA_0}\sum_{n=-2}^{\infty} n(n-1)\,\tilde{a}_{0,n}x^{n-2}\right\}$$

and

$$\frac{L_1[y_0]}{-xA_0(x)I_0(x)} = C_0\left[\frac{-1}{xA_0^3}\sum_{n=0}^{\infty}\sum_{i+j+k=n} A_{2,i}A_{0,j}B_{0,k}x^n + \frac{1}{xA_0}\sum_{n=0}^{\infty}B_{2,n}x^n\right].$$

The particular solution of (3.19) with C_0 as factor is

$$y_2^{(0)} = C_0 I_0(x)\int^x \frac{L_0[y_1^{(0)}(s)] + L_1[y_0(s)]}{-sA_0(s)I_0(s)}ds$$

$$= C_0 I_0(x)\left[\frac{a_{1,-4}}{x^4} + \frac{a_{1,-3}}{x^3} + \frac{a_{1,-2}}{x^2} + \frac{a_{1,-1}}{x} + a_{1,0}\ln x + \cdots + a_{1,n}x^n + \cdots\right.$$

$$\left. + (-A_{0,0}B_{0,0}A_{2,0}\tilde{\tilde{A}}_{0,0} + B_{2,0}\tilde{A}_{0,0})\ln x + b_1x + \cdots + b_nx^n + \cdots\right], \qquad (3.21)$$

where

$$a_{1,-4} = -\tfrac{1}{4}\left[(-5A_{0,0}B_{0,0} + B_{0,0}^2)\tilde{\tilde{A}}_{0,0} + 6\tilde{A}_{0,0}\right]\tilde{a}_{0,-2},$$

$$a_{1,-3} = -\tfrac{1}{3}\left\{\left[(-6A_{0,1}B_{0,0} - 4A_{0,0}B_{0,1} + 2B_{0,0}B_{0,1})\tilde{\tilde{A}}_{0,0}\right.\right.$$

$$\left. + (-5A_{0,0}B_{0,0} + B_{0,0}^2)\tilde{\tilde{A}}_{0,1} + 6\tilde{A}_{0,1}\right]\tilde{a}_{0,-2}$$

$$\left. + \left[(-3A_{0,0}B_{0,0} + B_{0,0}^2)\tilde{\tilde{A}}_{0,0} + 2\tilde{A}_{0,0}\right]\tilde{a}_{0,-1}\right\},$$

$$a_{1,n-4} = \frac{-1}{n-4}\left\{\sum_{j=0}^{n}\tilde{a}_{0,n-2-j}\left[\sum_{i=0}^{j}\sum_{p=0}^{j-i}\left((2n-2p-j-i-5)A_{0,p}B_{0,j-i-p}\right.\right.\right.$$

$$+ B_{0,p}B_{0,j-i-p}\Big]\tilde{\tilde{A}}_{0,i} + (n-2-j)(n-2-j-1)\tilde{A}_{0,j}\Big]$$

$$- \sum_{j=0}^{n-2} \tilde{a}_{0,n-4-j} \sum_{i=0}^{j}\left[\sum_{p=0}^{j-i}\left(\sum_{q=0}^{p} A_{0,q}B_{0,p-q}\right)A_{1,j-i-p}\right]\tilde{A}_{0,i}$$

$$- \sum_{j=0}^{n-2}\left[\sum_{p=0}^{n-2-j}(n-4-j-p)\tilde{a}_{0,n-4-j-p}A_{1,p}\right]\tilde{A}_{0,j}$$

$$+ \sum_{j=0}^{n-2}\left[\sum_{p=0}^{n-2-j}\tilde{a}_{0,n-4-j-p}B_{1,p}\right]\tilde{A}_{0,j}\Big\},$$

$(n = 2, 3, 5, \cdots; n \neq 4)$

$$a_{1,0} = \sum_{j=0}^{4}\tilde{a}_{0,2-j}\left[\sum_{i=0}^{j}\sum_{p=0}^{j-i}\Big[(3-2p-j-i)A_{0,p}B_{0,j-i-p} + B_{0,p}B_{0,j-i-p}\Big]\tilde{\tilde{A}}_{0,i}\right.$$

$$\left. + (2-j)(1-j)\tilde{A}_{0,j}\right] -$$

$$- \sum_{j=0}^{2}\tilde{a}_{0,-j}\sum_{i=0}^{j}\sum_{p=0}^{j-i}\left(\sum_{q=0}^{p}A_{0,q}B_{0,p-q}A_{1,j-i-p}\right)\tilde{A}_{0,i}$$

$$- \sum_{j=0}^{2}\left[\sum_{p=0}^{2-j}(-j-p)\tilde{a}_{0,-j-p}A_{1,p}\right]\tilde{A}_{0.j} + \sum_{j=0}^{2}\left[\sum_{p=0}^{2-j}a_{0,-j-p}B_{1,p}\right]\tilde{A}_{0.j}$$

and

$$b_n = \frac{1}{n}\left[\sum_{j=0}^{n}\sum_{i=0}^{n-j}\left(\sum_{p=0}^{i}A_{0,p}B_{0,i-p}\right)A_{2,n-j-i} + \sum_{j=0}^{n}\tilde{A}_{0,j}B_{2,n-j}\right].$$

It is only if

$$\tilde{a}_{1,0} \equiv a_{1,0} + A_{0,0}B_{0,0}A_{2,0}\tilde{\tilde{A}}_{0,0} + B_{2,0}\tilde{A}_{0,0} = 0 \qquad (3.22)$$

that $y_2(x)$ is analytic at $x=0$.
EQ (3.22) is the third necessary condition for the resonance of differential equation (3.1).
For the differential equation (2.1), the third necessary conditon reduces to

$$p_2 = -108\,;$$

for the special case $A(x,\epsilon) \equiv 1$, $B(x,\epsilon) \equiv g(x,\epsilon)$, it reduces to

$$(3B_{0,0} + B_{0,0}^2 + 2)\tilde{a}_{0,2} + 2(B_{0,1} + B_{0,0}B_{0,1})\tilde{a}_{0,1} + +2(-B_{0,2} - 2B_{0,1} - 3B_{0,0}$$
$$+ B_{0,0}B_{0,3} + B_{0,1}B_{0,2})\tilde{a}_{0,-1} + (-B_{0,4} - 3B_{0,3} - 5B_{0,2} - 7B_{0,1} - 9B_{0,0}$$
$$+ B_{0,2}^2 + 2B_{0,0}B_{0,4} + 2B_{0,1}B_{0,3})\tilde{a}_{0,-2} + \tilde{a}_{0,0}B_{1,0} + \tilde{a}_{0,-1}B_{1,1} + \tilde{a}_{0,-2}B_{1,2}$$
$$+ B_{2,0} = 0$$

where $B_{0,0}=g(0,0)$, $B_{0,i}=\frac{1}{i}\frac{\partial^i g(0,0)}{\partial x^i}$, $B_{1,0}=g_\epsilon(0,0)$, $B_{1,i}=\frac{1}{i!}\frac{\partial^i g_\epsilon(0,0)}{\partial x^i}$, . . . etc.
By the same process we can find the successive necessary condtion for resonance. If we find that

$$y_n^{(0)} = C_0 I_0(x)\sum_{i=-r}^{\infty}\tilde{a}_{n-1,i}x^i \qquad (3.23)$$

where $r \leqslant N = \dfrac{B(0,0)}{A(0,0)}$ (N: non-negative integer), with $\tilde{a}_{n,0}=0$, then we get the equation for y_{n+1} from

(3.5)

$$-xA_0(x)y'_{n+1} + B_0(x)y_{n+1} = L_0[y_n^{(0)}] + L_1[y_{n-1}^{(0)}] + \\ + \cdots + L_n[y_0] \equiv I_0(x)F_n(x)\,, \tag{3.24}$$

where L_0, L_1 are defined above by (3.10), (3.20) and $L_i(i=2,3, \cdots n)$ are defined by

$$L_i \equiv xA_{i+1}\frac{d}{dx} - B_{i+1}\,. \tag{3.25}$$

We can find that the solution of (3.24) with C_0 as factor is

$$y_{n+1}^{(0)} = C_0 I_0(x)\int^x \frac{F_n(s)}{-sA_0(s)}ds\,. \tag{3.26}$$

Expanding the integrand in (3.26) into power series of x at $x=0$, and equating the coeffiecient of x^{-1} to zero, then we get the next necessary condition for resonance.

From above we see that the resonance cases are exceptional. But if A and B are constants, and B/A is a non-negative integer, or if $B(x,\epsilon) \equiv 0$, then the coefficients $\tilde{a}_{n,0}, (n=0,1,2, \cdots)$ are all zero, and the whole sequence of necessary conditions for resonance is satisfied, and the equation exhibits resonance, which is in agreement with our known result.

The construction of the asymptotic solution of boundary value problems for differential equations of the type

$$\epsilon y'' - xA(x,\epsilon)y' + B(x,\epsilon)y = 0$$

has been given in the earlier paper [6] of the present author for both the resonant case and the non-resonant case. The aymptotic correctness of the solution has also been discussed. The problems of generalizing the method to the case of multiple turning points is still open.

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