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The r-rank of the groups of exceptional Lie type

The r-Rank of the Groups of Exceptional Lie Type

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In this note, we prove the following result, settling a question raised at the end of [BOREL & SERRE, 1953], cf. [BOREL, 1983 pp. 228 and 708]. A related result for Lie groups of type E_8 was recently proved by J.F. Adams.

THEOREM. *Let r be a prime and G a simple algebraic group of exceptional type over an algebraically closed field of characteristic $\neq r$. Let E be an elementary abelian r-subgroup of G of maximal rank. Then* rank(E)=Lie rank(G) *with the exception of $r=2$ and $G=G_2$, F_4, the adjoint E_7, and E_8, in which cases* rank(E)=Lie rank(G)+1. *Moreover, E is unique up to conjugacy.*

1. THE PRIME 2

In this section we prove the following

THEOREM. *Let G be an algebraic group of type G_2, F_4, E_6, adjoint E_7, simply connected E_7, or E_8 over an algebraically closed field of characteristic $\neq 2$, and let E be an elementary abelian 2-group in G of maximal order. Then $|E|=2^3,2^4,2^6,2^8,2^7,2^9$ in the respective cases. Moreover, in each case any two such elementary abelian subgroups are conjugate.*

PROOF. By a theorem of [SPRINGER & STEINBERG, 1970], due to [BOREL & SERRE, 1953] in the Lie group case, E is a subgroup of $N_G(T)$ for some maximal torus T of G. In particular, $|E| \leq 2^l.|W|$, where l is the Lie rank of G and $W=N_G(T)/T$, so E is finite. We shall deal with each case separately, although the arguments are similar. The idea is to produce a certain subgroup containing the preimage in N of a Sylow 2-subgroup of W.

G_2. Let J_1, J_2 be commuting (nonconjugate) fundamental SL_2's. We may take $T \leq D=J_1J_2$. Moreover $N_D(T)/T$ contains a Sylow 2-subgroup of $W=N_G(T)/T$, so we may assume $E \leq D$. Let $Z(D)=<e>$. Maximality of E then implies $E=<e,x_1x_2,y_1y_2>$, where $x_1,y_1 \in J_1$, $x_2,y_2 \in J_2$, $x_1^2=x_2^2=y_1^2=y_2^2=[x_1,y_1]=[x_2,y_2]=e$. It is clear that any two such groups are conjugate in D.

F_4. There is an involution in F_4 with centralizer D, the simply connected group of type B_4. We may take $T \leq D$ and check that $N_D(T)/T$ contains a Sylow 2-subgroup of $N_G(T)/T$. Hence, we may take $E \leq D$. An involution in SO_9 lifts to an involution in D if and only if the eigenspace for eigenvalue -1 has dimension a multiple of 4. A direct check then shows that D has 2-rank 5 and all elementary abelian subgroups of D of order 2^5 are conjugate.

E_6. Set $V=\Omega_2(T)=\{t \in T \mid t^2=1\}$. Then V is the natural module for $O^-(6,2)\cong W$. By §8 of [ASCHBACHER & SEITZ, 1976] W has 4 classes of involutions, represented by a_2, c_2, b_1, and b_3. Here the subscript is the dimension of the commutator space of the involution. The involutions in $\Omega^-(6,2)$ are conjugates of a_2 and c_2. Long root subgroups of W are generated by conjugates of a_2, the commutator space $[V,a_2]$ is totally singular, and $C_V(a_2)=[V,a_2]^\perp$. Finally, by (19.9)(ii) of [ASCHBACHER & SEITZ, 1976], applied to $O^-(6,2)\cong U_4(2)\cdot 2$, we have $C_W(b_3) \leq C_W(b_1)$ for suitable choice of b_1.

To prove the theorem for E_6 we may and shall assume that E is not contained in a maximal torus

of G, $|E| \geq 2^6$, and $E \leq N_G(T)$. Set $\bar{E}=ET/T$, and for $x \in E$, write $\bar{x}=xT$. If $\bar{E}$ centralizes b_1, then there is a fundamental SL_2 normalized by T, containing a preimage in $N(T)$ of b_1, and such that $E \leq SL_2 \circ SL_6$. A direct check then shows that E is necessarily contained in a maximal torus, a contradiction. Hence $\bar{E}$ does not centralize a b_1 involution. In particular, $\bar{E} \leq \Omega^-(6,2)$. Moreover, if $C_V(\bar{E})$ contains a nonsingular vector v, then $\bar{E}$ centralizes the unique involution b_1 of W satisfying $[V,b_1]=<v>$, a contradiction. Therefore, $C_V(\bar{E})$ is totally singular, $|E \cap T| \leq 4$, and $|\bar{E}| \geq 2^4$.

Let $\bar{E} \leq P$ be the stabilizer in $O^-(6,2)$ of a singular 1-space of V. Then $P=O_2(P)L$, where $L \cong \Omega^-(4,2)$ and $O_2(P)$ is the natural module for L. Since $|\bar{E}| \geq 2^4$, an easy argument shows that $\bar{E}=O_2(P)$ and so $|\bar{E}|=2^4$. Hence, $\bar{E}$ contains distinct a_2 involutions $\bar{x},\bar{y}$. Then $C_V(\bar{x})$ and $C_V(\bar{y})$ have distinct radicals, so the singular points of $C_V(\bar{x}) \cap C_V(\bar{y})$ span a subspace of dimension ≤ 1. Consequently, $|E| \leq |\bar{E}|.|E \cap T| \leq 2^4.2 < 2^6$. This contradiction finishes the proof of the E_6 case.

E_7. Fix a maximal torus T and corresponding system of root groups. Let Σ denote a maximal set of pairwise commuting fundamental SL_2's from this system. If we label the diagram as follows
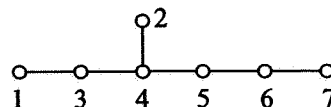
then we can take $\Sigma=\{J_1, \cdots, J_7\}$, where $J_i=<U_{\pm \beta_i}>$ and the β_i are as follows:

$$\beta_1 = 2234321, \quad \beta_2 = 0112221, \quad \beta_3 = 0000001, \quad \beta_4 = 0112100,$$
$$\beta_5 = 0000100, \quad \beta_6 = 0100000, \quad \beta_7 = 0010000.$$

Set $Z(J_i)=<e_i>$ and $J=J_1 \cdots J_7$. Then $\{e_1, \cdots, e_7\}$ is a set of commuting involutions which span $Z=Z(J)$.

LEMMA 1 (E_7).
(i) $N_G(J)/J \cong L_3(2)$ and $N_G(J)$ is 2-transitive on Σ, hence on $\{e_1, \cdots, e_7\}$.
(ii) If G is simply connected, the relations on $\{e_1, \cdots, e_7\}$ are spanned by $\{e_4e_5e_6e_7, e_2e_3e_6e_7, e_1e_2e_5e_6\}$. So $|Z|=2^4$.
(iii) If G is adjoint, the relations on $\{e_1, \cdots, e_7\}$ are spanned by $\{e_4e_5e_6e_7, e_2e_3e_6e_7, e_1e_2e_5e_6, e_1e_2e_3\}$. So $|Z|=2^3$.

PROOF. For each i, the centralizer $C_G(J_i)$ is of type D_6. Within D_6 a maximal commuting product of fundamental SL_2's corresponds to a decomposition of the usual orthogonal module into three perpendicular 4-spaces. One checks that S_4 is induced on such a commuting product, transitive on the 6 copies of SL_2. Hence, $N_G(J)$ is 2-transitive on $\{J_1, \cdots, J_7\}$, $N_G(J)/J$ has order 168, and (i) follows.

For (ii) and (iii) first check that $e_4e_5e_6e_7, e_2e_3e_6e_7, e_1e_2e_5e_6, e_1e_2e_3$ are each in $Z(G)$ (show that they centralize each root group corresponding to a fundamental root). Hence, in the simple group $|Z| \leq 2^3$. Equality must hold since $L_3(2)$ acts nontrivially on Z. This gives (iii). For (ii), view $E_7 \leq E_8$ and note that $e_4e_5e_6e_7, e_2e_3e_6e_7, e_1e_2e_5e_6$ are in $Z(E_8)=1$, while $e_1e_2e_3$ is not.

One can now list explicitly all relations on the e_i's, listing tuples of integers to indicate corresponding products of e_i's which are trivial.

CASE 2. $C_Z(E) \cong Z_2 \times Z_2$. Then E fixes 3 J_i's, but does not centralize Z. So we may assume E normalizes J_1, J_2, and J_3. No element of $L_3(2)$ fixes more than 3 elements of the usual module, so X is semiregular on $\{J_4, J_5, J_6, J_7\}$.

First assume $X \cong Z_2$. Then YZ/Z has order at least 2^5 and without loss of generality we may assume the nontrivial orbits of E on Σ to be $\{J_4, J_5\}$ and $\{J_6, J_7\}$. Now $Y \cap J_1 J_2 J_3$ is not contained in Z, so we may assume $x_1 x_2 x_3 \in Y$. If $Y \cap J_1 J_2 J_3 = <e_1, e_2, x_1 x_2 x_3>$, then the image of Y under projection to $J_4 J_5 J_6 J_7 Z / Z$ coincides with $<a_4 a_5, b_4 b_5, a_6 a_7, b_6 b_7>$ and this forces Y to be nonabelian. So assume $x_1 x_2 x_3$, $y_1 y_2 y_3$ are both in Y. As above, we may assume Y contains an element projecting to $a_4 a_5$, which again forces Y to be nonabelian. Thus $X \cong Z_2 \times Z_2$.

Hence E has a unique nontrivial orbit on Σ of size 4 and $|YZ/Z| \geq 2^4$. It follows that $|Y \cap J_1 J_2 J_3| \geq 2^4$, so we may assume $x_1 x_2 x_3$, $y_1 y_2 y_3 \in Y$. Now $N_G(J_1) \cap N_G(J_2) \cap N_G(J_3) = J_1 J_2 J_3 D$, where D° is simply connected of type D_4 (indeed, $Z(D) = <e_4 e_5, e_5 e_7>$). Take $h \in E - <Z, x_1 x_2 x_3, y_1 y_2 y_3>$. Since h commutes with $x_1 x_2 x_3$ and $y_1 y_2 y_3$, we may take $h \in D$. Now D has just 1 class of involutions in $D - Z(D)$, represented by e_4 (corresponding to involutions in SO_8 of type $(1)^4(-1)^4$). Hence $C_D(h)$ is D-conjugate to $C_D(e_4) = J_4 J_5 J_6 J_7$. Thus, $E \leq J_1 J_2 J_3 C_D(h)$, a D-conjugate of J. This completes the proof of Lemma 3.

LEMMA 4 (E_7). *Assume G is simply connected and $E \leq N_G(J)$ is an elementary abelian 2-group. Then $|E| \leq 2^7$, equality possible only if E is G-conjugate to a subgroup of J.*

PROOF. Assume $|E| \geq 2^7$ and $E \not\leq J$. Then, up to conjugacy in $N_G(J)/J \cong L_3(2)$, we have that $X = EJ/J$ is one of the groups listed in the table below, where a, b, c are elements of $N_G(J)/J$ inducing the permutations $(2,3)(6,7)$, $(4,5)(6,7)$, $(4,6)(5,7)$, respectively, on Σ. A direct check shows that, in each case, $C_Z(X)$ is as indicated in the table. Thus, the rank of $E \cap Z$ (a subgroup of $C_Z(X)$) is at most 3, 2, and 3, so that $|(E \cap J)Z/Z| \geq 2^3, 2^3$, and 2^2, in the respective cases.

On the other hand, if $q = q_1 \cdots q_7$, where $q_i \in Q_i Z/Z$, is an involution then the tuple of indices i with $q_i \neq 1$ is a 4-tuple of (*). Moreover, if q is centralized by X, this tuple must be invariant under the permutation action of X on Σ. In the table, under $inv(X)$, those tuples from (*) are listed which are X-invariant. It readily follows from the structure of $inv(X)$ that $(E \cap Q)Z/Z$ has size at most 2^2 in all three cases. Therefore, we must have $X = <b, c>$, $E \geq <e_1, e_2, e_3>$, and, without loss of generality, $(E \cap J)Z/Z = <x_4 x_5 x_6 x_7, y_4 y_5 y_6 y_7>Z/Z$. In particular $E \leq N_G(J_1) N_G(J_2) N_G(J_3)$, and we can finish as in the previous lemma.

X	$<c>$	$<a,b>$	$<b,c>$
$C_Z(X)$	$<e_1, e_2, e_3>$	$<e_1, e_2 e_3>$	$<e_1, e_2, e_3>$
$inv(X)$	4567, 1357, 1346	4567, 2367, 2345	4567

The E_7 case of the theorem follows from Lemmas 2, 3, and 4.

E_8. We proceed as for E_7. Again T is a maximal torus, and Σ a maximal set of pairwise commuting fundamental SL_2's. We label the diagram

$$
\begin{array}{ccccccccccc}
 & & & & & \overset{\circ}{} 2 & & & & & \\
\circ & \!\!-\!\!-\!\! & \circ & \!\!-\!\!-\!\! & \circ & \!\!-\!\!-\!\! & \circ & \!\!-\!\!-\!\! & \circ & \!\!-\!\!-\!\! & \circ \\
1 & & 3 & & 4 & & 5 & & 6 & & 7 \quad 8
\end{array}
$$

and take $\Sigma = \{J_1, \cdots, J_8\}$, where $(J_i)_{1 \leq i \leq 7}$ is as for E_7 and $J_8 = <U_{\pm \beta_8}>$, with $\beta_8 = 23465432$. Set $Z(J_i) = <e_i>$ and $J = J_1 \cdots J_8$. Then $\{e_1, \cdots, e_8\}$ is a set of commuting involutions spanning $Z = Z(J)$.

LEMMA 5 (E_8).
(i) $N_G(J)/J \cong Z_2^3 L_3(2)$ and $N_G(J)$ is 3-transitive on Σ, hence on $\{e_1, \cdots, e_8\}$.
(ii) The relations on $\{e_1, \cdots, e_8\}$ are given by the tuples of even length in (*) and the tuples obtained by joining 8 to the tuples of odd length in (*).

PROOF. For each $i \in \{1, \cdots, 8\}$, the group $C_G(J_i)$ is of type E_7, so the lemma is easily derived from Lemma 1.

LEMMA 6 (E_8). Let $E \leqslant J$ be an elementary abelian 2-group.
(i) There exist subgroups Q_i of J_i such that $Q_i = \langle x_i, y_i \rangle$ is quaternion of order 8, $N_{J_i}(Q_i)$ induces S_3 on Q_i, and $E \leqslant Q = Q_1 \cdots Q_8$.
(ii) $|E| \leqslant 2^9$.
(iii) There is a unique J-class of elementary abelian subgroups of order 2^9, represented by

$$\langle Z, x_4 x_5 x_6 x_7, x_2 x_3 x_6 x_7, x_1 x_2 x_5 x_6, x_1 x_2 x_3 x_8, y_1 \cdots y_8 \rangle.$$

(iv) Any 2-group in G is conjugate to a subgroup of $N_G(J)$.

PROOF. Similar to Lemma 2.

LEMMA 7 (E_8).
(i) Let $K = \langle e_j e_k | 1 \leqslant j, k \leqslant 8 \rangle$ and $R/J = O_2(N(J)/J)$. Then K is a hyperplane in Z and $R = N(J) \cap C(K)$.
(ii) $R - J$ contains a conjugate d of e_1 such that each involution in $R - J$ is $N(J)$-conjugate to an involution in dK.
(iii) If $ijkl$ is a 4-tuple as in Lemma 5(ii) and if x_i, x_j, x_k, x_l are elements of order 4 in J_i, J_j, J_k, J_l, respectively, then $x_i x_j x_k x_l \in e_1^G$.

PROOF. $N(J)$ acts on K since it permutes Σ, and clearly K is a hyperplane in Z. So $N_G(J)$ induces $L_3(2)$ on K and (i) follows. Observe that R/J acts regularly on Σ.

Let $z \in K^\#$. Then $J \leqslant D = C_G(z) = D_8$ (half-spin). Consider SO_{16} (an image of the covering group of D) and its subgroup $\tilde{D} = SO_{16} \cap (O_4)^4$. Set $(\tilde{D})^\circ = \tilde{J}$, a group corresponding to J. Choose reflections t_1, t_2, t_3, t_4, one from each O_4. The product of any two of these is in SO_{16}, and these products generate an elementary abelian group $\tilde{S}$ of order 8 which acts faithfully on the set $\tilde{\Sigma}$ of simple factors of $\tilde{J}$. Let $\tilde{R}$ denote the subgroup corresponding to R. Then $\tilde{S} \cap \tilde{R}$ is not contained in $\tilde{J}$. Since $t = t_1 t_2 t_3 t_4$ is the unique element in $\tilde{S}$ acting semiregularly on $\tilde{\Sigma}$, we have $t \in \tilde{R} - \tilde{J}$. In SO_{16}, t is conjugate to an involution in a fundamental SL_2. Translating this to D we conclude that there must exist an element $d \in (D \cap R) - J$, with d a conjugate of e_1.

To prove (ii) let t be any involution in $R - J$. Since $(R/J)^\#$ is fused in $N_G(J)$, we may assume $tJ = dJ$. Hence, J/Z is the direct product of simple groups permuted semiregularly by t. Therefore, all involutions in dJ are conjugates of those in dZ. Hence, we may assume $t \in dZ$. Also, $C_Z(d) = K$ (since $Z - K = \{e_1, \cdots, e_8\}$). So the only involutions in dZ are in fact in dK. This proves (ii).

For (iii) again consider D and choose $X \circ Y \leqslant D$ with X, Y of type D_4. Then X and Y are simply connected and we may take $J \leqslant X \circ Y$, where $J \cap X$ and $J \cap Y$ are each a product of 4 of the fundamental SL_2's. Say $J \cap X = J_r J_s J_u J_v$. One checks that $e_r e_s e_u e_v = 1$ so $rsuv$ is one of the 4-tuples of Lemma 5(ii). From 3-transitivity of $N(J)$ on Σ we may assume $\{r, s, u, v\} = \{i, j, k, l\}$. Set $x = x_i x_j x_k x_l$. The image of x in a quotient of X isomorphic to SO_8 is necessarily conjugate to the images of e_i, e_j, e_k, and e_l (by consideration of the action of this image on the orthogonal module). Without loss we may assume the kernel to the map is $\langle e_i e_j \rangle$. Hence, $x \sim e_i$ or $e_i(e_i e_j) = e_j$, proving (iii).

LEMMA 8 (E_8). Let $E \leqslant N_G(J)$ be an elementary abelian 2-group. Then $|E| \leqslant 2^9$, equality possible only if E is G-conjugate to a subgroup of J.
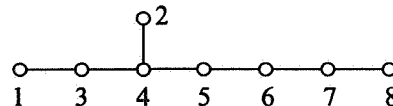
PROOF. Assume $|E| \geqslant 2^9$ and let $X = EJ/J$. If X has a fixed point, say J_8, on Σ, then $E \leqslant N(J_8) = J_8 E_7$ and we are done by reduction to E_7. Similarly, we may assume E centralizes no conjugate of e_1.

Assume $X \cap (R/J) = 1$, so $|X| \leqslant 4$. Involutions in $N(J)/J$ fixing a point in Σ fix exactly 4 points, so from the above paragraph we conclude X contains a regular involution, say x. Then $C_Z(x) \leqslant K$ (as $Z - K = \{e_1, \ldots, e_8\}$) and x is nontrivial on K (as $x \notin R$). Thus, $|E \cap Z| \leqslant |C_Z(E)| \leqslant 4$. But $|E \cap J| \geqslant 2^7$, whence $(E \cap J)Z$ is an elementary abelian group of order at least 2^9.

Apply Lemma 6. Replacing E by a J-conjugate, if necessary, we may assume $(E \cap J)Z / Z = \langle Z, x_4 x_5 x_6 x_7, x_2 x_3 x_6 x_7, x_1 x_2 x_5 x_6, x_1 x_2 x_3 x_8, y_1 \cdots y_8 \rangle$. However, x must centralize $(E \cap J)Z/Z$ and have no fixed points on Σ. Checking possible orbits of x we see this to be impossible.

We may now assume $X \cap (R/J) \neq 1$ and let $s \in (E \cap R) - J$. Lemma 7(ii) implies $sK = aK$ for some $a \in e_1^G$. From the first paragraph it follows that E does not centralize K. In particular, E is not contained in R. Let $f \in E - R$. Then $|E \cap Z| \leqslant |C_Z(E)| \leqslant |C_Z(s) \cap C_Z(f)| = |C_K(f)| = 4$.

It follows that $E \cap J$ must contain an element of the form $d = x_i x_j x_k x_l z$, where $ijkl$ is a tuple as in Lemma 5, x_r is of order 4 in J_r for $r \in \{i,j,k,l\}$, and $z \in Z$. Note that $\{i,j,k,l\}$ is necessarily a union of two orbits of $\langle s \rangle$. Also X must act on $\{i,j,k,l\}$ and also on its complement (as E centralizes s).

If $|X| \leqslant 4$, then as above $|(E \cap J)Z| \geqslant 2^9$ and we again obtain (recall that $|E \cap Z| \leqslant 4$) a contradiction using Lemma 6. Hence, $|X| \geqslant 8$. Restricting the abelian group X to $\{i,j,k,l\}$ we obtain an element $1 \neq x \in X$ fixing i, j, k, and l. Also X is transitive on either $\{i,j,k,l\}$ or its complement $\{i',j',k',l'\}$. Order considerations imply $E \cap J$ must also contain an element of the form $x_{i'} x_{j'} x_{k'} x_{l'} z'$, so we may assume X is transitive on $\{i,j,k,l\}$. Let $e \in E$ satisfy $eJ = x$. Then $d = d^e = (x_i x_j x_k x_l)^e z^e$. But e normalizes each of J_i, J_j, J_k, J_l, so $x_i^e = x_i e_i'$ for $t = 0,1$. Transitivity forces $x_r^e = x_r e_r'$ for each $r \in \{i,j,k,l\}$, and so $(x_i x_j x_k x_l)^e = x_i x_j x_k x_l$. Thus, $z = z^e$ and so $z \in C_Z(e) = \langle e_i, e_j, e_k, e_l \rangle$. Hence $d \sim x_i x_j x_k x_l$ (use an element of $\langle y_i, y_j, y_k, y_l \rangle$) and so by Lemma 7(iii), $d \in e_1^G$, contradicting the first paragraph.

The E_8 case of the theorem is now immediate from Lemmas 6 and 8.

COROLLARY. *Let q be an odd prime power. Then the 2-rank of $^2G_2(q)$, $G_2(q)$, $F_4(q)$, $E_6(q)$, $^2E_6(q)$, $E_7(q)$, $\hat{E}_7(q)$, $E_8(q)$ is 3,3,5,6,6,8,7,9 in the respective cases.*

PROOF. Let G be the algebraic group and let q be a power of the prime p. If σ is a field endomorphism, then it is immediate from the description given that the elementary abelian 2-groups of maximal rank can be taken in $O^{p'}(G_\sigma)$. Suppose G is of type E_6 and that $\sigma = q\tau$, where τ is a graph automorphism. Set $E = \Omega_2(T)$, where T is a σ-stable torus contained in a σ-stable Borel subgroup. Let $\dot{w}_0 \in N(T)$ represent the long word $w_0 \in W = N(T)/T$. Since $\tau \dot{w}_0$ acts on T by inversion it fixes E elementwise; hence $\sigma \dot{w}_0 = q\tau \dot{w}_0$ fixes E elementwise. The result follows since Lang's Theorem implies that σ and $\sigma \dot{w}_0$ are G-conjugate. Finally, consideration of the centralizer of an involution shows that the 2-rank of $^2G_2(q)$ is 3.

2. ODD PRIMES

In this section r is an odd prime and G is an algebraic group of exceptional type over an algebraically closed field of characteristic disinct from r. We begin with a general lemma.

LEMMA 9. *Let J be an algebraic group over an algebraically closed field of characteristic $p \neq r$. Suppose $J = T_s \circ J_1 \circ \cdots \circ J_k$, a central product of an s-dimensional torus and k groups isomorphic to SL_r. Then the r-rank of J is $s + k(r-1)$ and all elementary abelian r-subgroups of maximal rank are contained in a maximal torus of J.*

PROOF. Assume $2 < r \neq p$. J contains a maximal torus of rank $s + k(r-1)$, so the r-rank of J is at

least $s+k(r-1)$. Since the r-rank of both SL_r and PSL_r is easily checked to be $r-1$, the first assertion follows by induction, factoring out T_s and all but one of the SL_r's. These remarks also show that J/J_i has r-rank $s+(k-1)(r-1)$, for each $1 \leq i \leq k$. Let E be an elementary abelian r-subgroup of J having maximal rank and fix i ($1 \leq i \leq k$). By the above, $E \cap J_i$ has rank $r-1$, and since $E \cap J_i$ is abelian it is contained in a maximal torus T'_i of $J_i \cong SL_r$. Moreover, $C_{J_i}(E \cap J_i) = T'_i$. Hence, $E \leq \bigcap_i C_J(E \cap J_i) = T_s T'_1 \cdots T'_k$, a maximal torus of J. The lemma follows.

By the results of [SPRINGER & STEINBERG, 1970] every elementary abelian r-group in G can be embedded in a torus if $r>3$ for G_2 and F_4, $r>5$ for E_6 and E_7, and $r>7$ for E_8. Thus, we only consider the remaining odd primes. Let T be a maximal torus of G. The following subgroups D of G contain T and are such that $N_D(T)/T$ contains a Sylow r-group of $N_G(T)/T$.

G	$r=3$	$r=5$	$r=7$
G_2	A_2		
F_4	A_2A_2		
E_6	$(A_2A_2A_2)3$	T_2A_4	
E_7	$T_1(A_2A_2A_2)3$	T_3A_4	
E_8	$A_2(A_2A_2A_2)3$	A_4A_4	T_2A_6

Here, T_i stands for a torus of rank i, and A_{r-1} for a fundamental subgroup isomorphic to SL_r.

PROPOSITION. *If r is an odd prime and G is an algebraic group of exceptional Lie type over an algebraically closed field of characteristic $\neq r$, then the r-rank of G is the Lie rank of G. Moreover, all elementary abelian r-subgroups of maximal rank are conjugate and contained in a maximal torus of G.*

PROOF. Let E be an elementary abelian r-subgroup of G of rank at least the Lie rank of G. In view of the previous comments we may take r to be one of the primes in the table above and assume that $E \leq D$, where D is also given in the table. A dimension check shows that D° contains a maximal torus of G, so the result follows from Lemma 9 provided $E \leq D^\circ$.

Suppose there is $e \in E-D^\circ$. Then $r=3$, $G=E_6$, E_7, or E_8. Here D contains a normal subgroup S with $S \cong 1$, T_1, or A_2, respectively, $D/SZ(D)$ the wreath product of PSL_3 with $\mathbb{Z}_3$, and $C_{D/SZ(D)}(e) \cong PSL_3 \times \mathbb{Z}_3$. So the 3-rank of E is at most 3 plus the 3-rank of $C_{SZ(D)}(e)$. From the action of e it is clear that the latter is at most 2,3,4, respectively, so this is a contradiction.

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