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Department of Applied Mathematics Report AM-R8704 May

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Stochastic and Chaotic Relaxation Oscillations

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For relaxation oscillators stochastic and chaotic dynamics are investigated. The effect of random perturbations upon the period is computed. For an extended system with additional state variables chaotic behaviour can be expected. As an example the Van der Pol oscillator is changed into a third order system admitting period doubling and chaos in a certain parameter range. The distinction between chaotic oscillation and oscillation with noise is explored by studying a time series of one of the variables. Return maps, power spectra and Lyapunov exponents are analyzed for that purpose.

1980 Mathematics Subject Classification: 34C15, 34E15, 58F13, 60H10
Keywords & Phrases: relaxation oscillation, stochastic perturbation, chaos
Note: This report will be submitted for publication elsewhere.
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1. Introduction

Physical systems with periodic behaviour usually show fluctations in the period. One is inclined to ascribe this to perturbations coming from the environment (noise). The possibility of chaotic behaviour of a deterministic system of three or more nonlinear differential equations has turned the attention to a possible explanation in this direction.

In this paper we explore the distinction between stochastic and chaotic relaxation oscillations. A general theory for Van der Pol type relaxation oscillators with stochastic perturbations is formulated in section 3. We derive equations by which the probability distribution of the time between two jumps can be approximated. As an example we study in detail the Van der Pol oscillator with a random forcing and compute the mean and variance of the interjump time for small noise intensity. In section 4 it is shown that a generalized Van der Pol type oscillator may exhibit chaotic dynamics. As a prototype of such a system we consider the Van der Pol oscillator with an additional state variable. Finally, in section 5 this system is perturbed by noise and various methods for detecting the presence of a strange attractor from an output signal are discussed. We deal with power spectra, return maps and Lyapunov exponents. Among these characteristics the Lyapunov exponents yield the most definite information about the presence of a strange attractor.

The Van der Pol relaxation oscillator is the classical example of a nonlinear oscillator with the specific temporal structure as if it is periodically reset in its initial state. The equation reads,

$$\frac{d^2x}{d\tau^2} + \nu(x^2-1)\frac{dx}{d\tau} + x = 0,$$

which is equivalent, using the transformations

$$\tau = t\nu, \ \nu = 1/\sqrt{\epsilon}$$

appropriate to the singularly perturbed case, to the system

$$\epsilon\frac{dx}{dt} = y - \frac{1}{3}x^3 + x \tag{1.1a}$$

Report AM-R8704
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$$\frac{dy}{dt} = -x \tag{1.1b}$$

For $\epsilon \to 0$ the limit cycle makes two jumps: from A to B and from C to D, see fig. 1. At the arcs BC and DA the trajectory satifies $y = \frac{1}{3}x^3 - x$. Substitution in (1.1b) yields an expression for the period:

$$\frac{1}{2}T = \int_2^1 \frac{x^2-1}{-x}\, dx = \frac{3}{2} - \ln 2. \tag{1.2}$$

In a more refined approach of small but nonzero values of ϵ, where the jumps are replaced by boundary layer approximations, one obtains

$$T(\epsilon) \approx 3 - 2\ln 2 + 3\alpha\epsilon^{2/3} + \mathcal{O}(\epsilon \ln \epsilon) \quad (\epsilon \to 0) \tag{1.3}$$

where $-\alpha = -2.33811$ is the first zero of the Airy function, see GRASMAN (1987). Eq. (1.1) can be viewed as a representative of a large class of nonlinear oscillators. It is therefore worthwhile to analyse the effect of small changes which turn this oscillator into a stochastic or chaotic system.

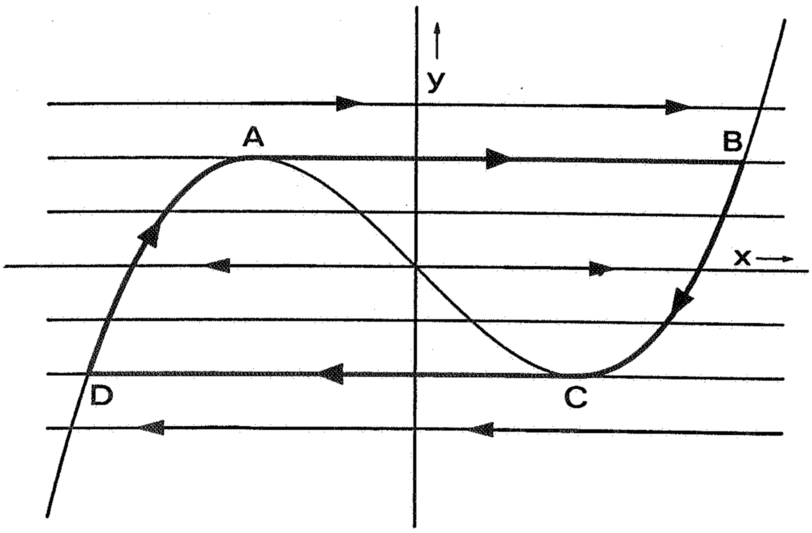


Fig.1 Trajectories of the system (1.1) in the limit $\epsilon \to 0$. For any starting value (the origin excluded) the solution approaches the limit cycle $ABCD$.

2. THE GENERALIZED VAN DER POL OSCILLATOR

The relaxation oscillations we consider are described by a system of differential equations of the form

$$\epsilon\frac{dx_i}{dt} = F_i(x,y;\epsilon),\ i=1,...,m, \tag{2.1a}$$

$$\frac{dy_j}{dt} = G_j(x,y;\epsilon),\ j=1,...,n, \tag{2.1b}$$

where ϵ is a small positive parameter. Here the $\{x_i\}$ represent the variables which undergo fast relaxation and the $\{y_i\}$ represent the slow variables. It is assumed that the system (2.1) has a relaxation oscillation (for the definition see GRASMAN (1987)) as a solution.

If we let $\epsilon \to 0$, the so-called *discontinuous approximation* of the oscillation is found which holds over a large phase of the cycle. The approximating trajectory satisfies

$$\frac{dy}{dt} = G(x,y;0) \tag{2.2}$$

and is restricted to the manifold

$$\mathcal{F} = \{(x,y)|F(x,y;0)=0\}. \tag{2.3}$$

A trajectory of (2.1) remains only near $\mathcal{F}$ if, for y fixed, the stationary solution $x=x_s$ of (2.1a) is stable with x_s satisfying $F(x_s,y;0)=0$ and the point (x_s,y) near the trajectory. When this is not the case, the trajectory will exhibit a large change in x over a short time interval of length $\mathcal{O}(\epsilon)$. It it is assumed that indeed a subset $\mathbb{S}$ of $\mathcal{F}$ exists for which the matrix

$$A = \{\frac{\partial F_i}{\partial x_j}\}_{m\times m} \tag{2.4}$$

has eigenvalues with negative real parts: the stable manifold. At the boundary of $\mathbb{S}$ only one eigenvalue may have a real part in the form of a simple zero resulting in $\det(A)=0$. When the approximating trajectory arrives at a point $p\epsilon\partial\mathbb{S}$, it leaves the manifold $\mathcal{F}$ and the solution jumps instantaneously to a point r lying in $\mathbb{S}$ with $x_r \neq x_p$ and $y_r \neq y_p$. Clearly, the equation $F(x,y,0)=0$ must be nonlinear in x.

As an example we consider relaxation oscillations of a system with one "fast variable" x and two "slow" variables y_1 and y_2. Expressing x as a function of y, we obtain for $\mathcal{F}$:

$$x = H(y). \tag{2.5}$$

This expression is locally valid: at different branches of $\mathcal{F}$ different representations are needed. In fig. 2 the manifold $\mathcal{F}$ is stable except for the middle branch at the fold. For any starting value away from $\mathcal{F}$ a trajectory of (2.1) will jump instantaneously (in the limit as $\epsilon \to 0$) to one of the stable branches. The periodic solution has a jump at $y=y_0$. Its period satisfies

$$T = \oint \frac{1}{G(H(y),y)}ds\ \ (\epsilon\to 0). \tag{2.6}$$

where the integral is over the closed curve of the periodic solution. Let us take a $(m-1)$-dimensional transversal intersection $\Sigma \subset \mathcal{F}$ of (2.2)-(2.3) with $\det(A)\neq 0$ for all $s\in\Sigma$. Approximating trajectories with jumps at $\partial\mathbb{S}$ generate a Poincaré-mapping

$$P:\Sigma\to\Sigma. \tag{2.7}$$

This mapping or a finite repetition of it may have a fixed point that corresponds to a periodic solution. When this fixed point is stable, existence of a periodic solution of (2.1) with $\epsilon>0$ can be proved, see MISHCHENKO and ROSOV (1980). As we will see in section 4, this mapping may also have chaotic solutions, corresponding to chaotic behaviour of the system.

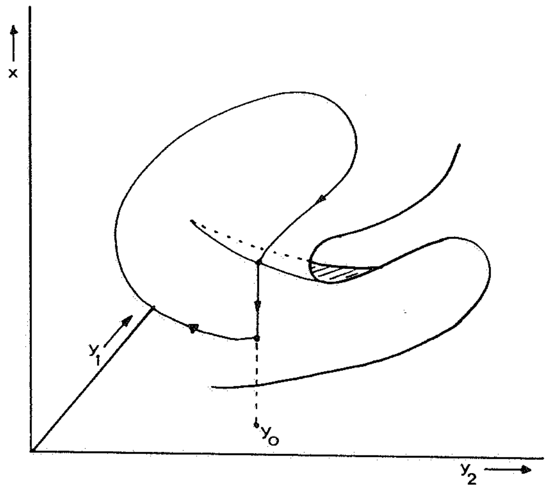


Fig.2. Relaxation oscillation in a system with one fast and two slow variables. At $y=y_o$ the limit solution ($\epsilon \to 0$) jumps from one point of the manifold $\mathcal{F}$ to a different point, where (2.1a) with $(y,\epsilon)=(y_0,0)$ is stable.

3. Randomly perturbed oscillators

First we will model the influence of random perturbations upon the generalized Van der Pol oscillator (2.1). Near $\mathbb{S}$, points in $\mathbb{R}^{m+n}$ with equal values of x will remain close to the trajectory starting in $\mathcal{F}$. Consequently, perturbations in the x_i-directions will not change the velocity in the direction along $\mathcal{F}$. Since we wish to study fluctuations in the period, we only take into account perturbations in the slow variables $\{y_i\}$. Thus we analyze the system of stochastic differential equations

$$\epsilon dX_i = F_i(X,Y)dt, \quad i=1,...,m, \tag{3.1a}$$

$$dY_j = G_j(X,Y)dt + \delta \sum_{k=1}^{p} \sigma_{jk}(X,Y)dW_k, \; j=1,...,n, \tag{3.1b}$$

where $W_1,..., W_p$ denote p independent Wiener processes. It is assumed that $0<\epsilon<<\delta<<1$.

In our perturbation analysis we let $\epsilon \to 0$ and consider a jumping periodic solution L_0 of (2.1) and its stochastic perturbation given by (3.1). We will apply methods for stochastic differential equations as described by Gardiner (1983).

In the limit $\epsilon \to 0$ we analyse the reduced system

$$dy_j = G_j(H(y),y)dt + \delta \sum_{k=1}^{p} \sigma_{jk}(H(y),y)dW_k, \; j=1,...,n \tag{3.2}$$

where $x=H(y)$ is a local solution of $F(x,y)=0$. The existence of a periodic solution L_0 for $\delta=0$ implies that the mapping P of (2.7) has a stable fixed point. This closed trajectory contains a number of jumps, say N. The slow action is on $\mathcal{F}$ from a return point r_k to a leaving point p_{k+1}. A jump is made from p_k to $r_k, k=1,...,N$. It is noted that r_N is connected to p_1 by an interval of slow action. Let $U_k \subset \mathcal{F}$ be a set of points in a neighbourhood of p_k satisfying det$(A)=0$, so $U_k \subset \partial\mathbb{S}$. Then a set of points $V_k \subset \mathcal{F}$ in a neighbourhood of r_k is defined by

$$V_k = \{(x,y)|(x,y)\in\mathbb{S},(\bar{x},y)\in U_k \text{ for some } \bar{x}\}. \tag{3.3}$$

We will analyse stochastic trajectories in a domain Ω_k of the y-space being a sufficiently large neighbourhood of the projected segment L_0' of L_0, which connects the return point r_{k-1} to the leaving point p_k, see fig. 3. [1] The calculation of the distribution of the corresponding time interval proceeds by considering the part of the boundary $\partial\Omega_k$ formed by the set U_k' to be absorbing. The remaining part of $\partial\Omega_k$ is assumed to be at a sufficiently large distance of L_0', so that is does not play any role in the stochastic analysis.

We consider stochastic trajectories of (3.2) starting in V'_{k-1} up to the moment they reach the boundary at U_k'. The time T_k needed for this is a stochastic variable, called the interjump time. A method for approximating its distribution will be developed. For that purpose we first need to have information about the distribution of points p_k at $U_k'(=V_k')$ for a stationary oscillation as it goes through the k^{th} jump.

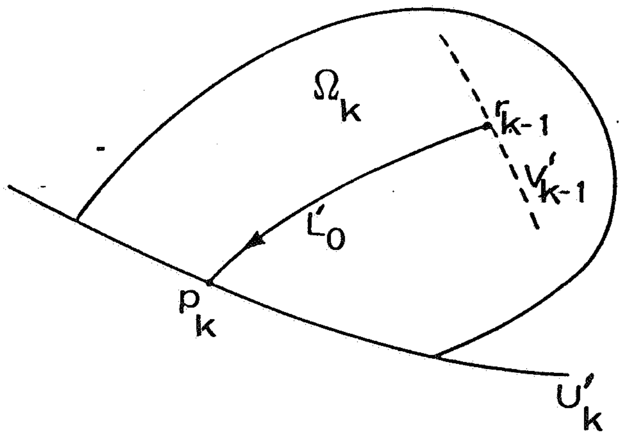


Fig.3 A segment of the discontinuous limit cycle L_0 between a return point and a leaving point is projected in y-space. In projection all trajectories leave Ω_k through U_k'.

Let for a trajectory starting at $y\in\Omega_k$ the probability of exit *before time t* through a surface element $dS_k(u)$, located at the point $u\in U_k'$, be denoted by

$$g_k(u,y,t)|dS_k(u)|. \tag{3.4}$$

The probability density of *ultimate exit* through $dS_k(u)$ is then given by

1. Here and in the following, S' is the projection of a set S onto the y-space, for any $S\in\mathbf{R}^{m+n}$.

$$\pi_k(u,y) = g_k(u,y,\infty) \tag{3.5}$$

and satisfies the Kolmogorov backward equation (GARDINER, 1983)

$$L_\delta \pi_k \equiv \sum_{i=1}^{n} G_i(y)\frac{\partial \pi}{\partial y_i} + \frac{1}{2}\delta^2 \sum_{i,j=1}^{n} a_{ij}(y)\frac{\partial^2 \pi}{\partial y_i \partial y_j} = 0 \text{ in } \Omega_k \tag{3.6a}$$

$$\pi_k = \delta_s(u-y) \text{ for } y \text{ on } \partial\Omega_k \text{ and } u \in U_k', \tag{3.6b}$$

where

$$a_{ij} = \sum_{k=1}^{p} \sigma_{ik}\sigma_{kj}, \tag{3.7}$$

and with $\delta_s(u-y)$ a delta function defined on $\partial\Omega_k$ such that for any test function f,

$$\int_{\partial\Omega_k} \delta_s(u-y)f(y)|dS_k(y)| = f(u), \quad u \in U_k'. \tag{3.8}$$

If the distribution of starting points at V'_{k-1} for a stationary relaxation oscillation is denoted by $f_{k-1}(y)$, then

$$f_k(u) = \int_{V'_{k-1}} \pi(u,y)f_{k-1}(y)|dS_{k-1}(y)| \tag{3.9}$$

is the distribution of arrival points at U_k' being also the stationary distribution of starting points at V'_k. From the set of N equations (3.9) with the index k taken modulo N we may determine $f_k(y), k=1,2,...,N$. For small noise intensity δ, the distribution is approximately of the form

$$f_k(y) = \frac{C_k}{\delta^{n-1}}\exp\{-\frac{(y-y_p^{(k)})^T A_k (y-y_p^{(k)})}{\delta^2}\}, \tag{3.10}$$

and one obtains a set of relations between $A_k, k=1,...,N$.

Next we consider the interjump time distribution. The time $T_k(y)$ needed to reach U'_k for the first time from a point y belonging to V'_{k-1} is a random variable with density

$$g_k(y,t) = \int_{U_k'} g_k(u,y,t)|dS_k(u)|. \tag{3.11}$$

Its first and second moments, $T_k^{(1)}(y)$ and $T_k^{(2)}(y)$, are defined by

$$T_k^{(n)}(y) = \int t^n \frac{\partial}{\partial t} g_k(y,t)dt, \quad n=1,2. \tag{3.12}$$

They satisfy

$$L_\delta T_k^{(1)} = -1 \text{ in } \Omega_k, \tag{3.13a}$$

$$T_k^{(1)} = 0 \text{ on } U_k', \tag{3.13b}$$

$$\partial T_k^{(1)}/\partial n = 0 \text{ on } \partial\Omega_k / U_k' \tag{3.13c}$$

and

$$L_\delta T_k^{(2)} = -2T_k^{(1)}, \tag{3.14a}$$

$$T_k^{(2)} = 0 \text{ on } U_k', \tag{3.14b}$$

$$\partial T_k^{(2)}/\partial n = 0 \text{ on } \partial\Omega_k / U_k', \tag{3.14c}$$

where the elliptic operator L_δ is defined by (3.6a). The unconditional probability density of the time T_k between jump $k-1$ and k is

$$g_k(t) = \int_{V_{k-1}'} g_k(y,t)f_k(y)|dS_k(y)| \tag{3.15}$$

with first and second moments given by

$$T_k^{(n)} = \int_{V_{k-1}'} T_k^{(n)}(y) f_k(y) |dS_k(y)|, \; n=1,2. \tag{3.16}$$

EXAMPLE: The Van der Pol oscillator with random forcing term.

We consider the following stochastically perturbed Van der Pol oscillator:

$$\epsilon dX = \{Y - \tfrac{1}{3}X^3 + X\}dt, \tag{3.17a}$$

$$dY = -Xdt + \delta dW. \tag{3.17b}$$

The local solutions of the equation

$$y = \tfrac{1}{3}x^3 - x \tag{3.18}$$

are $x = H_+(y)$ for $x > 1$ and $x = H_-(y)$ for $x < -1$, see fig. 1. In this case there are two jumps and, because of the symmetry, we only have to compute the distribution of one interjump time. The sets $\{U_k'\}$ each contain just one point, viz. $y = 2/3$ and $-2/3$, respectively.

Let us analyse the stochastic trajectories on the branch $x = H_-(y)$. The stochastic differential equation for the reduced problem ($\epsilon = 0$) reads

$$dY = -H_-(Y)dt + \delta dW, \tag{3.19a}$$

$$Y(0) = y, \; y < 2/3. \tag{3.19b}$$

The domain Ω is bounded by a reflecting boundary at $y = -\infty$ and an absorbing one at $y = 2/3$. For the problem (3.13) and (3.14) explicit solutions can be found:

$$T^{(1)}(y;\delta) = \frac{2}{\delta^2} \int_y^{2/3} \int_{-\infty}^{u} \exp[\frac{2}{\delta^2}\{R(u) - R(z)\}] dz du, \tag{3.20a}$$

$$T^{(2)}(y;\delta) = \frac{2}{\delta^2} \int_y^{2/3} \int_{-\infty}^{u} T^{(1)}(z;\delta) \exp[\frac{2}{\delta^2}\{R(u) - R(z)\}] dz du, \tag{3.20b}$$

where

$$R(y) = \int_{-\infty}^{y} H_-(u) du, \tag{3.21}$$

see GARDINER (1983). The integrals can be evaluated asymptotically for $0 < \delta << 1$,

$$T^{(n)}(y;\delta) = T_0^{(n)}(y) + \delta^2 T_1^{(n)}(y) + \cdots, \; n=1,2 \tag{3.22}$$

We find, temporarily suppressing the y-dependences,

$$T_0^{(1)} = -\tfrac{1}{2} + \tfrac{1}{2}H_-^2 - \ln(-H_-), \; T_0^{(2)} = \{T_0^{(1)}\}^2 \tag{3.23.a,b}$$

$$T_1^{(1)} = \tfrac{1}{4}(\frac{1}{H_-^2} - 1), \tag{3.23c}$$

$$T_1^{(2)} = -\tfrac{1}{4} - \tfrac{1}{4}H_-^2 + \frac{1}{2H_-^2} + 2\ln(-H_-). \tag{3.23d}$$

Consequently, the interjump time moments are found by substituting $y = -2/3$ and we have the following expected value and variance,

$$E(T) = (\tfrac{2}{3} - \ln 2) - \frac{3}{16}\delta^2 + O(\delta^4) \tag{3.24a}$$

$$Var(T) = (-\frac{9}{16} + \frac{13}{8}\ln 2)\delta^2 + \mathcal{O}(\delta^4) \tag{3.24b}$$

To verify this result a simulation of (3.17) was carried out by numerically solving the stochastic difference equations

$$X(t+h) = X(t) + \frac{h}{\epsilon}\{Y(t) - \tfrac{1}{3}X^3(t) + X(t)\} \tag{3.25a}$$

$$Y(t+h) = Y(t) - hX(t) + \delta\sqrt{h}\,G(t) \tag{3.25b}$$

where $G(t)$ is a generator of random numbers with a normal distribution $N(0,1)$. For $\epsilon=0.1$ and $\delta=0.25$ we find an agreement with (3.24) with an accuracy of 0.001 for a sample of about 100 interjump times. For δ sufficiently small a linear noise approximation can be made: the distribution of interjump times then has a normal distribution. For larger values of δ, however, there will be nonnegligible contributions from the tail of the normal distribution with negative values of the interjump time, which are physically impossible. In that case it is proposed to use the inverse Gaussian distribution

$$f(t;\mu,\lambda) = \sqrt{\frac{\lambda}{2\pi t^3}}\,\exp\{-\frac{\lambda(t-\mu)^2}{2\mu^2 t}\},\quad t \geqslant 0 \tag{3.26}$$

as an approximation. Its expected value and variance are

$$E(T) = \mu,\quad Var(T) = \mu^3/\lambda. \tag{3.27a,b}$$

If in (3.19a) H_- were a positive constant, this density function would be exact, see KARLIN and TAYLOR (1975). In fig. 4 the distribution of 197 interjump times, numerically computed by (3.25), is given in a histogram. For the values

$$\epsilon=0.1 \text{ and } \delta=0.75 \tag{3.28}$$

the asymptotic theory yields

$$E(T) = T_0(\epsilon) - \frac{3}{16}\delta^2 = \frac{3}{2} - \ln 2 + \frac{3}{2}\alpha\epsilon^{2/3} - \frac{3}{16}\delta^2 = 1.3, \tag{3.29a}$$

$$Var(T) = (-\frac{9}{16} + \frac{13}{8}\ln 2)\delta^2 = 0.3. \tag{3.29b}$$

Consequently, in the inverse Gaussian distribution function (3.26) we have to take

$$\mu=1.3 \text{ and } \lambda=7.4. \tag{3.30}$$

In fig. 4 this distribution is represented by a solid line. It is noted that it fits the data much better than the normal distribution with the same mean and variance (dashed line).

4. CHAOTIC RELAXATION OSCILLATIONS

From studies of LORENZ (1963), SMALE (1967), RUELLE and TAKENS (1971) and others it is known that continuous-time dynamical systems may exhibit chaotic behaviour. It is quite well possible to construct a system of type (2.1) possessing a limit solution being a strange attractor and having some of the qualitative properties of a relaxation oscillation. By example we demonstrate that a system that remains most of the time in a two-dimensional manifold, still may exhibit chaotic dynamics.

In fig. 5 we sketch trajectories of a system with one fast and two slow variables. It is seen that for $\epsilon \to 0$ the Poincaré mapping of the compact interval AB into itself agrees qualitatively with the logistic map. In this mapping the phenomena of period doubling and chaos are present, see MAY (1967) and COLLET and ECKMANN (1980). Examples of chaotic relaxation oscillations in fluid mechanics and biology are given by LOZI (1983). ARGÉMI and ROSSETTO (1983) model irregular electrical activity of nerve cells by a chaotic relaxation oscillator. More systematic studies of chaotic dynamics in constrained equations ($\epsilon=0$) are presented by USHIKI and LOZI (1986), OKA and KOKUBU (1985) and TAKENS (1986).

EXAMPLE: A chaotic Van der Pol oscillator.
The Van der Pol oscillator (1.1) can be augmented with a third state variable, such that the resulting system of equations will have chaotic dynamics, while the regular Van der Pol relaxation oscillator is recovered if the third variable is set to zero. This is achieved by construction of a vector field similar to the one of fig. 5. After some experimentation we found a system with the simplest polynomials

$$\epsilon\frac{dx}{dt} = y-\tfrac{1}{3}x^3 + x, \tag{4.1a}$$

$$\frac{dy}{dt} = -x-x^2z, \tag{4.1b}$$

$$\frac{dz}{dt} = (x+a)z^2. \tag{4.1c}$$

In fig. 6a we sketch the trajectories in the x,z-plane for $\epsilon\to 0$. The solution jumps from $x=1$ to $x=-2$ and from $x=-1$ to $x=2$. The Poincaré mapping P_0 of the line $(x,y,z)=(2,2/3,z)$ into itself is given in fig. 6b. For $a<a_1$ one periodic solution is found; repeated period doubling occurs at a_k with

$$a_1 = 1.46,\ a_2 = 1.59,\ a_3 = 1.64,.... \tag{4.2}$$

At $a_\infty=1.66$ the domain with chaotic solutions is entered.
For $a=1.75$ we computed the Lyapunov exponent of P_0, yielding $\lambda=.33$. Taking ϵ a fixed positive constant we may derive the Lyapunov exponents of a mapping P_ϵ of a transversal plane into itself. Independent of the particular choice of the plane that is made, we obtain

$$\lambda_1 = .51,\ \lambda_2 = -.08 \quad \text{for } \epsilon = .05. \tag{4.3}$$

Finally, using the scheme of WOLF *et al.* (1985) we compute the Lyapunov exponents of the flow of (4.1):

$$\lambda_1=.09,\ \lambda_2=0 \text{ and } \lambda_3=-26.45 \text{ for } a=1.75 \text{ and } \epsilon=.05. \tag{4.4}$$

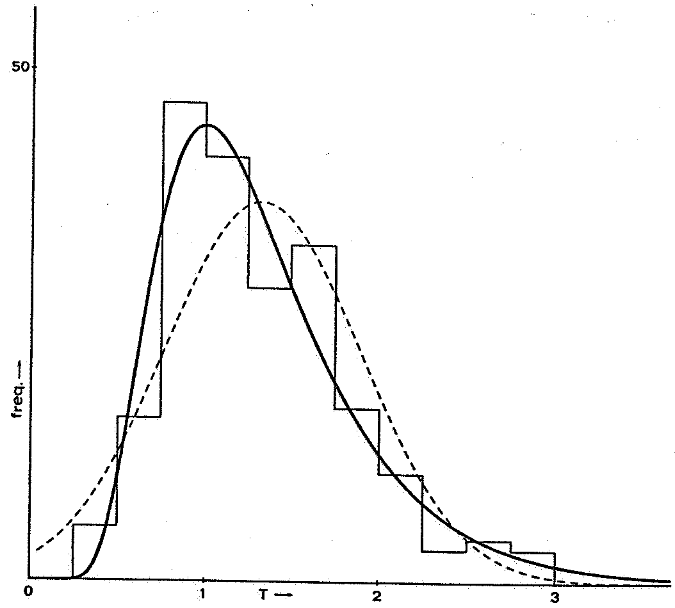


Fig.4 Distribution of 197 interjump times from a simulation run of the system (3.17) and its approximation by a normal distribution (dotted line) and by an inverse Gaussian distribution (solid line).

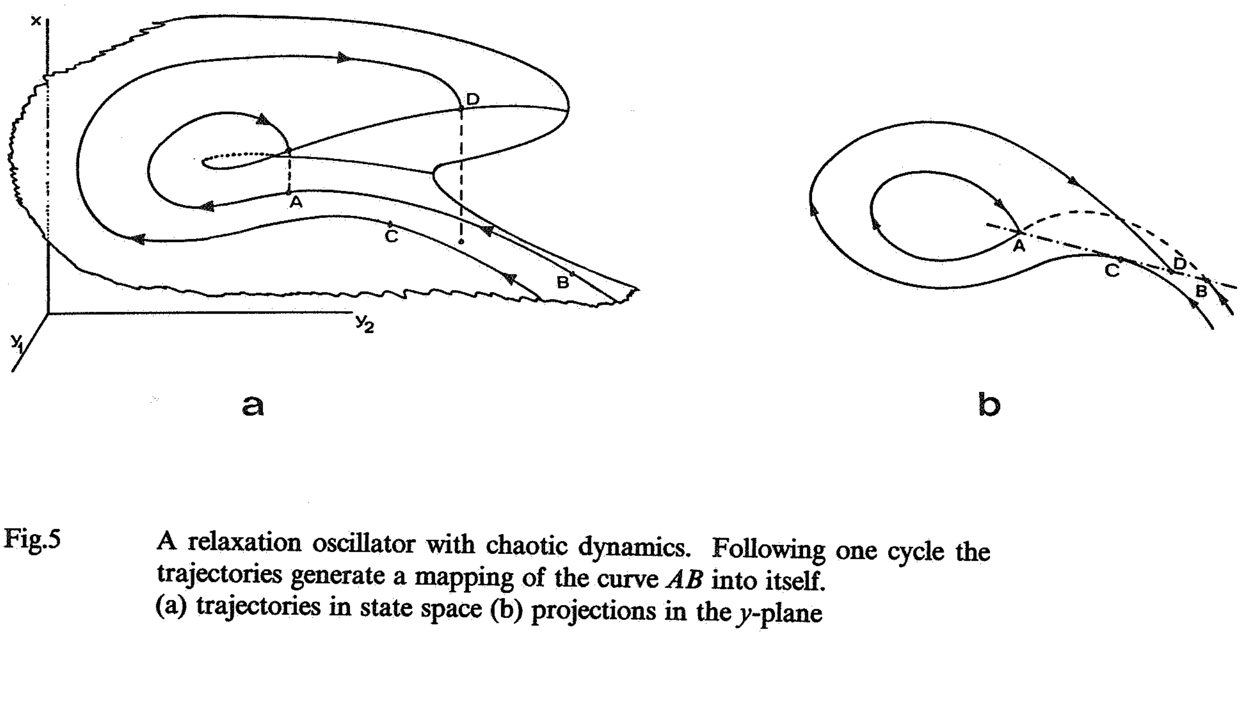
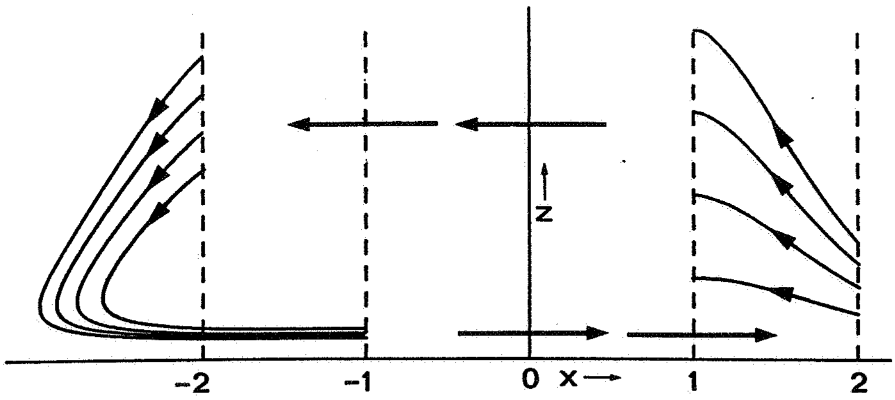


Fig.5 A relaxation oscillator with chaotic dynamics. Following one cycle the trajectories generate a mapping of the curve *AB* into itself.
(a) trajectories in state space (b) projections in the *y*-plane



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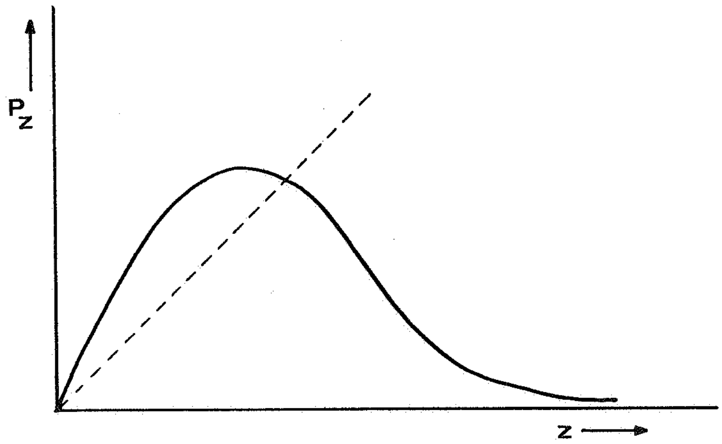

b

Fig.6 A Van der Pol type relaxation oscillator with a strange attractor in the limit $\epsilon \to 0$.
(a) trajectories projected in the x,z-plane (b) the mapping at $(x,y)=(2,2/3)$

5. Stochastic versus chaotic dynamics of relaxation oscillations

In the foregoing sections we analyzed the dynamics of explicitly given oscillators which either had a white noise input or possessed an intrinsic random behaviour. For physical systems the exact equations are not always known, so that one has to decide from an output signal about the internal dynamics of the system. When a physical system shows fluctations in the period, one is inclined to ascribe this to perturbations coming from the environment (noise).

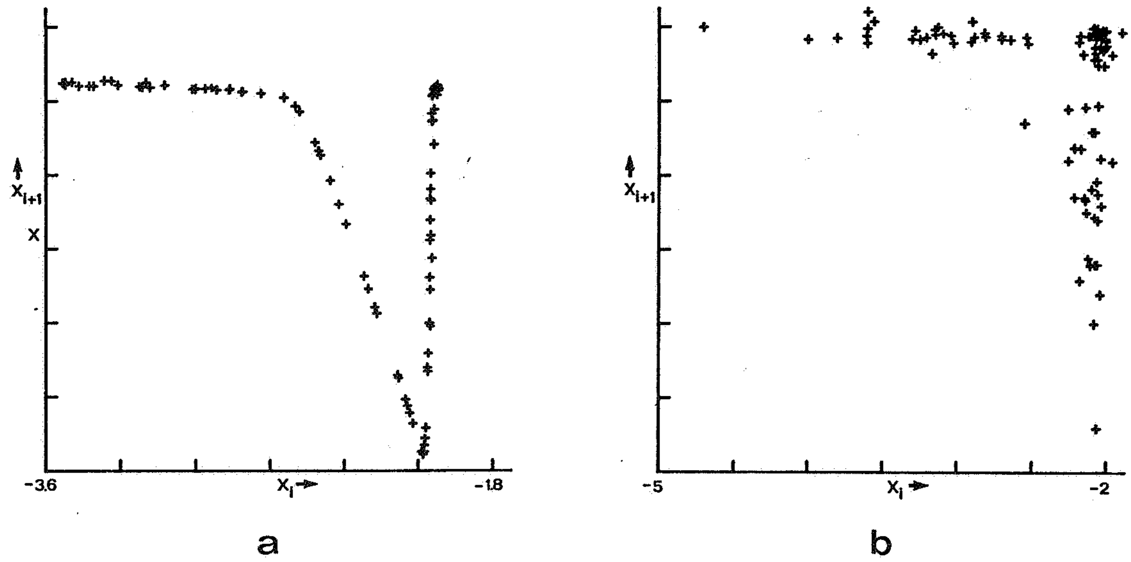

Fig.7 Return map of the lowest value of X in a cycle for the system (5.1).
(a) $\delta=0$ (b) $\delta=0.3$

In a chemical system an intermediate reactant, which is present for only a short time and therefore not noticed (hidden), may influence the dynamics and give rise to chaotic dynamics. The perturbation of the period has, in that case, a more fundamental cause, see Engel-Herbert *et al.* (1985). On the other hand noise may regularize chaotic dynamics and make it look periodic, see Matsumoto and Tsuda (1983) and Matsumoto (1984). They speak of hidden (chaotic) dynamics.

In order to explore the properties of stochastically perturbed systems with periodic or chaotic solutions, we analyse as a special case the chaotic Van der Pol oscillator of the preceding section with a small white noise input:

$$\epsilon dX = (Y-\tfrac{1}{3}X^3+X)dt, \tag{5.1a}$$

$$dY = (-X-X^2Z)dt + \delta dW, \tag{5.1b}$$

$$dZ = (X+a)Z^2 dt, \tag{5.1c}$$

where $W(t)$ is a Wiener process. We analyse its properties from the output signal $X(t)$ only. For different values of a and δ we take from the x-component a time series of 5000 points with a time step of 0.01.

First we consider the return map for the lowest x-value over each cycle, see fig. 7. For $a = 1.7$ and $\delta = 0$ the points form a Cantor set. For small noise a cloud of points around this set is expected. In fig. 7b we observe a breakdown of this cloud for $\delta = 0.3$. From the clustering of data around a single point one would be inclined to conclude that the system is more or less periodic with some stochastic distribution. The power spectrum (see fig. 8) confirms this observation: the peaks are higher and the area below the curve has relatively decreased, which is in agreement with the analysis of MATSUMOTO (1984).

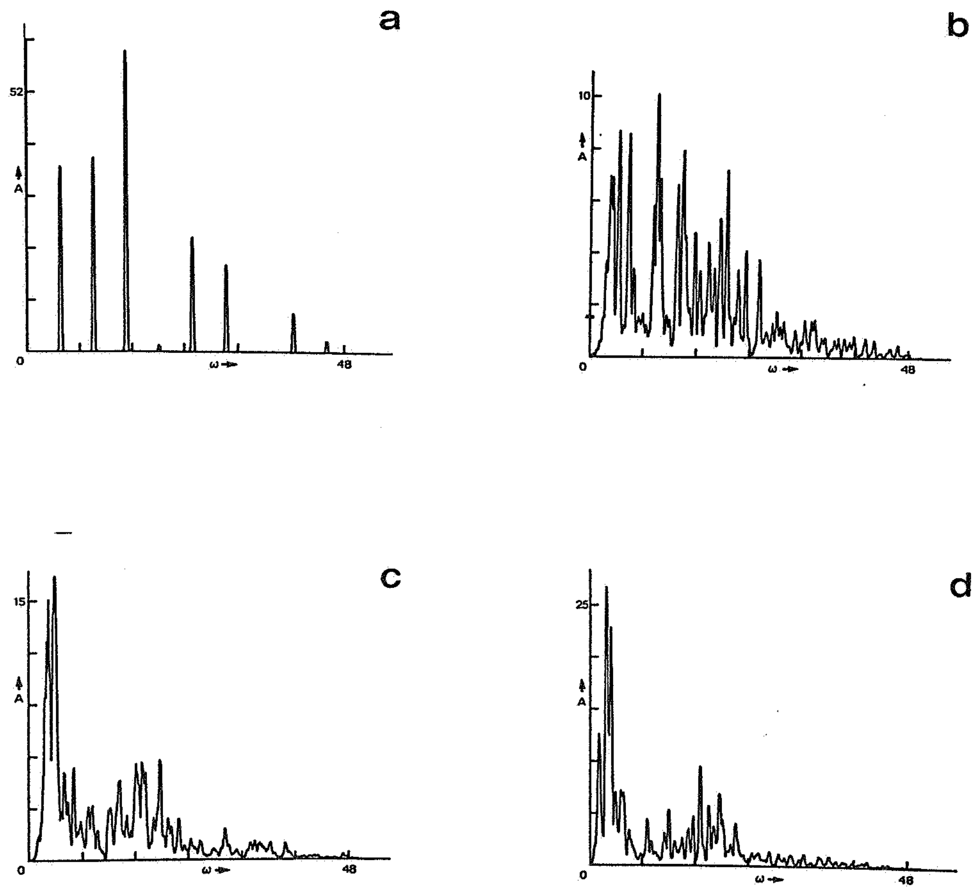

Fig.8 Power spectra of the system (5.1)
(a) $a = 1.4$ $\delta = 0$ (b) $a = 1.4$ $\delta = 0.3$
(c) $a = 1.7$ $\delta = 0$ (d) $a = 1.7$ $\delta = 0.3$

Finally, in fig. 9 we plot the largest Lyapunov exponent λ_1 as a function of the proportion of the time series that is used in the approximation process, see WOLF *et al.* (1985). The process converges to a value 0.0075 with fluctuations of magnitude 0.0025, both in the case without ($\delta = 0$) and with noise ($\delta = 0.3$). Variation of the tuning parameters of the approximation process does not have any significant influence upon the value of the largest Lyapunov exponent. Clearly, the presence of chaotic dynamics can be detected quite well from the output signal by the method of WOLF *et al.* (1985). This criterion is more dependable than an inspection of the power spectrum or a return map, particularly when the signal is corrupted by noise.

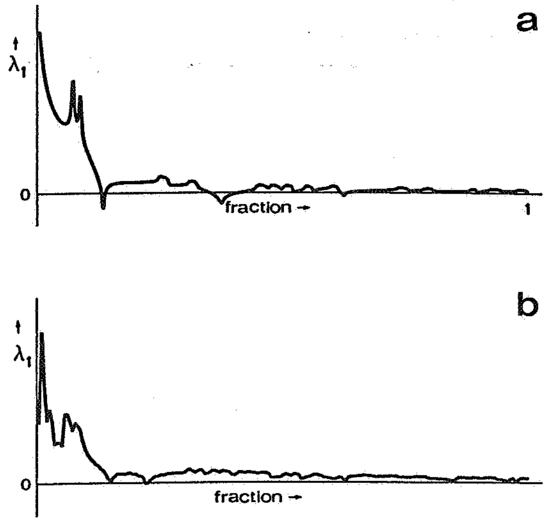

Fig.9 Largest Lyapunov exponent from the data set as a function of the fraction of the time series that is used in the approximation process. (a) $a = 1.7$ $\delta = 0$ (b) $a = 1.7$ $\delta = 0.3$

LITERATURE

ARGÉMI, J., and B. ROSSETTO (1983). *Solutions periodiques discontinues pour l' approximation singulière d'un modèle neurophysiologique dans R^4-une metaphore dans R^3 avec chaos,* J. Math. Biol. **17**, p. 67-92.

COLLET, P., and J.-P. ECKMANN (1980). *Iterated Maps on the Interval as Dynamical Systems,* Birkhäuser, Basel.

ENGEL-HERBERT, H., W. EBELING and H. HERZEL (1985). *The influence of fluctuations on sustained oscillations,* in Temporal Order, L. Rensing and N.I. Jaeger (eds.), Springer Verlag, Berlin, p. 144-152.

GARDINER, C.W. (1983). *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences,* Springer-Verlag, Berlin.

GRASMAN, J. (1987). *Asymptotic Methods for Relaxation Oscillations and Applications,* Springer-Verlag, New York, to appear.

KARLIN, S., and H.M. TAYLOR (1975). *A First Course in Stochastic Processes,* Acad. Press, New York.

LORENZ, E.N. (1963). *Deterministic nonperiodic flow,* J. Atmos. Sci, **20**, p. 130-141.

LOZI, R. (1983). *Modèles Mathematiques Qualitatifs Simples et Consistants pour L'Etude de Quelques Systemes Dynamiques Experimentaux,* Thesis, Univ. of Nice.

MATSUMOTO, K. and I. TSUDA (1983) *Noise-induced order,* J. Stat. Phys. **31**, p. 87-106.

MATSUMOTO, K. (1984). *Noise-induced order II,* J. Stat. Phys. **34**, p. 111-127.

MAY, R.M. (1976). *Simple mathematical models with very complicated dynamics,* Nature **216**, p. 459-

467.
MISHCHENKO, E.F., and N.K. ROSOV (1980). *Differential Equations with Small Parameters and Relaxation Oscillations,* Plenum Press, New York.
OKA, H. and H. KOKUBU (1985). *Constrained Lorenz-like attractors,* Japan J. of Appl. Math. **2,** p. 495-500.
RUELLE, D., and F. TAKENS (1971). *On the nature of turbulence,* Comm. Math. Phys. **20,** p. 167-192.
SMALE, S. (1967). *Differentiable dynamical systems,* Bull. Am. Soc. **73,** p. 747-817.
TAKENS, F. (1986). *Transitions from periodic to strange attractors in constrained equations,* Report ZW-8601, Univ. of Groningen, The Netherlands.
USHIKI, S., and R. LOZI (1986). *Confinor and anti-confinor in constrained "Lorenz" system,* preprint Univ. of Nice.
WOLF, A., J.B. SWIFT, H.L. SWINNEY and J.A. VASTANO (1985). *Determining Lyapunov exponents from a time series,* Physica **16D,** p. 285-317.