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# Boundary Value Problems for Ordinary Differential Equations with Multiple Turning Points

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In this paper we consider the boundary value problems for ordinary differential equations with multiple turning points by means of the method of multiple scales. The uniformly valid asymptotic expansions of solutions have been constructed both for resonant case and non-resonant case under certain conditions. The results in (1)-(3) have been generalized.

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*Key Words & Phrases:* turning point problem, asymptotic expansion

## 1. Introduction

Since 1970, there have been appeared many papers on the behavior of solutions of boundary value problems with a simple turning point. There are also a few publications dealing with multiple turning points. In 1975, Matkowsky [1] considered the boundary value problem of the form:

$$\begin{cases} \epsilon y'' - x^3(x^2-1)(x-2)^2 y' = 0, & -2 \leqslant x \leqslant b,\ b>1, (b \neq 2), \\ y(-2) = A, y(b) = B \end{cases}$$

and obtained the leading terms of the expansion of its solution. In 1980, de Groen [2] considered the boundary value problem of the form:

$$\begin{cases} \epsilon y'' - p(x) y' = 0, & -1 \leqslant x \leqslant 1, \\ y(-1) = A,\ y(1) = B \end{cases}$$

where $p(x)$ has several zeros in the interior of the interval (-1,1), and pointed out that there are transition layers at the absolute maxima of a primitive of $p(x)$; the asymptotic solution is almost constant elsewhere.

In this paper, we generalize their results by means of the method of multiple scales which have proved to be effective in the study of boundary value problems with a simple turning point [3].

## 2. Asymptotic Expansion of Solution

Let us first find the expansion of solution in the general case.

Consider the boundary value problem:

$$L_2 y \equiv \epsilon y'' - f(x,\epsilon) y' + g(x,\epsilon) y = 0,\ (-a \leqslant x \leqslant b), \tag{2.1}$$

$$y(-a) = \alpha,\ \ y(b) = \beta, \tag{2.2}$$

where $a,b$ are positive constants; $f(x,\epsilon), g(x,\epsilon)$ have the asymptotic expansions:

$$f(x,\epsilon) \sim f_0(x) + \epsilon f_1(x) + \epsilon^2 f_2(x) + ...,$$

$$g(x,\epsilon) \sim g_0(x) + \epsilon g_1(x) + \epsilon^2 g_2(x) + ...,$$

$f_i(x), g_i(x), (i = 0,1,2,...)$ are infinitely differentiable on $[-a,b]$ and $f_0(x)$ has several zeros in $(-a,b)$.

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Introduction of two variables with different scales:

$$\xi = \frac{U(x)}{\epsilon} \quad \eta = x, \tag{2.3}$$

transforms equation (2.1) into a partial differential equation with variables $\xi,\eta$:

$$(K_0+\epsilon K_1+\epsilon^2 K_2+...)y = 0, \tag{2.4}$$

where

$$\begin{cases} K_0 \equiv U_x^2\frac{\partial^2}{\partial\xi^2}-U_x f_0(\eta)\frac{\partial}{\partial\xi}, \\ K_1 \equiv 2U_x\frac{\partial^2}{\partial\xi\partial\eta}+U_{xx}\frac{\partial}{\partial\xi}-U_x f_1(\eta)\frac{\partial}{\partial\xi}-f_0(\eta)\frac{\partial}{\partial\eta}+g_0(\eta), \\ K_2 \equiv -U_x f_2(\eta)\frac{\partial}{\partial\xi}+\frac{\partial^2}{\partial\eta^2}-f_1(\eta)\frac{\partial}{\partial\eta}+g_1(\eta), \\ K_i \equiv -U_x f_i(\eta)\frac{\partial}{\partial\xi}-f_{i-1}(\eta)\frac{\partial}{\partial\eta}+g_{i-1}(\eta),(i\geqslant 3). \end{cases} \tag{2.5}$$

Let the asymptotic expansion of the solution be

$$y = y_0(\xi,\eta)+\epsilon y_1(\xi,\eta)+\epsilon^2 y_2(\xi,\eta)+.... \tag{2.6}$$

Substituting (2.6) into (2.4) and equating the terms with identical powers of $\epsilon$, we obtain the recurrent system of differential equations for $y_n,(n = 0,1,2,...)$:

$$K_0 y_0 \equiv U_x^2\frac{\partial^2 y_0}{\partial\xi^2}-U_x f_0(\eta)\frac{\partial y_0}{\partial\xi} = 0, \tag{2.7}$$

$$K_0 y_1 = -K_1 y_0, \tag{2.8}$$

$$K_0 y_i = -(K_1 y_{i-1}+K_2 y_{i-2}+...+K_i y_0),(i\geqslant 2). \tag{2.9}$$

In order to get the simplest formulas to determine $y_i(i = 0,1,2,...)$, we choose

$$U(x) = -\int_{x_0}^{x} f_0(t)dt,$$

where $x_0$ is an undetermined point, then the equation for the leading term $y_0(\xi,\eta)$ reduces to

$$\frac{\partial^2 y_0}{\partial\xi^2}+\frac{\partial y_0}{\partial\xi} = 0. \tag{2.10}$$

We can easily find its general solution:

$$y_0(\xi,\eta) = A_0(\eta)+B_0(\eta)e^{-\xi} = A_0(x)+B_0(x)\exp(\frac{1}{\epsilon}\int_{x_0}^{x} f_0(t)dt), \tag{2.11}$$

where $A_0(x),B_0(x)$ are arbitrary functions which will be determined later on.

Substituting (2.11) into (2.8), and putting its right-hand side to zero, we obtain the differential equations for $A_0(\eta)$ and $B_0(\eta)$,

$$-f_0\frac{dA_0}{d\eta}+g_0 A_0 = 0, \tag{2.12}$$

$$-f_0\frac{dB_0}{d\eta}+(-f_0'+f_0 f_1-g_0)B_0 = 0. \tag{2.13}$$

Solving these equations we get

$$A_0(\eta) = C_0\exp(\int_{\eta_1}^{\eta}\frac{g_0}{f_0}dt), \tag{2.14}$$

$$B_0(\eta) = \frac{D_0}{f_0(\eta)}\exp[\int_{\eta_2}^{\eta}(f_1-\frac{g_0}{f_0})dt]. \tag{2.15}$$

where $C_0$ and $D_0$ are constants determined by the boundary conditions; the points $\eta_1$ and $\eta_2$ denote the end points of the interval, they will be chosen later in order to determine the constants $C_0$ and $D_0$ more easily. Thus from (2.11) the leading term of the expansion is completely determined,

$$y_0(\xi,\eta) = C_0\exp(\int_{\eta_1}^{\eta}\frac{g_0}{f_0}dt)+\frac{D_0}{f_0(\eta)}\exp[\int_{\eta_2}^{\eta}(f_1-\frac{g_0}{f_0})dt]e^{-\xi}$$

$$\equiv C_0\exp(\int_{x_1}^{x}\frac{g_0}{f_0}dt)+\frac{D_0}{f_0(x)}\exp[\int_{x_2}^{x}(f_1-\frac{q_0}{f_0})dt]\exp(\frac{-1}{\epsilon}\int_{x_0}^{x}-f_0dt). \tag{2.16}$$

It is valid away from the zeros of $f_0(x)$.

Substitution of $y_0(\xi,\eta)$ reduces equation (2.8) to

$$\frac{\partial^2 y_1}{\partial\xi^2}+\frac{\partial y_1}{\partial\xi} = 0 \tag{2.17}$$

which is the same form as (2.10), its general solution is

$$y_1(\xi,\eta) = A_1(\eta)+B_1(\eta)e^{-\xi}, \tag{2.18}$$

where $A_1(\eta)$ and $B_1(\eta)$ are arbitrary functions which will be determined in the following step.

Substituting (2.18) into Eq. (2.9) (with $i=2$), and putting its right-hand side to zero, we obtain the differential equations governing $A_1$ and $B_1$:

$$-f_0\frac{dA_1}{d\eta}+g_0A_1 = -(A''_0-f_1A'_0+g_1A_0)\equiv F_1, \tag{2.19}$$

$$-f_0\frac{dB_1}{d\eta}+(-f'_0+f_0f_1-g_0)B_1 = B''_0-f_1B'_0+(-f_0f_1+g_1)B_0\equiv G_1, \tag{2.20}$$

Solving these equations we get

$$A_1(\eta) = C_1\exp(\int_{\eta_1}^{\eta}\frac{g_0}{f_0}dt)+\int_{\eta_1}^{\eta}\frac{-F_1(s)}{f_0(s)}exp(\int_{\eta}^{s}\frac{-g_0}{f_0}dt)ds, \tag{2.21}$$

$$B_1(\eta) = \frac{D_1}{f_0(\eta)}\exp[\int_{\eta_2}^{\eta}(f_1-\frac{g_0}{f_0})dt]+\int_{\eta_2}^{\eta}\frac{-G_1(s)}{f_0(s)}\exp[-\int_{\eta}^{s}(f_1-\frac{g_0}{f_0})dt]ds, \tag{2.22}$$

where $C_1$ and $D_1$ are constants determined by the boundary conditions. Substituting $A_1(\eta),B_1(\eta)$ into (2.18) yields the second term of the expansion. Proceeding in a similar way, we can obtain the expansion of the solution of boundary value problem (2.1)-(2.2) up to any order,

$$y_{as}^{(n)} = y_0(\xi,\eta)+\epsilon y_1(\xi,\eta)+...+\epsilon^n y_n(\xi,\eta), \tag{2.23}$$

where $\xi$ and $\eta$ are the variables of multiple scales defined by (2.3), which is also valid away from the zeros of $f_0(x)$.

From the terms of the expansion we can see that the asymptotic behavior of the solution mainly depends on the sign of the integrals

$$I(x;-a) = \int_{-a}^{x}-f_0dt, \text{ and } I(x;b) = \int_{b}^{x}-f_0dt. \tag{2.24}$$

If $I(x;-a)$ (or $I(x;b)$) is negative in the neighborhood of $x = -a$ (or $x=b$), the boundary layer does not exist at $x = -a$ (or $x=b$).

If on the contary, $I(x;-a)$ (or $I(x;b)$) is positive in the neighborhood of $x = -a$ (or $x=b$), the boundary layer generally appears at $x = -a$ (or $x=b$).

EXAMPLE 1. Consider the asymptotic behavior of the solution of the boundary value problem:

$$\begin{cases} \epsilon y'' + xy' + 2y = 0, \ (-a \leqslant x \leqslant b) \\ y(-a) = \alpha, y(b) = \beta, \end{cases}$$

where $a$ and $b$ are positive constants.

Since integrals $I(x;-a)$ and $I(x;b)$ are negative in the neighborhood of $x = -a$ and $x = b$ respectively, there is no asymptotic expansion of the form (2.6) for this boundary value problem. Moreover, it has been shown by O'MALLEY [4] that its solution is exponentially large in the interior of $(-a,b)$ as $\epsilon \to 0$.

EXAMPLE 2. Consider the asymptotic behavior of the boundary value problem:

$$\begin{cases} \epsilon y'' - xA(x,\epsilon)y' + B(x,\epsilon)y = 0, \ (-a \leqslant x \leqslant b), \\ y(-a) = \alpha, y(b) = \beta, \end{cases}$$

where $A(x,\epsilon)$ and $B(x,\epsilon)$ have the asymptotic expansions:

$$A(x,\epsilon) \sim A_0(x) + \epsilon A_1(x) + \epsilon^2 A_2(x) + \ldots,$$

$$B(x,\epsilon) \sim B_0(x) + \epsilon B_1(x) + \epsilon^2 B_2(x) + \ldots,$$

and $A_0(x) \geqslant \delta > 0$ in $(-a,b)$.

This problem has been considered by author in [3]. We point out here that, in the resonant case, the presence of boundary layers at the end points depends on the value of $b$. If $a>b$, then $I(x;-a)>0$ in whole interval $(-a,b)$, and the boundary layer generally exists at $x = -a$. If $a < b$, than $I(x;b)>0$ in whole interval $(-a,b)$ and the boundary layer generally exists at $x = b$. Finally, if $a = b$, then $I(b;a)>0$, or $I(0;-a) = I(0;b)>0$, the boundary layers generally exist at both end points $x = -a, x = b$. In the non-resonant case, the boundary layers generally exist at both end points too, since $I(x;-a)>0$ in the neighborhood of $x = -a$, and $I(x;b)>0$ in the neighborhood of $x = b$ whether $a > b$ or $a < b$.

## 3. Boundary Value Problems with Multiple Turning Points

From the above results we can easily obtain the asymptotic expansion of solution of boundary value problems with multiple turning points.

For the boundary value problem given by (2.1)-(2.2) two cases may arise: resonance and non-resonance. From [5] we know that that the resonant case is exceptional. If, however, $g(x,\epsilon) \equiv 0$, resonance generally occurs. The necessary conditions for resonance can be found by the process given in [5].

### *3.1. Non-resonant case*

In this case, the limit solution is trivial, and generally there are boundary layers at both end points. So, we must suppose that the following conditions are satisfied,

$$\begin{cases} I(x;-a) = \int_{-a}^{x} -f_0(t)dt > 0, \text{ in the neighborhood of } x = -a, \\ I(x;b) = \int_{b}^{x} -f_0(t)dt > 0, \text{ in the neighborhood of } x = b. \end{cases} \tag{3.1}$$

Under conditions in (3.1), from (2.16) we know that the leading term of the expansion of solution of boudary value problem (2.1)-(2.2) is

$$y_0 = \begin{cases} \Psi_0^{(1)}(x)\exp(\frac{-1}{\epsilon}\int_{-a}^{x} -f_0 dt), & \text{for } -a \leqslant x \leqslant 0, \\ \Psi_0^{(2)}(x)\exp(\frac{-1}{\epsilon}\int_{b}^{x} -f_0 dt), & \text{for } 0 \leqslant x \leqslant b, \end{cases} \tag{3.2}$$

where

$$\Psi_0^{(1)}(x) = \begin{cases} \frac{D_0^{(1)}}{f_0(x)} exp[\int_{-a}^{x}(f_1 - \frac{g_0}{f_0})dt], & \text{for } -a \leqslant x < -a + \frac{\eta}{2} \\ \phi_0^{(1)}(x), & \text{for } -a + \frac{\eta}{2} \leqslant x < -a + \eta, \\ 0, & \text{for } -a + \eta \leqslant x \leqslant 0. \end{cases}$$

$$\Psi_0^{(2)}(x) = \begin{cases} 0, & \text{for } 0 \leqslant x < b - \eta \\ \phi_0^{(2)}(x), & \text{for } b - \eta \leqslant x < b - \frac{\eta}{2} \\ \frac{D_0^{(2)}}{f_0(x)} \exp[\int_{b}^{x}(f_1 - \frac{g_0}{f_0})dt] & \\ & \text{for } b - \frac{\eta}{2} \leqslant x \leqslant b \end{cases}$$

It is supposed that $\eta$ is a positive number such that no zeros of $f_0(x)$ in $\eta$-neighborhoods of $x = -a$ and $x = b$ exist, and that

$$\frac{1}{f_0(x)}\exp(\frac{-1}{\epsilon}\int_{-a}^{x} -f_0 dt) = O(\epsilon^N), \text{ for } -a + \frac{\eta}{2} \leqslant x \leqslant -a + \eta,$$

$$\frac{1}{f_0(x)}\exp(\frac{-1}{\epsilon}\int_{b}^{x} -f_0 dt) = O(\epsilon^N), \text{ for } b - \eta \leqslant x \leqslant b - \frac{\eta}{2},$$

where $N$ is a positive integer depending on the required accuracy. The functions $\phi_0^{(1)}, \phi_0^{(2)}$ are smooth connecting functions, and the constants $D_0^{(1)}, D_0^{(2)}$ are determined by boundary conditions. In this problem,

$$D_0^{(1)} = \alpha f_0(-a), \; D_0^{(2)} = \beta f_0(b). \tag{3.3}$$

Moreover, from (2.17)-(2.19) we obtain the second term of the expansion of its solution,

$$y_1 = \begin{cases} \Psi_1^{(1)}(x)\exp(\frac{-1}{\epsilon}\int_{-a}^{x} -f_0 dt), & \text{for } -a \leqslant x \leqslant 0, \\ \Psi_1^{(2)}(x)\exp(\frac{-1}{\epsilon}\int_{b}^{x} -f_0 dt), & \text{for } 0 \leqslant x \leqslant b, \end{cases} \tag{3.4}$$

where

$$\Psi_1^{(1)}(x) = \begin{cases} \int\limits_{-a}^{x} \frac{-G_1(s)}{f_0(s)} \exp[-\int\limits_{x}^{s} (f_1 - \frac{g_0}{f_0}) dt] ds, & \text{for } -a \leqslant x < -a + \frac{\eta}{2} \\ \phi_1^{(1)}(x), & \text{for } -a + \frac{\eta}{2} \leqslant x < -a + \eta \\ 0, & \text{for } -a + \eta \leqslant x \leqslant 0, \end{cases}$$

$$\Psi_1^{(2)}(x) = \begin{cases} 0, & \text{for } 0 \leqslant x < b - \eta, \\ \phi_1^{(2)}(x), & \text{for } b - \eta \leqslant x < b - \frac{\eta}{2}, \\ \int\limits_{b}^{x} \frac{-G_1(s)}{f_0(s)} \exp[-\int\limits_{x}^{s} (f_1 - \frac{g_0}{f_0}) dt] ds, & \text{for } b - \frac{\eta}{2} \leqslant x \leqslant b, \end{cases}$$

with $G_1(x)$ defined by (2.19).

It is easily shown that the asymptotic expansion of its solution constructed above

$$y_{as}^{(1)} = y_0 + \epsilon y_1, \tag{3.5}$$

satisfies the equation (2.1) up to order $O(\epsilon)$, because in the intervals $(-a+\frac{\eta}{2}, -a+\eta)$ and $(b-\eta, b-\frac{\eta}{2})$ the derivatives of $\phi_i^{(1)}\exp(-\frac{1}{\epsilon}\int\limits_{-a}^{x} -f_0 dt)$ and $\phi_i^{(2)}\exp(\frac{-1}{\epsilon}\int\limits_{b}^{x} -f_0 dt)$ are of order $O(\epsilon^N)$.

Example 3. Consider the boundary value problem with a single multiple turning point $x = 0$,

$$\begin{cases} \epsilon y'' - x^{2n+1} A(x,\epsilon) y' + B(x,\epsilon) y = 0, \ (-a \leqslant x \leqslant b, n \neq 0) \\ y(-a) = \alpha, \ y(b) = \beta, \end{cases} \tag{3.6}$$

where $A(x,\epsilon), B(x,\epsilon)$ have the asymptotic expansions (2.26), (2.27) respectively, and

$$A_0(x) \geqslant \delta > 0, \ B_0(x,\epsilon) \not\equiv 0.$$

We can easily show that this boundary value problem is of non-resonant type, moreover, the conditions in (3.1) are satisfied. So, its solution has the expansion given by (3.2)-(3.5) with

$$f_0(x) = -x^{2n+1} A_0(x).$$

Example 4. Consider the boundary value problem

$$\begin{cases} \epsilon y'' - x^2(x^2-1)(x-2)^2 y' + By = 0, \ (-2 \leqslant x \leqslant b, b > 2) \\ y(-2) = \alpha, \ y(b) = \beta \end{cases} \tag{3.7}$$

where $B$ is a constant and $B \neq 0$. The case $B = 0$ had been considered by Matkowsky [1].

We also can easily show that it is of non-resonant type too. Besides, in the neighborhood of $x = -2$ and $x = b$ we have

$$I(x;-2) = \frac{1}{280}[\rho(-2) - \rho(x)] > 0, \ I(x;b) = \frac{1}{280}[\rho(b) - \rho(x)] > 0$$

respectively, where

$$\rho(x) = 35x^8 - 160x^7 + 140x^6 + 224x^5 - 280x^4.$$

So, its solution also has an expansion given by (3.2)-(3.5) with

$$f_0(x) = x^2(x^2-1)(x-2)^2.$$

*3.2. Resonant case*

As we have pointed out in [5], this case is exceptional; but if $g(x,\epsilon) \equiv 0$, resonance generally occurs. In the following, we only consider the boundary value problem in case $g(x,\epsilon) \equiv 0$. Let it be

$$\begin{cases} \epsilon y'' - f(x,\epsilon)y' = 0, \ (-a \leqslant x \leqslant b) \\ y(-a) = \alpha, \ y(b) = \beta \end{cases} \tag{3.8}$$

where $f(x,\epsilon)$ has the asymptotic expansion:

$$f(x,\epsilon) \sim f_0(x) + \epsilon f_1(x) + \epsilon^2 f_2(x) + \ldots,$$

and $f_0(x)$ has several zeros in $(-a,b)$.

Suppose that one of the following conditions is satisfied,

Condition 1. $I(x;-a) = \int_{-a}^{x} -f_0 dt > 0$, for $-a < x < b$. (3.9)

Condition 2. $I(x;b) = \int_{b}^{x} -f_0 dt > 0$, for $-a < x < b$. (3.10)

Condition 3. $\begin{cases} I(x;-a) > 0, \text{ in the neighborhood of } x = -a. \\ I(x;b) > 0, \text{ in the neighborhood of } x = b. \end{cases}$ (3.11)

If Condition 1 is satisfied, we know from (2.16) that the leading term of the expansion of solution is

$$y_0 = \beta + \Psi_0^{(1)}(x) \exp\left(\frac{-1}{\epsilon} \int_{-a}^{x} -f_0 dt\right), \ (-a \leqslant x \leqslant b), \tag{3.12}$$

where

$$\Psi_0^{(1)}(x) = \begin{cases} \dfrac{D_0^{(1)}}{f_0(x)} \exp\left(\int_{-a}^{x} f_1 dt\right), & \text{for } -a \leqslant x < -a + \dfrac{\eta}{2}, \\ \phi_0^{(1)}, & \text{for } -a + \dfrac{\eta}{2} \leqslant x < -a + \eta, \\ 0, & \text{for } -a + \eta \leqslant x \leqslant b, \end{cases} \tag{3.13}$$

with $D_0^{(1)}$ a constant determined by the boundary condition at $x = -a$:

$$D_0^{(1)} = (\alpha - \beta) f_0(-a). \tag{3.14}$$

If Condition 2 is satisfied, we have from (2.16)

$$y_0 = \alpha + \Psi_0^{(2)}(x) \exp\left(\frac{-1}{\epsilon} \int_{b}^{x} -f_0 dt\right), \ (-a \leqslant x \leqslant b), \tag{3.15}$$

where

$$\Psi_0^{(2)}(x) = \begin{cases} 0, & \text{for } -a \leqslant x < b-\eta, \\ \phi_0^{(2)}(x), & \text{for } b-\eta \leqslant x < b-\frac{\eta}{2}, \\ \frac{D_0^{(2)}}{f_0(x)} \exp(\int_b^x f_1 dt), & \text{for } b-\frac{\eta}{2} \leqslant x \leqslant b, \end{cases} \tag{3.16}$$

and

$$D_0^{(2)} = (\beta-\alpha) f_0(b). \tag{3.17}$$

If Condition 3 is satisfied, we have from (2.16)

$$y_0 = \begin{cases} C_0 + \Psi_0^{(1)}(x) \exp(\frac{-1}{\epsilon} \int_{-a}^x -f_0 dt), & \text{for } -a \leqslant x \leqslant 0, \\ C_0 + \Psi_0^{(2)}(x) \exp(\frac{-1}{\epsilon} \int_b^x -f_0 dt), & \text{for } 0 \leqslant x \leqslant b, \end{cases} \tag{3.18}$$

where

$$\Psi_0^{(1)}(x) = \begin{cases} \frac{D_0}{f_0(x)} \exp(\int_{-a}^x f_1 dt), & \text{for } -a \leqslant x \leqslant -a+\frac{\eta}{2} \\ \phi_0^{(1)}, & \text{for } -a+\frac{\eta}{2} \leqslant x < -a+\eta \\ 0, & \text{for } -a+\eta \leqslant x \leqslant 0 \end{cases} \tag{3.19}$$

$$\Psi_0^{(2)}(x) = \begin{cases} 0, & \text{for } 0 \leqslant x < b-\eta, \\ \phi_0^{(2)}(x), & \text{for } b-\eta \leqslant x < b-\frac{\eta}{2}, \\ \frac{D_0}{f_0(x)} \exp(\int_b^x f_1 dt), & \text{for } b-\frac{\eta}{2} \leqslant x \leqslant b, \end{cases} \tag{3.20}$$

where $C_0$ and $D_0$ are constants determined by the boundary conditions. If $f_0(-a) \neq f_0(b)$, we have

$$C_0 = \frac{\alpha f_0(-a) - \beta f_0(b)}{f_0(-a) - f_0(b)}, \quad D_0 = \frac{(\beta-\alpha) f_0(-a) f_0(b)}{f_0(-a) - f_0(b)}. \tag{3.21}$$

If $f_0(-a) = f_0(b)$, the boundary value problem reduces to the non-resonant case, and the leading term of the expansion of its solution will be given by (3.2)-(3.3) with $f_1 = 0$, $g_0 = 0$.

EXAMPLE 5. Consider the boundary value problem:

$$\begin{cases} \epsilon y'' - x^3(x^2-1)(x-2)^2 y' = 0, \\ y(-2) = \alpha, \; y(b) = \beta \end{cases} \quad (-2 \leqslant x \leqslant b, b > 1, b \neq 2),$$

which had been considered by MATKOWSKY [1].

We can easily show that for this boundary value problem resonance occurs, and

$$I(x;-2) = \int_{-2}^{x} -t^3(t^2-1)(t-2)^2 dt = \frac{1}{280}[\rho(-2)-\rho(x)]>0$$

in the neighborhood of $x = -2$, where $\rho(x)$ has been found in Example 5.

Let $b_0$ be the point where $I(x;-2)$ changes sign. If $b<b_0$, then

$$I(x;-2)>0, \text{ for } -2<x<b$$

and Condition 1 is satisfied. The leading term of the expansion is given by (3.12)-(3.14),

$$y_0 = \beta+\Psi_0^{(1)}(x)\exp\{\frac{-1}{280\epsilon}[\rho(-2)-\rho(x)]\},$$

where

$$\Psi_0^{(1)}(x) = \begin{cases} \dfrac{(\alpha-\beta)f_0(-2)}{f_0(x)}, & \text{for } -2\leqslant x<-2+\dfrac{\eta}{2}, \\ \phi_0^{(1)}(x), & \text{for } -2+\dfrac{\eta}{2}\leqslant x<-2+\eta, \\ 0, & \text{for } -2+\eta\leqslant x\leqslant b, \end{cases}$$

If $b>b_0$, then

$$I(x;b) = \int_{b}^{x} -f_0 dt>0, \text{ for } -2<x<b.$$

Thus Condition 2 is satisfied, and we have from (3.15)-(3.17)

$$y_0 = \alpha+\Psi_0^{(2)}(x)\exp\{\frac{-1}{280\epsilon}[\rho(b)-\rho(x)]\},$$

where

$$\Psi_0^{(2)}(x) = \begin{cases} 0, & \text{for } -2\leqslant x<b-\eta, \\ \phi_0^{(2)}(x), & \text{for } b-\eta\leqslant x<b-\dfrac{\eta}{2}. \\ \dfrac{(\beta-\alpha)f_0(b)}{f_0(x)}, & \text{for } b-\dfrac{\eta}{2}\leqslant x\leqslant b, \end{cases}$$

If $b = b_0$, then

$$I(x;-2) = I(x;b)>0.$$

Again Condition 3 is satisfied, and we have from (3.18)-(3.21)

$$y_0 = \begin{cases} C_0+\Psi_0^{(1)}(x)\exp\{\dfrac{-1}{280\epsilon}[\rho(-2)-\rho(x)]\}, & \text{for } -2\leqslant x\leqslant 0, \\ C_0+\Psi_0^{(2)}(x)\exp\{\dfrac{-1}{280\epsilon}[\rho(-2)-\rho(x)]\}, & \text{for } 0\leqslant x\leqslant b, \end{cases}$$

where

$$\Psi_0^{(1)}(x) = \begin{cases} \dfrac{D_0}{f_0(x)}, & \text{for } -2\leqslant x\leqslant -2+\dfrac{\eta}{2}, \\ \phi_0^{(1)}, & \text{for } -2+\dfrac{\eta}{2}\leqslant x<-2+\eta, \\ 0, & \text{for } -2+\eta\leqslant x\leqslant 0, \end{cases}$$

$$\Psi_0^{(2)}(x) = \begin{cases} 0, & \text{for } 0 \leqslant x < b-\eta \\ \phi_0^{(2)}(x), & \text{for } b-\eta \leqslant x < b-\frac{\eta}{2}, \\ \frac{D_0}{f_0(x)}, & \text{for } b-\frac{\eta}{2} \leqslant x \leqslant b. \end{cases}$$

Since $f_0(-2) \neq f_0(b)$, so that we have from (3.21)

$$C_0 = \frac{\alpha f_0(-2) - \beta f_0(b)}{f_0(-2) - f_0(b)}, \quad D_0 = (\beta - \alpha) \frac{f_0(b) f_0(-2)}{f_0(-2) - f_0(b)}.$$

This is in agreement with the results derived by MATKOWSKY [1].

From the above examples we see that the conclusion given by DE GROEN [2] that "there are transition layers at the absolute maxima of a primitive of $p(x) = f_0(x)$" is not always true if only one of the Conditions 1-3 mentioned above is satisfied. If none of the three Conditions is satisfied, the asymptotic behavior of the solution of boundary value problem (2.1)-(2.2) is rather complicated, it may be exponentially large within the interval, or it approaches a bounded limit, which has been demonstrated by O'MALLEY [4] for the case of a single simple turning point.

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