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1. INTRODUCTION

In the following we survey a number of results pertaining to a novel class of completely integrable N-particle systems, which generalize the well-known Calogero-Moser (CM) systems, and which were discovered in collaboration with H. Schneider [1)]. The new systems are relativistically invariant, and the CM systems result upon taking a parameter to ∞ which may be regarded as the speed of light. They can be quantized in such a fashion that integrability is preserved [2)]. However, we shall restrict ourselves here to the classical level, and present results from Refs. [1)-4)] with a bias towards relations with soliton PDE, in keeping with the spirit of the conference. Related results concerning external fields and Toda systems, and a Lax pair formulation for the elliptic case can be found in Refs. [5)-7)], resp. (See also F. Calogero's contribution to these proceedings with regard to the last-mentioned topic.)

In Section 2 we describe our systems and their relation to the CM systems, and discuss their relativistic invariance and complete integrability. Section 3 is concerned with a matrix whose symmetric functions yield the N independent commuting Hamiltonians. This matrix is an essential tool in the construction of the action-angle map, which is discussed in Section 4. This section also prepares the ground for Section 5, which deals with the relation of the particle systems to various well-known soliton equations. It turns out that the solutions consisting of solitons, antisolitons, breathers,... may all be viewed as manifestations of underlying point particle dynamics associated with our systems. The relation leads in particular to a natural concept of soliton space-time trajectory, which is presented and discussed in Section 6.

2. A DESCRIPTION OF THE SYSTEMS

Recall that the Hamiltonian of the CM system reads

$$H_{nr} \equiv \sum_{i=1}^{N} \theta_i^2/2 + g^2 \sum_{1 \leq i < j \leq N} V(q_i - q_j). \tag{1}$$

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Here, the particle mass equals 1, and the pair potential V equals the Weierstrass $\wp$-function (elliptic case) or one of its degenerate cases, viz., $1/sh^2$ (hyperbolic case), $1/\sin^2$ (trigonometric case), $1/q^2$ (rational case). (More precisely, this defines the CM system associated with the Lie algebra A_{N-1}, cf. the review [8].) Supplementing H_{nr} with the functions

$$P_{nr} \equiv \sum_{i=1}^{N} \theta_i \tag{2}$$

(space translation generator) and

$$B_{nr} \equiv -\sum_{i=1}^{N} q_i \tag{3}$$

(boost generator) leads to a representation of the nonrelativistic space-time symmetry group (the Galilei group) in one space dimension.

The integrable systems to be discussed here lead to a representation of the relativistic space-time symmetry group (the Poincaré group). The generators are given by

$$H \equiv c^2 \sum_{i=1}^{N} ch(\theta_i/c) \prod_{j\neq i} f(q_i - q_j) \tag{4}$$

$$P \equiv c \sum_{i=1}^{N} sh(\theta_i/c) \prod_{j\neq i} f(q_i - q_j) \tag{5}$$

$$B \equiv -\sum_{i=1}^{N} q_i. \tag{6}$$

Here, c is the speed of light and f satisfies

$$f^2(q) = a + b\wp(q), \tag{7}$$

where a,b are arbitrary constants. Picking

$$a = 1,\ b = g^2/c^2, \tag{8}$$

it is obvious that one obtains the CM systems in the nonrelativistic limit. That is, one has

$$H_{nr} = \lim_{c\to\infty} (H - Nc^2) \tag{9}$$

$$P_{nr} = \lim_{c\to\infty} P \qquad B_{nr} = \lim_{c\to\infty} B \tag{10}$$

It is not supposed to be obvious that H,P,B satisfy the Poincaré Lie algebra

$$\{H,P\} = 0,\ \{H,B\} = P,\ \{P,B\} = H/c^2. \tag{11}$$

More precisely, the vanishing of the first bracket (translation invariance) is not obvious.. One readily verifies that for even f its vanishing is equivalent to the functional equation

$$\sum_{i=1}^{N} \partial_i \prod_{j\neq i} f^2(q_i - q_j) = 0. \tag{12}$$

This equation is trivially satisfied for $N = 2$, but for $N = 3$ it amounts to the well-known functional equation of the $\wp$-function. Thus, (7) results.

The fact that the $\wp$-function satisfies (12) for arbitrary N appears to be new. A direct proof is given in Appendix A of Ref. [1]. It is also shown there that (12) entails that the functions

$$S_k \equiv \sum_{\substack{|I|=k \\ I\subset\{1,\ldots,N\}}} \exp(\sum_{i\in I}\theta_i) \prod_{\substack{i\in I \\ j\notin I}} f(q_i - q_j), \quad k = 1,\ldots,N \tag{13}$$

commute with H. (Here and henceforth we set $c \equiv 1$.) Moreover, it is proved that these integrals are in involution if and only if f^2 satisfies

$$\sum_{\substack{|I|=k \\ I\subset\{1,\ldots,N\}}} \sum_{i\in I} \partial_i \prod_{\substack{i\in I \\ j\notin I}} f^2(q_i-q_j) = 0, \ \forall k\in\{1,\ldots,N\}, \ \ \forall N>1 \tag{14}$$

The validity of these more general functional equations in the hyperbolic case has been established in Ref. [1] by invoking scattering theory. The fact that they still hold in the elliptic case is proved in Ref. [2]. They follow by taking a limit in new functional equations for the Weierstrass σ-function, which express the fact that a suitably chosen quantization preserves integrability. The limit involved here may be viewed as taking Planck's constant to 0 [2].

3. LAX MATRICES AND EXPLICIT SOLUTIONS

In the hyperbolic case it is convenient to use the parametrization

$$f(q) = [1-sh^2\frac{\mu}{2}/sh^2\frac{q}{2}]^{\frac{1}{2}}. \tag{15}$$

Here and in the sequel we shall take

$$\mu \in i(0,\pi]. \tag{16}$$

The commuting functions $S_1,\ldots,S_N$ are then equal to the symmetric functions of the $N\times N$ matrix with elements

$$L_{ij}(q,\theta) \equiv d_i(q,\theta)C_{ij}(q)d_j(q,\theta) \tag{17}$$

where

$$d_j \equiv e^{\theta_j/2}\prod_{j\neq i} f(q_i-q_j)^{\frac{1}{2}} \tag{18}$$

$$C_{ij} \equiv sh\frac{\mu}{2}/sh\frac{1}{2}(\mu+q_i-q_j). \tag{19}$$

This assertion can be verified by making suitable use of Cauchy's identity,

$$|(\frac{1}{x_i-y_j})| = \prod_i \frac{1}{x_i-y_i}\prod_{i<j}\frac{(x_i-x_j)(y_i-y_j)}{(x_i-y_j)(y_i-x_j)} \tag{20}$$

In the elliptic case there seems to be no matrix generalizing (17) whose symmetric functions are equal to $S_1,\ldots,S_N$. However, a matrix whose symmetric functions are proportional to $S_1,\ldots,S_N$ does exist; the proportionality follows from a generalization of Cauchy's identity. This matrix depends on an extra parameter λ and leads to Krichever's [9] Lax matrix in the nonrelativistic limit, so that the integrability of the CM systems follows as a corollary. Also, the hyperbolic Lax matrix (17) results by taking one period of $\wp$ to ∞, and then sending the spectral parameter λ to ∞ [2]. Bruschi and Calogero independently arrived at the same elliptic Lax matrix, and also obtained a matrix M such that $\{S_1,L\} = [L,M]$ [7)–10)].

It is not yet clear whether the elliptic Lax matrix can be used to obtain explicit solutions, as a generalization of Krichever's results in the nonrelativistic case [2)–9)]. However, in the hyperbolic case this problem has been solved. For instance, the solutions $q_1(t),\ldots,q_N(t)$ to Hamilton's equations for S_1 are equal to the logarithms of the eigenvalues of the positive matrix $e^{Q(0)/2}e^{tL(0)}e^{Q(0)/2}$, where

$$Q \equiv diag(q_1,\ldots,q_N). \tag{21}$$

This is proved in Ref. [1)] by exploiting Rayleigh-Schrödinger perturbation theory. This result generalizes the results of Olshanetski and Perelomov in the nonrelativistic case [8)].

More generally, the Lax matrix (17) can be used to construct the action-angle transformation. This transformation is crucial for making contact with soliton solutions. We proceed by describing it, without entering into the technicalities associated with its construction and with the proof of its canonicity.

4. THE HYPERBOLIC ACTION-ANGLE MAP

The key to the construction of the action-angle map is the commutation relation

$$\frac{1}{2}cth\frac{\mu}{2}[e^{Q},L] = e^{Q/2}d\otimes e^{Q/2}d-\frac{1}{2}(e^{Q}L+Le^{Q}) \tag{22}$$

which is readily verified, cf. (17-19), (21). From this one can infer that L has positive and simple spectrum for any (q,θ) in the phase space

$$\Omega \equiv \{(q,\theta)\in\mathbb{R}^{2N}|q_1<...<q_N\}, \tag{23}$$

and that there exists a uniquely determined unitary U satisfying

$$UL(q,\theta)U^{-1} = diag(e^{\hat{\theta}_1},...,e^{\hat{\theta}_N}) \tag{24}$$

$$Ue^{Q(q)}U^{-1} = \bar{L}(\hat{\theta},\hat{q}), \tag{25}$$

where $(\hat{q},\hat{\theta})$ belongs to the action-angle phase space

$$\hat{\Omega} \equiv \{(\hat{q},\hat{\theta})\in\mathbb{R}^{2N}|\hat{\theta}_1<...<\hat{\theta}_N\}. \tag{26}$$

(This generalizes results obtained for the nonrelativistic $1/q^2$-system by Airault, McKean and Moser [11)].)

It is a matter of simple linear algebra to show that one obtains a bijection $\Phi:\Omega\rightarrow\hat{\Omega}$ in this way, but to show that Φ is a symplectic diffeomorphism and hence may be regarded as the action-angle map involves a lot more work [3)]. Once this is proved, it is obvious from (24-25) that Φ linearizes the flows generated by Hamiltonians of the form

$$H_h = Tr\ h(lnL),\ \ h\in C^{\infty}_{\mathbb{R}}(\mathbb{R}). \tag{27}$$

Indeed, the transform of H_h to $\hat{\Omega}$ reads

$$(H_h\circ\Phi^{-1})(\hat{q},\hat{\theta}) \equiv \hat{H}_h(\hat{q},\hat{\theta}) = \sum_{i=1}^{N}h(\hat{\theta}_i), \tag{28}$$

so that the nonlinear flow

$$(q,\theta)\rightarrow(q(t),\theta(t)) \equiv \exp(tH_h)(q,\theta) \tag{29}$$

maps into the linear flow

$$(\hat{q},\hat{\theta})\rightarrow\exp(t\hat{H}_h)(\hat{q},\hat{\theta}) = (\hat{q}_1+th'(\hat{\theta}_1),...,\hat{q}_N+th'(\hat{\theta}_N),\hat{\theta}). \tag{30}$$

More generally, consider the multi-parameter flow

$$(q,\theta)\rightarrow(q(t_0,...,t_m),\theta(t_0,...,t_m)) \equiv exp(\sum_{k=0}^{m}t_kH_{h_k})(q,\theta). \tag{31}$$

Let us introduce

$$Q(t_0,...,t_m) \equiv diag(q_1(t_0,...,t_m),...,q_N(t_0,...,t_m)) \tag{32}$$

and

$$A(t_0,...,t_m) \equiv \bar{L}(\hat{\theta},\hat{q}_1 + \sum_{k=0}^{m} t_k h_k'(\hat{\theta}_1),...,\hat{q}_N + \sum_{k=0}^{m} t_k h_k'(\hat{\theta}_N)), \tag{33}$$

where $(\hat{q},\hat{\theta}) \equiv \Phi(q,\theta)$. Then it readily follows from the above that the matrices $A(\vec{t})$ and $\exp(Q(\vec{t}))$ are similar, so that

$$ln(\det(1+A(\vec{t}))) = \sum_{i=1}^{N} ln(1+\exp(q_i(\vec{t}))) \tag{34}$$

$$Tr(ArctgA(\vec{t})) = \sum_{i=1}^{N} Arctg(exp(q_i(\vec{t}))). \tag{35}$$

As will be explained below, for $m = 1$ and suitable μ these equalities can be used to arrive at a picture of N-soliton solutions as being deformations of an elastic medium, which hides an underlying motion of N point particles. The arbitrary m case encodes the relation of our systems to the commuting flows of various soliton hierarchies.

The scattering theory for the special Hamiltonian S_1 (for which $h(u)=e^u$) is a crucial ingredient in our proof that Φ is canonical. In fact, Φ^{-1} is very closely related to the $t\to\infty$ Møller transformation. For the case at hand there is only one scattering channel; physically speaking, the interparticle forces are repulsive, so that all particles move freely for asymptotic times. (Cf. also (42) below, with $h_0(u)=e^u$, $x\equiv 0$.

However, let us now make the substitutions $q_j \to q_j + i\pi$, $1 \leqslant n < j \leqslant N$, in the Hamiltonians $H,P,S_1,...,S_{N-1}$ of Section 2. Then we get again integrable N-particle systems, since the functional equations expressing integrability can be analytically continued, and since the resulting functions are real-valued (recall (15-16)). In the nonrelativistic case it is obvious that the substitutions lead to attraction between the particles $1,...,n$ (solitons) and $n+1,...,N$ (antisolitons), since (e.g.) the repulsive potential $1/sh^2\frac{1}{2}(q_1-q_N)$ turns into the attractive potential $-1/ch^2\frac{1}{2}(q_1-q_N)$ (as observed first by Calogero [12]).

It is not obvious, but true, that the substitutions have the same physical consequence for our more general systems. Correspondingly, the phase space of initial conditions splits up into a number of components which are invariant under S_1 and which correspond to various scattering channels, and an exceptional set which is nowhere dense and has measure zero. These channels correspond to asymptotics of freely evolving solitons, antisolitons and k soliton-antisoliton pairs (breathers); one has $k \in \{0,...,\min(n,N-n)\}$ and all possibilities occur. The exceptional set contains channels with many-body bound states, and also states that do not admit a scattering interpretation.

As suggested by our terminology, the systems just discussed can be used to describe solutions to soliton PDE with similar characteristics. For instance, the various particle-like solutions of the sine-Gordon equation can be obtained via (the generalization of) (35) with $m=1$ and $\mu=i\pi$.

Unfortunately, a more precise description of the latter relation cannot be given without detailing the action-angle map in the general case, something which is well beyond the scope of our survey. Here, we only mention that (the analytic continuation of) the commutation relation (22) is the starting-point in the general case, too. Again, it can be exploited to derive spectral properties of L. In contrast to the pure soliton case, these properties depend on the region of phase space involved, and this gives rise to the splitting mentioned above. For instance, the no-bound-state channel corresponds to L having simple and real spectrum, whereas the states without scattering interpretation correspond to points in phase space where L is not diagonalizable. For the full story we refer to our forthcoming paper [4].

5. THE RELATION TO SOLITON EQUATIONS

We shall begin by describing the relation of our N-particle systems to the KdV equation

$$\dot{u}-6uu'+u'''=0. \tag{36}$$

Set

$$\tau(\vec{t}) \equiv \det(1+A(\vec{t})), \tag{37}$$

where $A(\vec{t})$ is given by (33), and specialize to

$$\mu = i\pi,\ m=1,\ t_0=t,\ t_1=-x. \tag{38}$$

Now choose

$$h_0(u)=e^u,\ h_1(u)=\frac{1}{3}e^{3u}. \tag{39}$$

Then

$$u(t,x) \equiv -2\partial_x^2(ln\tau(t,x)) \tag{40}$$

is an N-soliton solution to (36). This can be seen from Hirota's pioneering paper [13] by invoking Cauchy's identity (20) [1]. Moreover, the multi-parameter KdV τ-function of the Kyoto school can be obtained analogously by taking $\mu = i\pi$ and suitable Hamiltonians $h_1,\ldots,h_m$ in (37), cf. pp. 960, 963 in the review [14].

More generally, picking

$$\mu_n = 2i\pi/n,\ \ n=2,3,4,\ldots \tag{41}$$

one obtains the τ-function of the $A_n^{(1)}$-reduction of the KP-hierarchy. In particular, the case $n=3$ corresponds to the Boussinesq hierarchy [14].

The N-soliton solutions of the Toda lattice can also be obtained via (37,38): One needs only take $x\in\mathbf{Z}$ and make another choice of h_0,h_1. Similarly, the solitons of the modified KdV and sine-Gordon equations can be written in terms of $A(t,x)$ by picking suitable h_0,h_1. For these cases the function $Tr\ ArctgA$ plays the role of $\ln(det(1+A))$. We refer to [1] for the details.

Relations between the above-mentioned soliton PDE and motions of point particles have been known for over a decade. The idea that there might be such connections was first put forward by Kruskal, who proposed to study the pole motions of soliton solutions [15]. Subsequent work in this area includes Refs. [16)−20]. We mention in particular the work by the Choodnovsky's [17], Airault, McKean and Moser [11], and Krichever [9], who discovered relations with the nonrelativistic CM systems.

It is likely that some of the latter results may be viewed as degenerate cases of the relations detailed in [1,3,4]. However, for function-theoretic reasons one cannot obtain the known results concerning elliptic solutions and the elliptic nonrelativistic CM systems from the hyperbolic context considered by us. Rather, the latter relations suggest that the action-angle map for the relativistic elliptic systems (which has not been constructed yet) can be used to parametrize the quasi-periodic solutions to the above soliton equations, as an analogue of the situation in the hyperbolic case. More generally, there are several hints which lead us to conjecture that the KP hierarchy can be tied in with integrable particle systems generalizing the ones at issue here.

6. SOLITON SPACE-TIME TRAJECTORIES

Let us now return to the context of N-soliton solutions in a two-dimensional space-time. As we have seen in the previous section, the r.h.s. of (34) and (35), with $m=1$, $t_0=t$, $t_1=-x$, gives rise to various well-known soliton functions, provided we choose appropriate μ and space-time translation generators $Tr\ h_\nu(lnL)$, $\nu = 0,1$. For the moment, let us only assume

$$\mu \in i(0,\pi],\ h_0''>0. \tag{42}$$

This assumption is satisfied in all cases of interest, up to flipping the sign of h_0, which amounts to time reversal.

The point of this assumption is, that it already suffices to determine the long-time asymptotics of the functions $q_i(t,x)$. In view of the fact that $A(t,x)$ has eigenvalues $exp(q_i(t,x))$, this boils down to a determination of spectral asymptotics. The result of this reads [3]

$$\underset{N-i+1}{q_i}(t,x) \sim \hat{q}_i + th_0'(\hat{\theta}_i) - xh_1'(\hat{\theta}_i) \mp \frac{1}{2}\Delta_i(\hat{\theta}),\ t\to\pm\infty \tag{43}$$

Here, the phase shift Δ_i is given by

$$\Delta_i = -\sum_{j<i}\delta(\hat{\theta}_i-\hat{\theta}_j) + \sum_{j>i}\delta(\hat{\theta}_i-\hat{\theta}_j) \tag{44}$$

and δ is the pair phase shift,

$$\delta(x) = ln(1-sh^2\frac{\mu}{2}/sh^2\frac{x}{2}). \tag{45}$$

This result amounts to a unified derivation of soliton scattering, and is crucial in justifying a notion of soliton space-time trajectory we are about to describe [1]. For concreteness we shall specialize from now on to the sine-Gordon equation

$$\phi''-\ddot{\phi} = \sin\phi. \tag{46}$$

Its N-soliton solutions can be written

$$\phi(t,x) = 4Tr(ArctgA(t,x)) = 4\sum_{i=1}^{N} Arctg(exp(q_i(t,x))), \tag{47}$$

provided one sets

$$\mu = i\pi,\ h_0(u) = chu,\ h_1(u) = shu. \tag{48}$$

This entails $H_{h_0} = H$, $H_{h_1} = P$, so that

$$\dot{q}_i = sh\theta_i\prod_{j\neq i}f(q_i-q_j) \tag{49}$$

$$q_i' = -ch\theta_i\prod_{j\neq i}f(q_i-q_j) \tag{50}$$

by virtue of Hamilton's equations.

Let us now fix t and require

$$q_i(t,x_i(t)) = 0,\ \ i = 1,\dots,N. \tag{51}$$

Since $q_i'<-1$, this yields uniquely determined space-time trajectories $x_1(t),\dots,x_N(t)$. It is not hard to see that these trajectories are Poincaré invariant, and (49-51) entail

$$\dot{x}_i(t) = th(\theta_i(t,x_i(t))). \tag{52}$$

Hence, the speeds $|\dot{x}_i|$ are always smaller than the speed of light. (Recall $c \equiv 1$.) Finally, for $|t| \to \infty$ the particle trajectories coincide with the N maxima of $\phi'(t,x)$, as a consequence of (43). We conclude with a picture which illustrates this state of affairs.

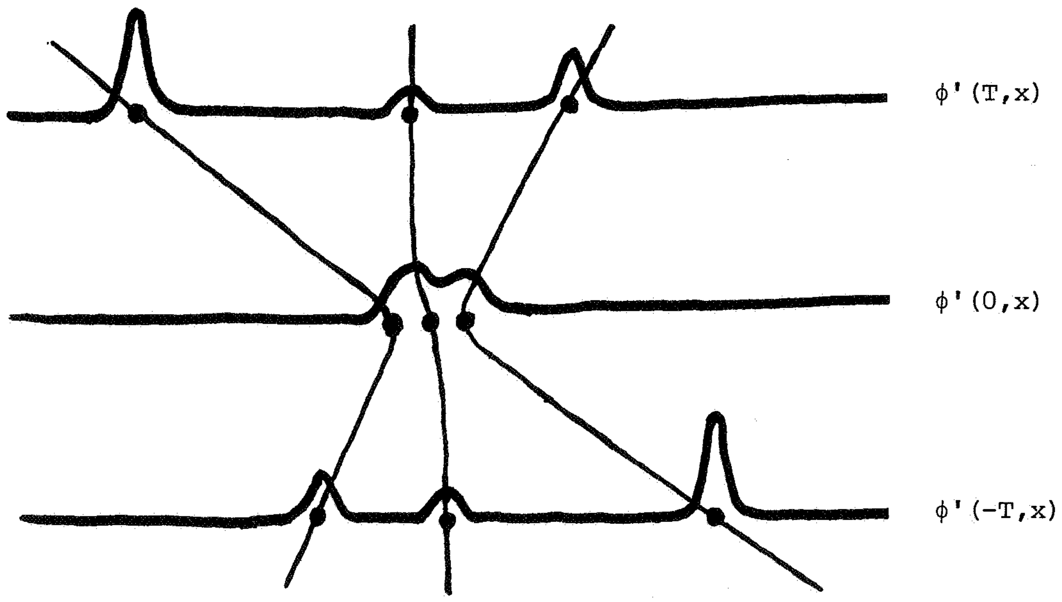

Fig. 1. Particle trajectories for a 3-soliton solution to the sine-Gordon equation $\phi'' - \ddot{\phi} = \sin\phi$.

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