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Adaptive exponential stabilization of a first order continuous-time system

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Adaptive Exponential Stabilization

of a First Order Continuous-time System

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An algorithm for adaptive pole-placement for the class of first order continuous-time systems is proposed. The asymptotic properties of the algorithm do not depend on the presence of persistently exciting signals. Excitation is used only initially to avoid that parameter estimates become non-controllable. The main result is that the asymptotic behaviour of the system equals the behaviour one would have obtained on the basis of the true system. Identification of the system-parameters however, does not necessarily take place. The reason that such a result can be obtained without identification of the true system is the following: Suppose we have a wrong estimate of the system, and that based on that estimate we generate the controls, and that the incorrectness of the estimate is not revealed by the resulting closed-loop behaviour of the system, then the inputs are exactly equal to the ones we would have applied if we had known the system.

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1. Introduction

This note deals with the stabilization of a first order continuous-time deterministic system of which the parameters are completely unknown. The problem of adaptive stabilization has received considerables attention in the last few years, see [1,3,4,5,6,7] The final answer, at least at a theoretical level, is provided by [1] and [3]. In [3] it is proved that the order of a dynamic stabilizing compensator is sufficient information about a plant to be able to stabilize it, and in [1] it is stated that this information is also necessary.

The problem we want to treat here, is not only stabilizing the system, but stabilizing it with a prescribed rate of stability. We will restrict attention to the first order case. The proposed algorithm is the continuous-time version of the algorithm presented in [6] for discrete-time systems. It is based on closed-loop identification of the system parameters and the certainty-equivalence principle. Hitting non-controllable pairs is avoided by applying special (large) inputs if the estimate comes too close to non-controllable. Our main result is that the adaptively controlled systems behaves asymptotically the same as if the parameters were known. This result is obtained by proving that the parameter estimates converge, in general not to the true system parameters, to a pair that gives rise to same control-law as the desired one. This implies that asymptotically the applied inputs are as desired.

The paper is organized as follows. First we will give the class of systems and the problem formulation. Then we will describe the algorithm and its geometrical and asymptotic properties. This will lead to the proof of the forementioned claim. We will then comment upon the implementability of the algorithm and finally we will draw some conclusions.

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2. The algorithm

We will first give the system description and the problem statement. Then we will give the algorithm in a form suitable for analysis and relatively easy to understand. In the next section we show that the algorithm is implementable.
Let the true system be given by:

$$\dot{x} = a_0 x + b_0 u, \ x_0, \tag{2.1}$$

where (a_0,b_0) is fixed but unknown. The only assumption we make is that $b_0 \neq 0$. Choose any $\alpha \in \mathbb{R}$, and let the desired closed-loop behaviour be:

$$\dot{x} = \alpha x. \tag{2.2}$$

The unique feedback-law that achieves the control objective is:

$$u(t) = f(a_0,b_0)x(t) \tag{2.3}$$

where f is defined by:

$$f(a,b) = \frac{\alpha - a}{b} \tag{2.4}$$

Our algorithm will be based on closed-loop identification of (a_0,b_0) and the certainty-equivalence principle. On the basis of the available data an estimation $(a(t),b(t))$ is made of (a_0,b_0), and then we take $u(t) = f(a(t),b(t))x(t)$. This can of course only be done when $b(t) \neq 0$. Therefore our algorithm consists of two parts. Roughly speaking it works as follows. At every time instant t a check is made whether $b(t)$ is not too close to zero. If not we calculate the input according to certainty-equivalence. In the other case we apply a special input for a certain time period to bias $b(t)$ away from zero. The crucial observation is that within finite time $b(t)$ will be bounded away from zero and hence within finite time we will use certainty-equivalence for ever.

The algorithm

Choose sequences $\{\epsilon_k\}_{k \in \mathbb{N}}$ and $\{C_k\}_{k \in \mathbb{N}}$ such that:

$$\epsilon_k \downarrow 0 \text{ and } C_k \uparrow \infty \tag{2.5}$$

Choose $(a(0),b(0))$ arbitrarily and $j = 0$, $z(0) = 0$. Define the functions $a(t)$, $b(t)$, $z(t)$, $u(t)$ and the sequence τ_k by the following conditional differential equations:

$$\dot{a} = \frac{x}{x^2+u^2}(\dot{x} - ax - bu) \tag{2.6.a}$$

$$\dot{b} = \frac{u}{x^2+u^2}(\dot{x} - ax - bu) \tag{2.6.b}$$

$$\dot{z} = -I_{\{z>0\}} \tag{2.6.c}$$

if $z(t) > 0$ then:

$$u(t) = C_j x(t) \tag{2.6.1}$$

else:

if $|b(t)| \geqslant \epsilon_j$ then:

$$u(t) = f(a(t),b(t))x(t) \tag{2.6.2.a}$$

else (if $|b(t)| < \epsilon_j$)

$$j := j+1 \tag{2.6.2.b}$$

$$z(t) := j \tag{2.6.2.c}$$

$$\tau_j := t \tag{2.6.2.d}$$

$$u(t) := C_j x(t) \tag{2.6.2.e}$$

COMMENT The equations for a and b are the continuous versions of the projection algorithm for discrete time ([2] and [6]). They can be derived from the discrete time equations by using infinitesimal arguments. As in [6], the sequence $\{\tau_k\}_{k\in\mathbb{N}}$ can be seen as a sequence of stopping times. As soon as $t = \infty$ it is easy to see that:

$$\tau_1 = \inf\{t \mid |b(t)| \leqslant \epsilon_1\} \tag{2.7.a}$$

$$\tau_{k+1} = \inf\{t \geqslant \tau_k + k \mid |b(t)| \leqslant \epsilon_{k+1}\} \tag{2.7.b}$$

The infimum is understood to be infinity if the set over which it is taken, is empty. Define:

$$I_k := [\tau_k, \tau_k + k) \tag{2.8}$$

Outside I_k, $u(t)$ is calculated according to certainty-equivalence. For $t \in I_k$ we take: $u(t) = C_k x(t)$. Before we analyze the algorithm formally, let us try to explain intuitively that $\tau_k = \infty$ for some finite k. Suppose $b_0 > 0$ and $b(0) < 0$. Since $b(t)$ tries to estimate b_0, it can be expected that $b(t)$ has to pass through the set $\{b = 0\}$. If $b(t) = 0$, we cannot calculate $f(a(t),b(t))$. Now as soon as $b(t)$ comes too close to zero (measured by the sequence $\{\epsilon_k\}$), we start to apply special inputs: from time τ_k to time $\tau_k + k$ we take $u(t) = C_k x(t)$. Assume for the moment that this alternative procedure has to be started infinitely often (i.e. for every k: $\tau_k < \infty$). Then due to the growing inputs and the increasing time interval during which they are applied, it can be proven that $b(\tau_k + k)$ will converge to b_0. Then, using the geometrical properties of the trajectory of $(a(t),b(t))$, we prove that since by assumption $|b_0| > 0$, $|b(t)|$ will eventually be bounded from below by some $\epsilon > 0$. This implies that, since ϵ_k tends to zero, the alternative procedure will not be started again as soon as $\epsilon_k < \epsilon$. This contradicts the assumption, hence after some finite time instant t_0 we will always apply $u(t) = f(a(t),b(t))x(t)$. The idea of using growing inputs when things become dangerous was first developed in [6]. The formal proof of the above reasoning will be given in Lemma 2.3.
Define:

$$g(a(t),b(t)) = C_k \text{ if } t \in I_k \tag{2.9.a}$$

$$= f(a(t),b(t)) \text{ otherwise} \tag{2.9.b}$$

then: for all t:

$$u(t) = g(a(t),b(t))x(t) \tag{2.10}$$

LEMMA 2.1 $(a_0 - a(t))^2 + (b_0 - b(t))^2$ is non-increasing.

PROOF Define $V(a,b) := (a_0 - a)^2 + (b_0 - b)^2$, then along trajectories of $(a(t),b(t))$, it follows from (2.1), (2.6.a), (2.6.b) and (2.10) that:

$$\frac{1}{2}\dot{V}(a,b) = \frac{-1}{g^2+1}(a_0 - a + (b_0 - b)g)^2 \leqslant 0 \tag{2.11}$$

From Lemma 2.1 we conclude that $(a(t),b(t))$ converges to a circle with radius $R \geqslant 0$ for some R and centre (a_0,b_0). For $t \in [\tau_k + k, \tau_{k+1})$, the trajectory of (a(t),b(t)) has the following interesting geometrical property:

LEMMA 2.2 For all k and for all $t \in [\tau_k + k, \tau_{k+1})$:

$$(a(t) - \alpha)^2 + b(t)^2 = (a(\tau_k + k))^2 + (b(\tau_k + k))^2 =: r_k^2 \tag{2.12}$$

Proof Define $W(a,b) := (a-\alpha)^2 + b^2$. Along trajectories of $(a(t),b(t))$ for $t\in[\tau_k+k,\tau_{k+1})$ we have:

$$\frac{1}{2}\dot{W}(a,b) = (a-\alpha)\dot{a} + b\dot{b} \tag{2.13}$$

$$= \left(\frac{a-\alpha}{1+f(a,b)^2} + \frac{bf(a,b)}{1+f(a,b)^2}\right)\left(\frac{\dot{x}}{x} - a - bf(a,b)\right) = 0 \tag{2.14}$$

The statement follows.

Comment Lemma 2.1 and 2.2 imply that for $t\in[\tau_k+k,\tau_{k+1})$, $(a(t),b(t))$ moves on a circle S_k with radius r_k and centre $(\alpha,0)$, in such a way that the distance between $(a(t),b(t))$ and (a_0,b_0) is decreasing. Define l_0 as the line passing through $(\alpha,0)$ and (a_0,b_0), and denote by $(\overline{a}_k,\overline{b}_k)$ the point of intersection of S_k and l_0, which is closest to (a_0,b_0). Then $(a(t),b(t))$ moves along S_k into the direction of $(\overline{a}_k,\overline{b}_k)$. Since $(a(t),b(t))$ cannot leave this point along S_k without increasing the distance to (a_0,b_0), it follows that if $(a(t),b(t))$ reaches $(\overline{a}_k,\overline{b}_k)$ before τ_{k+1}, it will stay there for ever. In other words it will then converge to $(\overline{a}_k,\overline{b}_k)$ and as a consequence τ_{k+1} will be infinity. In the next lemma we will prove that $(a(t),b(t))$ will indeed converge to $(\overline{a}_k,\overline{b}_k)$, for some k.

Lemma 2.3 $\{\tau_k \mid \tau_k < \infty\}$ is a finite set.

Proof The proof will rely heavily on the geometrical properties of the trajectory of $(a(t),b(t))$. See figure 1 for a pictorial explanation. Now suppose the claim is not true. From (2.6.b) and (2.6.1) it follows that on I_k we have:

$$\dot{b} = \frac{C_k}{1+C_k^2}(a_0 - a + (b_0-b)C_k) \tag{2.15}$$

From Lemma 2.1 we know that $\{a(t)\}$ is bounded and hence, since C_k tends to infinity and since also the length of I_k tends to infinity, we conclude that:

$$\lim_{k\to\infty} b(\tau_k+k) = b_0 \tag{2.16}$$

Suppose that:

$$\lim_{k\to\infty} (a_0-a(t))^2+(b_0-b(t))^2 = R^2 > 0 \tag{2.17}$$

then by (2.16):

$$\lim_{k\to\infty} (a_0-a(\tau_k+k))^2 = R^2. \tag{2.18}$$

Hence at least one of the points $a_0\pm R$ is a limit point of $a(\tau_k+k)$. Assume without loss of generality:

$$\lim_{k\to\infty} a(\tau_k+k) = a_0-R \tag{2.19}$$

For $\tau_k+k \leqslant t < \tau_{k+1}$, $(a(t),b(t))$ moves along S_k , hence by (2.16) and (2.18) we conclude that:

$$\lim_{k\to\infty} r_k = [b_0^2+(\alpha-a_0+R)^2]^{\frac{1}{2}} =: r \tag{2.20}$$

In other words, the sequence of circles S_k converges to a circle S with center $(\alpha,0)$ and radius r. Define $(\overline{a},\overline{b})$ as the point of intersection of S and l_0 which is closest to (a_0,b_0), then:

$$\lim_{k\to\infty} \overline{a}_k = \overline{a} \text{ and } \lim_{k\to\infty} \overline{b}_k = \overline{b} \tag{2.21}$$

Straightforward calculations give:

$$\bar{b}^2 = (1+(\frac{a_0-\alpha}{b_0})^2)^{-1}r^2 \tag{2.22}$$

This implies that: $|\bar{b}| > 0$. One may also check that $\mathrm{sign}(\bar{b}) = \mathrm{sign}(b_0)$, and hence it follows that in going from (a_0-R,b_0) to $(\bar{a},\bar{b})$ along S and without increasing the distance to (a_0,b_0), we do not pass through $\{b = 0\}$. Let P_k be the path from $(a(\tau_k+k),b(\tau_k+k))$ to $(\bar{a}_k,\bar{b}_k)$ along S_k, then it follows that there exist $\epsilon > 0$ and k_0 such that for all $k \geqslant k_0$:

$$\inf\{\ |b|\ \mid \exists a \text{ such that: } (a,b)\in P_k\ \} \geqslant \epsilon \tag{2.23}$$

Now choose $k_1 \geqslant k_0$ such that for all $k \geqslant k_1$: $\epsilon_k < \epsilon$. Then in going along P_{k_1}, $|b(t)|$ will never become smaller than ϵ_{k_1+1}, and hence $(a(t),b(t))$ will converge to $(\bar{a}_{k_1},\bar{b}_{k_1})$. From (2.7) it follows that $\tau_{k_1+1} = \infty$. This contradicts the assumption and the statement follows.

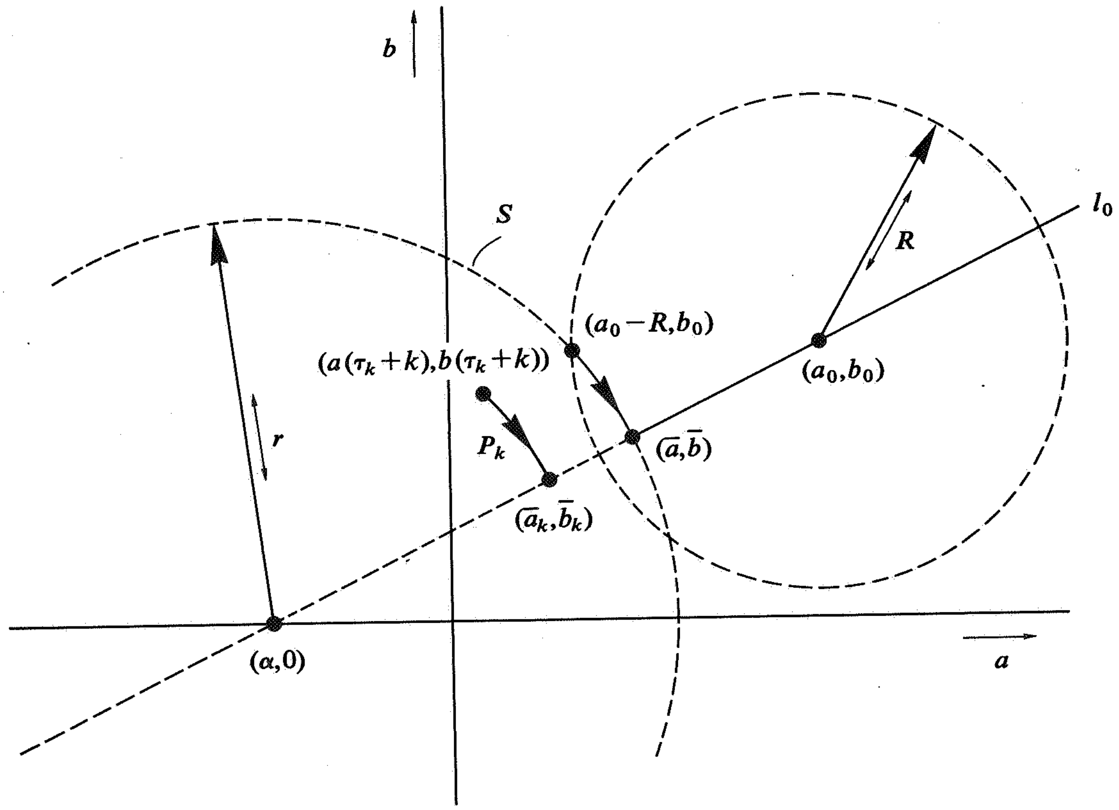

FIGURE 1: Behaviour of $(a(t),b(t))$.

COROLLARY 2.4 There exists t_0 and $\epsilon > 0$, such that for all $t \geqslant t_0$:

$$(i)\ |b(t)| \geqslant \epsilon \tag{2.24}$$

$$(ii)\ u(t) = f(a(t),b(t))x(t) \tag{2.25}$$

$$(iii)\ \mathrm{sign}(b(t)) = \mathrm{sign}(b_0) \tag{2.26}$$

We will now study the asymptotic behaviour of $(a(t),b(t))$. From now on we will assume that $t \geq t_0$.

THEOREM 2.5 $\lim_{t\to\infty} f(a(t),b(t)) = f(a_0,b_0)$

PROOF From Lemma 2.3 we know that: $\lim_{t\to\infty}(a(t),b(t)) = (\bar{a},\bar{b}) \in l_0$. Now $l_0 = \{(a,b) \mid a+b\frac{\alpha-a_0}{b_0} = \alpha\}$. One may check that $(a,b)\in l_0$ and $b \neq 0$ implies: $\frac{\alpha-a}{b} = \frac{\alpha-a_0}{b_0}$, which is equivalent with $f(a,b) = f(a_0,b_0)$. Since $\bar{b} \neq 0$, the statement follows.

COMMMENT Theorem 2.5 states that asymptotically the applied feedback-law equals the desired one. We do not claim that $\lim_{t\to\infty}(a(t),b(t)) = (a_0,b_0)$. The reason that nevertheless 2.5 holds lies in the following observations. Since identification takes place in closed-loop, the following set of parameter-values is certainly invariant under the algorithm:

$$G := \{(a,b) \mid a+bf(a,b) = a_0+b_0f(a,b)\} \tag{2.27}$$

By substituting for f we see that:

$$G = \{(a,b) \mid a+b\frac{\alpha-a_0}{b_0} = \alpha,\ b \neq 0\} \tag{2.28}$$

$$\subset l_0 = \{(a,b) \mid f(a,b) = f(a_0,b_0)\} \cup \{(\alpha,0)\} \tag{2.29}$$

It is not surprising that $(a(t),b(t))$ converges to G, since G is invariant under the algorithm. It is however due to the functional form of f that every point of G corresponds to the desired control-law. It can be proven that apart from $f(a,b)$=constant, there are no other control-laws with that property. We will now characterize the asymptotic closed-loop behaviour of the controlled system.

THEOREM 2.6 There exists a function $\delta : \mathbb{R} \to \mathbb{R}$, such that:

$$(i)\ \ \dot{x} = (\alpha+\delta(t))x \tag{2.30}$$

$$(ii)\ \ \lim_{t\to\infty}\delta(t) = 0 \tag{2.31}$$

PROOF (i) Take $\delta(t) = b_0(f(a(t),b(t))-f(a_0,b_0))$, then: $a_0 + b_0f(a(t),b(t)) = \alpha+\delta(t)$,which gives (2.30).
(ii) This follows from Theorem 2.5.

REMARK Note that Theorem 2.6 holds whether or not $\alpha < 0$. This shows that the adaptation part of the algorithm does not depend on the stability of the closed-loop system. If we assume that $\alpha < 0$, we obtain the following:

COROLLARY 2.7 If $\alpha < 0$, then the system is exponentially stable: for all $\epsilon > 0$ there exists t_ϵ such that for all $t \geq t_\epsilon$:

$$|x(t)| \leq e^{(\alpha+\epsilon)t} \tag{2.32}$$

3. Implementation

The form in which the algorithm has been described in the previous section is not directly implementable, since it depends on the derative of the output. In this section we will show that differentiating the output can be avoided. First we will show this for $t \in [\tau_k + k, \tau_{k+1})$, and then for $t \in I_k$.
For $t \in [\tau_k + k, \tau_{k+1})$, the equation for a can be written as:

$$\dot{a} = \frac{b^2}{b^2 + (\alpha - a)^2}\left(\frac{\dot{x}}{x} - \alpha\right) \tag{3.1}$$

From (2.12) we know that for $t \in [\tau_k + k, \tau_{k+1})$: $b^2 + (\alpha - a)^2 = r_k^2$, hence (3.1) can be written as:

$$\dot{a} = \frac{r_k^2 - (\alpha - a)^2}{r_k^2}\left(\frac{\dot{x}}{x} - \alpha\right) \tag{3.2}$$

Or, equivalently:

$$\frac{r_k^2\, \dot{a}}{r_k^2 - (\alpha - a)^2} = \frac{\dot{x}}{x} - \alpha \tag{3.3}$$

Integrating both sides of (3.3) gives:

$$\frac{1}{2} r_k \log\left(\frac{r_k + a - \alpha}{r_k - a + \alpha}\right) = \log|x| - \alpha t + c_k \tag{3.4}$$

where c_k is determined by $a(\tau_k + k)$, $b(\tau_k + k)$, and $x(\tau_k + k)$. Taking exponentials at both sides of (3.4) and solving for a gives, for $t \in [\tau_k + k, \tau_{k+1})$:

$$a(t) = [1 + (d_k e^{-\alpha t}|x|)^{\frac{-r_k}{2}}]^{-1}\, [\alpha - r + (r_k + \alpha)(d_k e^{-\alpha t}\, |x|)^{\frac{-r_k}{2}}] \tag{3.5}$$

where $d_k = e^{c_k}$, and:

$$c_k = \frac{1}{2} r_k \log\left(\frac{r_k + a(\tau_k + k) - \alpha}{r_k - a(\tau_k + k) + \alpha}\right) - \log|x(\tau_k + k)| - (\tau_k + k)\alpha \tag{3.6}$$

$$r_k = [(a(\tau_k + k) - \alpha)^2 + b(\tau_k + k)^2]^{\frac{1}{2}} \tag{3.7}$$

Finally, from (2.12) we derive:

$$b(t) = \mathrm{sign}(b(\tau_k + k))\, [r_k^2 - (a(t) - \alpha)^2]^{\frac{1}{2}} \tag{3.8}$$

We will now study the equations for $t \in I_k$. For $t \in I_k$, we have:

$$\dot{a} = \frac{1}{1 + C_k^2}\left(\frac{\dot{x}}{x} - a - bC_k\right) \tag{3.9}$$

$$\dot{b} = \frac{C_k}{1 + C_k^2}\left(\frac{\dot{x}}{x} - a - bC_k\right) \tag{3.10}$$

From which we deduce:

$$a(t) = a(\tau_k) + \frac{1}{1 + C_k^2}\left(\log|x| - \int_{\tau_k}^{t} a(s)ds - C_k \int_{\tau_k}^{t} b(s)ds\right) \tag{3.11}$$

$$b(t) = b(\tau_k) + \frac{C_k}{1 + C_k^2}\left(\log|x| - \int_{\tau_k}^{t} a(s)ds - C_k \int_{\tau_k}^{t} b(s)ds\right) \tag{3.12}$$

Hence both parts of the algorithm are implementable.

4. Conclusions

We have presented and analyzed an adaptive pole-assignment algorithm for first order deterministic continuous time systems. We have proved that the algorithm is self-tuning in the sense that the desired control-law is identified asymptotically without the identification of the system parameters. We hope to report on a generalization of the algorithm for higher order systems in the near future.

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