CWI

Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

P.J.C. Spreij

Selfexciting counting process systems with finite state space

Selfexciting counting process systems with finite state space

P.J.C. Spreij

Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

Stochastic systems with counting process output and a finite state space are considered. This leads to studying processes with finite state space that are Markovian with respect to the flow of σ-algebras, that is generated by the counting process. It appears that there is a close relationship between the transition intensities of the Markov process and the intensity of the counting process. Some consequences for a stochastic realization problems are then studied.

1. INTRODUCTION

Suppose that the dynamic behaviour of some phenomenon may be modelled by means of a counting process. It is then attractive to model the intensity of this counting process as a Markov process evolving on a finite state space. A practical situation where this model shows a very satisfactory behaviour is reported in e.g. KEMP [7]. In the case that one can observe the counting process, but not the associated finite state Markov process there exists a finite dimensional filter that estimates the Markov process. The existence of such a filter is one of the advantages of this model.

On the other hand it has been argued, see BOEL [1], that it is probably better to model a counting process as a selfexciting process, than to model it as a doubly stochastic poisson process that involves some non-observable state process.

Here we adopt both these points and the question arises whether this yields an interesting model. To put it a little bit more precise, we want to characterize the class of counting processes that admit an intensity, which is a function of a finite state process which is Markov with respect to the flow of σ-algebras generated by such a counting process. Or, to formulate it in terms of a stochastic realization problem, given a counting process, under what conditions can it be represented as the output of a stochastic system, where the state proces assumes finitely many values, and is Markov with respect to the filtration generated by the output.

The purpose of this paper is to present a solution of the above stated problems. In particular a detailed investigation will be made of finite state process which are Markov with respect to some given counting process.

The paper is organized as follows. In section 2 preliminary results for counting processes are reviewed. Section 3 contains results for finite state Markov processes. In particular, finite state Markov processes are characterized as solutions of certain stochastic differential equations. Section 4 reports the main results. A characterizaton of finite state processes which are Markov with respect to a counting process is given. In section 5 the results of section 4 will be used to solve a stochastic realization problem.

2. Basic results for counting processes

Let (Ω, F, P) be a complete probability space. let $n: \Omega \times [0, \infty) \to \mathbb{N}_0$ be a counting process and let $\mathcal{F}_t^n = \sigma\{n_s, s \leqslant t\}$ be the σ-algebra generated by the collection $\{n_s, s \leqslant t\}$. Write $\mathbb{F}^n = \{\mathcal{F}_t^n, t \geqslant 0\}$. Assume that n admits the minimal decomposition

$$dn_t = \lambda_t dt + dm_t$$

where $\lambda: \Omega \times [0, \infty) \to \mathbb{R}_+$ is the $\mathbb{F}^n$-predictable intensity process of n and $m: \Omega \times [0, \infty) \to \mathbb{R}$ is a $\mathbb{F}^n$-adapted martingale.

The following lemma, known as the martingale representation theorem, plays a crucial role.

LEMMA 2.1: *Let $M: \Omega \times [0, \infty) \to \mathbb{R}$ be a $\mathbb{F}^n$-adapted martingale. Then there exists a $\mathbb{F}^n$-predictable process $k: \Omega \times [0, \infty) \to \mathbb{R}$ such that for all $t \geqslant 0$*

$$M_t = M_0 + \int_0^t k_s(dn_s - \lambda_s ds)$$

The process k is $P(d\omega)\lambda_t(\omega)$ a.e. uniquely defined and for all $t \geqslant 0$

$$\int_0^t k_s \lambda_s ds < \infty \quad \text{a.s.}$$

For a proof see BRÉMAUD [2, p. 76]

COROLLARY 2.2: *Let $S: \Omega \times [0, \infty) \to \mathbb{R}$ be a $\mathbb{F}^n$-adapted semi-martingale of the form $S_t = S_0 + V_t + M_t$. Here V is a process of bounded variation which is assumed to be continuous, $V_0 = 0$ and M is a $\mathbb{F}^n$-adapted martingale.*

i) *S can jump only when n jumps i.e. $\Delta S_t \neq 0 \Rightarrow \Delta n_t = 1$*

ii) *If moreover S is a pure jump process (which is the case if it takes its values in a countable set), then*

$$S_t = S_0 + \int_0^t k_s dn_s$$

and V is absolutely continuous satisfying

$$V_t = \int_0^t k_s \lambda_s ds$$

where k is as in lemma 2.1.

PROOF:

i) From lemma 2.1 we know that $M_t = \int_0^t k_s(dn_s - \lambda_s ds)$ for some $\mathbb{F}^n$-predictable process k. But then from the assumption that V is continuous $\Delta S_t = \Delta M_t = k_t \Delta n_t$.

ii) Now $\quad S_t - S_0 = \Sigma_{u \leqslant t} \Delta S_u = \Sigma_{u \leqslant t} k_u \Delta n_u = \int_0^t k_s dn_s \quad$ and $\quad V_t = S_t - S_0 - M_t = \int_0^t k_s dn_s - \int_0^t k_s(dn_s - \lambda_s ds) = \int_0^t k_s \lambda_s ds$. a.s.

REMARK: The assumption that V is continuous implies that the given decomposition of S is $\mathbb{F}^n$ unique, since S is now a forteriori a special semi-martingale.

In section 4 the relation between two counting processes n and $\tilde{n}$ will be investigated. The following proposition will turn out to be useful there.

PROPOSITION 2.4: *Let n and $\tilde{n}$ be two counting processes and let λ and $\tilde{\lambda}$ be their $\mathbb{F}^n$-, respectively $\mathbb{F}^{\tilde{n}}$-, predictable intensities. Equivalent are*

(i) $\mathscr{F}_t^{\tilde{n}} \subset \mathscr{F}_t^n$, and $\mathscr{F}_\infty^n$ and $\mathscr{F}_t^n$ are conditionally independent given $\mathscr{F}_t^{\tilde{n}}$.

(ii) $\tilde{n}_t = \int_0^t 1_{\{\tilde{\lambda}_s > 0\}} dn_s$ and $\tilde{\lambda}_t = 1_{\{\tilde{\lambda}_t > 0\}} \lambda_t$.

In the proof we will use

LEMMA 2.5 [BRÉMAUD & YOR, 3]: *Consider two filtrations $\mathbb{F}$ and $\mathbb{G}$, such that for all $t \geq 0 : \mathscr{F}_t \subset \mathscr{G}_t$. Then there is equivalence between*

(i) *Any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale*

(ii) $\mathscr{F}_\infty$ and $\mathscr{G}_t$ are conditionally independent given $\mathscr{F}_t$.

PROOF OF PROPOSITION 2.4:

(i)$\Rightarrow$(ii): Write $d\tilde{n}_t = \tilde{\lambda}_t dt + d\tilde{m}_t$, the Doob-Meyer decomposition of $\tilde{n}$ with respect to $\mathbb{F}^{\tilde{n}}$. From lemma 2.5 $\tilde{m}$ is also a $\mathbb{F}^n$-martingale. Hence $\tilde{m}_t = \int_0^t h_s dm_s$ for a $P(d\omega) \otimes dn_t(\omega)$ a.e. unique process h. Then $(d\tilde{n}_t = (\tilde{\lambda}_t - \lambda_t h_t) dt + h_t dn_t$, which gives $d\tilde{n}_t = h_t dn_t$ and $\tilde{\lambda}_t = h_t \lambda_t$. Therefore on the jump times T_k of n we have $h_{T_k}^2 = h_{T_k}$. Hence we can also write $d\tilde{n}_t = h_t d\tilde{n}_t = h_t \tilde{\lambda}_t + h_t d\tilde{m}_t$. From the fact that predictable intensities are unique, we find $\tilde{\lambda}_t = h_t \tilde{\lambda}_t$ a.s., which implies that $h_t 1_{\{\tilde{\lambda}_t > 0\}} = 1_{\{\tilde{\lambda}_t > 0\}}$. An obvious choice of h that satisfies this relation is $h'_t = 1_{\{\tilde{\lambda}_t > 0\}}$. It is certainly $\mathbb{F}^n$-predictable and

$$E \int_0^\infty 1_{\{h_t \neq 1_{\{\tilde{\lambda}_t > 0\}}\}} dn_t = E \sum_{n \geq 1} 1_{\{h_{T_n} \neq 1_{\{\tilde{\lambda}_{T_n} > 0\}}\}} =$$
$$= E \sum_{n \geq 1} [1_{\{h_{T_n} = 1, \tilde{\lambda}_{T_n} = 0\}} + 1_{\{h_{T_n} = 0, \tilde{\lambda}_{T_n} > 0\}}] = 0,$$

which can be seen as follows. It $h_{T_n} = 1$, then $\tilde{n}$ jumps at T_n, so that $\tilde{\lambda}_{T_n} > 0$, and if $h_{T_n} = 0$, then $\lambda_{T_n} = 0$ from $\tilde{\lambda}_t = h_t \tilde{\lambda}_t$. The uniqueness of the process h now gives the result.

(ii)$\Rightarrow$(i): Notice first that $\mathscr{F}_t^{\tilde{n}} \subset \mathscr{F}_t^n$, since by the assumption $\tilde{n}_t = \int_0^t 1_{\{\tilde{\lambda}_s > 0\}} dn_s$, the sequence $\{\tilde{T}_k\}$ of jump times of $\tilde{n}$ is contained in the sequence $\{T_k\}$. In view of lemma 2.5 it is now sufficient to prove that any $\mathbb{F}^{\tilde{n}}$-martingale is a $\mathbb{F}^n$-martingale. So let M be a $\mathbb{F}^{\tilde{n}}$-martingale. Then there is a $\mathbb{F}^{\tilde{n}}$-predictable process h such that $M_t = M_0 + \int_0^t h_s d\tilde{m}_s$. Now $\tilde{\lambda}_t dt + d\tilde{m}_t = d\tilde{n}_t = 1_{\{\tilde{\lambda}_t > 0\}} dn_t = 1_{\{\tilde{\lambda}_t > 0\}} \lambda_t dt + 1_{\{\tilde{\lambda}_t > 0\}} dm_t = \tilde{\lambda}_t dt + 1_{\{\tilde{\lambda}_t > 0\}} dm_t$ by assumption. Because of $\mathscr{F}_t^{\tilde{n}} \subset \mathscr{F}_t^n$ $1_{\{\tilde{\lambda}_t > 0\}}$ is $\mathbb{F}^n$-predictable, hence $\tilde{m}$ is also a $\mathbb{F}^n$-martingale. But then the same conclusion holds for M.

REMARK: The formulation of condition (ii) of proposition 2.4 can be replaced by

(ii)' There exists a $\mathbb{F}^n$-predictable process u such that

$$\tilde{n}_t = \int_0^t u_s dn_s \text{ and } \tilde{\lambda}_t = u_t \lambda_t.$$

later on one can identify u as $u_t = 1_{\{\tilde{\lambda}_t > 0\}}$, showing that if even becomes $\mathbb{F}^{\tilde{n}}$-predictable.

3. Markov processes with a finite state space

3.1 Recall first that a stochastic process X with state space $(E, \mathfrak{S})$ is Markov with respect to some filtration $\mathbb{F} = \{\mathfrak{S}_t, t \geq 0\}$, we will say that it is $\mathbb{F}$-Markov, if $\forall t \geq s, \forall B \in \mathfrak{S}$

$$P(X_t \in B | \mathfrak{S}_s) = P(X_t \in B | \sigma(X_s))$$

Or, equivalently, that for all bounded measurable functions f on E we have

$$E[f(X_t) | \mathfrak{S}_s] = E[f(X_t) | \sigma(X_s)]$$

From now on we specialize to the case where the state space E is finite, $E = \{c_1, \ldots, c_n\}$, and $\mathfrak{S}$ is the power set of E. Define $Y : \Omega \times [0, \infty) \to \{0, 1\}^n$ by its components $Y_{it} := 1_{\{X_t = c_i\}}$ Denote by $\Phi(t, s)$ the matrix of transition probabilities of X. That is for $t \geq s$, with the notation $z^+ = z^{-1} 1_{\{z \neq 0\}}$ and the understanding $\frac{0}{0} = 0$

$$\Phi_{ij}(t, s) = P(X_t = c_i | X_s = c_j) = (EY_{js})^+ E(Y_{js} Y_{it}).$$

Then we have the following well known facts. Semigroup property: $\Phi(t, s) = \Phi(t, u) \Phi(u, s)$ for $t \geq u \geq s$. Assume that for all $t \geq 0$ the following limit exists

$$A(t) := \lim_{h \downarrow 0} \frac{1}{h} [\Phi(t + h, t) - I]$$

So $A(t)$ has nonpositive diagonal elements, the other entries are nonnegative and the column sums are zero. Such a matrix will be called a Markov matrix. Then $\frac{\partial}{\partial t} \Phi(t, s) = A(t) \Phi(t, s)$ In particular $\frac{\partial}{\partial t} \Phi(t, 0) = A(t) \Phi(t, 0)$. From this equation we get $\det \Phi(t, 0) = \exp(\int_0^t tr A(s) ds)$. Hence, by definition of $A(t)$, we see that $\Phi(t, 0)$ is invertible for all $t \geq 0$.

PROPOSITION 3.1: *Define $Z : \Omega \times [0, \infty) \to \mathbb{R}^n$ by $Z_t = \Phi(t, 0)^{-1} Y_t$. Then Z is an $\mathbb{F}$-martingale and Y satisfies the stochastic differential equation*

$$dY_t = A(t) Y_t dt + \Phi(t, 0) dZ_t \tag{3.1}$$

PROOF: Using a representation of a conditional expectation when the conditioning σ-algebra is generated by a finite number of disjoint sets we get

$$E[Z_t | \mathfrak{S}_s] = \Phi(t, 0)^{-1} E[Y_t | \mathfrak{S}_s] = \Phi(t, 0)^{-1} E(Y_t | \sigma(X_s)) =$$
$$= \Phi(t, 0)^{-1} E[Y_t | \sigma(Y_s)] = \Phi(t, 0)^{-1} \sum_j E[Y_{js}]^+ E[Y_t Y_{js}] Y_{js} =$$
$$= \Phi(t, 0)^{-1} \Phi(t, s) Y_s = \Phi(s, 0)^{-1} Y_s = Z_s.$$

The second assertion can easily be proved by applying the stochastic differentiation rule to the product $Y_t = \Phi(t, 0) Z_t$. $\square$

Notice that $\int_0^t \Phi(s, 0) dZ_s$ appearing in (3.1) is again a $\mathbb{F}$-martingale since $\Phi(\cdot, 0)$ is trivially predictable. Proposition 3.1 thus gives us a representation of Markov processes in terms of a linear stochastic differential equation driven by a martingale. The next result gives a converse statement.

PROPOSITION 3.2: *Let $X : \Omega \times [0, \infty) \to \{c_1, \ldots, c_n\}$ be a stochastic process, $\mathbb{F}$-adapted, and let Y be associated with X as before. Assume that Y satisfies*

$$dY_t = A(t) Y_t dt + dm_t^Y \tag{3.2}$$

Here $A : [0, \infty) \to \mathbb{R}^{n \times n}$ is a Lebesque measurable function (deterministic !) and m^Y an $\mathbb{F}$-adapted

martingale. Then X and Y are $\mathbb{F}$-Markov processes.

PROOF: We have to prove that $E[f(X_t)|\mathcal{F}_s]=E[f(X_t)|\sigma(X_s)]$ for all $f:\{c_1,\ldots,c_n\}\to\mathbb{R}$. Since $f(X_t)=\Sigma_j f(c_j)Y_{jt}$ we will only prove $E[Y_t|\mathcal{F}_s]=E[Y_t|\sigma(x_s)]$.

Introduce the notation $B(t)=\exp(\int_0^t A(s)ds)$. Now we can write the solution Y_t of (3.2) as

$$Y_t = B(t)Y_0 + B(t)\int_0^t B^{-1}(s)dm_s^Y.$$

Notice again that $\int_0^t B^{-1}(s)dm_s^Y$ is an $\mathbb{F}$-martingale, $B(t)$ deterministic. Hence

$$E[Y_t|\mathcal{F}_s] = B(t)Y_0 + B(t)\int_0^s B^{-1}(u)dm_u^Y =$$
$$= B(t)Y_0 + B(t)[B^{-1}(s)Y_s - Y_0] = B(t)B^{-1}(s)Y_s$$

Since we have $\sigma(X_s)=\sigma(Y_s)\subset\mathcal{F}_s$ we get

$$E[Y_t|\sigma(X_s)] = E[E[Y_t|\mathcal{F}_s]|\sigma(Y_s)] = E[B(t)B^{-1}(s)Y_s|\sigma(Y_s)]$$
$$= B(t)B^{-1}(s)Y_s = E[Y_t|\mathcal{F}_s].\square$$

Concluding we see that the statement X and Y are $\mathbb{F}$-Markov is equivalent with saying that the indicator process Y satisfies equation (3.2).

Propositions 3.1 and 3.2 will play an important role in section 4. Here is another illustration of the usefulness of this result.

Applying propositions 3.2 and 3.2 to the case where X is a counting process and $\mathbb{F}=\mathbb{F}^n$, we easily obtain an intuitively appealing criterion, see also JACOBSEN [5], in terms of the predictable intensity, that shows whether or not a counting process is Markov. Of course we need a generalization of propositions 3.1, 3.2 to include processes that assume countably many values, but this is straightforward.

PROPOSITION 3.3: *Let n be a counting process, and λ its $\mathbb{F}^n$-predictable intensity process: $dn_t=\lambda_t dt+dm_t$.*
Equivalent are
(i) *n is ($\mathbb{F}^n$-) Markov*
(ii) *there exists a measurable $f:[0,\infty)\times\mathbb{N}_0\to[0,\infty)$ such that $\lambda_t=f(t,n_t-)$.*

PROOF: (i)$\Rightarrow$(ii): Let Y be the with n associated indicator process and $N^T=[0\ 1\ 2....]$. Then $n_t=N^T Y_t$ and Y satisfies by assumption

$$dY_t = A(t)Y_t dt + dm_t^Y.$$

On the other hand we have immediately from the definition of Y:

$$dY_t = (J - I)Y_{t-}dn_t,$$

where J is defined by its entries $J_{kl}=\delta_{k,l+1},k,l\geqslant 0$ and $I_{kl}=\delta_{kl}$, $k,l\geqslant 0$. Then

$$dY_t = (J - I)Y_t\lambda_t dt + (J - I)Y_{t-}dm_t.$$

Since each component of Y is a special semi martingale we have from the uniqueness of the decomposition for all t

$$\int_0^t (J - I)Y_{s-}\lambda_{s-}ds = \int_0^t A(s)Y_{s-}ds$$

Since all processes above are left continuous we have for all $t>0$: $(J-I)Y_{t-}\lambda_t=A(t)Y_{t-}$. After multiplying this equation by Y_{t-}^T we get

$$-\lambda_{t-} = Y_{t-}^T A(t)Y_{t-} = \sum_{i\geqslant 0} A_{ii}(t)Y_{it-}$$

Define now f by $f(t,n) = -a_{nn}(t)$, then

$$\lambda_{t-} = f(t,n_{t-}).$$

Then λ, being predictable, is indistinguishable from $f(t,n_{t-})$.

(ii)$\Rightarrow$(i). Define $F(t) \in \mathbb{R}^{\mathbb{N}_0}$ by $\mathbb{F}_n(t) = f(t,n)$. Hence

$$\lambda_t = F(t)^T Y_t.$$

As in part (i) of the proof

$$dY_t = (J - I)Y_{t-} dn_t.$$

Hence

$$dY_t = (J - I)Y_t Y_t^T F(t)dt + (J - I)Y_{t-} dm_t.$$

$$= (J - I)diag(Y_t)F(t)dt + (J - I)Y_{t-} dm_t.$$

Define $A(t) \in \mathbb{R}^{\mathbb{N}_0 \times \mathbb{N}_0}$ by $A_{kl}(t) = (J-I)_{kl}F_l(t)$, then

$$A(t)Y_t = (J - I)diag(Y_t)F(t) \text{ and}$$

$$dY_t = A(t)Y_t dt + (J - I)Y_{t-} dm_t,$$

which is of the form as in proposition 3.2. $\square$

3.3 From the equivalence of $\mathbb{F}$-Markov processes and solutions of certain linear stochastic differential equations (Propositions 3.1 and 3.2) it is easy to see when functions of a Markov chain again yield a Markov chain.

To be specific let as before X be a $\mathbb{F}$-Markov chain with state space $E = \{c_1, \ldots, c_n\}$. Let H be another set and $f : E \to H$ a function. Clearly $f(X)$ is again Markov if f is injective. To avoid trivialities let us assume that $H = \{h_1, \ldots, h_m\}$, $m < n$ and that f is onto. Write $Z_t = f(X_t)$. Associate with Z the indicator process W as usual:

$$W: \Omega \times [0, \infty) \to \{0,1\}^m, \quad W_{it} = 1_{\{Z_t = h_i\}}.$$

Define $F \in \mathbb{R}^{m \times n}$ by $F_{ij} = 1_{\{f(c_j) = h_i\}}$. Notice that $1_m^T F = 1_n^T$, where 1_m is a column vector with as elements $+1$. Then $W_t = FY_t$. Notice that because f is onto F has rank m, i.e. it has full row rank. Let $K \in \mathbb{R}^{n \times (n-m)}$ be a fixed matrix such that it columns span $KerF$. Let as before $A(t)$ be the matrix of transition intensities of X. We have the following.

THEOREM 3.4: *Let X be $\mathbb{F}$-Markov with finite state space E. Let $f : E \to H$. Then $f(X)$ is again $\mathbb{F}$-Markov iff $FA(t)K = 0$ where the columns of K span $KerF$ and F is related to f as indicated above. If this condition is satisfied, then the matrix $B(t)$ of transition intensities of $f(X)$ is given by $B(t) = FA(t)\hat{F}$, where $\hat{F}$ is any right inverse of F.*

PROOF: We have $dY_t = A(t)Y_t dt + dm_t^Y$. Hence

$$dW_t = FA(t)Y_t dt + Fdm_t^Y$$

Now Z is $\mathbb{F}$-Markov iff $dW_t = B(t)W_t dt + dm_t^W$ for some matrix-valued function B and a $\mathbb{F}$-martingale m^W. Hence we have Z is $\mathbb{F}$-Markov if and only if there is a $B(\cdot)$ such that $FA(t) = B(t)F$. Let $\hat{F}$ be a fixed right inverse of F. It exists, since F has full row rank. Then the last equation implies $B(t) = FA(t)\hat{F}$. Of course for B to be well defined it should not depend on the particular choice of $\hat{F}$. Starting from $\hat{F}$ all other right inverses G of F are given by $G = \hat{F} + KX$, where $X \in \mathbb{R}^{(n-m) \times m}$ is an arbitrary matrix. Hence $B(t)$ is well defined iff $FA(t)\hat{F} = FA(t)(\hat{F} + KX)$ or iff $FA(t)K \equiv 0$. $\square$

REMARKS:
1. The determining condition $FA(t)K \equiv 0$ can be understood in two ways. Firstly for a given matrix $A(t)$ it tells us what functions f (if any) yield a Markov process $f(X)$. Secondly if one wants $f(X)$ to be Markov it gives a condition on $A(t)$ when this is indeed the case.
2. The result as such is not new but can be found in a slightly different form in KEMENY and SNELL [6, p. 126] where Markov chains in discrete time are considered. However the proof given here is shorter.

4. $\mathbb{F}^n$-MARKOV PROCESSES

4.1 We will combine the results of corollary 2.2. and propositions 3.1, 3.2 applied to the situation where $\mathbb{F} = \mathbb{F}^n$ in order to find an integral representation of a finite state $\mathbb{F}^n$-Markov process in terms of its infinitesimal characteristics and the intensity of the counting process. Let as before $\lambda_t^+ = \frac{1}{\lambda_t} 1_{\{\lambda_t > 0\}}$, with the understanding that $\frac{0}{0} = 0$.

THEOREM 4.1: *Let X be an $\mathbb{F}^n$-Markov process with state space $\{c_1, \ldots, c_n\}$ and let Y be the indicator process associated to X as before. Then*

$$Y_t = Y_0 + \int_0^t \lambda_s^+ A(s) Y_{s-} \, dn_s \qquad (4.1)$$

PROOF: Y is a pure jump process satisfying $Y_t = Y_0 + \int_0^t A(s) Y_s ds + m_t^Y$ where m^Y is a $\mathbb{F}^n$-martingale. Hence a multivariate extension of corollary 2.2 applies: $Y_t = Y_0 + \int_0^t k_s dn_s$, where $k : \Omega \times [0, \infty) \to \mathbb{R}^n$ is $\mathbb{F}^n$-predictable. In the notation of this corollary we have $V_t = \int_0^t A(s) Y_s ds$. So k satisfies for all $t \geq 0$

$$\int_0^t A(s) Y_s ds = \int_0^t k_s \lambda_s ds. \qquad (4.2)$$

Hence, in order to ensure $\mathbb{F}^n$-predictability of k we have

$$A(t) Y_{t-} = k_t \lambda_t \qquad (4.3)$$

Now define $\tilde{k}$ by $\tilde{k}_t = k_t 1_{\{\lambda_t > 0\}}$. Then $\tilde{k}$ is $\mathbb{F}^n$-predictable and

$$0 \leq E \int_0^\infty 1_{\{k_t \neq \tilde{k}_t\}} \lambda_t dt \leq E \int_0^\infty 1_{\{\lambda_t = 0\}} \lambda_t dt = 0.$$

Hence k and $\tilde{k}$ are $P(d\omega)\lambda_t(\omega)$ a.e. the same. From the uniqueness result of lemma 2.1 we may use $\tilde{k}$ as well as k. So we have

$$A(t) Y_{t-} = \tilde{k}_t \lambda_t.$$

Hence

$$A(t) Y_{t-} \lambda_t^+ = \tilde{k}_t \lambda_t \lambda_t^+ = \tilde{k}_t.$$

Now drop the wiggle on $\tilde{k}$ and the proof is complete. $\square$

COROLLARY 4.2: *We have the following explicit expression for Y:*

$$Y_t 1_{\{T_k \leq t < T_{k+1}\}} = \prod_{l=1}^k (\lambda_{T_l}^+ A(T_l) + I) Y_0 1_{\{T_k \leq t < T_{k+1}\}}$$

8

PROOF: Immediate from theorem 4.1 by noting that $Y_{T_{l^-}} = Y_{T_{l-1}}$ and the fact that Y is right continuous. $\square$

EXAMPLE 4.1: Assume that the intensity process λ does not depend on t. Then $\lambda_t(\omega) = \lambda$ for some non random constant λ since $\lambda_0(\cdot)$ is $\mathscr{F}_0$-measurable. Assume $\lambda > 0$. Assume further that X is a homogeneous Markov process. Then

$$Y_t 1_{\{T_k \leqslant t < T_{k+1}\}} = (\lambda^{-1}A + I)^k Y_0 1_{\{T_k \leqslant t < T_{k+1}\}}$$

or

$$Y_t = (\lambda^{-1}A + I)^n Y_0$$

Since Y_t is a unit vector for all t, $\lambda^{-1}A + I$ is a semi-permutation matrix in the sense that each of its columns has exactly one $+1$ entry and the other entries are zero. Of course two $+1$ entries may occur in the same row. Consequently all the diagonal elements A_{ii} of A are either zero or equal to $-\lambda$. If some $A_{ii} = -\lambda$ then there is in the i-th column A_i of A exactly one A_{ji} equal to $+\lambda$. All the other entries of A_i are zero. If $A_{ii} = 0$ for some i then the whole column $A_i = 0$.

A similar remark applies to the general expression in corollary 4.2. We have for all i $A_{ii}(T_l) \leqslant 0$. Then if $A_{ii}(T_l) < 0$ there is exactly one $j = j(i, T_l)$ such that $A_{ji}(T_l) = -A_{ii}(T_l)$. Since T_l can assume any value > 0, we have that for each i and t there is exactly one $j = j(i, t)$ such that $A_{ji}(t) = -A_{ii}(t)$, all the other entries in the column $A_i(t)$ being zero.

4.2 The objective of this subsection is to study how λ and A are related. Equation (4.3) relates the intensity λ_t of the counting process with the matrix $A(t)$ of transition intensities of X by means of the intermediate process k. In this subsection we will study this relation a little further.
Multiply (4.3) by Y_{t-}^T to obtain

$$\lambda_t Y_{t-}^T k_t = Y_{t-}^T A(t) Y_{t-} \tag{4.4}$$

At a jump time T_n of the counting process there are two possibilities. If X also jumps then $Y_{T_n} \neq Y_{T_n^-} = Y_{T_{n-1}}$ and $Y_{T_{n-1}}^T k_{T_n} = Y_{T_{n-1}}^T (Y_{T_n} - Y_{T_{n-1}}) = -1$. If X does not jump then $Y_{T_{n-1}}^T k_{T_n} = 0$. So assuming that X jumps we get from (4.4)

$$\lambda_{T_n} = -Y_{T_{n-1}}^T A(T_n) Y_{T_{n-1}} \tag{4.5}$$

This last equation (4.5) suggests the following connection between λ and A:

$$\lambda_t = -Y_{t-}^T A(t) Y_{t-}$$

This connection will be studied in the sequel. First we need a definition. Define $\tilde{n}: \Omega \times [0, \infty) \to \mathbb{R}$ by $\tilde{n}_t = \frac{1}{2}[Y^T, Y]_t$. Here $[Y^T, Y]$ is the optional quadratic variation process of Y. It satisfies

$$Y_t^T Y_t = Y_0^T Y_0 + 2\int_0^t Y_{s-}^T dY_s + [Y^T, Y]_t \tag{4.6}$$

We now have the following proposition.

PROPOSITION 4.3:
i) $\tilde{n}$ is an $\mathbb{F}^n$ and $\mathbb{F}^Y$-adapted counting process with $\mathbb{F}^n$ and $\mathbb{F}^Y$-predictable intensity $\tilde{\lambda}_t = -Y_t^T A(t) Y_{t-}$
ii) $n - \tilde{n}$ is also a counting process. It is only $\mathbb{F}^n$-adapted and has $\mathbb{F}^n$-predictable intensity $\lambda_t + Y_{t-}^T A(t) Y_{t-}$
iii) $\tilde{n}_t = \int_0^t 1_{\{\tilde{\lambda}_s > 0\}} dn_s$ and $\tilde{\lambda}_t = 1_{\{\tilde{\lambda}_t > 0\}} \lambda_t$
iv) $\mathscr{F}_t^n$ and $\mathscr{F}_\infty^{\tilde{n}}$ are conditionally independent given $\mathscr{F}_t^{\tilde{n}}$

PROOF:

i) In view of eq (4.6) we have $0 = Y_t^T Y_t - Y_s^T Y_s = 2\int_s^t Y_{u-}^T dY_u + [Y^T, Y]_t - [Y^T, Y]_s =$

$$= 2\int_s^t Y_u^T A(u) Y_u du + 2\int_s^t Y_{u-}^T dm_u^Y + 2(\tilde{n}_t - \tilde{n}_s).$$

By observing that $\int_0 Y_{u-}^T dm_u^Y$ is again a $\mathbb{F}^n$ and $\mathbb{F}^Y$ martingale we get the desired result according to the definition of intensity.

ii) From known results in stochastic calculus we get $2\tilde{n}_t = [Y^T, Y]_t = \int_0^t k_s^T k_s dn_s = 2\int_0^t 1_{\{Y_s \neq Y_{s-}\}} dn_s$ because we need only knowledge of k at the jump times T_n. If X does not jump when n does then $k_{T_n} = 0$, and if it jumps then $k_{T_n}^T k_{T_n} = 2$. Hence $\tilde{n}_t \leqslant n_t$ for all t and $n_t - \tilde{n}_t = \int_0^t 1_{\{Y_s = Y_{s-}\}} dn_s$ which yields in view of (i) that $n - \tilde{n}$ has the described intensity.

iii) Notice that $1_{\{Y_{T_k} \neq Y_{T_{k-1}}\}} = 1_{\{\lambda_{T_k} > 0, \tilde{\lambda}_{T_k} > 0\}} = 1_{\{\tilde{\lambda}_{T_k} > 0\}}$, since $\tilde{\lambda}_{T_k} \leqslant \lambda_{T_k}$. Hence $d\tilde{n}_t = 1_{\{\tilde{\lambda}_t > 0\}} dn_t$. But then $d\tilde{n}_t = 1_{\{\tilde{\lambda}_t > 0\}} \lambda_t dt + 1_{\{\tilde{\lambda}_t > 0\}} dm_t$, which shows that $1_{\{\tilde{\lambda}_t > 0\}} \lambda_t$ is the $\mathbb{F}^n$-intensity of $\tilde{n}$ which is then also equal to $\tilde{\lambda}_t$ by part (i).

iv) This is an alternative formulation of (iii) in view of proposition 2.4.

An important corollary of this proposition occurs when all the $A_{ii}(t)$ are strictly negative. It is stated as the next

THEOREM 4.4:

i) Let all the $A_{ii}(t)$ be strictly negative. Then $n = \tilde{n}$, $\mathscr{F}_t^Y = \mathscr{F}_t^n$ for all $t > 0$ and $\lambda_t = -Y_{t-}^T A(t) Y_{t-}$

ii) $k_t = -(Y_{t-}^T A(t) Y_{t-})^{-1} A(t) Y_{t-} = -\Sigma \dfrac{A_i(t)}{A_{ii}(t)} Y_{it-}$

PROOF:

i) From eq. (4.4) we have $\lambda_{T_n}^T Y_{T_{n-1}}^T k_{T_n} = Y_{T_{n-1}}^T A(T_n) Y_{T_{n-1}} = \Sigma_i A_{ii}(T_n) 1_{\{X_{T_{n-1}} = c_i\}} < 0$. Hence $\lambda_{T_n} > 0$ and $k_{T_n} \neq 0$, which means that X always jumps as soon as n jumps. Hence $n = \tilde{n}$. Since always $\mathscr{F}_t^{\tilde{n}} \subset \mathscr{F}_t^Y \subset \mathscr{F}_t^n$ we now also have $\mathscr{F}_t^n = \mathscr{F}_t^Y$. Finally $n = \tilde{n}$ implies $\lambda_t = \tilde{\lambda}_t = -Y_{t-}^T A(t) Y_{t-}$.

ii) This is a simple consequence of formula (4.1) and part i) of the theorem.

It is appropriate to inspect the results of proposition 4.3 and theorem 4.4 a little closer. In general we have for all $t \geqslant 0$ $\mathscr{F}_t^{\tilde{n}} \subset \mathscr{F}_t^Y \subset \mathscr{F}_t^n$. In the case described in theorem 4.4, we get equality of those σ-algebra's. Since now n is also the total number of jumps (or transitions) of the Markov chain and $\mathscr{F}_t^Y = \mathscr{F}_t^n$ it seems logical to expect that we have in the general situation (where $\tilde{n}$ counts the transitions of the chain) $\mathscr{F}_t^{\tilde{n}} = \mathscr{F}_t^Y$, which means in words that if we have a Markov chain adapted to a counting process then it is also adapted to the counting process that describes the total number of transitions of the the chain. One could say that $\tilde{n}$ is sort of "minimal" counting processes to which X is adapted. Next we show that the claim $\mathscr{F}_t^{\tilde{n}} = \mathscr{F}_t^Y$ holds true. It is a consequence of

THEOREM 4.5: *Let X be finite state $\mathbb{F}^n$-Markov, then Y_t is $\mathscr{F}_t^{\tilde{n}}$-measurable.*

PROOF: Let $\tilde{T}_1, \tilde{T}_2, \ldots$ be the possibly finite sequence of jump times of $\tilde{n}$. From the discussion leading to (4.5) we see that $\lambda_{\tilde{T}_j} = -Y_{\tilde{T}_j-1}^T A(\tilde{T}_j) Y_{\tilde{T}_{j-1}} > 0$. Consider first $\tilde{T}_1$. Then $\lambda_{\tilde{T}_1}$ is a (measurable) function of $\tilde{T}_1$ only. Hence from $Y_{\tilde{T}_1} = (\lambda_{\tilde{T}_1}^{-1} A(\tilde{T}_1) + I) Y_0$ $Y_{\tilde{T}_1}$ is also a measurable function of $\tilde{T}_1$ only. But then by induction we find that $Y_{\tilde{T}_n} = (\lambda_{\tilde{T}_n}^{-1} A(\tilde{T}_n) + I) Y_{\tilde{T}_{n-1}}$ is a measurable function of $\tilde{T}_1, \ldots, \tilde{T}_n$, say $Y_{\tilde{T}_n} = y_n(\tilde{T}_1, \ldots, \tilde{T}_n)$.

Consequently, by right continuity of Y, we get with $y_0 = Y_0$

$$Y_t = Y_0 + \sum_{n=1}^{\infty} y_n(\tilde{T}_1, \ldots, \tilde{T}_n) 1_{\{\tilde{T}_n \leqslant t < \tilde{T}_{n+1}\}}.$$

Notice that y_n is $\tilde{\mathscr{F}}_{T_n}$-measurable since $\tilde{\mathscr{F}}_{T_n} = \sigma(\tilde{T}_1, \ldots, \tilde{T}_n)$. Now we invoke the fact that $\tilde{\mathscr{F}}_{T_n} \cap \{\tilde{T}_n \leqslant t < \tilde{T}_{n+1}\} = \tilde{\mathscr{F}}_t^{\tilde{n}} \cap \{\tilde{T}_n \leqslant t < \tilde{T}_{n+1}\}$ (see BRÉMAUD [2, p. 308]) to see that indeed Y_t is $\tilde{\mathscr{F}}_t^{\tilde{n}}$ measurable. □

The statement of the theorem is sometimes immediately seen in specific cases. Consider for example the case where $\lambda_t \equiv \lambda > 0$ and A is a constant matrix (example 4.1). Then we have in fact $Y_t = (\lambda^{-1}A + I)^{\tilde{n}_t} Y_0$.

COROLLARY 4.6: *If X is a finite state $\mathbb{F}^n$-Markov process, then it is also $\mathbb{F}^{\tilde{n}}$ Markov.*

PROOF: Since a process that is Markov with respect to some filtration is also Markov with respect to any other smaller filtration to which is adapted, this is an immediate consequence of theorem 4.5.

Thusfar we have seen the following results. Given the fact that we have a $\mathbb{F}^n$-Markov process X, X is also $\mathbb{F}^{\tilde{n}}$-Markov and $\tilde{n}$ has intensity $\tilde{\lambda}_t = -Y_{t-}^T A(t)Y_{t-}$, where $\tilde{n}$ is as before the process that counts all the transitions of X. As such these results form necessary conditions that follow from the existence of such processes. One might raise the question how to formulate sufficient conditions on a given Markov matrix function $A(\cdot)$ such that there exists an associated $\mathbb{F}^n$-Markov chain X.
Secondly, given that a process X is $\mathbb{F}^{\tilde{n}}$-Markov, what other counting processes n do exist such that X is also $\mathbb{F}^n$-Markov.
Answering the first question will be postponed until section 5. Concerning the second one we have - as a converse of previous results -

PROPOSITION 4.7: *Let X be $\mathbb{F}^{\tilde{n}}$-Markov. Let n be another counting process with $\mathbb{F}^n$-predictable intensity λ such that*

(i) $\tilde{n}_t = \int_0^t 1_{\{\tilde{\lambda}_s > 0\}} dn_s$

(ii) $\tilde{\lambda}_t = 1_{\{\tilde{\lambda}_t > 0\}} \lambda_t$
Then X is also $\mathbb{F}^n$-Markov.

PROOF: From proposition 2.4, we see that $\tilde{\mathscr{F}}_t^{\tilde{n}} \subset \mathscr{F}_t^n$ and that $\mathscr{F}_\infty^{\tilde{n}}$ and $\mathscr{F}_t^n$ are conditionally independent given $\mathscr{F}_t^{\tilde{n}}$. Hence X is certainly $\mathbb{F}^n$-adapted.
Observe first that $\tilde{\lambda}_t = 0 \Leftrightarrow Y_{t-}^T A(t)Y_{t-} = 0$ implies $A(t)Y_{t-} = 0$ as a result of the fact that $A(t)$ is a Markov-matrix. Since X is $\mathbb{F}^{\tilde{n}}$-Markov: $dY_t = \tilde{\lambda}_t^+ A(t)Y_{t-} d\tilde{m}_t$ (theorem 4.1). Hence

$$dY_t = \tilde{\lambda}_t^+ \lambda_t A(t)Y_t dt + \tilde{\lambda}_t^+ A(t)Y_{t-} d\tilde{m}_t$$
$$= A(t)Y_t dt + \tilde{\lambda}_t^+ A(t)Y_{t-} d\tilde{m}_t.$$

From the conditional independence relation and lemma 2.5, the last term is a $\mathbb{F}^n$-martingale. Therefore application of proposition 3.2 completes the proof.

REMARK: In view of the remark following the proof of proposition 2.4 one can replace conditions (i) and (ii) in proposition 4.7 by $\tilde{n}_t = \int_0^t u_s dn_s$ and $\tilde{\lambda}_t = u_t \lambda_t$ for some $\mathbb{F}^n$-predictable process u.

Until now we have studied processes X that are $\mathbb{F}^n$-Markov and thus $\mathbb{F}^n$-adapted. As mentioned before, one of the results is then, that X is also $\mathbb{F}^{\tilde{n}}$-Markov (corollary 4.6). Knowing this, one can prove all the results mentioned in the foregoing, such as $\tilde{\lambda}_t = Y_{t-}^T A(t)Y_{t-}$ etc.
An interesting question is to see whether a process which is Markov with respect to its own flow of σ-algebras and which is $\mathbb{F}^n$-adapted, shares the same properties. In general this is not true. For instance if n is standard poisson process and X is defined by $X_t = n_{\frac{1}{2}t}$, then X is $\mathbb{F}^X$-Markov, but not

$\mathbb{F}^n$-Markov. Theorem 4.8 gives a sufficient condition for an affirmative answer. Let us first remark that any bounded process that is adapted to some filtration is a special semi-martingale with respect to this filtration. See DELLACHERIE & MEYER [4, VII.25]

THEOREM 4.8: *Let X be a finite state $\mathbb{F}^X$-Markov chain and assume that X is adapted to $\mathbb{F}^n$ - for some counting process n. Assume moreover that the indicator process Y, being a $\mathbb{F}^n$ - special semi martingale, admits a decomposition such that the predictable process of finite variation is continuous. Then $\mathscr{F}_t^X = \mathscr{F}_t^{\tilde{n}} \forall t \geqslant 0$ and X is $\mathbb{F}^{\tilde{n}}$-adapted and thus $\mathbb{F}^{\tilde{n}}$-Markov.*

PROOF: From corollary 2.2 we know that $dY_t = k_t dn_t$ for some $\mathbb{F}^n$-predictable process k. By definition of $\tilde{n}$ we have $d\tilde{n}_t = \frac{1}{2}[Y^T, Y]_t = \frac{1}{2}k_t^T k_t dn_t$. So $\Delta\tilde{n}_t = 0$ iff $k_t = 0$. Therefore we can write $dY_t = k_t d\tilde{n}_t$. Observe that $\tilde{n}$ is $\mathbb{F}^Y$-adapted. As in BRÉMAUD [2, p.2.13], we can interpret k_t as a Radon-Nikodym derivative $\dfrac{dY_t}{d\tilde{n}_t}$ on the $\mathbb{F}^Y$-predictable sets. Therefore we may take k to be $\mathbb{F}^Y$-predictable. For $\tilde{n}$ we have by its definition

$$d\tilde{n}_t = -Y_{t-}^T dY_t = -Y_{t-}^T A(t) Y_{t-} dt - Y_{t-} dm_t^Y$$

so

$$dY_t = k_t d\tilde{n}_t = -k_t Y_{t-}^T A(t) Y_{t-} dt - k_t Y_{t-}^T dm_t^Y \qquad (4.7)$$

on the other hand

$$dY_t = A(t) Y_t dt + dm_t^Y \qquad (4.8)$$

Since all processes in (4.7) and (4.8) are $\mathbb{F}^Y$-adapted, we have from the uniqueness of the decomposition of a special semi martingale that $-k_t Y_{t-}^T A(t) Y_{t-} = A(t) Y_{t-}$ a.s., which then leads to $k_t = -(Y_{t-}^T A(t) Y_{t-})^+ A(t) Y_{t-}$. As in the proof of the theorem 4.5 we can conclude that Y is $\mathbb{F}^{\tilde{n}}$-measurable. Therefore $\mathscr{F}_t^Y \subset \mathscr{F}_t^{\tilde{n}} \subset \mathscr{F}_t^Y$. Hence X is $\mathbb{F}^X$-Markov is now equivalent to X is $\mathbb{F}^Y = \mathbb{F}^{\tilde{n}}$-Markov. $\square$

REMARK: The statement of theorem 4.8 indicates why $n_{\frac{1}{2}t}$ cannot be $\mathbb{F}^n$-Markov. This is immediately seen by noting that $n_{\frac{1}{2}t}$ is $\mathbb{F}^n$-predictable. Hence its dual predictable projection with respect to $\mathbb{F}^n$ is the process itself, which is discontinuous.

4.3 In this subsection we mention some consequences of the foregoing for the case where X is a homogeneous chain.

COROLLARY 4.9: *Assume that X is a homogeneous chain*
i)   *If $A_{ii} < 0$, then in the corresponding column A_i of A there is exactly one $j = j(i)$ such that $A_{ji} = -A_{ii}$ and all other A_{ki}'s are zero. If $A_{ii} = 0$ then the whole column $A_i = 0$.*
ii) *k is now a left continuous piecewise constant process and satisfies*

$$k_t 1_{\{T_n \leqslant t \leqslant T_{n+1}\}} = -\sum_i A_{ii}^+ A_i 1_{\{X_{T_n} = c_i\}} 1_{\{T_n < t \leqslant T_{n+1}\}}$$

iii) *The sampled chain $\hat{X}_n := X_{T_n}$ is now a deterministic process and completely known given the initial state $\hat{X}_0 = X_0$.*
iv) *If there are no absorbing states, then the process λ assumes only a finite number of values. Specifically $\lambda_t \in \{-A_{11}, \ldots, -A_{nn}\}$.*

PROOF: i), iii) iv) follow immediate from the explicit expression in corollary 4.2. ii) requires a little work. Recall that we have $k_t = \lambda_t^+ A Y_{t-}$. Let T be the absorption time of the chain. Then

$AY_t - 1_{\{t>T\}} \equiv 0$. Hence $\tilde{\lambda}_t > 0 \Leftrightarrow t \le T$. Therefore $\lambda_t 1_{\{t \le T\}} = \tilde{\lambda}_t 1_{\{t \le T\}} = -Y_{t-}^T AY_t - 1_{\{t \le T\}}$. Hence $k_t = -\Sigma_i A_{ii}^+ A_i Y_{it} - 1_{\{t \le T\}} = -\Sigma_i A_{ii}^+ A_i Y_{it-}$, because $A_i Y_{it} - 1_{\{t>T\}} = 0$.

At this point one might raise the question in virtue of corollary 4.9 iv) whether λ is also a Markov process. Clearly this is the case if all the A_{ii} are different or when they are all the same. Interesting is the case when there exists at least one pair (i,j) such that $A_{ii} = A_{jj}$. We will answer this question by means of theorem 3.4. Assume that there are $2 \le m \le n-1$ distinct values among the A_{ii}. Call these $a_1, \ldots, a_m$ and denote for all $j = 1, \ldots, m$ by E_i the set of of all j such that $A_{jj} = a_i$. Define $F \in \mathbb{R}^{m \times n}$ by $F_{ij} = 1_{\{j \in E_i\}}$ We have the following result in the terminonology of theorem 3.3.

PROPOSITION 4.10: *In the terminology of theorem 3.4:* λ *is an* $\mathbb{F}^n$*-Markov chain iff* $FAK = 0$. *If the last condition is satisfied then the matrix B of transition intensities of* λ *is given by FAF.*

EXAMPLE 4.2
i) If

$$A = \begin{bmatrix} -a & 0 & 0 & b \\ a & -b & 0 & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & a & -b \end{bmatrix}$$

then λ is Markov with $B = \begin{bmatrix} -a & b \\ a & -b \end{bmatrix}$ and state space $\{-a, -b\}$. Here we should take $F = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

ii) If

$$A = \begin{bmatrix} -a & 0 & 0 & b \\ a & -a & 0 & 0 \\ 0 & a & -b & 0 \\ 0 & 0 & b & -b \end{bmatrix}$$

then λ is not Markov, which is seen by calculating $FAK = \begin{bmatrix} a-b \\ -a & b \end{bmatrix}$ with, $F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $K^T = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

REMARKS:
i) Although it might happen that λ is not Markov of course (λ, X_-) is jointly Markov.
ii) Since it follows from proposition 4.5 iv that the number of values that λ can assume is always at most the number of states that X can assume, we see that a necessary condition for a process X to be Markov is, that it takes values in a set which is at least as big as the set of values of $\lambda: n \ge \# \{\lambda_t : t \ge 0\}$. Hence a homogeneous chain X cannot have a finite state space if λ has a continuously varying component. In the same way as checking, whether λ is $\mathbb{F}^n$-Markov one can investigate whether there exist Markov processes X^1 with a smaller state space than X by considering all possible choises of F. Thus obtaining a description of a "minimal" Markov process. This is of some relevance in connection with the stochastic realization problem to be posed in section 5.
iii) The case where λ is $\mathbb{F}^n$-Markov itself implies here that it changes value as soon as n jumps. Thus we can immediately see from the A-matrix whether λ is $\mathbb{F}^n$-Markov or not. In the ex. 4.2.i we see

that at jump times λ switches from a to b or conversely, which is in agreement with the fact that it is Markov. In the ex. 4.2.ii we see that it is possible that λ stays in a even when n jumps.

4.4: In the previous subsection we have seen that the existence of a homogeneous $\mathbb{F}^n$-Markov chain X does not necessarily imply that λ is also $\mathbb{F}^n$-Markov. Here after we describe some consequences of the situation where indeed λ is a $\mathbb{F}^n$-Markov process with finite state space. Since in this case λ assumes only a finite number of values it follows that λ (being predictable) may be taken as a left continuous process. Write $X_t = \lambda_{t+}$, the right continuous version of λ. We will apply the previous results to this particular choice of X.

Denote by $\{\lambda_1, \ldots, \lambda_n\}$ the state space of X. If there are no absorbing states then $A_{ii} < 0$ and we have that $\lambda_i = -A_{ii}$ for all i in view of corollary 4.5.iv. So all $\lambda_i > 0$.

For reasons of completeness we will show what happens if some of the A_{ii} are equal to zero or if one of the λ_i equals zero. The latter case clearly implies that the corresponding $A_{ii} = 0$. Hence this case is covered by the first one. Define $B \subset \{1, \ldots, n\}$ to be the set of integers i such that λ_i is an absorbing state. Define also $T = \inf\{t \geq 0 : X_t \in \{\lambda_i, i \in B\}\}$.

Notice that $T < \infty$ a.s. if and only if $B \neq \varnothing$, and for $i \in B$ we have $A_{ii}(t) \equiv 0$, and hence the whole column $A_i(t) \equiv 0$. The principle result of this subsection is the next proposition which tells that for $t \leq T$ we can more or less identify the intensity λ_t as one of the $A_{ii}(t)$'s, and that $A_{ii}(t)$ only assumes the values $-\lambda_i$ or 0.

PROPOSITION 4.11: *Assume that λ is $\mathbb{F}^n$-Markov with state space $\{\lambda_1, \ldots, \lambda_n\}$ and transition intensity matrix $A(t)$. Let T be the absorption time as defined above and B the set of integers corresponding to the absorbing states. Then*

$$\lambda_t = \lambda_T 1_{\{t > T\}} + \sum_{i \in B^c} \lambda_i 1_{\{\lambda_t = \lambda_i\}} 1_{\{A_{ii}(t) = 0\}} - \sum_{i \in B^c} A_{ii}(t) 1_{\{\lambda_t = \lambda_i\}}$$

and for $i \in B^c : A_{ii}(t) = -\lambda_i$ if $A_{ii}(t) < 0$.

PROOF: Let $X_t = \lambda_{t+}$, then $Y_{it} = 1_{\{X_t = \lambda_i\}}$ and $Y_{it-} = 1_{\{\lambda_t = \lambda_i\}}$. In the notation that we have used previously, $\tilde{n}$ has rate $\tilde{\lambda}_t = -Y_{t-}^T A(t) Y_{t-} = -\Sigma_{i \in B^c} A_{ii}(t) 1_{\{\lambda_t = \lambda_i\}} 1_{\{t \leq T\}}$ Since $\tilde{\lambda}_t = {}_{\{\tilde{\lambda}_t > 0\}} \lambda_t$ (proposition 4.3. iii) we have

$$\tilde{\lambda}_t 1_{\{t \leq T\}} = 1_{\{\tilde{\lambda}_t > 0\}} 1_{\{t \leq T\}} \lambda_t + 1_{\{\tilde{\lambda}_t > 0\}} 1_{\{t > T\}} \lambda_t$$

$$= 1_{\{\tilde{\lambda}_t > 0\}} \lambda_t,$$

since $\tilde{\lambda}_t > 0$ implies $t \leq T$ and conversely $t > T$ implies $\tilde{\lambda}_t = 0$. Hence

$$-\sum_i A_{ii}(t) 1_{\{\lambda_t = \lambda_i\}} 1_{\{t \leq T\}} = 1_{\{\tilde{\lambda}_t > 0\}} \sum_i \lambda_i 1_{\{\lambda_t = \lambda_i\}}$$

Now let $i \in B^c$. Then

$$-A_{ii}(t) 1_{\{\lambda_t = \lambda_i\}} 1_{\{t \leq T\}} = 1_{\{\tilde{\lambda}_t > 0\}} 1_{\{\lambda_t = \lambda_i\}} \lambda_i.$$

Observe that

$$1_{\{\lambda_t > 0\}} 1_{\{\lambda_t = \lambda_i\}} = 1_{\{A_{ii}(t) < 0\}} 1_{\{\lambda_t = \lambda_i\}}$$

and for $i \in B^c$ $\lambda_t = \lambda_i$ implies $t \leq T$. Hence we get

$$-A_{ii}(t) 1_{\{\lambda_t = \lambda_i\}} = 1_{\{A_{ii}(t) < 0\}} 1_{\{\lambda_t = \lambda_i\}} \lambda_i.$$

Since we may assume that $P(\lambda_t = \lambda_i) > 0$ we now get by taking expectations

$$-A_{ii}(t) = 1_{\{A_{ii}(t) < 0\}} \lambda_i$$

14

which proves the second assertion of the proposition. Furthermore

$$\lambda_t = \lambda_T 1_{\{t > T\}} + \lambda_t 1_{\{t \leqslant T\}} =$$

$$= \lambda_T 1_{\{t > T\}} + 1_{\{t \leqslant T\}} \sum_{i \in B^c} 1_{\{\lambda_t = \lambda_i\}} \lambda_i$$

$$= \lambda_T 1_{\{t > T\}} + \sum_{i \in B^c} 1_{\{\lambda_t = \lambda_i, A_{ii}(t) = 0\}} \lambda_i + \sum_{i \in B^c} 1_{\{\lambda_t = \lambda_i, A_{ii}(t) < 0\}} \lambda_i$$

$$= \lambda_T 1_{\{t > T\}} + \sum_{i \in B^c} 1_{\{\lambda_t = \lambda_i, A_{ii}(t) = 0\}} \lambda_i - \sum_{i \in B^c} A_{ii}(t) 1_{\{\lambda_t = \lambda_i\}}$$

which proves the first assertion. $\square$

REMARKS

1. If λ is a homogeneous $\mathbb{F}^n$-Markov chain, then A is a constant matrix and we have for $i \in B^c$ the identity $A_{ii}(t) \equiv -\lambda_i$. Hence

$$\lambda_t = \lambda_T 1_{\{t > T\}} - \sum_{i \in B^c} A_{ii} 1_{\{\lambda_t = \lambda_i\}}.$$

And of course if there are no absorbing states or if the value zero is the only one, then $A_{ii}(t) \equiv -\lambda_i$ for all i and $\lambda_t = -\Sigma_{i=1}^n A_{ii} 1_{\{\lambda_t = \lambda_i\}}$

2. Now it is easy to see that for any function f which is not injective or constant $f(\lambda)$ cannot be a $\mathbb{F}^n$-Markov chain, since we have tacitly assumed that all the λ_i are different. Hence the number of states of λ is now the minimal number of elements that a set should have in order that it can serve as a state space for some $\mathbb{F}^n$-Markov process. In this sense one can say that λ, if it is $\mathbb{F}^n$-Markov, is the minimal $\mathbb{F}^n$-Markov chain.

5. STOCHASTIC REALIZATION

The purpose of this section is to solve a certain stochastic realization problem, to be stated in subsection 5.2. The solution involves a technical result on the existence of $\mathbb{F}^n$-Markov processes which is formulated in subsection 5.1.

5.1. It is known that given a Markov-matrix function $A : [0, \infty) \to \mathbb{R}^{n \times n}$, one can always construct a probability space $(\Omega, \mathcal{F}, P)$ and a Markov process $X : \Omega \times [0, \infty) \to \{1, \ldots, n\}$, such that its transition probabilities are generated by A.

In this section we are concerned with a restrictive version of this problem, namely given a complete probability space $(\Omega, \mathcal{F}, P)$ a counting process $n : \Omega \times [0, \infty) \to \mathbb{N}_0$ and a Markov matrix function $A : [0, \infty) \to \mathbb{R}^{n \times n}$, does there exist a $\mathbb{F}^n$-Markov process $X : \Omega \times [0, \infty) \to \{1, \ldots, n\}$ such that A generates its transition probabilities. We know from previous results that given such a process we have the identities $\tilde{\lambda}_t = -Y_{t-}^T A(t) Y_{t-}$ and $\tilde{\lambda}_t = \lambda_t 1_{\{\tilde{\lambda}_t > 0\}}$ and that for each (i, t) such that $A_{ii}(t) < 0$, there exists only one j such that $A_{ji}(t) = -A_{ii}(t)$. Hence for the existence of such a process X this imposes some necessary conditions on the matrix $A(t)$. In theorem 5.1 we present a set of sufficient conditions that implies the existence of such a desired process X, and we also give a construction for X. Before stating the theorem let us emphasize that one should not overestimate its content, since in a sense it looks like a tautology. On the other hand it shows how one can extract a $\mathbb{F}^n$-Markov process that is hidden in a suitable matrix function A. After having proved the theorem we give an example, how to use the construction of X.

THEOREM 5.1: *Given a counting process n with $\mathbb{F}^n$-predictable intensity λ and a Markov matrix function $A : [0, \infty) \to \mathbb{R}^{n \times n}$. There exists a $\mathbb{F}^n$-Markov process $X : \Omega \times [0, \infty) \to \{1, \ldots, n\}$ with A as its infinitesimal generator if there is a unique sequence of random variables $\{x_m\}_{m \geqslant 0}, x_m : \Omega \to \{1, \ldots, n\}$ such that the following two conditions hold*

a) $A_{x_m x_m}(T_m)(A_{x_m x_m}(T_m)+\lambda_{T_m}I)=0, \forall m.$

b) If $A_{x_m x_m}(T_m)<0$ then x_{m+1} is such that $A_{x_{m+1} x_m}(T_m)=-A_{x_m x_m}(T_m)$ and if $A_{x_m x_m}(T_m)=0$, then $x_{m+1}=x_m.$

PROOF: Let us define a process $Y^-:\Omega\times[0,\infty)\to\{0,1\}^n$ by requiring that $Y_t^- 1_{\{T_{m-1}<t\leqslant T_m\}}=Y_{T_m}^- 1_{\{T_{m-1}<t\leqslant T_m\}}$ and $Y_{iT_m}^-=1_{\{x_m=i\}}$. Then

$$\lambda_{T_m}^+\sum_j A_{ij}(T_m)Y_{jT_m}^- = \lambda_{T_m}^+\sum_j A_{ij}(T_m)1_{\{x_m=j\}} =$$

$$= \lambda_{T_m}^+ A_{ix_m}(T_m) = \lambda_{T_m}^+ A_{x_{m+1}x_m}(T_m)1_{\{x_{m+1}=i\}} +$$

$$\lambda_{T_m}^+ A_{x_m x_m}(T_m)1_{\{x_m=i\}} + \lambda_{T_m}^+ A_{ix_m}(T_m)1_{\{x_m\neq i, x_{m+1}\neq i\}}$$

$$= -\lambda_{T_m}^+ A_{x_m x_m}(T_m)1_{\{x_{m+1}=i\}} 1_{\{A_{x_m x_m}(T_m)<0\}} +$$

$$\lambda_{T_m}^+ A_{x_m x_m}(T_m)1_{\{x_m=i\}} + 0$$

$$= -\lambda_{T_m}^+ A_{x_m x_m}(T_m)[1_{\{x_{m+1}=i\}} - 1_{\{x_m=i\}}]$$

$$= 1_{\{x_{m+1}=i\}} - 1_{\{x_m=i\}} = Y_{iT_{m+1}}^- - Y_{iT_m}^-.$$

So in vector notation we have

$$Y_{T_{m+1}}^- - Y_{T_m}^- = \lambda_{T_m}^+ A(T_m)Y_{T_m}^- \tag{5.1}$$

Notice that $\lambda_{T_m}=0$ implies $A(T_m)Y_{T_m}^-=0$. Therefore with the usual convention that $\frac{0}{0}=0$ we have from (5.1)

$$Y_{T_{m+1}}^- - Y_{T_m}^- = \lambda_{T_m}^{-1} A(T_m)Y_{T_m}^-. \tag{5.2}$$

Define now $Y:\Omega\times[0,\infty)\to\{0,1\}^n$ by $Y_t=Y_{t+}^-$. Then $Y_{T_{m+1}}^-=Y_{T_m}$. Hence (5.1) reads

$$Y_{T_m} - Y_{T_{m-1}} = \lambda_{T_m}^{-1} A(T_m)Y_{T_{m-1}} \tag{5.3}$$

which can be rephrased as

$$dY_t = \lambda_t^{-1} A(t)Y_{t-}dn_t$$

or

$$dY_t = A(t)Y_{t-}dt + \lambda_t^+ A(t)Y_{t-}dm_t$$

We now want to apply proposition 3.2. Therefore we have to verify that Y_{t-} is $\mathbb{F}^n$-predictable. Observe that

$$Y_{t-}1_{\{T_m<t\leqslant T_{m+1}\}} = Y_{T_m}1_{\{T_m<t\leqslant T_{m+1}\}} \tag{5.5}$$

Now the sequence $\{x_m\}_{m\geqslant 0}$ is such that x_{m+1} is selected on the basis of knowing x_m and T_m, or iteratively is selected on the knowledge of $\{T_1,\ldots,T_m\}$. Therefore $Y_{iT_m}=Y_{iT_{m+1}}^-=1_{\{x_{m+1}=i\}}$ only depends on $\{T_1,\ldots,T_m\}$. From (5.5) and lemma 2.3 we now find the desired result. $\square$

EXAMPLE 5.1: Let λ be constant between the jump times T_i and envolve according to $\lambda_1,\lambda_2,\lambda_3,\lambda_1,\lambda_2,\lambda_3\cdots$ etc. Let

$$A_1 = \begin{bmatrix} -\lambda_1 & 0 & \lambda_2 \\ \lambda_1 & -\lambda_3 & 0 \\ 0 & \lambda_3 & -\lambda_2 \end{bmatrix}.$$

Then we see that A_1 cannot be a transition matrix of a $\mathbb{F}^n$-Markov chain $X:\Omega\times[0,\infty)\to\{1,2,3\}$. Because from condition a) of the theorem we see that $X_t=1$ iff $\lambda_t=\lambda_1$, $X_t=3$ iff $\lambda_t=\lambda_1$ and $X_t=2$ iff $\lambda_t=\lambda_3$. From $X_t=1$ it can only jump to 2 according to A_1. But from the given sequence of λ's it should jump from 1 to 3. However

$$A_2 = \begin{bmatrix} -\lambda_1 & 0 & \lambda_3 \\ \lambda_1 & -\lambda_2 & 0 \\ 0 & \lambda_2 & -\lambda_3 \end{bmatrix}$$

is compatible with the sequence of λ's as one can easily verify and thus A_2 can act as the transition matrix of a $\mathbb{F}^n$-Markov chain $X:\Omega\times[0,\infty)\to\{1,2,3\}$.

5.2 In this section we will adress a certain stochastic realization problem, and see how we can solve it by means of theorem 5.1. Let us state the problem precisely.

We are given a complete filtered probability space $(\Omega,\mathfrak{F},\mathbb{F}^n,P)$, where the filtration $\mathbb{F}^n$ is generated by a counting process satisfying $dn_t=\lambda_t dt+dm_t$, where λ is the $\mathbb{F}^n$-predictable intensity process and m a $\mathbb{F}^n$-martingale.

We pose the following question. Does there exist a homogeneous $\mathbb{F}^n$-Markov process X with finite state space E and a (measurable) function $f:E\to\mathbb{R}^+$ such that $\lambda_t=f(X_{t-})$?

One can reformulate this question in terms that are used in stochastic realization theory. The concepts involved are then stochastic system, state process, output process. However it seems that there is no consensus on how to define in abstract terms, what a stochastic system is. One approach can be found in VAN SCHUPPEN [8]. We will not touch upon all the difficulties that are inherent to this problem.

We will give a definition that suffices for our purpose. Suppose that we are given an object, to be called a stochastic system, with output process y. Then from the intuitive interpretation of state a process X that should play the role of state process has to satisfy at least the following requirement: The conditional distribution of $X_{t+v}, v>0$ given all X_s and y_s for $s\leqslant t$ is the same as the conditional distribution of X_{t+v} given X_t alone.

Here we are interested in systems with a counting process output only. The above considerations are captured in the next definition, which is probably not the most general one.

DEFINITION 5.2: A stochastic state space system with counting process output is a complete filtered probability space $(\Omega,\mathfrak{F},\mathbb{F},P)$ together with an adapted stochastic process $X:\Omega\times[0,\infty)\to E$, an adapted counting processes n and a measurable function $f:\mathbb{R}\times E\to\mathbb{R}_+$ such that $\sigma\{X_s,s\geqslant t\}$ and $\mathfrak{F}_t$ are conditionally independent given $\sigma(X_t)$ and $f(t,X_{t-})$ is the $\mathbb{F}$- predictable intensity of n.

In this section we are concerned with state processes X that assume finitely many values and with selfexciting counting processes n, like in previous sections. This amounts to studying stochastic systems such that X is $\mathbb{F}^n$-adapted. By taking $\mathbb{F}=\mathbb{F}^n$ in definition 5.2, we have that the state X is even $\mathbb{F}^n$-Markov, as follows from the next proposition, the proof of which is obvious. As already mentioned in the introduction, the property that the state process is $\mathbb{F}^n$-Markov was a motivation for studying $\mathbb{F}^n$-Markov processes.

PROPOSITION 5.2: *Let X be a stochastic process. Equivalent are*
(i) *X is $\mathbb{F}$-Markov*
(ii) *For all $t\geqslant0$ $\sigma\{X_s,s\geqslant t\}$ and $\mathfrak{F}_t$ are conditionally independent given $\sigma(X_t)$ and X_t is $\mathfrak{F}_t$ measurable.*

REMARK: Observe that we can take n as a state process if and only if n is Markov, which is the case if and only if its predictable intensity is of the form $f(t,n_{t-})$ (proposition 3.3).

An alternative formulation of the question that we posed in the beginning of this section is the

following. Given a counting process n on $(\Omega, \mathcal{F}, P)$ can we find a stochastic system on $(\Omega, \mathcal{F}, \mathbb{F}^n, P)$ such that its state process X is homogeneous and has finite state space E and such that the output processes is n with $\mathbb{F}^n$- predictable intensity $f(X_{t-})$ for some $f : E \to \mathbb{R}_+$.

Let us suppose that we can affirmatively answer this question. From corollary 4.9 we see that the sequence $\{\lambda_{T_m}\}$ is eventually constant or periodic. This observation also gives us a sufficient condition for solving the problem, which is the content of the next theorem.

THEOREM 5.3: *There exists on* $(\Omega, \mathcal{F}, \mathbb{F}^n, P)$ *a finite state* $\mathbb{F}^n$-*Markov process* X *with state space* E *and a function* $f : E \to \mathbb{R}^+$ *such that* $\lambda_t = f(X_{t-})$ *if and only if there exist a jump time* T_k *of* n *such that the sequence* $\{\lambda_{T_n}\}$ *for* $T_n \geq T_k$ *is either constant or periodic.*

PROOF: We only have to prove that this condition on λ is sufficient for the existence of X.

(i) Consider first the case where $\{\lambda_{T_m}\}$ is eventually constant. Let N be the smallest integer such that $\lambda_{T_m} = \lambda_{T_N}$ for all $n \geq N$. Now we can construct a $\mathbb{F}^n$-Markov process X with state space $\{1, \ldots, N+1\}$ as follows. Define $A \in \mathbb{R}^{(N+1) \times (N+1)}$ as follows $A_{ii} = -\lambda_{T_{i-1}}$, $A_{i+1,i} = -A_{ii} = \lambda_{T_i-1}$, for $i = 1, \ldots, N$ and all other A_{ij}'s equal to zero.

$$A = \begin{bmatrix} -\lambda_0 & & & \\ +\lambda_0 & & & \\ & & -\lambda_{N-1} & \\ & & +\lambda_{N-1} & 0 \end{bmatrix}$$

This matrix clearly satisfies the conditions of theorem 5.1, which yields the existence of the desired X. The function $f : \{1, \ldots, N+1\} \to \mathbb{R}^+$ we are looking for is of course defined by $f(i) = \lambda_{T_{i-1}}$, $i = 1, \ldots, N+1$.

(ii) Consider now the case where $\{\lambda_{T_n}\}$ is eventually cyclic, which means that there exist integers N' and p' such that $\lambda_{T_{i+p'}} = \lambda_{T_i}$ for $i \geq N'$. Let N and p be the smallest of such integers. Now we can construct a $\mathbb{F}^n$-Markov process X with state space $\{1, \ldots, N+p\}$ as follows. Define $A \in \mathbb{R}^{(N+p) \times (N+p)}$ by $A_{ii} = -\lambda_{T_{i-1}}$ for $i = 1, \ldots, N+p$, $A_{i+1,i} = -A_{ii} = \lambda_{T_{i-1}}$ for $i = 1, \ldots, N+p-1$ and $A_{N+1,N+p} = \lambda_{T_{N+p-1}}$. All other A_{ij} are zero.

$$A = \begin{bmatrix} -\lambda_0 & & & & & \\ \lambda_0 & & & & & \\ & & -\lambda_{T_N} & & +\lambda_{T_{N+p-1}} & \\ & & +\lambda_{T_N} & & & \\ & & & & -\lambda_{T_{N+p-2}} & \\ & & & & +\lambda_{T_{N+p-2}} & -\lambda_{T_{N+p-1}} \end{bmatrix}$$

As in the first case the existence of the X we are looking for is guaranteed by theorem 5.1 and f is defined by $f(i) = \lambda_{T_{i-1}}$, $i = 1, \ldots, N+p$ $\square$

REMARK: The behaviour of the system for $t \leq T_N$ (T_N as defined in the proof of theorem 5.3) can be considered as the transient behaviour of the system. If one would assume that time runs from minus infinity, instead from zero, then the necessary and sufficient condition in theorem 5.3 would read: The sequence $\{\lambda_{T_n}\}$ is either periodic or constant.

One other problem that remains to be solved is that of minimality of the solution of the realization problem. In our context minimality means minimality of the number of elements of the state space E. We have the following result.

COROLLARY 5.4: *The solution of the stochastic realization problem as presented in the proof of theorem 5.3 is minimal.*

PROOF: In principle one can prove the corollary by applying the $FAK=0$ criterion of theorem 3.4. Here we give an alternative proof. Consider first case (i). Assume that there exists a function g such that $g(X)$ is Markov and a function h such that $h(g(X_t))=f(X_t)=\lambda_t$. Consider a state j of $X, j \leqslant N$. Then there is no $i \leqslant j-1$ such that $g(i)=g(j)$, otherwise the sequence $\{\lambda_{T_n}\}$ would reach a loop, which is forbidden by assumption. Similarly there is no $i \leqslant N$ such that $g(i)=g(N+1)$, otherwise the absorption time would be smaller than T_N, which is minimal by construction. This shows that g is injective, so that E is minimal. A similar argument applies to case (ii). Assume again that there is a function g such that $g(X)$ is Markov. For the transient states we have the same argument as in case (i). For the cyclic part of the chain we have for each recurrent state j that there is by definition no transient state $i < j$ such that $g(i)=g(j)$, but also no recurrent state $i < j$ such that $g(i)=g(j)$, because that would contradict the minimality of the number (period) p. Again g is injective. $\square$

6. CONCLUSIONS

The object that we have studied in this paper was a stochastic process X that is $\mathbb{F}^n$-Markov, where $\mathbb{F}^n$ denotes the filtration that was generated by some given counting process n, and has finite state space. The additional requirement that X is homogeneous resulted in the fact that then X has to be eventually either cyclic or constant. Consequently the idea of viewing n as the output of a stochastic system, with such a process X as state process, leads to a rather restricted class of counting processes that satisfy this requirement. This partly negative result answers a question posed in the introduction, namely whether we get an interesting class of counting processes that obeys the afore mentioned conditions.

REFERENCES.
1. R.K. BOEL (1985), Modelling, estimation and prediction for jump processes, Advance in statistical signal processing, volume 1, JAI Press.
2. P. BRÉMAUD, (1981) Point processes and Queues, Springer.
3. P. BRÉMAUD, & M. YOR (1978), Changes of filtration of probability measures, Zeitschrift f. Wahrsch. verw. Geb. 45, 269-295.
4. C. DELLACHERIE & P.A. MEYER (1980), Probabilités et potential, Ch. V-VIII, Hermann.
5. M. JACOBSEN (1982), Statistical Analysis of counting processes, LNS 12, Springer.
6. J.G. KEMENY & J.L. SNELL (1960), Finite Markov chains, Van Nostrand.
7. B. KEMP (1986), An optimal monitor of the rapid-eye-movement brain state, Biol. Cybern. 54, pp. 133-139.
8. J.H. VAN SCHUPPEN (1979), Stochastic Filtering theory, A discussion of concepts, methods and results, in "Stochastic Control theory and stochastic differential systems", M. Kohlmann & W. Vogel eds, pp. 209-226.