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Algebraic Specifications for Parametrized Data Types: the Case of Minimal Computable Algebras and Parameters with Equality

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For minimal algebras, and under certain assumptions on the domain of parameters, it is shown that a persistent parametrized data type with computable parameters is effective iff it has a finite equational specification.

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INTRODUCTION

In a series of papers (see [BT2] for an enumeration), Bergstra and Tucker investigated the scope of algebraic methods for specifying abstract data types. BERGSTRA and KLOP ([BK1,2]) lifted these investigations one level, to consider parametrized data types. They established a necessary and sufficient condition for the existence of an algebraic specification, for persistent parametrized data types whose domain consists of all the semicomputable algebras in some quasi-variety. I propose to prove a similar theorem for persistent parametrized data types whose parameters come from certain classes of computable data types. Consideration will be limited to *minimal* algebras.

1. PRELIMINARIES

By a *signature* I shall understand a finite set of symbols, some of which are marked as *sort* symbols, while the rest are *function* symbols. Moreover, for each function symbol f a fixed nonempty sequence of sorts is given, which I call the *type* of f.

Let Γ be a signature. An *algebra* A of signature Γ (short: a Γ-algebra) consists of a sequence of nonempty sets, one for each sort symbol $\sigma\in\Gamma$ (the *carrier* of σ), and for each function symbol $f\in\Gamma$ an operation $f^{\mathbf{A}}$ subject to the following condition: if $<\sigma_1,...,\sigma_n>$ is the type of f, and $A_i(1\leq i\leq n)$ is the carrier of σ_i, then $f^{\mathbf{A}}$ is a function from $A_1\times...\times A_{n-1}$ to A_n. If it is sufficiently clear which algebra is under discussion, I will use the same symbol for f and $f^{\mathbf{A}}$.

In general, I shall denote the carrier of sort σ in an algebra A by A_σ — and likewise for the other letters in the alphabet. (As appeared above, this convention may be overruled by esthetic considerations.) I shall write A for the disjoint union of the carriers of A.

If A is an algebra of signature Δ, and Γ is a subsignature of Δ, I shall denote the reduct of A to Γ by $\mathbf{A}\upharpoonright\Gamma$. $\mathbf{A}\upharpoonright\Gamma$ is obtained by dropping the carriers of sorts in $\Delta-\Gamma$, and forgetting the operations corresponding with function symbols in $\Delta-\Gamma$. Stretching the carrier-notation just introduced: $A\upharpoonright\Gamma$ may be a proper subset of A.

Let Γ be a signature. From the function symbols, and an unlimited supply of variables of each sort, terms over Γ may be constructed as usual. A term is *closed* if it does not contain any variables. I shall denote the set of all closed terms over Γ by $T(\Gamma)$.

Every term has a unique sort: if x is a variable of sort σ (which one may signal by writing x^σ), then

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σ is the sort of x; and if $\sigma_1,...,\sigma_{n-1}$ are the sorts of terms $t_1,...,t_{n-1}$ respectively, and the type of f is as above, then σ_n is the type of $ft_1...t_{n-1}$. I will always assume that there exist closed terms of every sort.

Terms over Γ are interpreted as usual. I write $[\![t]\!]_\rho^{\mathbf{A}}$ for the denotation of t in **A** under the assignment ρ to the variables; if t is closed, the assignment can be suppressed.

If s and t are terms over Γ of the same sort, we may consider the *equation* (or equality) $s=t$. Such equations are interpreted in Γ-algebras in the usual way. If s and t are closed, $s=t$ is a *simple* equation. A Γ-algebra **A** is *minimal* if it has no proper Γ-subalgebras. By the assumption that there are closed terms of every sort, every element of A must then be the interpretation of a closed term over Γ. Since Γ is finite, minimal Γ-algebras are countable. I shall write $ALG(\Gamma)$ for the class of all minimal Γ-algebras. Observe that if **A** and **B** are minimal Γ-algebras, there can be at most one homomorphism from **A** to **B**; and if there are homomorphisms from **A** to **B** and from **B** to **A**, then $\mathbf{A}\cong\mathbf{B}$.

The set $T(\Gamma)$ of all closed Γ-terms naturally gives rise to a minimal Γ-algebra, the algebra of closed terms, which I shall denote by $\mathbf{T}(\Gamma)$. Every $\mathbf{A}\in ALG(\Gamma)$ is a homomorphic image of $\mathbf{T}(\Gamma)$.

1.1 Coding

For each signature Γ, I assume a fixed, effective, bijective coding (Gödel numbering) $gn_\Gamma:T(\Gamma)\to\mathbb{N}$, with inverse $tm_\Gamma:\mathbb{N}\to T(\Gamma)$ ("the n-th closed term over Γ"). (This convention is of course impossible if $T(\Gamma)$ is finite. For this trivial case, assume that gn_Γ is a bijection to an initial segment $\{0,...,n\}$ of $\mathbb{N}$, and $tm_\Gamma(m)=tm_\Gamma(n)$ for $m>n$. Then still $tm_\Gamma gn_\Gamma(t)=t$.) When confusion is unlikely, I write $\lceil t\rceil$ for $gn_\Gamma(t)$ and $\bar{n}$ for $tm_\Gamma(n)$.

The set S of all pairs (m,n) such that $tm_\Gamma(n)=tm_\Gamma(n)$ is a meaningful equation (i.e. for which $tm_\Gamma(m)$ and $tm_\Gamma(n)$ belong to the same sort) may be assumed (primitive) recursive. I shall code the meaningful simple equations by a primitive recursive function $\pi_\Gamma:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ such that the restriction $\pi_\Gamma\restriction S$ is a bijection onto $\mathbb{N}$. (I shall drop the subscript Γ when confusion is unlikely. Again, some stipulation must be made for the case that $T(\Gamma)$ is finite.) π has inverses π_1 (projection to the first coordinate) and π_2 (to the second). For $n\in\mathbb{N}$, $\hat{n}$ shall be the equation $\pi_1(n)=\pi_2(n)$; conversely, $\pi(\lceil t_1\rceil,\lceil t_2\rceil)$ may be written $\lceil t_1=t_2\rceil$. When the signatures must be kept in mind, I shall write $me_\Gamma(n)$ and $gn_\Gamma(t_1=t_2)$ respectively.

Through the Gödel numbering gn_Γ, certain binary relations over $T(\Gamma)$ correspond with sets of natural numbers. Thus, if we have a fixed signature Γ in mind, properties of such relations correspond with properties of sets $X\subseteq\mathbb{N}$. In particular, I shall call $X\subseteq\mathbb{N}$ an *equivalence* if

1° $\forall n\in\mathbb{N}\,.\,\pi(n,n)\in X$
2° $\forall m,n\in\mathbb{N}:\pi(m,n)\in X\Rightarrow\pi(n,m)\in X$
3° $\forall k,l,m\in\mathbb{N}:\pi(k,l)\in X\;\&\;\pi(l,m)\in X\Rightarrow\pi(k,m)\in X$;

and a *congruence* (with respect to gn_Γ) if moreover

4° if f is a k-ary function symbol in Γ, and ϕ is the partial function $(\lceil t_1\rceil,...,\lceil t_k\rceil)\mapsto\lceil ft_1...t_k\rceil$, then for all $m_1,...,m_k$, $n_1,...,n_k\in\mathbb{N}$, if $(m_1,...,m_k)$ and $(n_1,...,n_k)$ belong to the domain of ϕ, and $\pi(m_1,n_1),...,\pi(m_k,n_k)\in X$, then $\pi(\phi(m_1,...,m_k),\phi(n_1,...,n_k))\in X$.

For $X\subseteq\mathbb{N}$, χ_X is the characteristic function of X:

$$\chi_X(n) = 1 \text{ if } n\in X,$$
$$= 0 \text{ otherwise.}$$

A function $f:\mathbb{N}\to2$ will be called a congruence if f is the characteristic function of a subset of $\mathbb{N}$ that is a congruence in the above sense.

Let $\mathbf{A}\in ALG(\Gamma)$. I shall denote the characteristic function of

$\{\lceil e \rceil \mid e$ is a simple equation over Γ and $A \models e\}$

by χ_A. In other words, $\chi_A(n)$ equals 1 if $A \models \hat{n}$, and $\chi_A(n)=0$ otherwise. A is *computable* if χ_A is recursive.

Through the coding, the congruences with regard to gn_Γ correspond with congruence relations over $\mathbf{T}(\Gamma)$. Hence the congruences, with regard to gn_Γ are precisely the functions χ_A for $A \in ALG(\Gamma)$.

1.2 Algebraic specification of abstract data types

Abstract data types may be identified with isomorphism classes of algebras. I shall speak of specifications of algebras, with the understanding that specification is always modulo isomorphism.

In general, a specification is a pair $\mathbb{S}:=(\Gamma,\Phi)$ of a signature Γ and a set Φ of formulas of some kind over Γ. I shall only consider specifications that are *finite* and *equational*, i.e. in which Φ is finite and consists entirely of equations.

If $\mathbb{S}=(\Gamma,\Phi)$ and $\mathbb{S}'=(\Gamma',\Phi')$ are specifications, then $\mathbb{S}\cup\mathbb{S}'$ will be shorthand for $(\Gamma\cup\Gamma',\Phi\cup\Phi')$.

Let $\mathbb{S}=(\Gamma,\Phi)$ be a specification. We define $ALG(\mathbb{S})$ as the class of all minimal Γ-algebras that are models of Φ. If $\mathbb{S}$ is equational, there exists a congruence relation $\sim_\Phi$ on $\mathbf{T}(\Gamma)$ such that for all $s,t \in T(\Gamma)$, $s \sim_\Phi t$ iff $\Phi \vdash s=t$. I shall write $\mathbf{T}(\mathbb{S})$ for the quotient $\mathbf{T}(\Gamma)/\sim_\Phi$. Every minimal Γ-algebra that is a model of Φ is a homomorphic image of $\mathbf{T}(\mathbb{S})$; thus, $\mathbf{T}(\mathbb{S})$ is initial in the category $ALG(\mathbb{S})$, with Γ-homomorphisms for arrows. $\mathbb{S}$ is a flat *initial algebra specification* of $A \in ALG(\Gamma)$ if $A \cong \mathbf{T}(\mathbb{S})$ — in other words, if A is initial in $ALG(\mathbb{S})$. (The qualification 'flat' is to distinguish this notion from specification with hidden sorts and functions, to be discussed shortly.)

There is another notion of specification, categorically dual to initial algebra specification. Let $ALG_0(\mathbb{S})$ be the class of all nontrivial algebras in $ALG(\mathbb{S})$. Then $\mathbb{S}$ is a flat *final algebra specification* of $A \in ALG_0(\Gamma)$ if A is final in $ALG_0(\mathbb{S})$, i.e. A is a homomorphic image of every element of $ALG_0(\mathbb{S})$.

I shall call $\mathbb{S}$ a *full* flat specification of A if it is both a flat initial algebra specification of A and a flat final algebra specification of A. It is a simple fact of universal algebra that, if Φ consists of equations, $\mathbb{S}$ is a full flat specification of A iff

(i) $\Phi \vdash e$ iff $A \models e$, for all simple equations e over Γ, and

(ii) for any equation $s=t$ over Γ, $A \not\models s=t$ iff every simple equation over Γ is deducible from $\Phi \cup \{s=t\}$.

(Cf. [BT1].) These are the criteria that will be used below.

The scope of these methods greatly increases if we allow the use of *hidden* sorts and functions. With $\mathbb{S}$ as above, suppose the algebra A to be specified belongs to $ALG(\Gamma_0)$ for some subsignature Γ_0 of Γ. Then $\mathbb{S}$ is a (initial/final/full) specification of A if $\mathbb{S}$ is a flat (initial/final/full) specification of some minimal Γ-algebra $\mathbf{B}$, and $\mathbf{B}\restriction\Gamma_0 \cong A$.

If $\mathbb{S}$ specifies A, this can be used to decide simple equations in A, as follows. $A \models e$ iff $\mathbf{T}(\mathbb{S}) \models e$ iff $\Phi \vdash e$. On the other hand, fix some simple equation e_0 that is false in A. Then $A \not\models e$ iff $\mathbf{T}(\mathbb{S}) \not\models e$ iff $\Phi \cup \{e\} \vdash e_0$. The method consists in simultaneously generating deductions from Φ and from $\Phi \cup \{e\}$, waiting for e or e_0 to appear as a conclusion.

1.3 Recursion theory

The partial recursive functionals may be defined in various ways (see [R] or [S]); but the general idea is easily sketched, as follows. A partial recursive functional $F:\mathbb{N}^{\mathbb{N}} \times \mathbb{N} \rightarrow \mathbb{N}$ is determined by an algorithm A that takes as input a natural number n, and that may demand, at any stage of the computation, the value of a function at some natural number argument k. $F(f,n)$ is defined iff A terminates with input n, $f(k)$ being offered when A asks for a function value at k; and the output of A, in this case, is $F(f,n)$. Note that $F(f,n)$, if it exists, is determined by a finite part of f, since in a finite computation finitely many values will be asked.

A class $\mathcal{C}$ of (total) recursive functions is said to have a *recursively dense base* B if B is a recursively enumerable set of natural numbers such that

$$n \in B \Rightarrow \{n\} \in \mathcal{C};$$
$$\{e\} \in \mathcal{C} \Rightarrow \exists n \in B\ \forall x \leq k\,.\ \{e\}(x) = \{n\}(x).$$

(Here $\{n\}$ is the recursive function with index n. It will always be clear from the context whether an expression such as $\{e\}$ stands for a function or a singleton set.)

Let $\mathcal{C}$ be a class of recursive functions, and $F:\mathcal{C}\times\mathbb{N}\rightarrow\mathbb{N}$ a functional. F is *effective* if there exists a partial recursive function ϕ such that

$$\{e\} \in \mathcal{C} \Rightarrow F(\{e\},n) \simeq \phi(e,n).$$

Now the theorem of Kreisel, Lacombe and Shoenfield (cf. [KLS]) may be formulated as follows:

PROPOSITION. *Let $\mathcal{C}$ have a recursively dense base; and $F:\mathcal{C}\times\mathbb{N}\rightarrow\mathbb{N}$ be an effective operation. Then F is the restriction to $\mathcal{C}\times\mathbb{N}$ of some partial recursive functional.*

2. Recursively dense bases

We want to apply the Kreisel-Lacombe-Shoenfield theorem to algebras A through their characteristic functions χ_A. For this, we must single out the classes of computable minimal algebras that are suitable for such an application.

DEFINITION. Let Γ be a signature; and $\mathbf{K} \subseteq ALG(\Gamma)$ a class of computable minimal algebras. Then $\mathbf{K}$ *has a recursively dense base* if $\{\chi_A | A \in \mathbf{K}\}$ has a recursively dense base.

Classes of algebras with a recursively dense base exist. There are effective methods for deciding the satisfiability of finite sets of simple equalities and simple inequalities, that in fact construct computable models of such sets in a uniform fashion (see NELSON & OPPEN [NO] for a recent algorithm). So given a recursively enumerable set $\{f_i | i \in \mathbb{N}\}$ of finite partial functions $f_i:\mathbb{N}\rightarrow 2$, one can construct a recursively enumerable set $B = \{e_i | i \in \mathbb{N}\}$, with $\{e_i\} \supseteq f_i$, for all $i \in \mathbb{N}$, and $\{e_i\} = X_A$ for the algebra A satisfying $\{\hat{n} | f_i(n)=1\} \cup \{\neg\hat{n} | f_i(n)=0\}$ constructed by a method as mentioned above.

A simple application of this construction is as follows. Suppose a finite set Φ of simple equalities over a signature Γ is given. Φ corresponds, by the coding of simple equations, with a finite partial function $f_0:\mathbb{N}\rightarrow 2$ ($f_0(n)=1$ iff $\hat{n} \in \Phi$). Of finite extensions of Φ (with simple equalities and inequalities) it can be decided whether they are satisfiable. So we can construct an enumeration $(f_i)_{i\in\mathbb{N}}$ of all finite partial functions that correspond with satisfiable finite extensions of Φ. Then B, constructed from $(f_i)_i$ as above, is a recursively dense base for the collection of all recursive congruences extending f_0. By the connection between congruences and functions X_A noted at the end of §1, we see that the class of all computable elements of $ALG(\Gamma,\Phi)$ has a recursively dense base. In particular, for $\Phi = \varnothing$:

PROPOSITION. *For any signature Γ, the class of all computable minimal Γ-algebras has a recursively dense base.*

3. Algebras with equality

If an algebra A is computable, we can obtain negative information about it (for which simple equations e, $A \not\models e$) as well as positive (for which e, $A \models e$). A parametrized data type F may depend on both kinds of information; so both kinds will have to find their way into an algebraic specification of F. (The pertinent definitions are in the next section, but I suppose this point is easily imagined anyway.) Now, that some equation holds in A can be expressed algebraically, simply by the equation itself. It is less easy, however, to channel negative information into an algebraic specification; and the less since this must be done uniformly, irrespective of actual specifications of input algebras. Here we shall get around the difficulty by stipulating that our parameters are of a special type, that allows stating simple inequalities as equalities.

3.1 DEFINITION. Let A be an algebra of some signature Γ. A is an *algebra with equality* if the following hold:

(i) Γ contains a sort B (booleans), and function symbols

$T, F : B$

$\sim : B \rightarrow B$

$\& : B \times B \rightarrow B$

$Eq_\sigma : \sigma \times \sigma \rightarrow B$ for every sort $\sigma \neq B$ (equality functions);

(ii) the function symbols listed in (i) are *all* the function symbols of Γ that have B in their types;

(iii) $A_B = \{F, T\}$, and $T \neq F$ (I write T for $[\![T]\!]$, F for $[\![F]\!]$);

$\sim F = T$, $\sim T = F$;

$a \& b = T$ iff $a = b = T$;

$Eq_\sigma(a,b) = T$ iff $a = b$.

Clearly, for A as above and closed terms s and t of sort σ, $A \not\models s = t$ iff $A \models Eq_\sigma(s,t) = F$.

Other truth functions can be defined from $\sim$ and $\&$, as usual. In particular, $p \vee q := \sim(\sim p \& \sim q)$, $p \Rightarrow q := \sim p \vee q$, and $p \Leftrightarrow q := (p \Rightarrow q) \& (q \Rightarrow p)$. $\Leftrightarrow$ is the equality function for B.

3.2 DEFINITION. If Γ is a signature containing the sort B, then Γ^0 is the subsignature of Γ that results from deleting from Γ the sort B and all function symbols that have B in their types.

From the definitions one easily infers:

3.3 LEMMA. *If* A *and* B *are* Γ*-algebras with equality, then* $A \restriction \Gamma^0 \cong B \restriction \Gamma^0$ *implies* $A \cong B$.

3.4 Let Γ be the signature of some algebra with equality. If we have reasonable Gödel numberings of $T(\Gamma)$ and $T(\Gamma^0)$, the following will hold:

LEMMA. *There exists a (primitive) recursive function g such that, whenever* $\{n\} = \chi_A \restriction \Gamma^0$ *for some algebra with equality* $A \in ALG(\Gamma)$, $\{g(n)\} = \chi_A$.

3.5 Algebras with equality may seem rather special; in particular, the validity of the law $\forall x^B (x = T \vee x = F)$ might be thought to make recursively dense bases for classes of algebras with equality awkwardly rare. It is a consequence of the above lemma that they are not:

COROLLARY. *Let* $\mathbf{K} \subseteq ALG(\Gamma)$ *be a class of computable algebras with equality. If* $\{A \restriction \Gamma^0 | A \in \mathbf{K}\}$ *has a recursively dense base, then so has* $\mathbf{K}$.

PROOF. If B is a recursively dense base for $\{\chi_A \restriction \Gamma^0 | A \in \mathbf{K}\}$, then $\{g(n) | n \in B\}$ is a recursively dense base for $\{\chi_A | A \in \mathbf{K}\}$. □

4. SPECIFICATIONS OF PARAMETRIZED DATA TYPES

With a few more definitions, we will be ready to formulate and prove the theorem we have been after. The proof is rather involved. To keep its structure visible, I have divided it into a series of lemmas, and removed the longer subproofs to separate sections.

4.1 DEFINITION. Let Γ and Δ be signatures such that $\Gamma \subseteq \Delta$. Suppose $\mathbf{K} \subseteq ALG(\Gamma)$ is a class of computable algebras. An operation $F : \mathbf{K} \rightarrow ALG(\Delta)$ will be called *persistent* if $\forall A \in \mathbf{K}.\ F(A) \restriction \Gamma \cong A$.

Thus, if F is persistent, $F(A)$ is an expansion of A, modulo isomorphism.

4.2 DEFINITION. Let F be as above. F is *effective* if there exists a pair (γ,ϵ) of effective operations such that for every $\mathbf{A}\in\mathbf{K}$, for every full specification $\mathbb{S}=(\Sigma,E)$ of $\mathbf{A}$, $(\gamma(\mathbb{S}),\epsilon(\mathbb{S}))$ is a full specification of $F(\mathbf{A})$.

4.3 DEFINITION. Let $F{:}\mathbf{K}\rightarrow ALG(\Delta)$ be a persistent operation; $\mathbf{K}\subseteq ALG(\Gamma)$ a class of computable algebras. A specification $\mathbb{S}'=(\Sigma',E')$ is a *specification of F* if for every $\mathbf{A}\in\mathbf{K}$ and each full specification $\mathbb{S}=(\Sigma,E)$ of $\mathbf{A}$ such that $\Sigma\cap\Sigma'\subseteq\Gamma$, $\mathbb{S}\cup\mathbb{S}'$ is a full specification of $F(\mathbf{A})$.

4.4 THEOREM. *Let* Γ *and* Δ *be signatures, such that* $\Gamma\subseteq\Delta$. *Let* $\mathbf{K}\subseteq ALG(\Gamma)$ *be a class of computable algebras with equality, that has a recursively dense base. Let* $F{:}\mathbf{K}\rightarrow ALG(\Delta)$ *be a persistent operation. Then F is effective iff F has a (finite, equational) specification.*

One direction of the equivalence is easy.

4.5 LEMMA. *If* $F{:}\mathbf{K}\rightarrow ALG(\Delta)$ *is a persistent operation, and F has a specification, then F is effective.*

PROOF. One easily formulates a procedure for changing the hidden signature of an input algebra in such a way that the specification of F can be safely appended. □

The rest of our efforts will be aimed at the other direction: to extract a specification from an effective procedure.

4.6 LEMMA. *Let* Γ,Δ *be signatures, with* $\Gamma\subseteq\Delta$, *and* Γ *a signature for algebras with equality. Then full specifications for algebras* $\mathbf{A}\in ALG(\Delta)$ *such that* $\mathbf{A}\upharpoonright\Gamma$ *is an algebra with equality uniformly determine algorithms for* $\chi_{\mathbf{A}}$.

PROOF. Let $\mathbb{S}=(\Sigma,\Phi)$ be any full specification of $\mathbf{A}$. Then we can calculate $\chi_{\mathbf{A}}$ by the procedure sketched at the end of §1.2, once we know a simple equation e_0 that is false in $\mathbf{A}$. Since $\mathbf{A}\upharpoonright\Gamma$ is an algebra with equality, we can be sure that $\mathbf{A}\not\models F=T$; so we can take $F=T$ for e_0 uniformly. □

The next lemma could be deduced from BERGSTRA & TUCKER's proof that an algebra is computable iff it has a full specification (see [BT1]). Instead, I shall give a direct proof that is shorter and rather elementary. Since it is still too long, most of it will be relegated to §5.

4.7 LEMMA. *Let* Γ *be a signature. There is uniform effective procedure for constructing full specifications for computable minimal* Γ*-algebras* $\mathbf{A}$ *from indices for* $\chi_{\mathbf{A}}$.

Bergstra and Tucker, in the paper just referred to, encode a given computable algebra in one of its sorts. Here we want to do something similar, but this time we need a procedure that works for several algebras at once. Consequently, we cannot assume that one of the sorts of Γ is suitable for encoding — e.g., in $\mathbf{A}_1$ the carrier of σ_1 may be infinite, that of σ_2 finite, while in $\mathbf{A}_2$, σ_1 is finite and σ_2 infinite. For this reason, we shall have to *add* a sort of codes. Since we have been coding with natural numbers all along, we take an extra sort $\mathbb{N}$ of natural numbers, with suitable functions. We use a simple lemma from [BT2]. In its statement, 0 stands for the constant zero, and S for the successor function $n\mapsto n+1$.

4.7 LEMMA. *Let* $0,S,f_1,...,f_m$ *be a list of primitive recursive functions that for every* $i\,(1\leqslant i\leqslant m)$ *contains all the functions occurring in some primitive recursive derivation of* f_i. *Then there exists a finite equational specification* $\mathbb{S}$ *such that*

$$\mathrm{T}(\mathbb{S})\cong(\mathbb{N};0,S,f_1,...,f_m).$$

We need the following primitive recursive functions:

- the predecessor function P, for which we assume the equations

$$(\alpha)\quad \begin{aligned} P0&=0\\ PSx&=x \end{aligned}$$

as a definition;

- addition $(+)$ and multiplication $(\cdot)$;
- the characteristic function T of Kleene's T-predicate, and the value extraction function U:

$$T(w,y,x)\leqslant 1,$$

$$\exists x.T(w,y,x)=1 \textit{ iff } \{w\}(y) \text{ converges},$$

$$T(w,y,x)=1\Rightarrow\{w\}(y)=Ux;$$

- a primitive recursive bijection $j:\mathbb{N}\times\mathbb{N}\rightarrow\mathbb{N}$ (for §6).

Moreover, to deal with a given signature Γ, I shall use

- π_Γ (assumed total and primitive recursive);
- for each Γ-sort σ, a function σ with

$$\begin{aligned}\sigma(n)&=1 \ \text{ if } tm_\Gamma(n)\in T(\Gamma)_\sigma\\ &=0 \ \text{ otherwise}\end{aligned}$$

(so σ is the characteristic function of the codes of closed terms of sort σ);

- for each function symbol f of Γ, ϕ_f such that

$$\phi_f(\lceil t_1\rceil,\ldots,\lceil t_k\rceil)=\lceil ft_1\ldots t_k\rceil$$

whenever $t_1,\ldots,t_k$ are of the right sorts.

Let N_Γ be the algebra with carrier $\mathbb{N}$, and the operations listed above, with $0,S$, and sufficient auxiliary operations (as in lemma 4.7.1). To prove lemma 4.7, we shall extend Γ with the signature of N_Γ and 'interpretation' functions connecting $\mathbb{N}$ with the sorts of Γ, to a new signature Σ_Γ. Then, for any index l of $\chi_{\mathbf{A}}$, we construct a system E_Γ^l of equations over Σ_Γ that translate the action of l (through the T-predicate and the interpretation functions) to simple equations over Γ, in such a way that, with $\mathbb{S}_\Gamma^l:=(\Sigma_\Gamma,E_\Gamma^l)$, $\mathrm{T}(\mathbb{S}_\Gamma^l)$ is final in $ALG_0(\mathbb{S}_\Gamma^l)$, and $\mathrm{T}(\mathbb{S}_\Gamma^l)\!\restriction\!\Gamma\cong\mathbf{A}$. This plan will be executed in §5.

4.8 **Lemma.** *Suppose Γ,Δ are signatures, with $\Gamma\subseteq\Delta$; and $\mathbf{K}\subseteq ALG(\Gamma)$ consists of computable algebras with equality. Let $\mathcal{K}:=\{\chi_{\mathbf{A}}|\mathbf{A}\in\mathbf{K}\}$. Then every effective persistent operation $F:\mathbf{K}\rightarrow ALG(\Delta)$ determines an effective operation $F':\mathcal{K}\times\mathbb{N}\rightarrow\mathbb{N}$ such that $\forall\mathbf{A}\in\mathbf{K}\ \forall x\in\mathbb{N}:F'(\chi_{\mathbf{A}},x)=\chi_{F(\mathbf{A})}(x)$.*

Proof: Let $F:\mathbf{K}\rightarrow ALG(\Delta)$ be an effective persistent operation. Suppose $\mathbf{A}\in\mathbf{K}$, and l is some index of $X_{\mathfrak{G}}$. We can construct a full specification $\mathbb{S}_\Gamma^l$ of $\mathbf{A}$ by lemma 7. Now suppose F is determined by the pair (γ,ϵ) of effective operations. Then $F(\mathfrak{G})$ will be specified by $\mathbb{S}:=(\gamma(\mathbb{S}_\Gamma^l),\epsilon(\mathbb{S}_\Gamma^l))$. By persistence, $F(\mathbf{A})\!\restriction\!\Gamma\cong\mathbf{A}$, which is an algebra with equality; so lemma 6 may be applied to extract from $\mathbb{S}$ an algorithm for $\chi_{F(\mathbf{A})}$, say with index l'.

By the effective nature of the conversion of l into l' there exists a partial recursive function ψ such that, whenever l is an index for $\chi_{\mathbf{A}}$ for some $\mathbf{A}\in\mathbf{K}$, $\chi_{F(\mathbf{A})}=\{\psi(l)\}$. Now ϕ, defined by $\phi(x,y)\simeq\{\psi(x)\}(y)$, determines a suitable operation F'. □

Finally, we need a lemma about constructing specifications from partial recursive functionals. The following will be proven in §6, by a variation on the pattern of §5:

4.9 **Lemma.** *Let Γ and Δ be signatures, $\Gamma\subseteq\Delta$, and suppose $\mathbf{K}\subseteq ALG(\Gamma)$ consists of computable algebras with equality. Let $F:\mathbf{K}\rightarrow ALG(\Delta)$ be a persistent operation, and suppose there exists a partial recursive* *functional F'' such that $\forall A\in K: F''(\chi_A,n)=\chi_{F(A)}(n)$. Then F has a finite equational specification.*

Now we may finish the proof of the theorem. Let $\Gamma,\Delta,\mathbf{K}$ and F be as stated, and suppose F is effective. Let $\mathcal{K}:=\{\chi_A|A\in\mathbf{K}\}$. By lemma 8, we have an effective operation $F':\mathcal{K}\times\mathbb{N}\to\mathbb{N}$ such that $\forall A\in\mathbf{K}\ \forall n\in\mathbb{N}$: $F'(\chi_A,n)=\chi_{F(A)}(n)$. That $\mathbf{K}$ has a recursively dense base just means that $\mathcal{K}$ has a recursively dense base in the sense of §1.3. So by the Kreisel-Lacombe-Shoenfield theorem, $F'=F''\restriction(\mathcal{K}\times\mathbb{N})$ for some partial recursive functional F''. By lemma 9, F has a finite equational specification. □

4.10 As was pointed out in §2, the class of all computable algebras of a fixed signature Σ always has a recursively dense base. In particular, this holds for $\Sigma=\Gamma^0$ in case Γ is a signature for algebras with equality. From the definition of algebras with equality, it is clear that a minimal Γ-algebra A with equality is computable iff $A\restriction\Gamma^0$ is computable. Hence, by corollary 3.5, the class of all computable minimal Γ-algebras with equality has a recursively dense base. So the theorem specializes as follows:

COROLLARY. *Let Γ be a signature for algebras with equality. Suppose $\mathbf{K}\subseteq ALG(\Gamma)$ is the class of all minimal computable Γ-algebras with equality. Let Δ be a signature extending Γ, and $F:\mathbf{K}\to ALG(\Delta)$ a persistent operation. Then F is effective iff F has a finite equational specification.*

5. Uniform specification of computable minimal algebras

We are to prove lemma 4.7. Let a signature Γ be fixed. Let N_Γ be as in §4. By lemma 4.7.1, N_Γ has a flat initial algebra specification $\mathfrak{T}_\Gamma=(\Theta_\Gamma,D_\Gamma)$. Because of the predecessor function, $\mathfrak{T}_\Gamma$ is final as well:

5.1 LEMMA. *$\mathfrak{T}_\Gamma$ is a full specification of* N_Γ.

PROOF. Clearly, the equations (α) (under 4.7.1) are valid in N_Γ; so all their simple instances follow from D_Γ. Now let A be a proper homomorphic image of N_Γ — so in particular $A\models D_\Gamma$. There must be $k,l\in\mathbb{N}$ with $k>l$ such that $A\models S^k0=S^l0$. Applying the scheme (α) l times, we get $S^{k-1}0=PS^k0=PS^l0=S^{l-1}0$, down to $S^{k-l}0=0$. Continuing, we find $S^i0=0$ for all $i\leq k-l$. Reapplying S and using the transitivity of equality, and by the existence of a surjective homomorphism from N_Γ to A, we get $a=[\![0]\!]^A$ for all $a\in A$, so A is trivial. □

Now I shall describe $\mathfrak{S}^l_\Gamma$, for any natural number l. Fix for each Γ-sort σ a term $\uparrow_\sigma\in T(\Gamma)_\sigma$.

For each sort $\sigma\in\Gamma$, let F_σ be a new function symbol of type $\mathbb{N}\to\sigma$; H_σ of type $\mathbb{N}\times\sigma\to\sigma$; and N_σ of type $\mathbb{N}\times\sigma\times\sigma\to\mathbb{N}$. Let

$$\Sigma_\Gamma:=\Theta_\Gamma\cup\Gamma\cup\{F_\sigma,H_\sigma,N_\sigma|\sigma \text{ is a sort in } \Gamma\}$$

(the three sets on the right side are assumed disjoint). E^l_Γ is the finite equational theory obtained by adding to D_Γ the equations

(β) $H_\sigma(Sy,x)=x$

(γ) $H_\sigma(0,x)=\uparrow_\sigma$

(δ) $H_\sigma(\sigma u,F_\sigma u)=F_\sigma u$

(ϵ) $H_\sigma(t_\sigma\cdot U_x,F_\sigma u)=H_\sigma(t_\sigma\cdot U_x,F_\sigma v)$, where $t_\sigma\equiv t_\sigma(u,v,x)\equiv\sigma u\cdot\sigma v\cdot T(S^l0,\pi_\Gamma(u,v),x)$ (here $\equiv$ stands for syntactical equality, and the notation $t_\sigma(u,v,x)$ is to ease substitution)

(ζ) $H_\sigma(\sigma_1u_1\cdot\ldots\cdot\sigma_ku_k,F_\sigma\phi_fu_1\ldots u_k)=H_\sigma(\sigma_1u_1\cdot\ldots\cdot\sigma_ku_k,f(F_{\sigma_1}u_1,\ldots,F_{\sigma_k}u_k))$

(η) $N_\sigma(0,y,z) = 0$; $N_\sigma(Sx,y,y) = S0$

(θ) $N_\sigma(t_\sigma \cdot Sy, F_\sigma u, F_\sigma v) = t_\sigma \cdot Ux$, with t_σ as in (ϵ)

for each sort σ of Γ and each function symbol $f:\sigma_1 \times ... \times \sigma_k \to \sigma$ of Γ.

In the sequel, I will often simply write n instead of $S^n 0$ (1 instead of $S0$, etc.).

5.2 LEMMA. *Suppose* $\mathbf{B} \in ALG(\Sigma_\Gamma)$, $l \in \mathbb{N}$, *and*
(i) $\mathbf{B} \restriction \Gamma \in ALG(\Gamma)$ *and* $\{l\} = \chi_{\mathbf{B} \restriction \Gamma}$;
(ii) $\mathbf{B} \restriction \Theta_\Gamma = \mathbf{N}_\Gamma$.
Let t_σ *be as in* (ϵ) *above, and suppose* ρ *is an assignment into* $\mathbf{B}$. *Then* $[\![t_\sigma]\!]_\rho^{\mathbf{B}}$ *is 0 or 1; and if* $[\![t_\sigma]\!]_\rho = 1$, *then* $\rho(u), \rho(v) \in T(\Gamma)_\sigma$, *and* $[\![t_\sigma \cdot Ux]\!]_\rho^{\mathbf{B}} = 1$ *iff* $\mathbf{B} \models \overline{\rho(u)} = \overline{\rho(v)}$.

PROOF. t_σ is a product of factors that are either 0 or 1. If $[\![t_\sigma]\!]_\rho = 1$, then $\sigma(\rho(u)) = \sigma(\rho(v)) = 1$; which implies, by the definition of $\mathbf{N}_\Gamma$, that $\rho(u), \rho(v) \in T(\Gamma)_\sigma$. Then $[\![t_\sigma \cdot Ux]\!]_\rho = 1$ iff $T(l, \pi(\rho(u), \rho(v)), \rho(x)) = U(\rho(x)) = 1$, which implies that $\{l\}(\pi(\rho(u), \rho(\sigma))) = 1$. Since $\{l\} = \chi_{\mathbf{B} \restriction \Gamma}$, this means that $(\mathbf{B} \restriction \Gamma) \models \overline{\rho(u)} = \overline{\rho(v)}$. Likewise, $[\![t_\sigma \cdot Ux]\!]_\rho = 0$ implies $\{l\}(\pi(\rho(u), \rho(v))) = 0$, whence $(\mathbf{B} \restriction \Gamma) \not\models \overline{\rho(u)} = \overline{\rho(v)}$. □

We will be done once we have established the following:

5.3 PROPOSITION. *Suppose* $\mathbf{A} \in ALG(\Gamma)$ *and* $\chi_{\mathbf{A}} = \{l\}$. *Then* $\mathbb{S}_\Gamma^l$ *specifies* $\mathbf{A}$.

PROOF. We can straightforwardly combine $\mathbf{A}$ and $\mathbf{N}_\Gamma$ to an algebra $\mathbf{A} \cup \mathbf{N}_\Gamma$. Let $\mathbf{B}$ be the expansion of $\mathbf{A} \cup \mathbf{N}_\Gamma$ with functions

$$H_\sigma : (n+1, a) \mapsto a \quad , \qquad (0,a) \mapsto [\![\uparrow_\sigma]\!]^{\mathbf{A}} \ ;$$

$$F_\sigma : n \mapsto [\![\overline{n}]\!]^{\mathbf{A}} \ (\overline{n} = tm_\Gamma(n)) \text{ if } \overline{n} \text{ is of sort } \sigma, \qquad \mapsto [\![\uparrow_\sigma]\!]^{\mathbf{A}} \text{ otherwise;}$$

$$\text{and } N_\sigma : (0,a,b) \mapsto 0 \ , \qquad (n+1,a,b) \mapsto 1 \text{ if } a = b, \qquad \mapsto 0 \text{ otherwise;}$$

for all sorts $\sigma \in \Gamma$. Clearly, $\mathbf{B} \restriction \Gamma = \mathbf{A}$; so it will suffice to show that $\mathbb{S}_\Gamma^l$ is a flat specification of $\mathbf{B}$. For this we must show that (a) $\mathbf{B} \models E_\Gamma^l$, and for any simple equation e over Σ_Γ, (b) $\mathbf{B} \models e$ implies $E_\Gamma^l \vdash e$, and (c) $\mathbf{B} \not\models e$ implies $E_\Gamma^l \cup \{e\} \vdash e'$ for every simple equation e' over Σ_Γ.

(a) First, $\mathbf{B} \models D_\Gamma$ because $\mathbf{B} \restriction \Theta_\Gamma = \mathbf{N}_\Gamma$, and $\mathbf{N}_\Gamma \models D_\Gamma$.

The axioms (β) and (γ) are valid in $\mathbf{B}$ by the definition of H_σ. As for (δ), for $n \in \mathbb{N}$, $\sigma(n) \in \{0,1\}$. $H_\sigma(1, F_\sigma u) = F_\sigma u$ is just a substitution instance of (β); so only $H_\sigma(0, F_\sigma(k)) = F_\sigma(k)$ stands in need of proof ($k \in \mathbb{N}$ arbitrary). By (γ), $H_\sigma(0, F_\sigma(k)) = [\![\uparrow_\sigma]\!]$. Since $\sigma(k) = 0$ by assumption, $k \notin T(\Gamma)_\sigma$; so $F_\sigma(k) = [\![\uparrow_\sigma]\!]$.

To prove (ϵ), let ρ be an assignment into $\mathbf{B}$. If $[\![t_\sigma \cdot Ux]\!]_\rho = 1$, then $\mathbf{B} \models \overline{\rho(u)} = \overline{\rho(v)}$ by lemma 2; hence $\mathbf{B}, \rho \models F_\sigma u = F_\sigma v$. If $[\![t_\sigma \cdot Ux]\!]_\rho = 0$, then by ($\gamma$) there is nothing to prove.

For (ζ), again let ρ be an assignment into $\mathbf{B}$. If $[\![\sigma_1 u_1 \cdot ... \cdot \sigma_k u_k]\!]_\rho = 0$, there is nothing to prove. If $[\![\sigma_1 u_1 \cdot ... \cdot \sigma_k u_k]\!]_\rho = 1$, we must prove that $F_\sigma(\phi_f(\rho(u_1), ..., \rho(u_k))) = f(F_{\sigma_1}(\rho(u_1)), ..., F_{\sigma_k}(\rho(u_k)))$. In fact,

$$F_\sigma(\phi_f(\rho(u_1), ..., \rho(u_k))) = [\![\overline{\phi_f(\rho(u_1), ..., \rho(u_k))}]\!] \text{ by the definition of } F_\sigma$$

$$= [\![f(\overline{\rho(u_1)},...,\overline{\rho(u_k)})]\!] \text{ by the definition of } \phi_f,$$
$$\text{because } \overline{\rho(u_i)} \in T(\Gamma)_{\sigma_i}, \text{ when } 1 \leqslant i \leqslant k$$
$$= f(F_{\sigma_1}(\rho(u_1)),...,F_{\sigma_k}(\rho(u_k))).$$

The validity of (η) is by definition of N_σ. To check (θ), take ρ as before. There are two cases: $[\![t_\sigma]\!]_\rho=0$ and $[\![t_\sigma]\!]_\rho=1$. In the first case, $[\![t_\sigma \cdot Ux]\!]_\rho=0$, as by (η) it should. In the other case,

$$[\![N_\sigma(t_\sigma \cdot Sy, F_\sigma u, F_\sigma v)]\!]_\rho = N_\sigma(\rho(y)+1, F_\sigma(\rho(u)), F_\sigma(\rho(v))),$$

which is 0 or 1 depending on whether $F_\sigma(\rho(u))=F_\sigma(\rho(v))$; and by lemma 2, $[\![t_\sigma \cdot Ux]\!]_\rho=1$ iff $F_\sigma(\rho(u))=\overline{\rho(u)}=\overline{\rho(v)}=F_\sigma(\rho(v))$.

To prove (b) and (c), we first show that E^l_Γ allows us to reduce any closed Σ_Γ-term to either a numeral or a closed Γ-term.

5.3.1 Lemma. *Let s be a closed term over* Σ_Γ. *If s is of sort* $\mathbb{N}$, *then* $E^l_\Gamma \vdash s = S^n 0$ *for some* $n \in \mathbb{N}$. *If s is of a sort in* Γ, *then* $E^l_\Gamma \vdash s = t$ *for some* $t \in T(\Gamma)$.

Proof. By induction over terms. It will suffice to show that if $t_1, t_2 \in T(\Gamma)$ and $n \in \mathbb{N}$, then $F_\sigma S^n 0$ and $H_\sigma(S^n 0, t_1)$ reduce to Γ-terms, and $N_\sigma(S^n 0, t_1, t_2)$ to a numeral $S^m 0$. (Θ_Γ-terms reduce to numerals by lemma 1.)

The case of H_σ is immediate by (β) and (γ).

By D_Γ, $\sigma S^m 0$ reduces to $S0$ on 0 according as m is the Gödel number of a closed Γ-term of sort σ or not. Suppose that $s = f s_1 ... s_k$, with $f{:}\sigma_1 \times ... \times \sigma_k \to \sigma$; and $E^l_\Gamma \vdash F_{\sigma_i}\ulcorner s_i \urcorner = s_i$, $1 \leqslant i \leqslant k$. Then in E^l_Γ,

$$\begin{aligned} F_\sigma \ulcorner s \urcorner &= H_\sigma(1, F_\sigma \ulcorner s \urcorner) = H_\sigma(\sigma_1 \ulcorner s_1 \urcorner \cdot ... \cdot \sigma_k \ulcorner s_k \urcorner, F_\sigma \phi_f(\ulcorner s_1 \urcorner, ..., \ulcorner s_k \urcorner)) \\ &= H_\sigma(\sigma_1 \ulcorner s_1 \urcorner \cdot ... \cdot \sigma_k \ulcorner s_k \urcorner, f(F_{\sigma_1}\ulcorner s_1 \urcorner, ..., F_{\sigma_k}\ulcorner s_k \urcorner)) \text{ (by } (\zeta)) \\ &= H_\sigma(\sigma_1 \ulcorner s_1 \urcorner \cdot ... \cdot \sigma_k \ulcorner s_k \urcorner, f s_1 ... s_k) = H_\sigma(1, s) = s. \end{aligned}$$

If $\bar{n}$ is not of sort σ, then by (δ) and (γ),

$$E^l_\Gamma \vdash F_\sigma n = H_\sigma(\sigma n, F_\sigma n) = H_\sigma(0, F_\sigma n) = \uparrow_\sigma.$$

$E^l_\Gamma \vdash N_\sigma(0, t_1, t_2)=0$ by (η). If $n=Sk$, take m such that $t_\sigma(\ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner, m)=1$ (such m exist since $\{l\}$ is total). Then in E^l_γ, using that what has just been shown about F_σ,

$$\begin{aligned} N_\sigma(n, t_1, t_2) &= N_\sigma(n, F_\sigma \ulcorner t_1 \urcorner, F_\sigma \ulcorner t_2 \urcorner) \\ &= N_\sigma(t_\sigma(\ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner, m) \cdot Sk, F_\sigma \ulcorner t_1 \urcorner, F_\sigma \ulcorner t_2 \urcorner) \text{ by } D_\Gamma \\ &= t_\sigma(\ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner, m) \cdot Um, \end{aligned}$$

which is 0 or 1 by D_Γ. □

Now we can finish the proof of the proposition.

(b) By the lemma, any simple equation e over Σ reduces, in E^l_Γ, and therefore, by (a), in **B**, to a simple equation e_0 which is either of the form $S^m 0 = S^n 0$ or an equation over Γ. The only equations of the first form that are valid in **B** are those with $m=n$; they trivially follow from E^l_Γ. A simple Γ-equation e_0 holds in **B** only if $A \vDash e_0$. If $A \vDash e_0$, then $\{l\}(\ulcorner e_0 \urcorner)=1$. Let us say that $T(l, \ulcorner e_0 \urcorner, m)=1$, with $U(m)=1$. Suppose $e_0 \equiv (s=t)$, with $s, t \in T(\Gamma)_\sigma$. Then in E^l_Γ,

$$\begin{aligned} H_\sigma(t_\sigma(\ulcorner s \urcorner, \ulcorner t \urcorner, m) \cdot Um, F_\sigma \ulcorner s \urcorner) &= H_\sigma(t_\sigma(\ulcorner s \urcorner, \ulcorner t \urcorner, m) \cdot Um, F_\sigma \ulcorner t \urcorner) \\ H_\sigma(1, F_\sigma \ulcorner s \urcorner) &= H_\sigma(1, F \ulcorner t \urcorner), \text{ by } D_\Gamma \\ F_\sigma \ulcorner s \urcorner &= F_\sigma \ulcorner t \urcorner \end{aligned}$$

$s = t$, by the proof of the lemma.

(c) Suppose $s,t \in T(\Sigma_\Gamma)$, and $\mathbf{B} \not\models s=t$. If s and t are of sort $\mathbb{N}$, then by the lemma there are natural numbers m and n such that $E^l_\Gamma \vdash s=S^m 0$ and $E^l_\Gamma \vdash t=S^n 0$. Since $\mathbf{B} \models E^l_\Gamma$, $m \neq n$. Since $E^l_\Gamma \supseteq D_\Gamma$, and $(\Theta_\Gamma, D_\Gamma)$ specifies $\mathbf{N}_\Gamma$, $E^l_\Gamma \cup \{S^m 0=S^n 0\} \vdash S^{m'} 0=S^{n'} 0$ for all $m',n' \in \mathbb{N}$; hence, again using the lemma, $E^l_\Gamma \cup \{s=t\} \vdash s'=t'$ for all $s',t' \in T(\Sigma_\Gamma)_{\mathbb{N}}$. In particular, $E^l_\Gamma \cup \{s=t\} \vdash 0=1$. For any $s' \in T(\Gamma)_\sigma$ we have, in $E^l_\Gamma \cup \{s=t\}$:

$$s' = H_\sigma(S0,s') = H_\sigma(0,s') = \uparrow_\sigma,$$

so by transitivity of $=$ and lemma 5.3.1, $E^l_\gamma \cup \{s=t\} \vdash s'=t'$ for all $s',t' \in T(\Sigma_\Gamma)_\sigma$.

If s and t are of a sort $\sigma \in \Gamma$, then $\mathbf{A} \not\models s=t$, so $\{l\}(\lceil s=t \rceil)=0$. Suppose $T(l,\lceil s=t \rceil,m)=1$ (then $U(m)=0$). Now in E^l_Γ, $t_\sigma(\lceil s \rceil,\lceil t \rceil,m)=1$, and

$$\begin{aligned} N_\sigma(S0,s,t) &= N_\sigma(t_\sigma(\lceil s \rceil,\lceil t \rceil,m)\cdot S0, F_\sigma\lceil s \rceil, F_\sigma\lceil t \rceil) \\ &= t_\sigma(\lceil s \rceil,\lceil t \rceil,m)\cdot Um \\ &= 1\cdot 0 = 0. \end{aligned}$$

But in $E^l_\Gamma \cup \{s=t\}$, $N_\sigma(S0,s,t)=N_\sigma(S0,s,s)=1$. Thus we have $0=1$, and this case reduces to the earlier one. □

6. THE SPECIFICATION LEMMA FOR PARAMETRIZED DATA TYPES REPRESENTED BY A PARTIAL RECURSIVE FUNCTIONAL

We want to prove lemma 4.9. So let signatures Γ and Δ be given, with $\Gamma \subseteq \Delta$, and a class $\mathbf{K} \subseteq ALG(\Gamma)$ of algebras with equality; a persistent operation $F{:}\mathbf{K} \rightarrow ALG(\Delta)$, and a partial recursive functional F'' such that $\forall \mathbf{A} \in \mathbf{K}\, \forall n \in \mathbb{N}. F''(\chi_{\mathbf{A}},n)=\chi_{F(\mathbf{A})}(n)$. We shall produce a finite equational specification, and show that it specifies F. The construction closely parallels that of §5, but there are some complications. We begin with a triviality for later reference.

6.1 JOINT EXPANSION LEMMA.

Let Σ_0, Σ_1 and Σ_2 be signatures, with $\Sigma_0=\Sigma_1 \cap \Sigma_2$; and $\mathbf{A}_i \in ALG(\Sigma_i)$ ($i \leq 2$) such that $\mathbf{A}_1 \upharpoonright \Sigma_0 = \mathbf{A}_0 = \mathbf{A}_2 \upharpoonright \Sigma_0$. Then there is a unique joint expansion $\mathbf{A}_1 \cup \mathbf{A}_2 \in ALG(\Sigma_1 \cup \Sigma_2)$ of $\mathbf{A}_1$ and $\mathbf{A}_2$, with $(\mathbf{A}_1 \cup \mathbf{A}_2) \upharpoonright \Sigma_i = \mathbf{A}_i$, for $i \in \{1,2\}$.

(If you recognize a minimal algebra with empty signature, you will find an application of this lemma in the previous section.)

The partial recursive functional F'' is determined by an algorithm A that computes $F''(f,n)$ from n and a finite part $\{(n_1,f(n_1)),\ldots,(n_k,f(n_k))\}$ of f. From A we may extract an algorithm A′ that works on numbers m as follows. First, A′ looks for numbers m' and m'' such that $j(m',m'')=m$. (Since j is a bijection, these will in fact be found.) Next, A′ checks if $m'=gn_\Delta(t)$ for some closed boolean term $t \equiv (\sim)Eq_{\sigma_1}(s_1,t_1)\&\ldots\&(\sim)Eq_{\sigma_l}(s_l,t_l)$, with s_i,t_i ($1 \leq i \leq l$) closed terms over Γ. If m' is not the right kind of Gödel number, A′ fails (never terminates, say). If m' is a proper Gödel number, we set A to work on n and $\{gn_\Gamma(s_1=t_1),j_1),\ldots,(gn_\Gamma(s_l=t_l),j_l)\}$, with j_i equal to 0 or 1 according as $\sim$ appeared immediately before $Eq_{\sigma_i}(s_i,t_i)$ or not. (We assume, as is reasonable, that proper Gödel numbers can be effectively converted to codes of simple equations over Γ.) Of course, A may well ask for a function value at an argument that t gives no information about; then, again, A′ fails.

The algorithm A′ has an index p. By the construction of A′, there exists for each $\mathbf{A} \in \mathbf{K}$ and $n \in \mathbb{N}$ a number m' such that $F''(\chi_{\mathbf{A}},n)=\{p\}(j(m',n))$, and $\mathbf{A} \models tm_\Delta(m')=T$ ($tm_\Delta(m')$ is in fact a closed term over $\Gamma \subseteq \Delta$). Conversely, if $\{p\}(j(m',n))$ is defined, then $F''(\chi_{\mathbf{A}},n)=\{p\}(j(m',n))$ for each $\mathbf{A} \in \mathbf{K}$ in which $tm_\Delta(m')=T$ holds.

Let Σ be the signature Σ_Δ (defined as in §5), extended with a new function symbol $N{:}B\to\mathbb{N}$. Fix for each sort σ of Δ a closed term $\uparrow_\sigma \in T(\Delta)_\sigma$. Let E be D_Δ, extended by

(ι) $N(T)=1$, $N(F)=0$;

(β),(γ) and (δ) of §5, for each sort σ of Δ (and with possibly different $\uparrow_\sigma$);

(ϵ^*) $H_\sigma(t_\sigma\cdot Ux,F_\sigma u)=H_\sigma(t_\sigma\cdot Ux,F_\sigma v)$, where $t_\sigma\equiv t_\sigma(y,u,v,x)\equiv NF_By\cdot\sigma u\cdot\sigma v\cdot T(p,j(y,\pi(u,v)),x)$ for each Δ-sort σ;

(ζ) and (η) of §5, for all function symbols f and sorts σ of Δ;

(θ^*) $N_\sigma(t_\sigma\cdot Sz,F_\sigma u,F_\sigma v)=t_\sigma\cdot Ux$, with t_σ as in (ϵ^*), for each sort σ of Δ.

We are going to show that $\mathbb{S}:=(\Sigma,E)$ specifies F.

Let $\mathbf{A}\in\mathbf{K}$; suppose $\mathbb{S}_0=(\Sigma_0,E_0)$ is a finite equational specification of $\mathbf{A}$, such that $\Sigma_0\cap\Sigma\subseteq\Gamma$. We must prove that $\mathbb{S}_0\cup\mathbb{S}$ specifies $F(\mathbf{A})$.

Let $\mathbf{B}_0$ be the result of expanding $\mathbf{N}_\Delta\cup F(\mathbf{A})$ with operations H_σ,N_σ and F_σ as in the proof of proposition 5.3 (but for Δ instead of Γ, and $F(\mathbf{A})$ instead of $\mathbf{A}$), and

$$N{:}T\mapsto 1$$
$$F\mapsto 0.$$

Note that, since $\mathbf{A}$ is an algebra with equality and F is persistent, $\{F,T\}$ is the entire boolean domain of $\mathbf{B}_0$. Since, of course, $\Delta\cap\Theta_\Delta=\varnothing$, $\mathbf{B}_0{\restriction}\Delta=F(\mathbf{A})$. By the persistence of F, it follows that $\mathbf{B}_0{\restriction}\Gamma=\mathbf{A}$.

Suppose $\mathbb{S}_0$ is a flat specification of $\mathbf{A}'$, with $\mathbf{A}'{\restriction}\Gamma=\mathbf{A}$. Since $\Sigma_0\cap\Sigma=\Gamma$, we may take the joint expansion $\mathbf{B}=\mathbf{B}_0\cup\mathbf{A}'$. As $\mathbf{B}$ is a minimal algebra, and $\mathbf{B}{\restriction}\Delta=F(\mathbf{A})$, it will suffice to prove

6.2 **Proposition.** *(a)* $\mathbf{B}\nvDash E_0\cup E$, *and for any simple equation e over* Σ, *(b)* $\mathbf{B}\models e$ *implies* $E_0\cup E\vdash e$, *and (c)* $\mathbf{B}\nvDash e$ *implies* $E_0\cup E\cup\{e\}\vdash e'$ *for every simple equation e′ over* Σ.

6.2.1 **Lemma.** *Let* ρ *be an assignment into* $\mathbf{B}$. *Then* $[\![t_\sigma]\!]^{\mathbf{B}}_\rho$ *and* $[\![t_\sigma\cdot Ux]\!]^{\mathbf{B}}_\rho$ *are 0 or 1; and if* $[\![t_\sigma]\!]^{\mathbf{B}}_\rho=1$, *then* $\rho(u),\rho(\sigma)\in T(\Delta)_\sigma$, *and* $[\![t_\sigma\cdot Ux]\!]^{\mathbf{B}}_\rho=1$ *iff* $F(\mathbf{A})\models\rho(u)=\rho(v)$.

Proof. The first part of the lemma is evident: the terms involved are products of factors that are either 0 or 1.

Suppose $[\![t_\sigma]\!]_\rho=1$; then we must have $\sigma(\rho(u))=\sigma(\rho(v))=1$, so by the definition of Θ_Δ, $tm_\Delta(\rho(u))$ and $tm_\Delta(\rho(v))$ are of sort σ. Also, $T(p,j(\rho(y),\pi_\Delta(\rho(u),\rho(v)),\rho(x))=1$, which may be taken to mean that the algorithm A′, described above, terminates on the argument $j(\rho(y),\pi_\Delta(\rho(u),\rho(v)))$. So $\rho(y)=gn_\Delta(t)$ for some closed Γ-term $t\equiv$

$$(\sim)Eq(s_1,t_1)\&...\&(\sim)Eq(s_k,t_k).$$

The algorithm A, working on the corresponding partial function $g_0:=$

$$\{(gn_\Gamma(s_1=t_1),j_1),...,(gn_\Gamma(s_k=t_k),j_k)\}$$

terminates, with output $U(\rho(x))$. So $F''(\chi_{\mathbf{C}},\pi_\Delta(\rho(u),\rho(v)))=U(\rho(x))$ for any $\mathbf{C}\in\mathbf{K}$ such that $g_0\subseteq\chi_{\mathbf{C}}$ - and the latter is equivalent to $\mathbf{C}\models t=T$. Now finally, $N(F_B(\rho(y)))=1$, whence $t=\rho(y)=F_B(\rho(y))=T$. So $\mathbf{A}\models t=T$, and consequently $F''(\chi_{\mathbf{A}},\pi(\rho(u),\rho(v)))=U(\rho(x))$. Now $[\![t_\sigma\cdot Ux]\!]_\rho=1$ iff $U(\rho(x))=1$ iff $F(\mathbf{A})\models\rho(u)=\rho(v)$.

Proof of 6.2 (a): $\mathbf{B}\models E_0$ is immediate by $\mathbf{A}'\models E_0$; similarly $\mathbf{B}\models D_\Delta$.

The validity of (ι),(β) and (γ) is immediate. (δ) is proved valid as in 5.3.

To prove (ϵ^*), let ρ be an assignment into **B**. $[\![t_\sigma \cdot Ux]\!]_\rho^{\mathbf{B}}$ is either 0 or 1; the first case is trivial. Suppose, then, that $[\![t_\sigma \cdot Ux]\!]_\rho = 1$. Since obviously $[\![t_\sigma]\!]_\rho = 1$, we have, by the lemma,

$$[\![F_\sigma u]\!]_\rho^{\mathbf{B}} = [\![\overline{\rho(u)}]\!]^{F(A)} = [\![\overline{\rho(v)}]\!]^{F(A)} = [\![F_\sigma v]\!]_\rho^{\mathbf{B}}.$$

(ζ) and (η) are as in 5.3. To check (θ^*), let ρ be an assignment into **B**. If $[\![t_\sigma]\!]_\rho = 0$, all is trivial, as in 5.3. So suppose $[\![t_\sigma]\!]_\rho = 1$. Both sides of the equation are either 0 or 1. By lemma 6.2.1 and the definitions of $F_\sigma^{\mathbf{B}}$ and $N_\sigma^{\mathbf{B}}$, $[\![t_\sigma \cdot Ux]\!]_\rho = 1$ iff $[\![F_\sigma u]\!]_\rho = [\![\overline{\rho(u)}]\!]^{F(A)} = [\![\overline{\rho(v)}]\!]^{F(A)} = [\![F_\sigma v]\!]_\rho$ iff $[\![N_\sigma(t_\sigma \cdot Sz, F_\sigma u, F_\sigma v)]\!]_\rho = 1$.

6.2.2 LEMMA. *Suppose $t', t'' \in T(\Delta)_\sigma$. Then there exist a closed boolean term $s \in T(\Gamma)$ and $n \in \mathbb{N}$ such that $A \models s = T$ and $D_\Delta \vdash T(p, j(\lceil s \rceil, \pi(\lceil t' \rceil, \lceil t'' \rceil)), n) = 1$; and $D_\Delta \vdash Un = 1$ if $F(A) \models t' = t''$, $D_\Delta \vdash Un = 0$ otherwise.*

PROOF. If $F(A) \models t' = t''$, then $F''(\chi_A, \lceil t' = t'' \rceil) = 1$. The algorithm A that determines F'' works on a finite part of χ_A; say that $g = \{(n_1, j_1), \ldots, (n_k, j_k)\}$ suffices (the numbers j_i $(1 \leqslant i \leqslant k)$ are either 1 or 0). Let $\sim^0 t \equiv t$, $\sim^1 t \equiv \sim t$, for boolean terms t. Then take $s \equiv$

$$\sim^{1-j_1} \hat{n}_1 \& \ldots \& \sim^{1-j_k} \hat{n}_k.$$

A′ works on $j(\lceil s \rceil, \lceil t' = t'' \rceil)$ as A does on g and $\lceil t' = t'' \rceil$. Since p codes A′, $\{p\}(j(\lceil s \rceil, \lceil t' = t'' \rceil)) = 1$; that is, for some $n \in \mathbb{N}$, $T(p, j(\lceil s \rceil, \lceil t' = t' \rceil), n) = 1$ and $Un = 1$. Since D_Δ specifies $\mathbf{N}_\Delta$, these equations are provable in D_Δ:

$$D_\Delta \vdash T(p, j(\lceil s \rceil, \pi(\lceil t' \rceil, \lceil t'' \rceil)), n) = 1 \text{ and } D_\Delta \vdash Un = 1.$$

The case that $F(A) \not\models t' = t''$ is similar. □

6.2.3 LEMMA. *Let t be a closed term over Σ. If t is of sort B, then $E_0 \cup E \vdash t = F$ or $E_0 \cup E \vdash t = T$. If t is of sort $\mathbb{N}$, then $E_0 \cup E \vdash t = S^n 0$ for some $n \in \mathbb{N}$. If t is of a sort in Δ, then $E_0 \cup E \vdash t = t'$ for some $t' \in T(\Delta)$.*

PROOF. Induction over terms, similar to lemma 5.3.1; the main difference is the added concern over booleans. First note that, as in the proof of lemma 5.3.1, one can show that $E_0 \cup E \vdash F_\sigma m = \overline{m}$ if $\overline{m}$ is of sort σ in Δ, and $E_0 \cup E \vdash F_\sigma m = \uparrow_\sigma$ otherwise.

For $t \in T(\Delta)_B$, we have $F(A) \models t = T$ or $F(A) \models t = F$. Suppose for determinacy that $F(A) \models t = T$ ($t = F$ is handled the same way). By lemma 6.2.1, there are a boolean term s and a natural number m such that $A \models s = T$ and

$$D_\Delta \vdash T(p, j(\lceil s \rceil, \pi(\lceil t \rceil, \lceil T \rceil)), m) = 1 \text{ and } D_\Delta \vdash Um = 1.$$

By (ϵ^*), we have in $E_0 \cup E$

$$H_B(t_B(\lceil s \rceil, \lceil t \rceil, \lceil T \rceil, m) \cdot Um, F_B \lceil t \rceil) = H_B(t_B(\lceil s \rceil, \lceil t \rceil, \lceil T \rceil, m) \cdot Um, F_B \lceil T \rceil). \qquad (*)$$

Since D_Δ specifies $\mathbf{N}_\Delta$, $D_\Delta \vdash B \lceil t \rceil = B \lceil T \rceil = 1$. Now if $E_0 \cup E \vdash NF_B \lceil s \rceil = 1$ as well, we get $t_B(\lceil s \rceil, \lceil t \rceil \lceil T \rceil, m) \cdot Um = 1$, and we can substitute into (*):

$$t = F_B \lceil t \rceil = H_B(1, F_B \lceil t \rceil) = H_B(1, F_B \lceil T \rceil) = T$$

(using the reduction of $F_B m$ to $\overline{m}$, for $\overline{m} \in T(\Delta)_B$).

To show that $NF_B \lceil s \rceil = 1$ it suffices, by (ι), to prove $F_B \lceil s \rceil = T$. As before, $F_B \lceil s \rceil = s$. Since $A \models s = T$, and E_0 specifies A, $E_0 \vdash s = T$; so indeed $E_0 \cup E \vdash F_B \lceil s \rceil = T$.

Now it will suffice to show that terms consisting of a function symbol in $\Sigma - \Delta$ and T, F, numerals and closed Δ-terms of sorts other than B, can be reduced as required. If the function symbol is N, this is immediate by (ι). F_σ has been dealt with above. H_σ is eliminated by (β) and (γ). $N_\sigma(0, t', t'')$ reduces by (η). This leaves only the case of $N_\sigma(Sk, t', t'')$.

If $F(A) \models t' = t''$, then there are a boolean term s such that $A \models s = t$, hence $E_0 \vdash s = T$, and a numeral n, such that D_Δ proves $T(p,j(\lceil s\rceil,\pi(\lceil t'\rceil,\lceil t''\rceil)),n)=1$ and $Un=1$. Then in $E_0\cup E$, calculating $t_\sigma\cdot Sk$ as before,

$$\begin{aligned}N_\sigma(Sk,t',t'') &= N_\sigma(t_\sigma(\lceil s\rceil,\lceil t'\rceil,\lceil t''\rceil,n)\cdot Sk,F_\sigma\lceil t'\rceil,F_\sigma\lceil t''\rceil)\\ &= t_\sigma(\lceil s\rceil,\lceil t'\rceil,\lceil t''\rceil,n)\cdot Un \quad \text{by } (\theta^*)\\ &= 1.\end{aligned}$$

Similarly, $N_\sigma(Sk,t',t'')=0$ if $F(\mathrm{A})\not\models t'=t''$. □

Now we are ready to finish the proof of the proposition.
(b) By the last lemma, any simple equation e over Σ reduces, in $E_0\cup E$, and therefore, by (a), in **B**, to a simple equation e_0 which is of the form $S^m0=S^n0$ or $s_0=s_1$ with $s_0,s_1\in\{F,T\}$, or is an equation over Δ. The first cases are trivial, as under proposition 5.3. A simple Δ-equation e_0 holds in **B** precisely if $F(\mathrm{A})\models e_0$. If $F(\mathrm{A})\models e_0$, then by lemma 6.2.2 there exist $s\in T(\Gamma)_B$ and $k\in\mathbb{N}$ such that $\mathrm{A}\models s=t$ and $D_\Delta\vdash T(p,j(\lceil s\rceil,\lceil e_0\rceil),k)=Uk=1$. Suppose $e_0\equiv(t=t')$, with $t,t'\in T(\Delta)_\sigma$. Then in $E_0\cup E$,

$$\begin{aligned}H_\sigma(t_\sigma(\lceil s\rceil,\lceil t\rceil,\lceil t'\rceil,k)\cdot Uk,F_\sigma\lceil t\rceil) &= H_\sigma(t_\sigma(\lceil s\rceil,\lceil t\rceil,\lceil t'\rceil,k)\cdot Uk,F_\sigma\lceil t'\rceil)\\ H_\sigma(1,F_\sigma\lceil t\rceil) &= H_\sigma(1,F_\sigma\lceil t'\rceil) \quad \text{(calculating in } D_\Delta)\\ t = F_\sigma\lceil t\rceil &= F_\sigma\lceil t'\rceil = t' \quad \text{(using the proof of the last lemma).}\end{aligned}$$

(c) Suppose e is a simple equation over Σ, and $\mathbf{B}\not\models e$. We are to show that $E_0\cup E\cup\{e\}\vdash e'$ for every simple equation e' over Σ. By the last lemma, we may assume that e is either $S^m0=S^n0$ for distinct natural numbers m and n; or $F=T$; or $t=t'$ for Δ-terms of a sort other than B. In each case we can derive $0=1$ — by D_Δ, (ι), or by (η) and (θ^*); and from $0=1$ every simple equation follows as in the proof of proposition 5.3. In the last case, we get $N_\sigma(1,t,t')=1$ by (η); and since $F(\mathrm{A})\not\models t=t'$, there are $s\in T(\Gamma)_B$ with $\mathrm{A}\models s=T$, and $k\in\mathbb{N}$, such that D_Δ proves $T(p,j(\lceil s\rceil,\pi(\lceil t\rceil,\lceil t'\rceil)),k)=1$ and $Uk=0$ (lemma 6.2.2), which will make $N_\sigma(1,t,t')=0$ by (θ^*). □

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