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Analysis of Smoothing Matrices for the Preconditioning of Elliptic Difference Equations

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Smoothing techniques have been used for stabilizing explicit time integration of parabolic and hyperbolic initial-boundary value problems. Similar techniques can be used for the preconditioning of elliptic difference equations. Such techniques are analysed in this paper. It is shown that the spectral radius of the Jacobian matrix associated with the system of equations can be reduced considerably by this type of preconditioners, without much computational effort. Theoretically, this results in a much more rapid convergence of function iteration methods like the Jacobi type methods. The use of smoothing techniques is illustrated for a few one-dimensional and two-dimensional problems, both of linear and nonlinear type. The numerical results show that the use of rather simple smoothing matrices reduce the number of iterations by at least a factor 10.

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1. Introduction

In a number of papers smoothing techniques have been used for stabilizing explicit integration methods in order to solve efficiently *parabolic* and *hyperbolic* initial-boundary-value problems (cf.[2,3,4,6,8]). In this paper, we shall analyse similar smoothing techniques for accelerating iteration methods in order to solve *elliptic* boundary-value problems. The resulting iteration methods can be interpreted as *residue smoothing methods,* and belong, in this respect, to the same class of methods as the well-known multigrid methods and the unigrid method of McCormick and Ruge [1]. However, unlike these methods, the smoothed iteration methods discussed here are extremely simple to implement on a computer and turn out to be rather effective.

Our starting point is the system of (nonlinear) equations

$$\mathbf{f}(\mathbf{u}) = \mathbf{0} \tag{1.1}$$

obtained by discretizing the elliptic boundary-value problem. A number of iteration methods for solving (1.1) express the $(n+1)$st iterate $\mathbf{u}_{n+1}$ *explicitly* in terms of one or more preceding iterates and the corresponding residue vectors $\mathbf{f}(\mathbf{u}_n),\ldots$. For instance, the Jacobi-type iteration methods such as

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \omega \mathbf{f}(\mathbf{u}_n), \tag{1.2}$$

ω being a relaxation parameter, and many "time-stepping" methods are of such a form (cf. [5,p.221] for other types of Jacobi methods). The convergence of these explicit iteration methods may be rather slow if the condition number of the Jacobian matrix $\partial \mathbf{f}/\partial \mathbf{u}$ is large, i.e. the value of ρ/δ, ρ and δ denoting the magnitude of the largest and smallest eigenvalue of $\partial \mathbf{f}/\partial \mathbf{u}$, is much bigger than 1. It is the purpose of this paper to analyse smoothing techniques for "preconditioning" the system (1.1) such that the condition number associated with the preconditioned system is much smaller than that of the original system. The smoothing operators analysed here are of the form

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$$S := P_k(D), \tag{1.3}$$

where P_k is a polynomial of degree k satisfying $P_k(0) = 1$ and D is a difference matrix with eigenvalues on the unit disk.

Thus, instead of (1.1) we shall solve the preconditioned equation

$$\mathbf{f}^*(\mathbf{u}) := S\mathbf{f}(\mathbf{u}) = \mathbf{0}. \tag{1.4}$$

The difference operator D should be such that for any test vector $\mathbf{v} = (v_j) := (w(\mathbf{x}_j))$, $w(\mathbf{x})$ being a sufficiently smooth function of $\mathbf{x}$, we have $D\mathbf{v} \to \mathbf{0}$ as the grid is refined. Then the smoothing matrix S will converge to the identity matrix I by virtue of our condition $P_k(0) = 1$.

EXAMPLE 1.1. Consider a one-dimensional problem (two-point boundary-value problem)

$$(e^u)_{xx} + g(u) = 0,\ 0 \leqslant x \leqslant 1$$
$$u(0) = 0,\ u(1) = 1.$$

Standard symmetric discretization yields the system

$$\begin{cases} f_0(\mathbf{u}) := u_0 = 0, \\ f_j(\mathbf{u}) := \dfrac{1}{\Delta^2}(e^{u_{j-1}} - 2e^{u_j} + e^{u_{j+1}}) + g(u_j) = 0,\ j = 1,\ldots,M, \\ f_{M+1}(\mathbf{u}) := u_{M+1} - 1 = 0, \end{cases} \tag{1.1'}$$

where $\Delta := 1/(M+1)$. As we shall show in Section 3 (cf. Table 3.4) the Jacobi process (1.2) converges extremely slow for this problem. Acceleration of convergence can be obtained by solving (1.4) with

$$D := \frac{1}{4}\begin{bmatrix} 0 & & & & 0 \\ 1 & -2 & 1 & & \\ & & & & \\ & & 1 & -2 & 1 \\ 0 & & & & 0 \end{bmatrix}_{(M+2)\times(M+2)}, \quad P_k(z) := 1 + z. \tag{1.5}$$

With this choice the "preconditioning" matrix $S := P_k(D) = I + D$ assumes the form of an "averaging" matrix:

$$S = \frac{1}{4}\begin{bmatrix} 4 & 0 & & & 0 \\ 1 & 2 & 1 & & \\ & & & & \\ & & 1 & 2 & 1 \\ 0 & & & 0 & 4 \end{bmatrix}_{(M+2)\times(M+2)}. \quad \square \tag{1.6}$$

In Section 2 we derive optimal polynomials $P_k(z)$ for the *model situation:*

$$D := \frac{1}{\rho}\frac{\partial \mathbf{f}}{\partial \mathbf{u}},\ \rho := \rho\left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right) \tag{1.7}$$

where $\partial \mathbf{f}/\partial \mathbf{u}$ has its eigenvalues in the negative interval $[-\rho, 0)$. We emphasize, however, that in actual application we shall not use the difference matrix D defined by (1.7), because it is much too expensive in a general nonlinear case. Instead, we shall employ the "optimal" polynomials $P_k(z)$ together with a "cheap" matrix D possessing the same type of spectrum as $\rho\partial \mathbf{f}/\partial \mathbf{u}$. In fact, we want to use matrices D that are to a large extent *independent of the problem to be solved.* For instance, in all one-dimensional problems with Dirichlet boundary conditions, we employ the matrix (1.5), and in all two-dimensional elliptic problems with Dirichlet conditions we employ the two-dimensional analogue of (1.5), i.e. a

matrix with zero rows if the row correspond to boundary points and with rows of the form

$$\frac{1}{8}(0,\ldots,0,1,0,\ldots,0,1,-4,1,0,\ldots,0,1,0,\ldots,0) \tag{1.8}$$

in nonboundary points. Smoothing matrices $S = P_k(D)$ using matrices D of this form leave all components of the residue vector $\mathbf{f}(\mathbf{u}_n)$ corresponding to boundary points unchanged. Hence, the "boundary" components of $f(\mathbf{u}_n)$ are fixed during the smoothing process. This approach is satisfactory in the case of Dirichlet conditions (cf. Section 3; we observe that in this case $\mathbf{f}$ has zero-boundary components, hence the diagonal elements of the corresponding rows in D can be replaced by nonzero values so that it becomes a nonsingular matrix).

2. Construction of smoothing matrices

2.1. The model situation

We start with the analysis of smoothing matrices of the form $S = P_k(D)$ where D is defined by (1.7). From now on we will assume that $\partial\mathbf{f}/\partial\mathbf{u}$ has *negative* eigenvalues. We want a polynomial $P_k(z)$ such that $P_k(D)\partial\mathbf{f}/\partial\mathbf{u}$ has also *negative* eigenvalues and the smallest possible condition number.

THEOREM 2.1. *Let $\partial\mathbf{f}/\partial\mathbf{u}$ have its eigenvalues in the negative interval $[-\rho,-\delta]$ and let D be defined by (1.7). Then, of all polynomials $P_k(z)$ with $P_k(0) = 1$ and $P_k(z) \geqslant 0$ on $[-1,0]$, the polynomial*

$$P_k(z) = \frac{T_{k+1}(w_0+w_1 z)-T_{k+1}(w_0)}{w_1 T'_{k+1}(w_0)z}, \quad w_0 := \frac{\rho+\delta}{\rho-\delta}, \; w_1 = w_0+1 \tag{2.2}$$

generates a smoothing matrix such that the condition number of $S\partial\mathbf{f}/\partial\mathbf{u}$ is minimal. This condition number is given by ρ^*/δ^*, where

$$\rho^* = (\rho-\delta)\frac{1+T_{k+1}(w_0)}{2T'_{k+1}(w_0)}, \; \delta^* = (\rho-\delta)\frac{-1+T_{k+1}(w_0)}{2T'_{k+1}(w_0)}. \; \square \tag{2.3}$$

The proof of this theorem can straightforwardly be given by using well-known properties of the Chebyshev polynomial $T_{k+1}(x)$. In fact, the polynomial (2.2) resembles the polynomials employed in Chebyshev iteration (cf. [7]).

For elliptic problems the condition number ρ/δ of $\partial\mathbf{f}/\partial\mathbf{u}$ is usually very large. In such cases, the smoothing matrix defined by (2.1) and (2.2) is rather effective, because the condition number of $S\partial\mathbf{f}/\partial\mathbf{u}$ can be made as small as we want by increasing k:

$$\frac{\rho^*}{\delta^*} = \frac{1+T_{k+1}(w_0)}{-1+T_{k+1}(w_0)} \approx \frac{2+2\frac{\delta}{\rho}T'_{k+1}(1)}{2\frac{\delta}{\rho}T'_{k+1}(1)} \approx \frac{1}{(k+1)^2}\frac{\rho}{\delta}. \tag{2.4}$$

2.2. The nonmodel situation

As we already remarked in the Introduction we do not want to define D by (1.7). In this section, we assume that D is a matrix with *negative* eigenvalues in the interval $[-1,0]$. Let these eigenvalues be denoted by μ and suppose that the residue vector $\mathbf{f}(\mathbf{u}_n)$ is expanded in the eigenvectors of D. Then, by applying the smoothing matrix $S = P_k(D)$ to $\mathbf{f}$, these eigenvectors are multiplied by

$$P_k(\mu) = \frac{T_{k+1}(w_0+w_1\mu)-T_{k+1}(w_0)}{w_1 T'_{k+1}(w_0)\mu}. \tag{2.5}$$

In Figure 2.1 the polynomial $P_k(\mu)$ is plotted for $k = 5$ and $\delta/\rho << 1$. From this picture we see that the smoothing matrix has a strong damping effect on all eigenvector components of $\mathbf{f}(\mathbf{u}_n)$ with eigenvalues close to -1. Since, usually, these eigenvector components represent the high frequencies of $\mathbf{f}(\mathbf{u}_n)$ we conclude that S reduces the high frequencies of the residue vector.

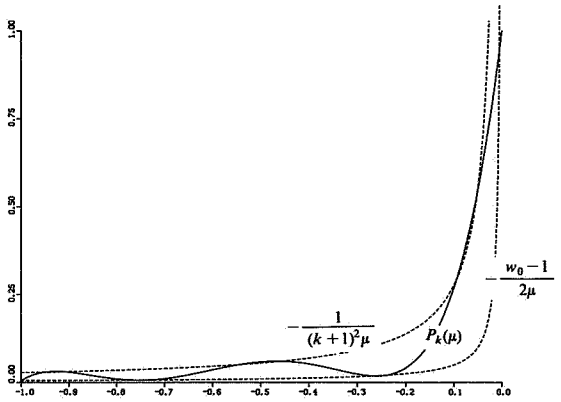

FIGURE 2.1. Behaviour of $P_k(\mu)$ for $\delta/\rho << 1$.

2.3. Generation of smoothed residues by recursion

In order to generate the smoothed residue $\mathbf{f}^* = P_k(D)\mathbf{f}$ we may employ the following recursion:

$$\begin{cases} \mathbf{f}_0 = \mathbf{f},\ \mathbf{f}_1 = 2(2w_0 + w_1 D)\mathbf{f}, \\ \mathbf{f}_{j+1} = 2(w_0 + w_1 D)\mathbf{f}_j - \mathbf{f}_{j-1} + 2T_{j+1}(w_0)\mathbf{f},\ j = 1,\ldots,k-1, \\ \mathbf{f}^* = \mathbf{f}_k / T'_{k+1}(w_0). \end{cases} \tag{2.6}$$

This recursion is easily derived from the three-terms Chebyshev recursion. It is numerically stable if D has its eigenvalues in the interval $[-1,(w_0-1)/(w_0+1)]$.

In our experiments we have always set $w_0 = 1$ and $w_1 = 2$. The matrix D defined by (1.5) (or its two-dimensional analogue) can then be used without danger of instability. However, a consequence of this choice is that $P_k(D)$ may have zero-eigenvalues (see Figure 2.1). This may give problems in the convergence when using an iteration method on $\mathbf{f}^*(\mathbf{u}) = \mathbf{0}$. In Section 3 we will discuss this aspect for the Jacobi method.

The recursion (2.6) requires k matrix-vector multiplications and is extremely simple to implement for problems in one or more dimensions with irregular boundaries. Moreover, the storage requirements are rather modest.

2.4. Generation of smoothed residues by factorization

The smoothed residue $\mathbf{f}^*$ can be obtained by far less than k matrix-vector multiplications if $k = 2^q - 1$ for some positive integer q and if $w_0 = w_1 - 1 = 1$. Let the matrices F_j be defined by

$$F_1 = I + D,\ F_{j+1} = (I - 2F_j)^2,\ j \geqslant 0. \tag{2.7a}$$

Then for all matrices D the smoothing matrix $S = P_k(D)$ can be factorized according to

$$S = F_q \cdot F_{q-1} \cdot \ldots \cdot F_1. \tag{2.7b}$$

Thus, only $q \approx^2 \log(k+1)$ matrix-vector multiplications are required to generate $\mathbf{f}^* = S\mathbf{f}$. The proof of this factorization property follows from a similar property of Chebyshev polynomials (cf. [2,

Lemma 3.2]).

2.4.1. One-dimensional problems

The factorization (2.7) requires the precomputation of the factor matrices F_j. In one-dimensional problems, this offers no difficulties. We easily find in the case (1.7):

$$F_1 = \frac{1}{4}\begin{bmatrix} 4 & 0 & & & & 0 \\ 1 & 2 & 1 & & & \\ & & 1 & 2 & 1 & \\ 0 & & & 0 & 4 & \end{bmatrix}, \quad F_2 = \frac{1}{4}\begin{bmatrix} 4 & 0 & & & & & & & & 0 \\ 2 & 1 & 0 & 1 & & & & & & \\ 1 & 0 & 2 & 0 & 1 & & & & & \\ & & 1 & 0 & 2 & 0 & 1 & & & \\ & & & 1 & 0 & 1 & 2 & & & \\ 0 & & & & & 0 & 4 & & & \end{bmatrix},$$

$$F_j = \frac{1}{4}\begin{bmatrix} 4 & & & & & & & & & & & & & & \\ 2 & 2 & & & & & -1 & 0 & 1 & & & & & & \\ \cdot & & \cdot & & & & & & & \cdot & & & & & \\ 2 & & & 2 & & -1 & & & & & & 1 & & & \\ 2 & & & & 1 & & & & & & & & 1 & & \\ 2 & & & -1 & & 2 & & & & & & & & 1 & \\ \cdot & & \cdot & & & & & \cdot & & & & & & & \cdot \\ 2 & -1 & & & & & 2 & & & & & & & 1 & \\ 1 & & & & & & & 2 & & & & & & & 1 \\ & \cdot & & & & & & & \cdot & & & & & & \cdot \\ & & 1 & & & & & & & 2 & & & & & 1 \\ & & & & & \cdot & \cdot & \cdot & & & & & & & \end{bmatrix},$$

where $j \geq 3$ and where the position of the element -1 in the second row is at the 2^{j-1}-th column.

Notice that only a few nonzero elements occur on each row.

2.4.2. Two-dimensional problems

In two or more dimensions the derivation of the matrices F_j defined by (2.7a) is not attractive. Therefore, we consider an alternative which only uses one-dimensional smoothing matrices.

We confine ourselves to two-dimensional problems. Let the residue vector $\mathbf{f}$ be arranged in a two-dimensional array in the natural way. First we compute an intermediate array $\mathbf{f}^*$ by applying to all rows of $\mathbf{f}$ the one-dimensional smoothing matrix S discussed in the preceding subsection. Next, we do the same with all columns of $\mathbf{f}^*$ to obtain the array $\mathbf{f}^{**}$. The preconditioned system of equations is then given by

$$\mathbf{f}^{**}(\mathbf{u}) := \tilde{S}\mathbf{f}(\mathbf{u}) = \mathbf{0}. \tag{2.8}$$

Of course, the corresponding smoothing matrix $\tilde{S}$ is essentially different from S and it is of interest to know the damping effect of S on high frequencies in $\mathbf{f}$.

In order to get some insight into the properties of $\tilde{S}$ we expand $\mathbf{f}$ in a discrete Fourier series:

$$\mathbf{f} = \sum_{\omega} c(\omega)\mathbf{e}(\omega), \quad \mathbf{e}(\omega) := (exp(i\omega \cdot \mathbf{x}_j)) \tag{2.9}$$

where ω represents the frequency vector and $\mathbf{x}_j$ runs through the grid points on which $\mathbf{f}$ is defined. Let $\{\mathbf{x}_j\}$ be a uniform grid $\{j\Delta,l\Delta\}$ with square meshes. Applying the smoothing matrix $\tilde{S}$ to $\mathbf{f}$ has the effect that the components $c(\omega)\mathbf{e}(\omega)$ of $\mathbf{f}$ are essentially multiplied by the factor $P_k(\mu_x)P_k(\mu_y)$, where μ_x and μ_y are the eigenvalues of the difference matrices D_x and D_y, respectively used in the row-smoothing of $\mathbf{f}$ and the column-smoothing of $\mathbf{f}^*$. For (μ_x,μ_y) away from the origin the estimate

$$P_k(\mu_x)P_k(\mu_y)\leqslant\frac{1}{(k+1)^4\mu_x\mu_y},\quad -1\leqslant\mu_x,\mu_y<0 \tag{2.10}$$

gives an indication of the increased damping power of the smoothing matrix $\tilde{S}$ (cf. Figure 2.2). It should be remarked, however, that this does not automatically imply an increased damping of the iteration error when $\tilde{S}$ is combined with, e.g., Jacobi iteration (cf. Section 3).

We conclude our discussion of the matrix S by deriving an estimate for the spectral radius ρ^{**} of $\partial\mathbf{f}^{**}/\partial\mathbf{u}$ in the model situation where

$$\frac{1}{2}(D_x+D_y)=\frac{1}{\rho}\frac{\partial\mathbf{f}}{\partial\mathbf{u}},\quad \rho:=\rho(\frac{\partial\mathbf{f}}{\partial\mathbf{u}}). \tag{2.11}$$

Let $w_0=w_1-1=1$ in (2.2), then the eigenvalues of $\tilde{S}\partial\mathbf{f}/\partial\mathbf{u}$ are given by

$$\lambda^{**}:=P_k(\mu_x)P_k(\mu_y)\frac{\mu_x+\mu_y}{2}\rho=\frac{T_{k+1}(1+2\mu_x)-1}{2(k+1)^2\mu_x}\,\frac{T_{k+1}(1+2\mu_y)-1}{2(k+1)^2\mu_y}\,\frac{\mu_x+\mu_y}{2}\rho, \tag{2.12}$$

where $-1\leqslant\mu_x,\mu_y\leqslant 0$ and $\mu_x+\mu_y\leqslant-2\delta/\rho$.

For small values of k the value of $\rho^{**}:=\max|\lambda^{**}|$ can straightforwardly be determined. For instance,

$$k=1\colon\ \rho^{**}=\frac{4\rho}{27}\ \text{assumed at}\ \mu_x=\mu_y=-\frac{1}{3} \tag{2.13a}$$

$$k=2\colon\ \rho^{**}=\frac{192\rho}{3125}\ \text{assumed at}\ \mu_x=\mu_y=-\frac{3}{20}.$$

For larger values of k we investigated λ^{**} numerically: we found

$$k\geqslant 3\colon\ \rho^{**}\approx.55\frac{\rho}{(k+1)^2}\ \text{assumed at}\ \mu_x=\mu_y\approx-\frac{1.35}{(k+1)^2}. \tag{2.13b}$$

The behaviour of $\lambda^{**}(\mu_x,\mu_y)$ with $\mu_x=\mu_y$ is plotted in Figure 2.2.

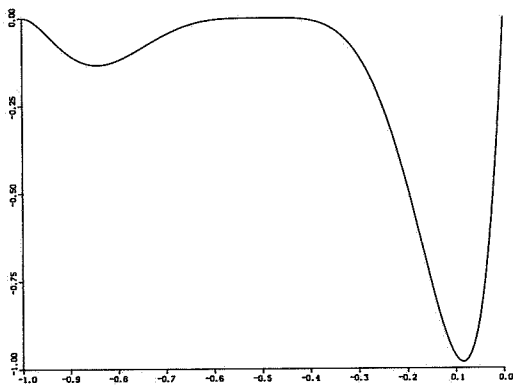

FIGURE 2.2. Eigenvalues λ^{**} along $\mu_x=\mu_y$ for $k=3$ and $\rho=(k+1)^2/.55$

3. Smoothed Jacobi iteration

We shall discuss the application of the Jacobi-type iteration method (1.2) to the systems (1.4) and (2.8) in the following two subsections.

3.1. The case $\mathbf{f}^*(\mathbf{u})=\mathbf{0}$

In first approximation, the error equation associated with the iteration process

$$\mathbf{u}_{n+1} = \mathbf{u}_n+\omega S\mathbf{f}(\mathbf{u}),\ S = P_k(D),\ n\geqslant 0 \tag{3.1}$$

reads

$$\mathbf{u}_{n+1}-\mathbf{u} = A_n(\mathbf{u}_n-\mathbf{u}),\ A_n := I+\omega P_k(D)\frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u}_n), \tag{3.2}$$

where $\mathbf{u}$ denotes the exact solution of (1.1).

In order to get an indication how the relaxation parameters ω should be chosen, we consider the model situation (1.7). Setting $w_0 = w_1-1 = 1$ in the expression (2.2) for $P_k(z)$, we find that the eigenvalues $\alpha_n(\mu)$ of A_n are given by

$$\alpha_n(\mu) = 1+\omega\rho\frac{T_{k+1}(1+2\mu)-1}{2(k+1)^2},\ -1\leqslant\mu\leqslant-\frac{\delta}{\rho}, \tag{3.3}$$

where μ denotes the eigenvalues of $\partial\mathbf{f}/\rho\partial\mathbf{u}$. From this expression it follows that $\alpha_n(\mu)$ equals 1 in all points $\mu\in[-1,-\delta/\rho]$ where $T_{k+1}(1+2\mu)$ equals 1 irrespective of the value of ω. This implies that we should not iterate with a fixed value of k. Therefore, we consider cyclic methods where $\omega = \omega(n)$ and $k = k(n)$, $\omega(n)$ and $k(n)$ being periodic functions of n: $\omega(n) = \omega(N+n)$, $k(n) = k(N+n)$ with N fixed. Instead of α_n we consider the "average" amplification factor

$$\alpha(\mu) = \Big|\prod_{n=0}^{N-1}\alpha_n(\mu)\Big|^{1/N}. \tag{3.4}$$

During a cycle of N iterations we shall impose the condition

$$\omega = \frac{2C(k+1)^2}{\rho} \tag{3.5}$$

where C is a constant. Furthermore, we shall require that $|\alpha_n(\mu)|$ is bounded by 1 for all n, i.e. we require $0<C\leqslant 1$. Thus, the iteration process assumes the form

$$\mathbf{u}_{n+1} = \mathbf{u}_n+\frac{C}{\rho}D^{-1}[T_{k+1}(I+2D)-I]\mathbf{f}(\mathbf{u}_n). \tag{3.1'}$$

It is easily verified that $\alpha'(0)$ is maximized for $C = 1$. Hence, for $C = 1$ we have a maximal damping in the neighbourhood of $\mu = 0$, however, at the same time, we have $\alpha(-1) = 1$. Furthermore, we found numerically that $C = \frac{1}{2}$ yields a maximal "overall" damping. In Figure 3.1 we have plotted these two extreme cases for $k(n) = 2^n-1$, $n = 0,...,4$ (notice that $C = 1$ yields an $\alpha(\mu)$ function which is symmetric w.r.t. $\mu = -\frac{1}{2}$. Part a of this figure clearly indicates that, except for a small region near $\mu = 0$, $C = \frac{1}{2}$ indeed leads to a substantially stronger damping than the $C = 1$ value. Part b of Figure 3.1 shows that only eigenvector components of the iteration error which correspond to eigenvalues λ of $\partial\mathbf{f}/\partial\mathbf{u}$ lying in the interval $\approx[-\cdot 004\rho,0)$ are stronger damped by choosing $C = 1$.

Next, we consider the average damping factor $\alpha(\mu)$ in the case where $k(n) = n$, $n = 0,1,...,15$. Figure 3.2 presents the analogue of Figure 3.1 and shows roughly the same picture.

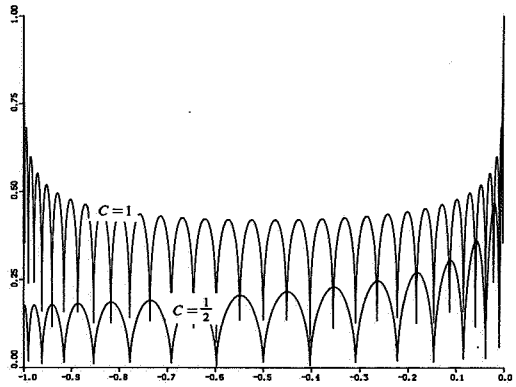

FIGURE 3.1a. Behaviour of $\alpha(\mu)$ on the interval $-1 \leqslant \mu \leqslant 0$ in the case $k(n) = 2^n - 1, n = 0,...,4$

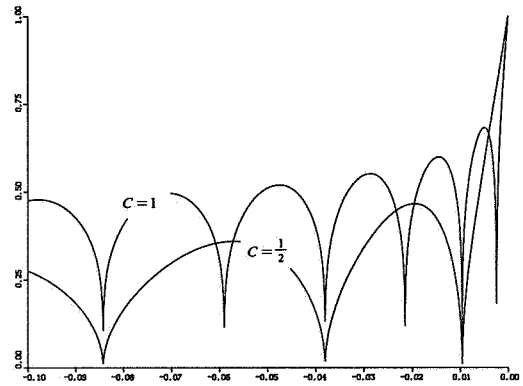

FIGURE 3.1b. Behaviour of $\alpha(\mu)$ on the interval $-\cdot 1 \leqslant \mu \leqslant 0$ in the case $k(n) = 2^n - 1, n = 0,...,4$

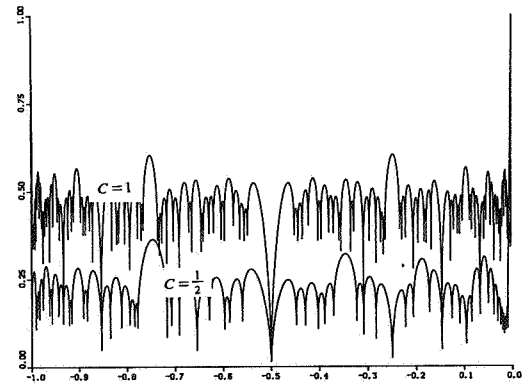

FIGURE 3.2a. Behaviour of $\alpha(\mu)$ on the interval $-1 \leqslant \mu \leqslant 0$ in the case $k(n) = n$, $n = 0,1,...,15$

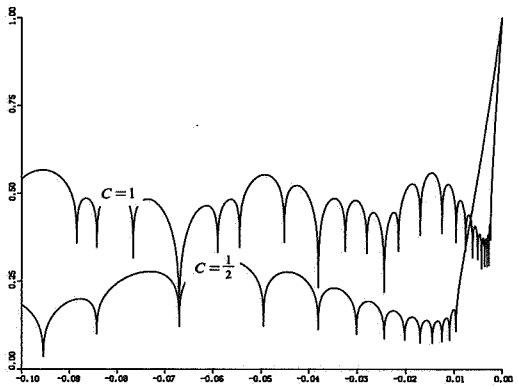

FIGURE 3.2b. Behaviour of $\alpha(\mu)$ on the interval $-\cdot 1 \leqslant \mu \leqslant 0$ in the case $k(n) = n$, $n = 0,...,15$

We illustrate the smoothed Jacobi method (3.1′) by two examples: a model problem and a nonmodel problem. We applied both the recursive smoothing process (2.6) with $k(n) = n$, $n = 0,...,N-1$, and the factorized smoothing process (2.7) with $k(n) = 2^n - 1$, $n = 0,...,N-1$. The resulting methods are denoted by RSJ(N,C) and FSJ(N,C), respectively (notice that RSJ(1,C) and FSJ(1,C) both represent the conventional Jacobi method). All methods started with the initial approximation (cf. Example 1.1)

$$\mathbf{u}_0 = ((M+1-j)\Delta \cdot u_0 + j\Delta \cdot u_{M+1})_{j=0}^{M+1}, \tag{3.6}$$

and stopped when the scaled residue

$$r(n) := \frac{\|\mathbf{f}(\mathbf{u}_n)\|_\infty}{\|\mathbf{f}(\mathbf{u}_0)\|_\infty} \tag{3.7}$$

dropped below a value specified in the tables of results.

TABLE 3.1. $n/(r(n))^{1/n}$ - values for the problem

$$u_{xx} - 20x^3 = 0,\ 0 \leqslant x \leqslant 1$$

$$u(0) = 0,\ u(1) = 1$$

$$\rho = 4/(\Delta x)^2,\ r(n) \leqslant 10^{-4}$$

Method	$\Delta x = 1/20$	$\Delta x = 1/40$	$\Delta x = 1/80$
RSJ(1,.95)	678/.986	n>2000	
RSJ(16,.95)	14/.50	29/.72	112/.92
FSJ(5,.95)	25/.68	30/.73	150/.94
RSJ(1,.5)	1290/.992	n>5000	
RSJ(16,.5)	15/.50	59/.85	221/.96
FSJ(5,.5)	15/.52	74/.88	295/.97

TABLE 3.2. $n/(r(n))^{1/n}$ values for the problem

$$(e^u)_{xx} - 5x^3 e^u(4+5u) = 0,\ 0 \leqslant x \leqslant 1$$

$$u(0) = 0,\ u(1) = 1$$

$$\rho = 4e/(\Delta x)^2,\ r(n) \leqslant 10^{-4}$$

Method	$\Delta x = 1/20$	$\Delta x = 1/40$	$\Delta x = 1/80$
RSJ(1,.95)	865/.989	n>3000	
RSJ(16,.95)	19/.61	41/.80	147/.94
FSJ(5,.95)	24/.67	49/.83	195/.95
RSJ(1,.5)	1645/.994	n>6000	
RSJ(16,.5)	36/.77	77/.89	283/.97
FSJ(5,.5)	44/.81	100/.91	380/.98

3.2. The case $\mathbf{f}^{**}(\mathbf{u}) = \mathbf{0}$

The error equation for the iteration process

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \omega \tilde{S}\mathbf{f}(\mathbf{u}) \tag{3.8}$$

(cf.(3.1)) is of the form (3.2) with $A_n := I + \omega \tilde{S}\partial \mathbf{f}/\partial \mathbf{u}(\mathbf{u}_n)$. Again considering the model situation, we find that the eigenvalues $\alpha_n(\mu_x, \mu_y)$ of A_n are given by

$$\alpha_n(\mu_x, \mu_y) = 1 + \omega \lambda^{**}(\mu_x, \mu_y), \tag{3.9}$$

where λ^{**} is defined by (2.12). As in the preceding subsection we shall require that $|\alpha_n| \leqslant 1$ for all n. This leads us to the condition $\omega \leqslant 2/\rho^{**}$. From (2.13) it follows that $\rho^{**} = c(k)\rho/(k+1)^2$, hence $\omega \leqslant 2(k+1)^2/c(k)\rho$; here, $c(k)$ assumes the values

$$c(0) = 1,\ c(1) = 16/27,\ c(2) = 1728/3125,\ c(k) \approx .55 \text{ for } k > 2.$$

In analogy with (3.5) we shall impose the condition

$$\omega = \frac{2C(k+1)^2}{c(k)\rho}, \tag{3.10}$$

where $k = k(n)$ is a periodic function of n and C is a constant in the interval $(0,1]$.

Let us define the average damping factor (cf. (3.4))

$$\alpha(\mu) = \Big(\underset{2\tilde{\mu}(\mu) \leqslant \mu_x \leqslant \mu}{Max} \prod_{n=0}^{N-1} |\alpha_n(\mu_x, 2\mu - \mu_x)| \Big)^{1/N},\ -1 \leqslant \mu \leqslant 0, \tag{3.11}$$

where $\tilde{\mu} = \mu$ if $\mu \geqslant -\frac{1}{2}$ and $\tilde{\mu} = -\frac{1}{2}$ if $\mu \leqslant -\frac{1}{2}$. This function was investigated numerically in the case where $k(n) = 2^n - 1$, $n = 0,...,4$. We found the best "overall" damping for values of C in the neighbourhood of .6. In Figure 3.3 the function $\alpha(\mu)$ is plotted for $C = 1$ and $C = .6$.

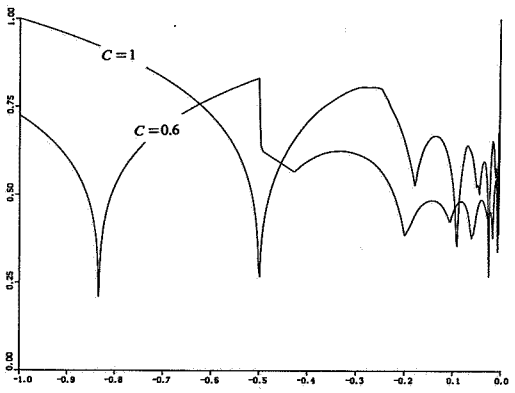

FIGURE 3.3a. Behaviour of $\alpha(\mu)$ on the interval $-1 \leqslant \mu \leqslant 0$ in the case $k(n) = 2^n - 1, n = 0,...,4$

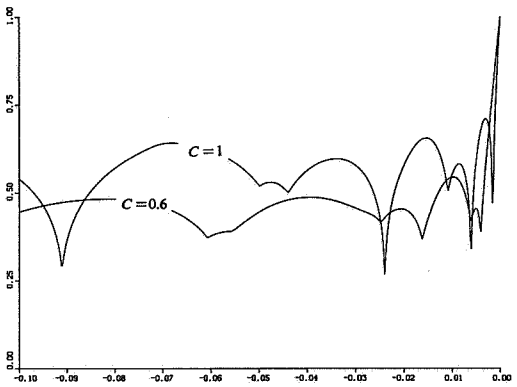

FIGURE 3.3b. Behaviour of $\alpha(\mu)$ on the interval $-\cdot 1 \leqslant \mu \leqslant 0$ in the case $k(n) = 2^n - 1, n = 0,...,4$

The performance of the smoothed Jacobi process ((3.8), (3.10)) is illustrated by a model and a non-model problem. As before, this method is denoted by FSJ(N,C). In addition, we applied the RSJ method employing the two-dimensional version of the matrix D defined in (1.5). The initial approximation $\mathbf{u}_0$ is defined by forming linear interpolations of the boundary values on $x = 0$, $x = 1$ and $y = 0$, $y = 1$, respectively, and by taking the average value of these functions; the iteration process was stopped if the value of $r(n)$ becomes less than a prescribed value which is specified in the tables of results.

TABLE 3.3. $n/(r(n))^{1/n}$-values for the problem

$$\Delta u - 6xy(x^2+y^2) = 0,\ 0 \leqslant x,y \leqslant 1$$

$$u = x^3y^3 \text{ along the boundary}$$

$$\rho = 8/(\Delta x)^2,\ \Delta x = \Delta y,\ r(n) \leqslant 10^{-4}.$$

method	$\Delta x = 1/20$	$\Delta x = 1/40$	$\Delta x = 1/80$
RSJ(1,.95)	468/.98	n>800	
RSJ(16,.95)	15/.54	16/.54	44/.81
FSJ(5,.95)	31/.74	27/.71	35/.76
RSJ(1,.5)	891/.99	n>800	
RSJ(16,.5)	13/.48	31/.74	
FSJ(5,.6)	16/.54	20/.62	54/.84

TABLE 3.4. $n/(r(n))^{1/n}$-values for the problem

$$e^u \Delta u - u^3 = 0,\ 0 \leqslant x,y \leqslant 1$$

$$u = x^3y^2 \text{ along the boundary}$$

$$\rho = 8e/(\Delta x)^2,\ \Delta x = \Delta y, r(n) \leqslant 10^{-3}.$$

method	$\Delta x = 1/20$	$\Delta x = 1/40$	$\Delta x = 1/80$
RSJ(1,.95)	252/.97	517/.987	
RSJ(16,.95)	11/.52	12/.54	14/.60
FSJ(5,.95)	18/.68	17/.66	17/.66
RSJ(1,.5)	481/.986	>800	
RSJ(16,.5)	26/.76	26/.76	27/.77
FSJ(5,.6)	36/.82	32/.80	32/.80

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