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A central limit theorem for linear combinations of a function of generalized order statistics with unbounded scores is established. The result supplements previous work of SILVERMAN (1983), SERFLING (1984) and AKRITAS (1986) concerning the asymptotic normality of generalized L-statistics. Our proof is patterned after the well-known Chernoff-Savage approach. A linear bound for the empirical distribution function of *U*-statistic structure is also derived and subsequently applied in the treatment of certain remainder terms.

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1. INTRODUCTION

For each $n \in N$ let $\xi_1, \cdots, \xi_n$ be independent and identically distributed (i.i.d) random elements with values in some measurable space X and let, for a fixed but arbitrary $m \in N, h: X^m \to \mathbb{R}$ be a measurabele mapping. For each of the $n(m)=n\cdot(n-1)\cdots(n-m+1)$ ordered m-tuples $(j(1), \cdots, j(m))$ of m distinct integers taken from $\{1, \cdots, n\}$ we form the random variable (r.v.) $h(\xi_{j(1)}, \cdots, \xi_{j(m)})$. Let $X_1, \cdots, X_{n(m)}$ be an enumeration of these r.v.'s and note that, although dependent in general, these r.v.'s are still identically distributed with common distribution function (d.f.) H, say. It will be assumed throughout that

$$H \textit{ is continuous on } \mathbb{R}\,, \tag{1.1}$$

so that for $i=1, \cdots, n(m)$ the transformed r.v.

$$H(X_i) \textit{has the uniform } (0,1) \textit{ distribution}\,. \tag{1.2}$$

The empirical d.f. of $H(X_1), \cdots, H(X_{n(m)})$ is as usual defined by

$$\hat{H}_{n(m)}(t) = \frac{1}{n(m)} \sum_{i=1}^{n(m)} 1_{[0,t]}(H(X_i)),\ t \in [0,1]\,. \tag{1.3}$$

$\hat{H}_{n(m)}$ is called the empirical df of U-statistic structure (cf. SERFLING (1984)). For $J_n:[0,1] \to \mathbb{R}$ with $c_{n(m),i} = J_n(i/n(m))$, and measurable $\Psi: \mathbb{R} \to \mathbb{R}$ with $\Psi_H = \Psi(H^{-1})$ let us consider the linear combination of the function Ψ applied to the $X_{i:n(m)}$, given by

$$T_n = \frac{1}{n(m)} \sum_{i=1}^{n(m)} c_{n(m),i}\ \Psi(X_{i:n(m)}) = \tag{1.4}$$

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$$= \int_0^1 J_n(\hat{H}_{n(m)}(t))\Psi_H(t)d\hat{H}_{n(m)}(t) \text{ ,a.s.}$$

For the special choice $\Psi_H = H^{-1}$ this class has been introduced by SERFLING (1984) as the class of generalized L-statistics, and asymptotic normality has been established in SILVERMAN (1983) and SERFLING (1984) for bounded scores $c_{n(m),i}$. A strong law for statistics of the general form (1.4), implying almost sure convergence of T_n to its natural limit $\mu(H) = \int_0^1 J(t)\psi_H(t)dt$ for some limiting score function J, is contained in Corrollary 3.2 of HELMERS *et al.* (1985). We also refer to JANSSEN *et al.* (1984) for the asymptotic normality of a related class of statistical functions..

In this note we will prove asymptotic normality for a class of functions Ψ_H and for unbounded scores. The order of magnitude that we obtain by our method is $\mathcal{O}(n^{\alpha}) = \mathcal{O}(n(m)^{a/m})$, for some $\alpha \in (0, \frac{1}{2})$ depending on the choice of Ψ_H. Although one might hope to find the order $\mathcal{O}(n(m)^{\alpha})$, it is in the line of the results in AERTS et al. (1985, Section 4) that the order is likely to depend on the structure of the kernel h and will be probably hard to specify. It should, however be noted that statistics T_n, which are, up to first order, asymptotically equivalent with normal scores are included and that for $m=1$, i.e. the classical i.i.d. case with sample size $n(m)=n$, the result reduces to the almost optimal order $\mathcal{O}(n^{\alpha})$.

Our representation in (1.4) is the natural starting point for the Chernoff-Savage approach which is also employed in e.g. MOORE (1968), RUYMGAART & VAN ZUIJLEN (1977) and BEIRLANT *et al.* (1982). From a theoretical point of view this method is not so attractive since it doesn't seem to yield theoretically refined results like e.g. the one in MASON (1981). On the other hand the method automatically leads to simple centering constants and the method is of indirect theoretical importance as it hinges on some interesting properties of the empirical d.f. involved. The two properties that we need, one of which seems to be new, are presented in Section 2. In Section 3 we return to the asymptotic normality of the statistics in (1.4).

Results related to our Theorems 2 and 3 were very recently obtained by AKRITAS (1986). However, rather then considering the empirical df $\hat{H}_{n(m)}$ of U-statistic structure and GL-statistics $T_n = T_n(\hat{H}_{n(m)})$ of the form (1.4), Akritas investigated the closely related empirical df$V_{n(m)}$ of von Mises structure (see SERFLING (1980), page 174) and modified GL-statistics $T_n = T_n(V_{n(m)})$. His purpose is to deal with such statistics in the multi-sample case in the presence of random censoring. An important drawback of Akritas results is that he requires, using a different method of proof, a rather restrictive condition on the kernel function h which we are able to avoid. On the other hand, as in our Theorem 3, unbounded scores are permitted.

2. PROPERTIES OF THE EMPIRICAL D.F. OF U-STATISTIC STRUCTURE.

Throughout this section we can and will assume without loss of generality that $n = m \cdot \nu$ for some $\nu \in \mathbb{N}$, although we will write $n \to \infty$ rather than $\nu \to \infty$. Let $\Re(m)$ denote the set of all ordered m-tuples $(j(1), \cdots, j(m))$ and take $r \in \mathbb{R}(m)$. As in SILVERMAN (1983) we define $\hat{H}_{\nu,r}$ to be the empirical d.f. of the ν r.v.'s. $H(h(\xi_{r(mj+1)}, \xi_{r(mj+2)}, \cdots, \xi_{r(mj+m)})$ for $j=0, \cdots \nu-1$. Note that this subset of ν r.v.'s consists of i.i.d. elements so that for the corresponding empirical process

$$U_{\nu,r}(t) = \nu^{\frac{1}{2}}(\hat{H}_{\nu,r}(t) - t), \ t \in [0,1] \ , \tag{2.1}$$

the usual properties hold true. The empirical process U_n of U-statistic structure based on all the $X_1, \cdots, X_{n(m)}$ is related to the i.i.d empirical processes in (2.1) according to

$$U_n(t) = n^{\frac{1}{2}}(\hat{H}_{n(m)}(t) - t) = \frac{m^{\frac{1}{2}}}{n(m)} \sum_{r \in \Re(m)} U_{\nu,r}(t), t \in [0,1] \, . \tag{2.2}$$

For arbitrary $\delta \in (0, \frac{1}{2})$ let

$$q_\delta(t) = [t(1-t)]^{\frac{1}{2}-\delta}, \ t\in(0,1). \tag{2.3}$$

The first property that we need is implied by SILVERMAN (1983; Theorem A) and says that

$$sup_{t\in(0,1)}|U_n(t)|/q_\delta(t) = \mathcal{O}_p(1), \ as \ n\to\infty . \tag{2.4}$$

In order to prepare for the second property we need the following probability inequality for the i.i.d. empirical processes in (2.1); a proof (for arbitrary dimension) can be found in EINMAHL (1986). Let $0\leqslant a<b\leqslant 1$ be arbitrary but fixed and let $a\leqslant s<t\leqslant b$. Then we have

$$P(sup_{a\leqslant s<t\leqslant b} -(U_{\nu,r}(t)-U_{\nu,r}(s))\geqslant\lambda)\leqslant \tag{2.5}$$
$$\leqslant C\exp(\frac{-(1-\epsilon)\lambda^2}{2(b-a)}), \ \lambda\geqslant 0,$$

for each $\epsilon>0$ with $C=C(\epsilon)\in(0,\infty)$. Note that we consider $-(U_{\nu,r}(t)-U_{\nu,r}(s))$ rather than $|U_{\nu,r}(t)-U_{\nu,r}(s)|$ since this suffices for our purposes. For this tail the bound appears to be smaller and easier to handle. Using a moment generating function technique to be found in SERFLING (1980), Section 5.6), see also HELMERS et al. (1985), it will be shown that an analogous inequality holds true for the processes U_n in (2.2). *Throughout the remainder of this section the symbols A and C will be used as generic constants in* (o,∞) *that are independent of all the relevant parameters* $(n,\lambda,a$ *and b).*

THEOREM *1. Fix arbitrary* $0\leqslant a<b\leqslant 1$. *Then we have*

$$P(sup_{a\leqslant s<t\leqslant b}-(U_n(t)-U_n(s))\geqslant\lambda)\leqslant \tag{2.6}$$
$$\leqslant C\exp\left[\frac{-(1-\epsilon)\lambda^2}{2m(b-a)}\right], \quad \lambda\geqslant 0.$$

PROOF. For each $x>0$ the probability in (2.6) is bounded above by

$$\exp(-\lambda x)E\exp(x sup_{a\leqslant s<t\leqslant b}-(U_n(t)-U_n(s))\leqslant \tag{2.7}$$
$$\leqslant\exp(-\lambda x)E\exp\left[\frac{xm^{\frac{1}{2}}}{n(m)}\sum_{x\in\mathbb{R}(m)} sup_{a\leqslant s<t\leqslant b}-(U_{\nu,r}(t)-U_{\nu,r}(s))\right]\leqslant$$
$$\leqslant\exp(-\lambda x)\int_0^\infty P\left[sup_{a\leqslant s<t\leqslant b}-(U_{\nu,r}(t))-U_{\nu,r}(s))\geqslant\frac{\log u}{xm^{\frac{1}{2}}}\right]du\leqslant$$
$$\leqslant\exp(-\lambda x)C\int_0^\infty\exp\left[\frac{-(1-\epsilon)(\log u)^2}{x^2m(b-a)}\right]du =$$
$$= C\exp(-\lambda x)\int_{-\infty}^\infty\exp\left[u-\frac{(1-\epsilon)u^2}{x^2m(b-a)}\right]du =$$
$$= C\exp\left[-\lambda x+\frac{x^2m(b-a)}{4(1-\epsilon)}\right]\left[\frac{\pi x^2m(b-a)}{1-\epsilon}\right]^{\frac{1}{2}}, \ \lambda\geqslant 0.$$

Minimizing the exponential factor as a function of x and taking into account the square root, we arrive at the exponential bound in (2.6). □.

Following the method of the proof in RUYMGAART & WELLNER (1982, Corollary 2.4) it is easy to see that with the aid of (2.6) we arrive at the global version

$$P(sup_{t\geqslant\nu}\frac{-U_n(t)}{t^{\frac{1}{2}}}\geqslant\lambda)\leqslant C\log(1/\nu)\exp(-A\lambda^2),\ \nu\in(0,1),\ \lambda\geqslant 0\ ; \tag{2.8}$$

see also EINMAHL & MASON (1985, Inequality 1). This inequality entails at once

$$P(sup_{t\geqslant\nu}\frac{-U_n(t)}{t}\geqslant\lambda)\leqslant C\log(1/\nu)\exp(-A\nu\lambda^2),\ \nu\in(0,1),\ \lambda\geqslant 0\ . \tag{2.9}$$

We may now formulate the second property of the empirical df of U-statistic structure.

THEOREM 2. *Let us choose* $\nu_n = c(\log n)/n$ *for some* $c\in(0,\infty)$ *and* $\beta\in(0,1)$ *arbitrary but fixed. For any choice of* $c\in(0,\infty)$ *we have*

$$P(\hat{H}_{n(m)}(t)\geqslant\beta\ t\forall t\in[\nu_n,1])\rightarrow 1,\ as\ n\rightarrow\infty\ . \tag{2.10}$$

For c sufficiently large we even have that $\{\hat{H}_{n(m)}(t)\geqslant\beta t$ for *all*$\forall t\in[\nu_n,1]\}$ *occurs infinitely often.*

PROOF. The complement of the event in (2.10) has probability

$$P(inf_{t\geqslant\nu_n}\hat{H}_{n(m)}(t)/t\leqslant\beta) = \tag{2.11}$$

$$= P(sup_{t\geqslant\nu_n}(t-\hat{H}_{n(m)}(t)/t\geqslant 1-\beta) =$$

$$= P(sup_{t\geqslant\nu_n}-U_n(t)/t\geqslant n^{\frac{1}{2}}(1-\beta))\ .$$

Both results now follow immediately from (2.9). □.

In the i.i.d. case ($m=1$) relation (2.10) is known to remain true with ν_n replaced by $H(X_{1:n(m)})$; see e.g. SHORACK (1972) and, for a.s. results, SHORACK & WELLNER (1978). Whether this is also the case for arbitrary m is an open question and the answer might depend on the structure of the kernel h; see also AERTS *et al.* (1985). In the i.i.d. case ($m=1$) almost sure results for non random ν_n can be found in WELLNER (1978).

3. APPLICATION TO GL-STATISTICS

Let us first formulate sufficient conditions on the functions J_n and Ψ_H in (1.4). It is convenient to first introduce the linear function

$$l_n(t) = \nu_n + (1-2\nu_n)t, t\in[0,1],\ \nu_n = c(\log n)/n\ , \tag{3.1}$$

for some $c\in,(0,\infty)$. The functions J_n will be derived from a fixed function $J:(0,1)\rightarrow\mathbb{R}$ according to

$$J_n(t) = J(l_n(t)),\ t\in[0,1]\ . \tag{3.2}$$

It will be assumed that

$$\begin{cases} J \text{ is continuously differentiable on } (0,1) \text{ with} \\ |J^{(i)}(t)|\leqslant C[t(1-t)]^{-\alpha-i},\ t\in(0,1),\ \alpha\in(0,1),\ i\in\{0,1\}\ , \end{cases} \tag{3.3}$$

where $J^{(0)}=J$, and that

$$\begin{cases} \Psi_H \text{ is of bounded variation on } (\epsilon,1-\epsilon) \text{ for any } \epsilon>0 \\ |\Psi_H(t)|\leqslant C[t(1-t)]^{-\beta},\ t\in(0,1),\ \beta\in(0,1)\ . \end{cases} \tag{3.4}$$

We finally assume that

$$\alpha + \beta<\frac{1}{2}\ . \tag{3.5}$$

THEOREM 3. *Let the conditions described in* (3.1) - (3.5) *be fulfilled. Then* $\sigma^2(H)>0$ *implies*

$$n^{\frac{1}{2}}(T_n-\mu(H))\xrightarrow{d}\mathfrak{N}(0,\sigma^2(H)),\ as\ n\to\infty\ , \tag{3.6}$$

where T_n *is defined in* (1.4), *and* $\mu(H) = \int_0^1 J(t)\Psi_H(t)dt,$

$$\sigma^2(H) = m^2\int_0^1\int_0^1(\min(s,t) - st)\, J(s)J(t)d\psi_H(t)d\psi_H(s) \tag{3.7}$$

PROOF. We may in principle follow the pattern of proof in BEIRLANT *et al.* (1982, pp. 427-430) or RUYMGAART & VAN ZUIJLEN (1977). Writing

$$Z_{0i} = J_n(H(X_i))\Psi_H(H(X_i)) \tag{3.8}$$

$$Z_{1i} = \int_0^1 (1_{[0,t]}(H(X_i))-t)J_n^{(1)}(t)\Psi_H(t)dt\ , \tag{3.9}$$

we shall first consider

$$A_n = \sum_{i=0}^{2} A_{in} \tag{3.10}$$

where

$$A_{0n} = n^{\frac{1}{2}}\frac{1}{n(m)}\sum_{i=1}^{n(m)}(Z_{0i}-EZ_{0i})\ , \tag{3.11}$$

$$A_{1n} = n^{\frac{1}{2}}\frac{1}{n(m)}\sum_{i=1}^{n(m)} Z_{1i} \tag{3.12}$$

and

$$A_{2n} = n^{\frac{1}{2}}(\int_0^1 J_n(t)\psi_H(t)dt-\mu(H)) \tag{3.13}$$

with $\mu(H)$ as in (3.7). Note that $EA_{0n}=EA_{1n}=0$ and that A_{2n} is non-random. Partial integration directly yields that

$$A_{0n}+A_{1n} = -n^{\frac{1}{2}}\frac{1}{n(m)}\sum_{i=1}^{n(m)}\int_0^1(I_{[0,t]}(H(X_i))-t)J_n(t)d\psi_H(t) \tag{3.14}$$

a U-statistic of degree m with a varying kernel

$$A^{(n)}(\xi_1,\cdots,\xi_m)=-\int_0^1(I_{[0,t]}(H(h(\xi_1,\cdots,\xi_m)))-t)J_n(t)d\psi_H(t) \tag{3.15}$$

depending on n. Let U_n denote the U-statistic of degree m, with fixed kernel, which is obtained from (3.14) by replacing J_n by J.

To establish the asymptotic normality of A_n (cf. (3.10)) we first note that the central limit theorem for U-statistics (see SERFLING (1980), page 192) directly yields that

$$U_n\to_d N(0,\sigma^2(H)), as\ n\to\infty, \tag{3.16}$$

with $\sigma^2(H)$ as in (3.7). In addition we shall prove that

$$A_{0n}+A_{1n}-U_n\to 0, as\ n\to\infty \tag{3.17}$$

and also that

$$A_{2n} \to 0, as\ n \to \infty \tag{3.18}$$

Together (3.16)-(3.18), combined with (3.10), gives the desired result: $A_n \to_d N(0,\sigma^2(H))$, as $n \to \infty$. To verify (3.17) we first apply Chebychev's inequality and the elementary inequality $\sigma^2(X+Y) \leq 2\sigma^2(X) + 2\sigma^2(Y)$ to find that it suffices clearly to show that both

$$\sigma^2(n^{\frac{1}{2}} n(m)^{-1} \sum_{i=1}^{n(m)} (J_n - J)(H(X_i))\psi_H(H(X_i))) \tag{3.19}$$

and

$$\sigma^2(n^{\frac{1}{2}} n(m)^{-1} \sum_{i=1}^{n(m)} \int_0^1 (I_{[0,t]}(H(X_i)) - t)(J_n^{(1)}(t) - J^{(1)}(t))\psi_H(t)dt) \tag{3.20}$$

tend to zero, as $n \to \infty$. With the aid of Lemma A on page 183 of SERFLING (1980) we easily check that the variances in (3.19) and (3.20) are respectively of the order

$$\mathcal{O}(\int_0^1 ((J_n - J)(s)\ \psi_H(s))^2 ds) \tag{3.21}$$

and

$$\mathcal{O}(\int_0^1 \int_0^1 (\min(s,\ t) - st)\ |J_n^{(1)}(s) - J^{(1)}(s)|\ |\ J_n^{(1)}(t) - J^{(1)}(t)\ |\ |\psi_H(t)\ |\ |\psi_H(s)\ |\ ds\ dt) \tag{3.22}$$

as $n \to \infty$. In view of the assumptions (3.1)-(3.5) one directly verifies that the quantities appearing in (3.21) and (3.22) both tend to zero, as $n \to \infty$, and (3.17) follows.

It remains to check (3.18). The same argument involving the assumptions (3.1)-(3.5) also yields that $n^{\frac{1}{2}} \int_0^1 (J_n(t) - J(t))\psi_H(t)dt \to 0$, as $n \to \infty$. Thus (3.18) indeed holds and the asymptotic normality of A_n is proved.

It remains to show that

$$\begin{aligned} B_n &= n^{\frac{1}{2}}(T_n - \mu(H)) - A_n = \\ &= n^{\frac{1}{2}} \int_0^1 \{J_n(\hat{H}_{n(m)}(t)) - J_n(t)\} \Psi_H(t) d\hat{H}_{n(m)} - A_{1n} \\ &\overset{p}{\to} 0,\ as\ n \to \infty\ . \end{aligned} \tag{3.23}$$

Let us briefly write

$$B_n = n^{\frac{1}{2}} \int_0^1 (\cdots) d\hat{H}_{n(m)} - n^{\frac{1}{2}} \int_0^1 (\cdots) dt\ . \tag{3.24}$$

To prove (3.23) is suffices to prove that each of the integrals

$$B_{1n} = n^{\frac{1}{2}} \int_0^v (\cdots) d\hat{H}_{n(m)}\ ,\ B_{2n} = n^{\frac{1}{2}} \int_{1-v} (\cdots) d\hat{H}_{n(m)}\ , \tag{3.25}$$

$$B_{3n} = n^{\frac{1}{2}} \int_0^v (\cdots) dt\ ,\ B_{4n} = n^{\frac{1}{2}} \int_{1-v}^1 (\cdots) dt \tag{3.26}$$

converges to 0 in probability as both $\nu \downarrow 0$ and $n \to \infty$, along with

$$n^{\frac{1}{2}} \int_{\nu}^{1-\nu} (\cdots) d\hat{H}_{n(m)} - n^{\frac{1}{2}} \int_{\nu}^{1-\nu} (\cdots) dt \xrightarrow{p} 0, \text{ as } n \to \infty , \tag{3.27}$$

for each $\nu \in (0, \frac{1}{2})$.

In order to illustrate the use of the properties (2.4) and (2.10) let us by way of an example consider B_{3n}. For any $\nu \in (0,1)$ we apply the mean value theorem to the factor within the brackets of the integral and find that

$$J_n(\hat{H}_{n(m)}(t)) - J_n(t)) = (\hat{H}_{n(m)}(t) - t)(1 - 2\nu_n) J^{(1)}(l_n(t_n)) , \tag{3.28}$$

where t_n is a random point between t and $\hat{H}_{n(m)}(t)$ and where l_n is defined in (3.1). By assumption (3.3) it is clear that

$$\begin{aligned} |J^{(1)}(l_n(t_n))| &\leq C[\nu_n(1-\nu_n)]^{-\alpha-1} \leq \\ &\leq C[t(1-t)]^{-\alpha-1} , \quad \text{for } 0 < t \leq \nu_n . \end{aligned} \tag{3.29}$$

Using C as a generic constant, the same assumption jointly with Theorem 2 yields that, for arbitrary fixed $\beta \in (0,1)$,

$$\begin{aligned} |J^{(1)}(l_n(t_n))| &\leq C[\beta t(1-\beta t)]^{-\alpha-1} \leq \\ &\leq C[t(1-t)]^{-\alpha-1} , \quad \text{for } \nu_n \leq t \leq \nu , \end{aligned} \tag{3.30}$$

with arbitrarily high probability for n sufficiently large. Property (2.4) entails that, with C generic again,

$$n^{\frac{1}{2}} (\hat{H}_{n(m)}(t) - t) \leq C[t(1-t)]^{\frac{1}{2}-\delta}, \text{ for } t \in (0,1) \text{ and } n \in \mathbb{N} , \tag{3.31}$$

with arbitrarily high probability .

Combining (3.28) - (3.31) and using assumption (3.4) it follows that

$$\begin{aligned} |B_{3n}| &\leq C \int_0^{\nu} [t(1-t)]^{-\alpha-1} [t(1-t)]^{-\beta} dt = \\ &= C \int_0^{\nu} [t(1-t)]^{-\frac{1}{2}-\alpha-\beta-\delta} dt , \text{ with arbitrarily high probability for n sufficiently large .} \end{aligned}$$

This last integral decreases to 0 as $\nu \downarrow 0$ provided that we choose δ so as to satisfy $0 < \delta < \frac{1}{2} - \alpha - \beta$; This can be done because of assumption (3.5). This proves that B_{3n} converges to 0 in probability as $n \to \infty$ and $\nu \downarrow 0$.

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