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Department of Operations Research and System Theory Report OS-R8719 November

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Decomposition of Graphs on Surfaces and a Homotopic Circulation Theorem

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Abstract. - We prove the following theorem. Let G be an eulerian graph embedded (without crossings) on a compact orientable surface S. Then the edges of G can be decomposed into cycles $C_1,\ldots,C_t$ in such a way that for each closed curve D on S:

$$\text{mincr}(G,D) = \sum_{i=1}^{t} \text{mincr}(C_i,D).$$

Here mincr(G,D) denotes the minimum number of crossings of G and $\tilde{D}$, among all closed curves $\tilde{D}$ homotopic to D (so that $\tilde{D}$ does not intersect vertices of G). Similarly, mincr(C,D) denotes the minimum number of crossings of $\tilde{C}$ and $\tilde{D}$, among all closed curves $\tilde{C}$ and $\tilde{D}$ homotopic to C and D, respectively.

As a corollary we derive the following 'homotopic circulation theorem'. Let G be a graph embedded on a compact orientable surface S, let $c:E\to\mathbb{Q}_+$ be a 'capacity' function, let $C_1,\ldots,C_k$ be cycles in G, and let $d_1,\ldots,d_k \in\mathbb{Q}_+$ be 'demands'. Then there exist circulations $x_1,\ldots,x_k$ in G so that each x_i decomposes fractionally into d_i cycles homotopic to C_i $(i=1,\ldots,k)$ and so that the total flow through any edge does not exceed its capacity, if and only if for each closed curve D on S not intersecting vertices of G we have that the sum of the capacities of the edges intersected by D (counting multiplicities) is not smaller than $\sum_{i=1}^{k} d_i\cdot\text{mincr}(C_i,D)$.

This applies to a problem posed by K. Mehlhorn in relation to the automatic design of integrated circuits.

1980 Mathematics Subject Classification (Revised 1985): Primary: 05Cxx, 57Mxx, 90C27. Secondary: 05C10, 05C38, 57N06.
Key Words and Phrases: Surface, homotopic, graph, curve, circulation, VLSI, crossing.

Report OS-R8719
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1. SURVEY OF RESULTS

In this paper we prove a number of theorems on the decomposition into cycles of the edges of graphs embedded on a compact orientable surface S. (Recall that a compact orientable surface is nothing else than a two-dimensional sphere added with a finite number of 'handles'.) As a main application we give a characterization of the existence of a circulation of a prescribed homotopy type in such graphs. The proof methods used are based on analyzing curves on surfaces and their crossings, and on some classical results in topology (due to Poincaré, Baer, von Kerékjártó, Brouwer) and linear algebra (Farkas' lemma).

The following result plays a crucial role.

THEOREM 1. *Let* G *be an eulerian graph embedded on a compact orientable surface* S. *Then the edges of* G *can be decomposed into cycles* $C_1,\ldots,C_t$ *in such a way that for each closed curve* D *on* S:

$$\text{(1)} \qquad \text{mincr}(G,D) = \sum_{i=1}^{t} \text{mincr}(C_i,D).$$

Here and in the sequel we use the following conventions and notation. The graph G has a finite number of vertices and edges, while loops and multiple edges are allowed. A graph is *eulerian* if all its degrees are even (where a loop at vertex v counts for two in the degree of v) - so connectedness is not required. *Embedding* a graph means embedding without intersecting edges. We identify an embedded graph with its

A *cycle* in G is a sequence

$$\text{(2)} \qquad (v_0,e_1,v_1,e_2,v_2,\ldots,e_\ell,v_\ell),$$

where $v_0,\ldots,v_\ell$ are vertices, with $v_0=v_\ell$, and where e_i is an edge connecting v_{i-1} and v_i $(i=1,\ldots,\ell)$. (If e_i is a loop, then we assume that an orientation of e_i is also specified.) *Decomposing the edges into cycles* $C_1,\ldots,C_t$ means that each edge occurs in exactly one of the C_i, while in each C_i all edges are different.

A *closed curve* on S is a continuous function $D:S_1 \longrightarrow S$, where S_1 denotes the unit circle $\{z \in \mathbb{C} \mid |z|=1\}$ in the complex plane. So each cycle (2) in G gives rise to a closed curve on S, which curve we identify with the cycle. Two closed curves D and D' on S are called *homotopic* (*on* S), denoted by $D \sim D'$, if there exists a continuous function $\Phi:S_1 \times [0,1] \longrightarrow S$ such that $\Phi(z,0)=D(z)$ and

$\Phi(z,1)=D'(z)$ for all $z \in S_1$. (This is sometimes called *freely homotopic* as we do not fix a 'base point'.)

We denote, if D is a closed curve on S not intersecting V:

$$(3)\qquad \begin{aligned} cr(G,D) &:= \left|\{z \in S_1 \mid D(z) \text{ belongs to } G\}\right|, \\ mincr(G,D) &:= \min\left\{cr(G,\tilde{D}) \mid \tilde{D} \sim D,\ \tilde{D} \text{ does not intersect } V\right\}. \end{aligned}$$

Moreover, if C and D are closed curves on S:

$$(4)\qquad \begin{aligned} cr(C,D) &:= \left|\{(y,z) \in S_1 \times S_1 \mid C(y)=D(z)\}\right|, \\ mincr(C,D) &:= \min\left\{cr(\tilde{C},\tilde{D}) \mid \tilde{C} \sim C,\ \tilde{D} \sim D\right\}. \end{aligned}$$

As is well-known, mincr(C,D) is finite.

Note that the inequality $\geq$ in (1) is trivial, and holds for any decomposition of the edges of G into cycles. The essence of the theorem is that there exists a decomposition which has equality in (1).

Note 1. We do not know if the orientability condition for S in Theorem 1 is necessary. In fact, the theorem is true also if we take for S the projective plane, which result is equivalent to a theorem of Lins [10].

Our proof below also works for nonorientable compact surfaces, if we restrict D in Theorem 1 to 'orientable' closed curves, i.e., to those closed curves for which left and right do not flip when making one orbit (these are the curves which generate an orientable covering surface, in the sense of Section 2). □

By means of the duality relation between graphs embedded on a surface we derive from Theorem 1:

THEOREM 2. *Let* $G = (V,E)$ *be a bipartite graph embedded on a compact orientable surface* S, *and let* $C_1,\ldots,C_k$ *be cycles in* G. *Then there exist closed curves* $D_1,\ldots,D_t : S_1 \rightarrow S$ *so that (i) no* D_j *intersects* V, *(ii) each edge of* G *is intersected by exactly one* D_j *and by that* D_j *only once, (iii) for each* $i=1,\ldots,k$:

$$(5)\qquad \text{minlength}_G(C_i) = \sum_{j=1}^{t} \text{mincr}(C_i,D_j).$$

Here we denote for any cycle C in G:

$$\text{(6)}\qquad \begin{aligned} \text{length}_G(C) &:= \ell, \text{ if } C = (v_0, e_1, v_1, \ldots, e_\ell, v_\ell), \\ \text{minlength}_G(C) &:= \min \left\{ \text{length}_G(\breve{C}) \mid \breve{C} \sim C,\ C \text{ cycle in } G \right\}. \end{aligned}$$

(Cycles $\breve{C}$ and C are allowed to pass one edge several times.)

<u>Note 2</u>. In fact, if G is a bipartite graph embedded on the compact orientable surface S so that each face (=component of $S \setminus G$) is simply connected, then there exist closed curves $D_1, \ldots, D_t$ satisfying (i) and (ii) of Theorem 2 so that $\text{minlength}_G(C) = \sum_{j=1}^{t} \text{mincr}(C, D_j)$ for *all* cycles C in G.

It is not difficult to see that this implies that if G is a bipartite graph embedded on the torus S, then also the above comclusion holds (as either each face is simply connected, or there is essentially only one cycle C in G to consider) - cf. [19].

This is not true for surfaces with more handles (so in general one must specify an arbitrary, but finite, number of cycles C_i in G in advance) - see Remark in Section 6.

In [19] it is also shown that if S is the torus, k=1 and $C := C_1$ is homotopic to a simple (i.e., not self-intersecting) closed curve, then we can delete the bipartiteness condition in Theorem 2. This result is equivalent to: let G=(V,E) be a graph embedded on the torus S, and let C be a simple closed curve on S. Then

$$\text{(7)}\qquad \text{minlength}_G(C) = \max \sum_{j=1}^{t} \text{mincr}(C, D_j),$$

where the maximum ranges over all collections of closed curves $D_1, \ldots, D_t$ on S not intersecting V and intersecting each edge of G at most once.

In terms of integer linear programming, this is equivalent to the 'total dual integrality' of the following system of linear inequalities in the variable x $\mathbb{R}^E$: (using notation (10) below):

$$\text{(8)}\qquad \begin{array}{lll} \text{(i)} & x_e \geqslant 0 & (e \in E). \\ \text{(ii)} & \sum_{e \in E} \chi^D(e) x_e \quad \text{mincr}(C,D) & (D \text{ closed curve on } S \setminus V). \end{array}$$

(Total dual integrality means that any linear program over (8) with integral objective function has integral primal and dual optimum solutions (cf. [18]). It can be shown that (8) can be restricted to only a finite number of inequalities.)

□

By means of the polarity relation between convex cones in euclidean space (i.e., Farkas' lemma), we deduce from Theorem 2 the following 'homotopic circulation theorem', where we use notation as follows. Let $G=(V,E)$ be embedded on a surface S. Let $C = (v_0,e_1,v_1,\ldots,e_\ell,v_\ell)$ be a cycle in G. Then the function $\chi^C: E \to \mathbb{Z}_+$ is defined by:

(9) $$\chi^C(e) = |\{j=1,\ldots,\ell \mid e=e_j\}| \qquad \text{for } e \in E.$$

Let $D: S_1 \to S$ be a closed curve on S, not intersecting V, and with $cr(G,D)$ finite. Then the function $\chi^D: E \to \mathbb{Z}_+$ is defined by:

(10) $$\chi^D(e) := |\{z \in S_1 \mid D(z) \text{ belongs to } e\}| \qquad \text{for } e \in E.$$

<u>THEOREM 3</u>. *Let* $G = (V,E)$ *be a graph embedded on a compact orientable surface* S, *and let* $c: E \to \mathbb{Q}_+$ *('capacity function'). Let* $C_1,\ldots,C_k$ *be cycles in* G, *pairwise not-homotopic, and let* $d_1,\ldots,d_k \in \mathbb{Q}_+$ *('demands'). Then there exist cycles* $\Gamma_1,\ldots,\Gamma_u$ *in* G *and* $\lambda_1,\ldots,\lambda_u \geq 0$ *so that:*

(11) $$\text{(i)} \quad \sum_{\substack{j=1 \\ \Gamma_j \sim C_i}}^{u} \lambda_j = d_i \qquad (i=1,\ldots,k),$$

$$\text{(ii)} \quad \sum_{j=1}^{u} \lambda_j \cdot \chi^{\Gamma_j}(e) \leq c(e) \qquad (e \in E),$$

if and only if for each closed curve D *on* S *not intersecting* V *we have:*

(12) $$\sum_{i=1}^{k} d_i \cdot \mathrm{mincr}(C_i,D) \leq \sum_{e \in E} c(e) \cdot \chi^D(e).$$

<u>Note 3</u>. In [19] and [20] it is shown that if each C_i is homotopic to a simple closed curve, all capacities and demands are integers, and *at least two* of the following conditions hold:

(13) (i) S is the torus;

(ii) $k = 1$;

(iii) the two sides in (12) have the same parity, for each D;

then we can take the λ_i in Theorem 3 integral. This implies that if each C_i

is homotopic to a simple closed curve, all capacities and demands are integers, and (13)(i) or (ii) holds, then we can take the λ_i to be half-integral.

If S is the torus and k=1, this is equivalent to the following min-max result. Let G=(V,E) be a graph embedded on the torus S, and let C be a simple closed curve on S. Then the maximum number of pairwise edge-disjoint cycles in G, each homotopic to C, is equal to the minimum value of

$$(14)\qquad \min_D \left\lfloor \frac{cr(G,D)}{mincr(C,D)} \right\rfloor,$$

where the minimum ranges over all closed curves D on $S \setminus V$ with $mincr(C,D) \geq 1$. ($\lfloor\ \rfloor$ denotes lower integer part.) In other words, system (8) has the 'integer decomposition property' - cf. [18].

In the Remark in Section 7 we give an example showing that (13)(iii) is not enough to imply integrality of the λ_i , for a general compact orientable surface S. Yet the following min-max *almost*-equality can be shown (cf. [20]). Let G=(V,E) be a graph embedded on a compact orientable surface S, and let C be a simple closed curve on S. Then the maximum number M of pairwise edge-disjoint cycles in G, each homotopic to C, satisfies

$$(15)\qquad \min_D \left\lceil \frac{cr(G,D)}{mincr(C,D)} \right\rceil -1 \leq M \leq \min_D \left\lfloor \frac{cr(G,D)}{mincr(C,D)} \right\rfloor,$$

where the minima range over all closed curves D on S V with $mincr(C,D) \geq 1$. ($\lceil\ \rceil$ denotes upper integer part.) □

As a consequence of Theorem 3 we derive a 'homotopic flow-cut theorem'.

<u>THEOREM 4</u>. *Let* G = (V,E) *be a planar graph embedded in the complex plane* $\mathbb{C}$. *Let* $I_1,\dots,I_p$ *be (the interiors of) some of the faces of* G, *including the unbounded face. Let* $P_1,\dots,P_k$ *be paths in* G *with end points on the boundary of* $I_1 \cup \dots \cup I_p$. *Then there exist paths* $P_1^1,\dots,P_1^{t_1},P_2^1,\dots,P_2^{t_2},\dots,P_k^1,\dots,P_k^{t_k}$ *in* G *and rationals* $\lambda_1^1,\dots,\lambda_1^{t_1},\lambda_2^1,\dots,\lambda_2^{t_2},\dots,\lambda_k^1,\dots,\lambda_k^{t_k} \geq 0$ *so that:*

(16) (i) $P_i^j \sim P_i$ *in* $\mathbb{C} \setminus (I_1 \cup \dots \cup I_p)$ $\qquad (i=1,\dots,k;j=1,\dots,t_i)$,

(ii) $\sum_{j=1}^{t_i} \lambda_i^j = 1$ $\qquad (i=1,\dots,k)$,

(iii) $\sum_{i=1}^{k} \sum_{j=1}^{t_i} \lambda_i^j \chi^{P_i^j}(e) \leq 1$ $\qquad (e \in E)$,

if and only if for each path $D:[0,1] \to \mathbb{C} \setminus (I_1 \cup \dots \cup I_p)$, *connecting two points* *on the boundary of* $I_1 \cup \ldots \cup I_p$*, not intersecting* V*, and intersecting* G *only a finite number of times, we have:*

(17) $$\sum_{i=1}^{k} \text{mincr}(P_i, D) \leq \text{cr}(G,D).$$

Here we use similar notation and terminology as before. A *path* in G is a sequence $(v_0, e_1, v_1, e_1, v_1, \ldots, e_\ell, v_\ell)$ where $v_0, \ldots, v_\ell$ are vertices, and where e_i is an edge connecting v_{i-1} and v_i $(i=1,\ldots,\ell)$. A *(topological) path* in a topological space T is a continuous function $D:[0,1] \to T$. The points D(0) and D(1) are the *end points* of D. So each path in G gives rise to a topological path on $\mathbb{C}$, which two paths we identify. Two paths $D,D':[0,1] \to T$ are called *homotopic (in* T), denoted by $D \sim D'$, if there exists a continuous function $\Phi:[0,1] \times [0,1] \to T$ so that $\Phi(x,0)=D(x)$, $\Phi(x,1)=D'(x)$, $\Phi(0,x)=D(0)$, $\Phi(1,x)= D(1)$ for all $x \in [0,1]$. (It follows that $D(0)=D'(0)$ and $D(1)=D'(1)$.) If C and D are paths in $\mathbb{C} \setminus (I_1 \cup \ldots \cup I_p)$, then:

(18) $$\begin{aligned} \text{cr}(C,D) &:= \left|\{(x,y) \in [0,1] \times [0,1] \mid C(x)=D(y)\}\right|, \\ \text{mincr}(C,D) &:= \min\{\text{cr}(\tilde{C},\tilde{D}) \mid \tilde{C} \sim C,\ \tilde{D} \sim D \text{ in } \mathbb{C} \setminus (I_1 \cup \ldots \cup I_p)\}. \end{aligned}$$

If $C=(v_0,e_1,v_1,\ldots,e_\ell,v_\ell)$ is a path in G, then $\chi^C:E \to \mathbb{Z}_+$ is defined by:

(19) $$\chi^C(e) := \left|\{j=1,\ldots,\ell \mid e=e_j\}\right| \qquad \text{for } e \in E.$$

If $D:[0,1] \to \mathbb{C} \setminus (I_1 \cup \ldots \cup I_p)$ is a path not intersecting V and intersecting E only a finite number of times, then $\chi^D:E \to \mathbb{Z}_+$ is defined by:

(20) $$\chi^D(e) := \left|\{x \in [0,1] \mid D(x) \text{ belongs to } e\}\right| \qquad \text{for } e \in E.$$

Note 4. It was shown by van Hoesel and Schrijver [7] that if p=2, we can take all λ_i^j in Corollary 3 equal to ½. More generally, the following was shown:

(21) *Let* $G=(V,E)$ *be a planar graph, embedded in* $\mathbb{C}$. *Let* O *denote the interior of the unbounded face, and let* I *be the interior of some other fixed face. Let* $P_1,\ldots,P_k$ *be paths in* G*, each with end points on the boundary of* $I \cup O$*, so that for each vertex* v *of* G *the number*

(22) $$\deg(v) + \left|\{i=1,\ldots,k \mid P_i \text{ begins in } v\}\right| + \left|\{i=1,\ldots,k \mid P_i \text{ ends in } v\}\right|$$

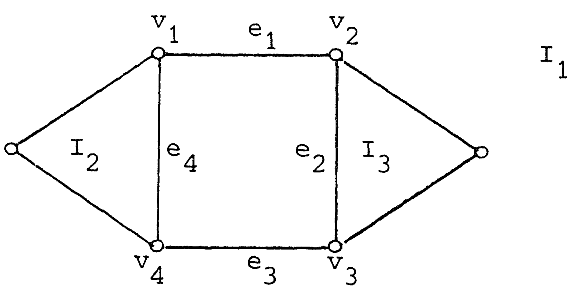
is even. Then there exist pairwise edge-disjoint paths $P'_1,\ldots,P'_k$ *in* G *so that* P'_i *is homotopic to* P_i *in* $\mathbb{C}\setminus(I\cup O)$ *for* $i=1,\ldots,k$, *if and only if for each path* $D:[0,1]\rightarrow\mathbb{C}\setminus(I\cup O)$ *connecting two points on the boundary of* $I\cup O$, *not intersecting* V *and intersecting* G *only a finite number of times, we have:*

$$\text{(23)}\qquad \sum_{i=1}^{k} \text{mincr}(P_i,D) \leq \text{cr}(G,D).$$

Here deg(v) denotes the degree of v in G.

This result was conjectured by K. Mehlhorn in relation to the automatic design of integrated circuits (see [8]). It generalizes the Okamura-Seymour theorem [14] . Result (21) cannot be extended to more than two faces (in the sense of Theorem 4), as is shown by the following example:

(24)



with $P_1 := (v_1,e_1,v_2,e_2,v_3)$ and $P_2 := (v_2,e_2,v_3,e_3,v_4)$. In fact, Theorem 4 answers affirmatively a question asked to me by Professor C.St.J.A. Nash-Williams in Oberwolfach, after discussing (21) and (24).

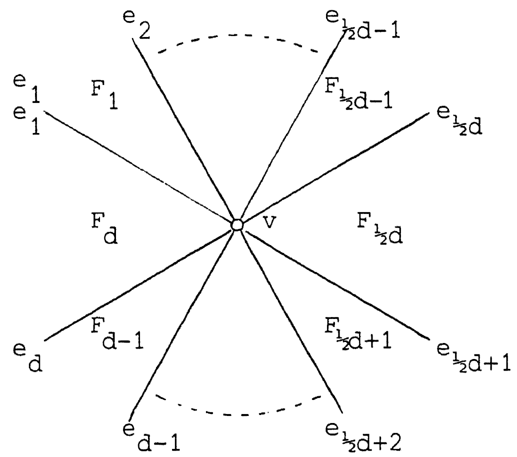
Integrality results for graphs obtained from the planar rectangular grid were obtained by Kaufmann and Mehlhorn [8]. An extension (derived with the help of Theorem 4) to 'straight-line' planar graphs is given in [21].

The Main Lemma and Theorem 5: the relation between minimally connected graphs and minimally crossing collections of curves.

As mentioned, we derive Theorem 4 from Theorem 3, Theorem 3 from Theorem 2, and Theorem 2 from Theorem 1. Theorem 1 itself is derived from a 'Main Lemma'. The content of this Main Lemma is the 'only if' part in our fifth theorem, whose 'if' part is derived from Theorem 3 (and hence from the Main Lemma).

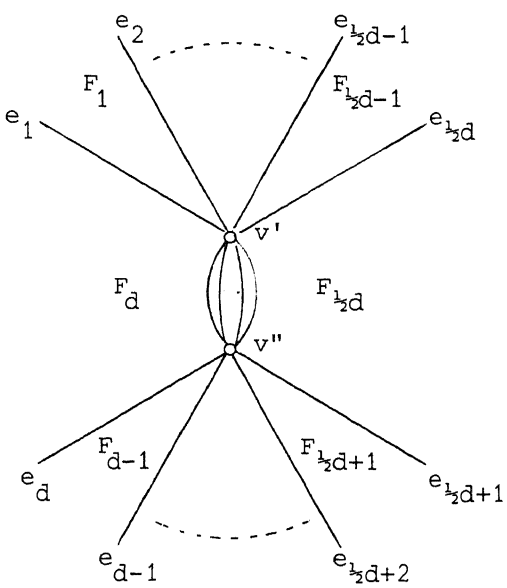
To formulate Theorem 5, we must introduce some terminology. Let G=(V,E) be an eulerian graph embedded on the compact orientable surface S. Consider a vertex v of G of degree at least 4, and a neighbourhood $N \simeq \mathbb{C}$ of v:

(25)



Choose two faces which are oppposite in v, say $F_{\frac12 d}$ and F_d (d being the degree of v). Now let G' be the graph obtained by replacing (25) in N by

(26)



where there are $\frac12 d-2$ parallel edges connecting the new vertices v' and v". We say that *the connectivity is preserved by opening* v *from* $F_{\frac12 d}$ *to* F_d if mincr(G',D) = mincr(G,D) for each closed curve D on S. We say that G is *minimally connected* if

(27) (i) for *no* choice of vertex v and faces F',F" opposite in v, the connectivity is preserved by opening v from F' to F";
(ii) no component of G is a null-homotopic circuit.

The *straight decomposition* of G is the decomposition of the edges of G into cycles as follows. In each vertex v of G we 'match' opposite edges. Thus in (25), edge e_1 is matched to $e_{\frac{1}{2}d+1}$, e_2 to $e_{\frac{1}{2}d+2}$, ..., $e_{\frac{1}{2}d}$ to e_d. This gives us a unique decomposition of the edges of G into cycles

$$(28)\qquad (v_0, e_1, v_1, \ldots, e_\ell, v_\ell)$$

so that for each $i=1,\ldots,\ell$, the edges e_i and e_{i+1} are matched in v_i (taking $e_{\ell+1} := e_1$). This decomposition is unique up to the choice of beginning (= end) point of the cycles and up to the direction of the cycles.

A closed curve $C: S_1 \to S$ is called *primitive* if C is not homotopic to D^n for some closed curve D and some $n \geq 2$ (where D^n is the closed curve with $D^n(z) := D(z^n)$ for $z \in S_1$). In particular, if C is primitive, C is not null-homotopic.

A collection of closed curves $C_1,\ldots,C_k$ is called *minimally crossing* if

(29) (i) $cr(C_i,C_j) = mincr(C_i,C_j)$, for $i,j=1,\ldots,k$; $i \neq j$;
(ii) $cr(C_i) = mincr(C_i)$ for $i=1,\ldots,k$.

Here

$$(30)\qquad \begin{aligned} cr(C) &:= \tfrac{1}{2}\left|\{(y,z) \in S_1 \times S_1 \mid C(y)=C(z),\ y \neq z\}\right|, \\ mincr(C) &:= \min\{cr(\tilde{C}) \mid \tilde{C} \sim C\}. \end{aligned}$$

Now finally we can formulate:

THEOREM 5. *Let G be an eulerian graph embedded on a compact orientable surface S. Then G is minimally connected if and only if the straight decomposition of G forms a minimally crossing collection of primitive closed curves.*

We first give in Section 2 a brief review of the topology of surfaces, and in Section 3 we study closed curves and their crossings. In Section 4 we prove the 'Main Lemma', after which we show Theorem 1-5 in Sections 5-9. Finally, in Section 10 we derive some further results on curves on surfaces.

3. SURFACES AND COVERING SURFACES.

In this paper we will assume some familiarity with the theory of surfaces, which belongs to the most basic and classical parts of topology, due to Poincaré, Dehn, Heegaard, Brouwer, Brahana, von Kerékjártó, Radó and Baer. We will list in this section the results we will use. For extended discussions we refer to the books by Ahlfors and Sario [1], von Kerékjártó [9], Massey [12], Moise [13] and Seifert and Threlfall [22].

A *surface* is any arc-connected Hausdorff space S in which each point x has a neighbourhood N_x homeomorphic to the complex plane $\mathbb{C}$. S is *orientable* if each N_x can be oriented so that if two N_x and N_y intersect, then their orientations coincide on the intersection.

Each compact orientable surface is homeomorphic to the space obtained from the 2-dimensional sphere by adding some finite number of 'handles' (possibly none) (see Dehn and Heegaard [5:p.197-198], Brahana [3]). Each compact orientable surface S has a *triangulation*, i.e., there exists a graph Δ embedded on S so that each face (= component of $S\setminus\Delta$) is homeomorphic to $\mathbb{C}$, and is bounded by a triangle of Δ. In fact, any graph G embedded on S can be extended to a triangulation (possibly adding new vertices on edges; see Radó [16]). (We use triangulations only in the proofs of Propositions 6 and 8 and Theorem 4.)

We now first settle some notation and terminology for curves, paths and homotopy. A *closed curve* on a surface S is a continuous function $D:S_1\to S$, where $S_1 := \{z\in\mathbb{C} \mid |z|=1\}$. An *open curve* on S is a continuous function $D:\mathbb{R}\to S$. A *path* on S is a continuous function $D:[0,1]\to S$. The path is said to *go from* D(0) *to* D(1), which two points are called the *end points* of D.

If $D:S_1\to S$ is a closed curve on S, then by $\text{path}D:[0,1]\to S$ we denote the path with:

$$\text{(31)} \qquad \text{path}D(x) := D(e^{2\pi ix}) \qquad \text{for } x\in[0,1].$$

If $D_1,D_2:S_1\to S$ are closed curves with $D_1(1)=D_2(1)$, then $D_1\cdot D_2$ is the closed curve with $(D_1\cdot D_2)(z):=D_1(z^2)$ if $\text{Im}z\geq 0$, and $:=D_2(z^2)$ if $\text{Im}z<0$. Similarly, $D_1\cdot\ldots\cdot D_n$ is defined. For $n\in\mathbb{Z}$, if $D:S_1\to S$ is a closed curve, then D^n is the closed curve with $D^n(z) := D(z^n)$ for $z\in S_1$.

Similarly, if $D_1,D_2:[0,1]\to S$ are paths with $D_1(1)=D_2(0)$, then $D_1\cdot D_2$ is the path with $(D_1\cdot D_2)(x):=D_1(2x)$ if $0\leq x\leq\frac{1}{2}$ and $:=D_2(2x-1)$ if $\frac{1}{2}<x\leq 1$.

Two closed curves $D,\tilde{D}:S_1\to S$ are said to be (*freely*) *homotopic*, denoted by $D\sim\tilde{D}$, if there exists a continuous function $\Phi:S_1\times[0,1]\to S$ such that $\Phi(z,0)=D(z)$ and $\Phi(z,1)=\tilde{D}(z)$ for all $z\in S_1$. (We say that Φ *transforms* D to $\tilde{D}$.)

This defines an equivalence relation between closed curves; the class containing closed curve D is denoted by hom(D). D is called *null-homotopic* if D is homotopic to some constant function.

Similarly, two paths $D,\check{D}:[0,1]\rightarrow S$ are said to be *homotopic*, denoted by $D\sim\tilde{D}$, if there exists a continuous function $\Phi:[0,1]\times[0,1]\rightarrow S$ so that $\Phi(x,0)=D(x)$, $\Phi(x,1)=\tilde{D}(x)$, $\Phi(0,x)=D(0)$, $\Phi(1,x)=D(1)$ for all $x\in[0,1]$. (We say that Φ *transforms* D to $\tilde{D}$.) It follows that $D(0)=\tilde{D}(0)$ and $D(1)=\tilde{D}(1)$. Again, homotopy of paths defines an equivalence relation; the class containing path D is denoted by hom(D). Moreover, for $p,q\in S$:

$$\text{(32)} \qquad \mathrm{Hom}(p,q) := \{\mathrm{hom}(D) \mid D \text{ is a path from } p \text{ to } q\}.$$

If $p,q,r\in S$ and $\lambda\in \mathrm{Hom}(p,q)$ and $\mu\in \mathrm{Hom}(q,r)$, then $\lambda\cdot\mu := \mathrm{hom}(P\cdot Q)$ for some arbitrary $P\in\lambda$ and $Q\in\mu$ (hom$(P\cdot Q)$ is easily seen to be independent of the choice of $P\in\lambda$, $Q\in\mu$). This operation makes Hom(p,p) to a group, the *fundamental group* $\pi_1(S)$ of S (as a group, it is independent of p). A path is called *null-homotopic* if it is homotopic to a constant function.

<u>The universal covering surface</u>.

Two 'covering surfaces' play an important role in our proof. The first one is as follows. Choose $p\in S$. The *universal covering surface* S' of S (*with respect to* p) is the space with point set:

$$\text{(33)} \qquad \{(q,\lambda) \mid q\in S,\ \lambda\in \mathrm{Hom}(p,q)\},$$

while a subset T of S' is open iff for each $(q,\lambda)\in T$ there is a neighbourhood N of q in S so that $N\simeq\mathbb{C}$ and so that for each r in N and for each path P in N from q to r the point $(r,\lambda\cdot\mathrm{hom}(P))$ belongs to T.

Now Poincaré [15] showed:

<u>Proposition 1</u>. *If S is a compact orientable surface, with at least one handle, then the universal covering surface S' of S is homeomorphic to* $\mathbb{C}$.

The *projection* $\pi:S'\rightarrow S$ is the continuous function defined by $\pi(q,\lambda):=q$. For any closed curve $D:S_1\rightarrow S$ with $D(1)=q$, say, and for any $\lambda\in\mathrm{Hom}(p,q)$, the *lifting of* D *to* S' *by* λ is the open curve $D':\mathbb{R}\rightarrow S'$ defined by:

$$\text{(34)} \qquad D'(x) := (D(e^{2\pi ix}),\lambda\cdot\mathrm{hom}(D(e^{2\pi ixy})_{y\in[0,1]})).$$

So $(\pi\circ D')(x)=D(e^{2\pi ix})$ for all x in $\mathbb{R}$.

Similarly, for any path $D:[0,1]\rightarrow S$ with $D(0)=q$, say, and for any $\lambda\in \mathrm{Hom}(p,q)$, the *lifting of* D *to* S' *by* λ is the path $D':[0,1]\longrightarrow S'$ defined by:

(35) $$D'(x) := (D(x),\lambda\cdot \mathrm{hom}(D(xy)_{y\in[0,1]})) \qquad \text{for } x\in[0,1].$$

So $\pi\circ D' = D$.

Note the symmetry of the universal covering surface: the universal covering surface and the liftings are essentially idependent of the choice of the point p. If S' and $\tilde{S}'$ are the universal covering surfaces of S with respect to the points p and $\tilde{p}$, respectively, and if $\mu\in \mathrm{Hom}(\tilde{p},p)$, then $F_{\mu}:S'\longrightarrow S'$ defined by $F_{\mu}(q,\lambda) := (q,\mu\cdot\lambda)$ is a homeomorphism. Moreover, if $D:S_1\longrightarrow S$ is a closed curve, with $q := D(1)$, if $\lambda\in \mathrm{Hom}(p,q)$, $\tilde{\lambda}\in \mathrm{Hom}(\tilde{p},q)$, and if D' and $\tilde{D}'$ denote the liftings of D to S' and $\tilde{S}'$, respectively, by λ and $\tilde{\lambda}$, respectively, then $\tilde{D}' = F_{(\tilde{\lambda}\cdot\lambda^{-1})}\cdot D'$.

<u>The covering surface generated by a closed curve.</u>

The other 'covering surface' we will use arises by 'rolling up' the universal covering space S' along a curve. Let $p\in S$, and let $D:S_1\longrightarrow S$ be a closed curve, with $D(1)=p$. The *covering surface* S" *generated by* D is the quotient space of the universal covering space S' with respect to p, obtained by identifying $(q,\lambda)\in S'$ and $(r,\mu)\in S'$ iff $q=r$ and $\lambda=\mathrm{hom}(\mathrm{path}(D^n))\cdot\mu$ for some $n\in\mathbb{Z}$. So the points of S" can be described as pairs $(q,\langle\lambda\rangle)$, where $q\in S$, $\lambda\in \mathrm{Hom}(p,q)$ and $\langle.\rangle$ denotes the class of . under the equivalence just defined. Let $\psi:S'\longrightarrow S''$ denote the quotient map. So $\psi(q,\lambda)=(q,\langle\lambda\rangle)$. The *projection* $\pi':S''\longrightarrow S$ is the function given by $\pi'(q,\langle\lambda\rangle):=q$.

The *lifting of* D *to* S" is the closed curve $D'':S_1\longrightarrow S''$ defined by:

(36) $$D''(z) := (D(z),\langle \mathrm{hom}(D(e^{2\pi ixy})_{y\in[0,1]})\rangle) \qquad \text{for } z\in S_1,$$

where x is so that $z = e^{2\pi ix}$ (note that (36) is invariant under replacing x by x+1). We now show:

<u>Proposition 2</u>. *Let* S *be a compact orientable surface, and let* $D:S_1\longrightarrow S$ *be a, not-null-homotopic,closed curve. Then the covering surface* S" *generated by* D *is homeomorphic to* $\mathbb{C}\setminus\{0\}$*, in such a way that the lifting* D" *of* D *to* S" *is homotopic to the unit circle in* $\mathbb{C}$.

(The *unit circle* here is the identical function $S_1 \rightarrow S_1$.)

Proof. I. We first show that the fundamental group $\pi_1(S'')$ of S" is isomorphic to the infinite cyclic group $(\mathbb{Z},+)$. Let $p'' := (p, \langle \underline{1} \rangle) \in S''$, where $\underline{1}$ denotes the identity in Hom(p,p). So $\pi_1(S'') \cong \mathrm{Hom}(p'',p'')$. Now pathD" is a path from p" to p". Moreover, pathD" is not null-homotopic, since D is not null-homotopic (if $\Phi: [0,1] \times [0,1] \rightarrow S''$ would transform pathD" to a constant function, then $\pi' \circ \Phi$ transforms $\pi' \circ$ pathD" = pathD to a constant function).

We next show that any path $P: [0,1] \rightarrow S''$ from p" to p" is homotopic to $(\mathrm{pathD}'')^n$ for some $n \in \mathbb{Z}$. Indeed, $\pi' \circ P: [0,1] \rightarrow S$ is a path from p to p. Let Q be the lifting of $\pi' \circ P$ to S' by $\underline{1}$. Then $\pi' \circ (\psi \circ Q) = \pi' \circ P$ and hence $\psi \circ Q = P$ (by elementary topology - see Massey [12:Ch.5,Lemma 3.1]). In particular, $(\psi \circ Q)(1) = P(1)$, implying that $\langle \mathrm{hom}(\pi' \circ P) \rangle = \langle \underline{1} \rangle$. Hence, by definition of $\langle . \rangle$, $\pi' \circ P \sim (\mathrm{pathD})^n = \pi' \circ (\mathrm{pathD}'')^n$ for some $n \in \mathbb{Z}$. Therefore, $P \sim (\mathrm{pathD}'')^n$ (again by elementary topology - see Massey [12:Ch.5,Lemma 3.3]).

II. Now by von Kerékjártó's classification theorem [9:5.Abschnitt] (see Richards [17] and Goldman [6] for more modern treatments), any orientable surface with infinite cyclic fundamental group is homeomorphic to $\mathbb{C} \setminus \{0\}$. By I. above, the fundamental group of S" is generated by hom(pathD"). So S" is homeomorphic to $\mathbb{C} \setminus \{0\}$. Clearly, the fundamental group of $\mathbb{C} \setminus \{0\}$ is generated by the unit circle. Hence we may assume D" to be homotopic to the unit circle. □

We finally mention the following classical theorem to be used [4]:

Brouwer's fixed point theorem. *Let* $K \subsetneq \mathbb{C}$ *be compact and convex, and let* $f: K \rightarrow K$ *be continuous. Then* $f(z) = z$ *for some* $z \in K$.

3. CURVES AND THEIR CROSSINGS

In this section we study, with the help of the covering surfaces treated in Section 2, how often curves on a compact orientable surface must cross. That is, we study the following concepts. If C and D are closed curves, let

$$(37)\qquad \begin{aligned} X(C,D) &:= \{(y,z)\ S_1\times S_1 \mid C(y)=D(z)\},\\ cr(C,D) &:= |X(C,D)|,\\ mincr(C,D) &:= \min\{cr(\tilde{C},\tilde{D}) \mid \tilde{C}\sim C,\ \tilde{D}\sim D\}. \end{aligned}$$

As is well-known, mincr(C,D) is finite. We also study self-crossings of one closed curve C. Let

$$(38)\qquad \begin{aligned} X(C) &:= \{(y,z)\in S_1\times S_1 \mid C(y)=C(z),\ y\neq z\},\\ cr(C) &:= \tfrac{1}{2}|X(C)|,\\ mincr(C) &:= \min\{cr(\tilde{C}) \mid \tilde{C}\sim C\}. \end{aligned}$$

Note that if X(C) is finite, it has an even number of elements (as $(y,z)\in X(C)$ if and only if $(z,y)\in X(C)$). Note also the difference between mincr(C) and mincr(C,C): in the latter we minimize over pairs of curves homotopic to C. We shall see that the following direct relation holds if C is primitive: mincr(C) = $\tfrac{1}{2}$mincr(C,C). (Recall that a curve C is *primitive* if there is no closed curve D and $n\geq 2$ so that $C\sim D^n$.)

We now first give two useful auxiliary results. We say that a closed curve $D:S_1\rightarrow S$ has *no null-homotopic parts* if there exist no $y,z\in\mathbb{R}$ with $0<|y-z|\leq 1$ so that the path $x\mapsto D(e^{2\pi i((1-x)y+xz)})$ for $x\in[0,1]$ is a null-homotopic path. In fact, the condition $|y-z|\leq 1$ can be deleted, which is the content of the following proposition.

Proposition 3. *Let* S *be a compact orientable surface. If the closed curve* $D:S_1\rightarrow S$ *has no null-homotopic parts, then any lifting of* D *to the universal covering surface* S' *is a one-to-one function.*

Proof. By the symmetry of the universal covering surface, we may assume that S' is the universal covering surface with respect to p := D(1), and that we consider the lifting D' of D to S' by $\underline{1}$. Let S" be the covering surface generated by D, and let $D'':S_1\rightarrow S''$ be the lifting of D to S". Since D has no null-homopic parts, also D" has no null-homotopic parts (as $D=\pi'\circ D''$). Hence D" is one-to-one. Proposition 2 then implies that for each $n\in\mathbb{Z}\setminus\{0\}$, the closed curve $(D'')^n$ has no null-homotopic parts. Therefore, D' is one-to-one. □

Next we show that if closed curve C is homotopic to D^n for some closed curve D and $n \geq 2$, then C can be decomposed as product of a curve $\tilde{D} \sim D$ and a curve $E \sim D^{n-1}$. More precisely:

Proposition 4. *Let* C *and* D *be closed curves on a compact orientable surface* S *and let* $n \in \mathbb{N}$ *so that* $C \sim D^n$. *Then there exists an orientation preserving homeomorphism* $\varphi : S_1 \to S_1$ *so that the closed curve* $C \circ \varphi$ *is equal to* $\tilde{D} \cdot E$, *where* $\tilde{D} \sim D$ *and* $E \sim D^{n-1}$.

[Homeomorphism $\varphi : S_1 \to S_1$ is called *orientation preserving* if φ is homotopic to the identity function on S_1, in the space $\mathbb{C} \setminus \{0\}$.]

Proof. Without loss of generality, $n \geq 2$. We may assume $C(1)=D(1)$. Let S' be the universal covering surface with respect to D(1), and let $C' : \mathbb{R} \to S'$ be the lifting of C to S' by $\underline{1}$. Let S" be the covering surface generated by D, let D" be the lifting of D to S", and let $C" := \pi' \circ D'$ (where $\pi' : S' \to S"$ is the projection function). By Proposition 2 we can identify S" with $\mathbb{C} \setminus \{0\}$, in such a way that D" is homotopic to the identity function $S_1 \to S$. So $C" \mid [0,1]$ is a cycle in $\mathbb{C} \setminus \{0\}$ going n times around the origin. What we have to show is that we can 'split off' a cycle going once around the origin (in the same direction).

To this end we can identify S' with $\mathbb{C}$ so that $\pi'(z) = e^{2\pi z} \in S"$ for all $z \in S'$. Then $C'(z+1) = C'(z) + \underline{i}n$ for all $z \in \mathbb{R}$. We first show:

(39) there exist $z', z" \in \mathbb{R}$ so that $z'-1 < z" < z'$ and $C'(z') = C'(z") + \underline{i}$.

Since $C"(z+1) = C"(z)$ for all $z \in \mathbb{R}$, there exist x and y in $\mathbb{R}$ minimizing and maximizing $|C"(z)|$, respectively. Hence

(40) $\mathrm{Re}(C'(x)) \leq \mathrm{Re}(C'(z)) \leq \mathrm{Re}(C'(y))$, for all $z \in \mathbb{R}$.

Without loss of generality, $y-1 < x \leq y$. Then the curves $P : [0,1] \to \mathbb{C}$ and $Q : [0,1] \to \mathbb{C}$, given by

(41) $P(\lambda) := C'((1-\lambda)x + \lambda y)$,
$Q(\lambda) := C'((1-\lambda)x + \lambda(y-1)) + \underline{i}$

for $\lambda \in [0,1]$, have a point in common (since by (40), $\mathrm{Re}(P(0)) \leq \mathrm{Re}(P(\lambda)) \leq \mathrm{Re}(P(1))$ and $\mathrm{Re}(Q(0)) \leq \mathrm{Re}(Q(\lambda)) \leq \mathrm{Re}(Q(1))$ for all $\lambda \in [0,1]$, and since $\mathrm{Im}(P(0)) = \mathrm{Im}(C'(x)) < \mathrm{Im}(C'(x))+1 = \mathrm{Im}(Q(0))$ and $\mathrm{Im}(P(1)) = \mathrm{Im}(C'(y)) >$

$\mathrm{Im}(C'(y))-n+1 = \mathrm{Im}(C'(y-1))+1 = \mathrm{Im}(Q(1)))$. That is, there exist $\lambda', \lambda'' \in [0,1]$ so that $P(\lambda')=Q(\lambda'')$. Let $z' := (1-\lambda')x+\lambda' y$ and $z'' := (1-\lambda'')x+\lambda''(y-1)$. Then $C'(z')=P(\lambda')=Q(\lambda'')=C'(z'')+\underline{i}$ and $x \leqslant z' \leqslant y$ and $y-1 \leqslant z'' \leqslant x$. Thus we have (39).

As $z'-1 < z'' < z'$, there exists an orientation preserving homeomorphism $\varphi: S_1 \to S_1$ so that $\varphi(1) = e^{2\pi z''}$ and $\varphi(-1) = e^{2\pi z'}$. Then the path $R:[0,1] \to S'$ defined by $R(\lambda) := C'((1-\lambda)z''+\lambda z')$ satisfies $R(1) = R(0)+\underline{i}$. So $\pi' \circ R:[0,1] \to S''$ is, as a cycle, homotopic to the identity function $S_1 \to S''=\mathbb{C}\setminus\{0\}$. Therefore, $C\circ\varphi = \tilde{D}\cdot E$ for cycles $\tilde{D}\sim D$ and $E\sim D^{n-1}$. □

Proposition 4 has the following consequence:

Proposition 5. *If* C *and* D *are closed curves on the compact orientable surface* S *and* $n\in\mathbb{N}$, *then* $\mathrm{mincr}(C,D^n) = n\cdot\mathrm{mincr}(C,D)$.

Proof. Clearly, $\mathrm{mincr}(C,D^n) \leqslant n\cdot\mathrm{mincr}(C,D)$, since there exist $\tilde{C}\sim C$ and $\tilde{D}\sim D$ so that $\mathrm{cr}(\tilde{C},\tilde{D}) = \mathrm{mincr}(C,D)$, whence $\mathrm{mincr}(C,D^n) \leqslant \mathrm{cr}(\tilde{C},\tilde{D}^n) = n\cdot\mathrm{cr}(\tilde{C},\tilde{D}) = n\cdot\mathrm{mincr}(C,D)$.

The reverse inequality is shown by induction on n, the cases n=0 and n=1 being trivial. Let $n \geqslant 2$, and let closed curves $\tilde{C}\sim C$ and $B\sim D^n$ be so that $\mathrm{cr}(\tilde{C},B) = \mathrm{mincr}(C,D^n)$. By Proposition 4, we may assume that $B = \tilde{D}\cdot E$ with $\tilde{D}\sim D$ and $E\sim D^{n-1}$. Then we obtain

(42) $$\begin{aligned}\mathrm{mincr}(C,D^n) &= \mathrm{cr}(\tilde{C},B) = \mathrm{cr}(\tilde{C},\tilde{D}) + \mathrm{cr}(\tilde{C},E) \geqslant \\ &\mathrm{mincr}(C,D) + \mathrm{mincr}(C,D^{n-1}) \geqslant \mathrm{mincr}(C,D) + (n-1)\cdot\mathrm{mincr}(C,D) = \\ &n\cdot\mathrm{mincr}(C,D).\end{aligned}$$ □

As before, define a collection $C_1,\dots,C_k$ of closed curves on S to be *minimally crossing* if (i) $\mathrm{cr}(C_i) = \mathrm{mincr}(C_i)$ for $i=1,\dots,k$; (ii) $\mathrm{cr}(C_i,C_j) = \mathrm{mincr}(C_i,C_j)$ for $i,j=1,\dots,k$, $i\neq j$. Closely related are the following conditions for closed curves $C_1,\dots,C_k$ on S (where S' denotes the universal covering surface of S):

(43)
(i) for each $i=1,\dots,k$, any lifting of C_i to S' is a not-self-intersecting curve;
(ii) for each $i=1,\dots,k$, any two liftings of C_i to S' either have the same image or intersect each other in at most one point;
(iii) for each $i,j=1,\dots,k$ with $i\neq j$, any lifting of C_i to S' has at most one point in common with any lifting of C_j to S'.

We call these conditions the *simplicity conditions*. What we will show below is that a collection of *primitive* curves $C_1,\ldots,C_k$ is minimally crossing if and only if it satisfies the simplicity conditions (43).

Let us first mention a basic theorem of Baer [2:Satz 2]:

Proposition 6. *For any collection of closed curves* $C_1,\ldots,C_k$ *on a compact orientable surface* S *there exist curves* $\tilde{C}_1 \sim C_1,\ldots,\tilde{C}_k \sim C_k$ *so that* $\tilde{C}_1,\ldots,\tilde{C}_k$ *satisfy the simplicity conditions* (43).

For the proof, based on Poincaré's representation of the universal covering surface as a hyperbolic plane, we refer to [2].

Remark. Baer showed moreover that if $C_1,\ldots,C_k$ satisfy the simplicity conditions (43), then for any lifting C_i' of any C_i, the set of points on C_i' which are also on a lifting C_j' of any C_j (possibly j=i) so that the image of C_j' is different from the image of C_i', does not have any point of accumulation in S'. Equivalently, the curves $C_1,\ldots,C_k$ intersect themselves and each other in a finite number of points of S.

We derive from Baer's theorem a formula for mincr(C,D). To this end, for any closed curve $D:S_1 \to S$ and $z,z' \in S_1$, let us call a path $P:[0,1] \to S$ a z-z'-*walk along* D if there exist $t,t' \in \mathbb{R}$ so that:

(44) $$z = e^{2\pi \underline{i} t},\ z' = e^{2\pi \underline{i} t'},$$
$$P(x) = D(e^{2\pi \underline{i}((1-x)t+xt')}) \qquad \text{for } x \in [0,1].$$

Now let $C,D:S_1 \to S$ be closed curves on S so that the set X(C,D) is finite, and so that if $(y,z) \in X(C,D)$ then C and D form crossing curves, if we restrict them to small neighbourhoods of y and z. Define the following equivalence relation on X(C,D):

(45) $(y,z) \approx (y',z')$ iff some y-y'-walk along C is homotopic to some z-z'-walk along D.

It is not difficult to see that this defines an equivalence relation. Let us call any class of this relation to be *odd* if it contains an odd number of elements. Let

(46) odd(C,D) := number of odd classes of $\approx$.

Proposition 6. mincr(C,D) = odd(C,D).

Proof. We use the theory of simplicial approximation (see Seifert and Threlfall [22: § 44]). Let $C_1 \sim C$ and $D_1 \sim D$ attain mincr(C,D), and let $\tilde{C} \sim C$ and $\tilde{D} \sim D$ be as given by Baer's theorem. We may assume that $C, C_1, \tilde{C}, D, D_1, \tilde{D}$ intersect each other and themselves only a finite number of times (cf. Remark above). Hence we may assume that C, C_1 and $\tilde{C}$ each follow the edges of a triangulation Γ of S, and that D, D_1 and $\tilde{D}$ each follow the edges of some other triangulation Δ of S, so that the vertices and edges of Γ and Δ have only a finite number of intersections.

Now one easily checks that odd(C,D) is invariant under the following modification of C: if C passes edge e of triangle T of Γ, replace e by the other two edges of T; similarly for D with respect to Δ. Since C_1 and D_1 arise from C and D by a series of these modifications and their reverses, we know:

$$\text{(47)} \qquad \text{mincr}(C,D) = \text{cr}(C_1,D_1) \geq \text{odd}(C_1,D_1) = \text{odd}(C,D).$$

On the other hand, for $\tilde{C}$ and $\tilde{D}$ we have that each class of $\approx$ consists of one element (this is exactly the property described by (43)(iii)), and hence:

$$\text{(48)} \qquad \text{mincr}(C,D) \leq \text{cr}(\tilde{C},\tilde{D}) = \text{odd}(\tilde{C},\tilde{D}) = \text{odd}(C,D).$$
□

A direct consequence is:

Proposition 7. *Let* S *be a compact orientable surface, with universal covering surface* S', *and let* $C,D: S_1 \rightarrow S$ *be closed curves on* S. *Then* cr(C,D) = mincr(C,D) *if and only if each lifting of* C *to* S' *intersects each lifting of* D *to* S' *at most once.*

Proof. Since odd(C,D) = cr(C,D) if and only if each lifting of C to S' intersects each lifting of D to S' at most once, the result follows directly from Proposition 6. □

Self-crossings of curves can be treated almost similarly. Let $C: S_1 \rightarrow S$ be a closed curve on the compact orientable surface S, so that cr(C) is finite and so that if $(y,z) \in X(C)$ then in neighbourhoods of y and z C forms a pair of crossing curves. Define the following equivalence relation on X(C):

(49) $(y,z) \approx (y',z')$ iff some y-y'-walk along C is homotopic to some z-z'-walk along C.

Again it is not difficult to see that this defines an equivalence relation. As $|X(C)|$ is even, we have that the number

(50) $\quad$ $\mathrm{odd}(C) :=$ the number of odd equivalence classes of $\overset{\sim}{\sim}$,

is even. Now:

<u>Proposition 8</u>. *If* C *is primitive, then* $\mathrm{mincr}(C) = \frac{1}{2}\mathrm{odd}(C)$.

<u>Proof</u>. Similar to the proof of Proposition 6. Note that the primitivity of is used in the fact that if $\tilde{C} \sim C$ satisfies (43)(ii), then X(C) is finite, implying $\mathrm{cr}(C) = \mathrm{odd}(C)$. □

Note that the formula in Proposition 9 generally does not hold for non-primitive closed curves. E.g., on the torus for each non-primitive closed curve C we have $\mathrm{odd}(C)=0$ while $\mathrm{mincr}(C) > 0$ (except if C is null-homotopic).

Analogous to Proposition 7 is the following:

<u>Proposition 9</u>. *Let* S *be a compact orientable surface, with universal covering surface* S', *and let* $C:S_1 \to S$ *be a closed curve on* S. *Then the following are equivalent:*

(51) $\quad$ *(i)* C *is primitive and* $\mathrm{cr}(C) = \mathrm{mincr}(C)$;

(ii) $\mathrm{cr}(C)$ *is finite, each lifting of* C *to* S' *is a not-self-intersecting curve, and each two liftings of* C *to* S' *intersect each other at most once, unless their images coincide.*

<u>Proof</u>. By the above, it suffices to show that (ii) implies that C is primitive. Suppose $C \sim D^n$ for some closed curve $D:S_1 \to S$ and $n \geq 2$. By Proposition 4 there exists an orientation preserving homeomorphism $\varphi:S_1 \to S_1$ so that $C \circ \varphi = \tilde{D} \cdot E$ for some $\tilde{D} \sim D$ and $E \sim D^{n-1}$. Without loss of generality, φ is the identity function, so that $C \circ \varphi = C$. Let S' be the universal covering surface with respect to $C(1) = C(-1)$. Consider the liftings C' and $\bar{C}'$ of C to S' by <u>1</u> and by $\lambda := \mathrm{hom}(\mathrm{path}(D))$, respectively. Let $\mu := \mathrm{hom}(\mathrm{path}(C))$. Then

(52)
$$C'(\tfrac{1}{2}) = (C(1),\lambda) = \bar{C}'(0),$$
$$C'(\tfrac{3}{2}) = (C(1),\lambda\cdot\mu) = \bar{C}'(1).$$

Hence, by (ii), the images of C' and $\bar{C}'$ coincide. This implies that $C'(y) =$

$\bar{C}'(z)$ for infinitely many pairs of distinct values of y,z. Hence $X(C)$ is infinite, contradicting the assumption. □

The following proposition gives a relation between the two function mincr we studied in this section.

Proposition 10. *If* C *is primitive then* $\mathrm{mincr}(C) = \frac{1}{2}\mathrm{mincr}(C,C)$.

Proof. Let C be such that $\mathrm{cr}(C) = \mathrm{mincr}(C)$. 'Draw' a curve $\tilde{C}$ parallel and close 'to the right of' C. One easily checks that $\mathrm{cr}(C,\tilde{C}) = 2\cdot\mathrm{cr}(C)$, and that each lifting of $\tilde{C}$ intersect each lifting of C at most once (if $\tilde{C}$ is close enough to C - 'close enough' can be made precise by considering a triangulation of S, and by lifting this triangulation to S'). So by Propositions 7 and 9, $\mathrm{mincr}(C,C) = \mathrm{cr}(C,\tilde{C}) = 2\cdot\mathrm{cr}(C) = 2\cdot\mathrm{mincr}(C)$. □

The above also implies:

Proposition 11. *Let* $C_1,\ldots,C_k$ *be primitive closed curves. Then* $C_1,\ldots,C_k$ *is a minimally crossing collection of curves if and only if* $C_1,\ldots,C_k$ *satisfy the simplicity condition* (43).

Proof. Directly from Propositions 7 and 9. □

We finally show an important property of minimally crossing collections of primitive closed curves:

Proposition 12. *Let* $C_1,\ldots,C_k$ *be a minimally crossing collection of closed curves on the compact orientable surface* S. *Then for any closed curve* D *on* S *there exists a closed curve* $\tilde{D}\sim D$ *so that* $\mathrm{cr}(C_i,\tilde{D}) = \mathrm{mincr}(C_i,D)$ *for every* $i=1,\ldots,k$.

[In fact it can be shown that if D is primitive, we can take $\tilde{D}$ so that moreover $\mathrm{cr}(\tilde{D}) = \mathrm{mincr}(D)$.]

Proof. Choose $\tilde{D}\sim D$ so that

$$(53)\qquad \sum_{i=1}^{k} \mathrm{cr}(C_i,\tilde{D})$$

is as small as possible. We may assume that $\tilde{D} = D$. We now first show that for each $n\in\mathbb{N}$ and $E\sim D^n$:

(54) $$\sum_{i=1}^{k} cr(C_i,E) \geqslant n\cdot\sum_{i=1}^{k} cr(C_i,D).$$

This is shown by induction on n, the case n=0 being trivial. If $n\geqslant 1$, by Proposition 4 we know that we may assume that $E=\check{D}\cdot B$ with $\check{D}\sim D$ and $B\sim D^{n-1}$. This implies:

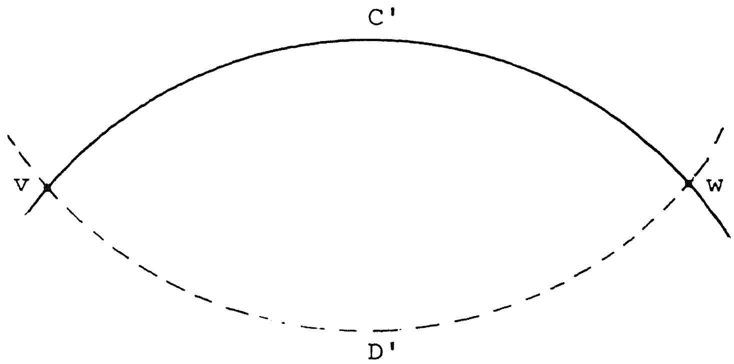
(55) $$\sum_{i=1}^{k} cr(C_i,E) = \sum_{i=1}^{k} cr(C_i,\tilde{D}) + \sum_{i=1}^{k} cr(C_i,B) \geqslant$$
$$\sum_{i=1}^{k} cr(C_i,D) + (n-1)\cdot\sum_{i=1}^{k} cr(C_i,D) = n\cdot\sum_{i=1}^{k} cr(C_i,D),$$

using the minimality of (53).

As we may assume that D has no null-homotopic parts, by Proposition 3 we know that any lifting of D to the universal covering surface S' of S is a not-self-intersecting curve.

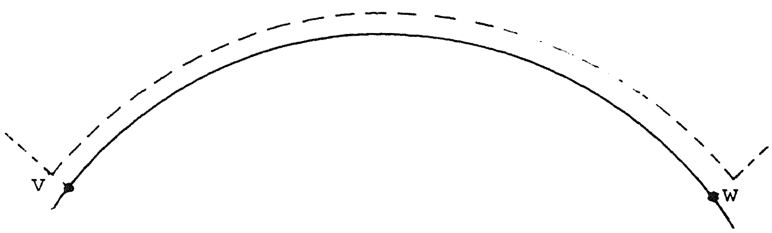
Suppose there is a lifting C_i' of some C_i to S' which has more than one intersection with some lifting D' of D to S'. Without loss of generality, i=1. Then part of D' makes with part of C_1' the following configuration in S':

(56)



where the uninterrupted curve is part of C' and the interrupted curve is part of D'. For each $x\in\mathbb{R}$, part $D'|[x,x+1)$ corresponds to one turn of cycle D. The number (53) is equal to the number of times $D'|[x,x+1)$ intersects any lifting of any C_i. Let x and y be so that D'(x)=v and D'(y)=w (cf. (56)). Without loss of generality, $0<x<y$. Choose $n\in\mathbb{N}$ so that $y<n$. Note that the image of D' is also the image of a lifting of D^n to S'. We can divert the image of D', and hence also D^n (through the projection function $\pi:S'\longrightarrow S$), along the outer side of the v-w-part of C':

(57)



Let this give curve $E \sim D^n$. Since the simplicity conditions (43) hold for $C_1,\ldots,C_k$ we know that this diversion decreases the number of crossings with $C_1,\ldots,C_k$. So

(58) $$\sum_{i=1}^{k} cr(C_i,E) < \sum_{i=1}^{k} cr(C_i,D^n) = n\cdot\sum_{i=1}^{k} cr(C_i,D),$$

contradicting (54). $\square$

In Section 10 we derive from our results two further propositions on curves on compact orientable surfaces:

Proposition 13. *For each not-null-homotopic closed curve* C *on a compact orientable surface* S *there exists a primitive closed curve* D, *unique up to homotopy, and a unique* $n \in \mathbb{N}$ *such that* $C \sim D^n$.

(This generalizes a result of Marden, Richards and Rodin [11].)

Proposition 14. *Let* B *and* C *be closed curves on a compact orientable surface* S *such that* mincr(B,D) = mincr(C,D) *for each closed curve* D. *Then* $B \sim C$ *or* $B \sim C^{-1}$.

(Curve $C^{-1}: S_1 \to S$ is defined by: $C^{-1}(z) := C(z^{-1})$ for $z \in S_1$.)

4. THE MAIN LEMMA.

Now we are ready to show our 'main lemma':

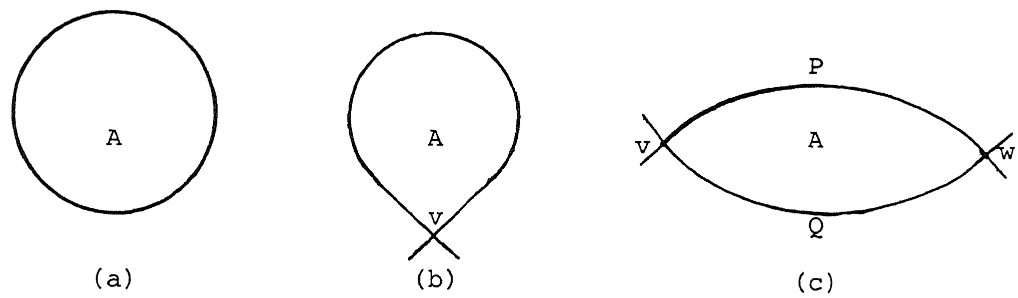
MAIN LEMMA. *Let* G = (V,E) *be a minimally connected eulerian graph, embedded on compact orientable surface* S. *Then the straight decomposition of* G *forms a minimally crossing collection of primitive closed curves.*

[In Theorem 5 we shall see that also the reverse implication holds.]

PROOF. Let $\mathfrak{D}$ be the straight decomposition of G. Let S' be the universal covering surface of S, with respect to some, for the moment arbitrary, point $p \in S$. Let $\pi: S' \rightarrow S$ be the projection. Now $\pi^{-1}[G]$ forms an infinite graph G', embedded on S', with vertex set $V' := \pi^{-1}[V]$ (note that this is a countable subset of S', without accumulation points). Again, at any vertex of G' we can 'match' opposite edges, yielding the straight decomposition $\mathfrak{D}'$ of the edges of G' into cycles and infinite paths - call them just 'curves'. Clearly, the curves in $\mathfrak{D}'$ are exactly the liftings of the cycles in $\mathfrak{D}$. So, by Propositions 7 and 9, it suffices to show that each curve in $\mathfrak{D}'$ is a not-self-intersecting infinite path, and that no two of these paths intersect each other more than once.

Suppose this last is not the case. Then there exist parts of curves in $\mathfrak{D}'$ forming one of the following configurations in S' ($\simeq \mathbb{C}$):

(59)



(a) (b) (c)

Each of these configurations consists of a not-self-intersecting cycle, made up from one or two parts of curves in $\mathfrak{D}'$, enclosing the open set A, say. A covers a finite number of faces of G'. Choose the configuration so that A covers the smallest possible number of faces of G'.

Claim 1. *The configuration is of type* (59)(c).

Proof of Claim 1. Suppose the configuration is of type (59)(a) or (b). Then

$\pi[A]$ forms a face of G. (Otherwise there is an edge of G' contained in A. Such an edge would be contained in a curve in $\mathfrak{D}'$, which would create a configuration of type (59) enclosing a smaller number of faces of G'.)

In particular, we de not have configuration (59)(a), since it would yield a component of G which is a null-homotopic circuit. This is excluded by the definition of 'minimally connected' (see (27)(ii)).

If we have configuration (59)(b), the connectivity is preserved by opening $\pi(v)$ from $\pi[A]$ to the face opposite in $\pi(v)$ to $\pi[A]$. However, again this is excluded by the definition of 'minimally connected' (see (27)(i)). □

We will use the notation v,w,P,Q as in (59)(c), where P and Q denote the *open* curves connecting v and w. Without loss of generality, v is equal to the point $(p,\underline{1})$ in S'.

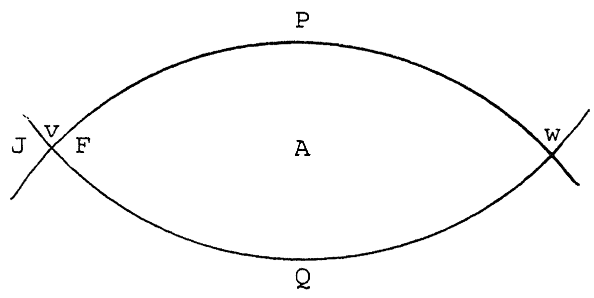
For any curve C in $\mathfrak{D}'$ intersecting A, if R is a component of this intersection, we call $\overline{R}$ a *chord* of A. Now a similar argument as used in proving Claim 1, is used in showing:

<u>Claim 2</u>. *Each chord of* A *is a not-self-intersecting path connecting* P *and* Q. *No two chords intersect each other more than once. Each edge of* G' *contained in* A *belongs to some chord.*

<u>Proof of Claim 2</u>. Otherwise we would obtain one of the configurations (59) with a smaller number of faces of G' enclosed. □

So restricted to $\overline{A}$ the curves in $\mathfrak{D}'$ are not too wild. In particular, no chord of A contains v or w. So the regions in a neighbourhood $N \simeq \mathbb{C}$ of v indicated by J and F in:

(60)



form intersections of faces of G' with N.

To avoid descriptional problems, let us call a closed curve $D:S_1 \to S$ *regular* if it intersects G a finite number of times and does not intersect $V \setminus \pi(v)$ (so it is allowed to intersect $\pi(v)$). Let us count each intersection with an edge for 1, and each intersection with $\pi(v)$ as follows. Let $F_1,\dots,F_d$ denote the faces of G incident to $\pi(v)$, in cyclic order. If D, at the intersection with $\pi(v)$ considered, connects F_i and F_j, then this intersection is counted for $\min\{|j-i|, d-|j-i|\}$. Summing up these numbers over all intersections, we obtain cr(G,D). Clearly,

(61) $$\mathrm{mincr}(G,D) = \min\{\mathrm{cr}(G,\tilde{D}) \mid \tilde{D} \sim D,\ \tilde{D} \text{ regular}\}.$$

Now by the minimal connectedness of G, there exists a regular closed curve $D:S_1 \to S$ so that

(62) $$\mathrm{cr}(G,D) - 2\tau(D) < \mathrm{mincr}(G,D),$$

where

(63) $\tau(D)$:= the number of times D intersects $\pi(v)$ so that in $\pi[N]$, by passing $\pi(v)$, D goes from $\pi(J)$ to $\pi(F)$ or from $\pi(F)$ to $\pi(J)$.

Choose this D so that:

(64) (i) $\mathrm{cr}(G,D) - 2\tau(D)$ is as small as possible,
(ii) under condition (i), $\tau(D)$ is as small as possible.

Define $n := \tau(D)$. So $n \geq 1$. We can assume that we can write:

(65) $$D = D_0 \cdot D_1 \cdot \ \dots \ \cdot D_{n-1}$$

where $D_0,\dots,D_{n-1}$ are closed curves in S with $D_0(1)=\dots=D_{n-1}(1)=v$, so that for each $j=0,\dots,n-1$:

(66) either (i) D_{j-1} ends via $\pi(J)$ and D_j starts via $\pi(F)$,
or (ii) D_{j-1} ends via $\pi(F)$ and D_j starts via $\pi(J)$.

Here we say that closed curve C *starts via* L if $\exists \varepsilon > 0\ \forall \delta \in (0,\varepsilon): C(e^{i\delta}) \in L$, and that C *ends via* L if $\exists \varepsilon > 0\ \forall \delta \in (0,\varepsilon): C(e^{-i\delta}) \in L$. Moreover, we let $D_{mn+j} := D_j$ if $m \in \mathbb{Z}$ and $0 \leq j \leq n-1$.

We may assume:

(67) (i) no part of D is null-homotopic;

(ii) the D_j intersect $\pi(v)$ only at their beginning and end points (i.e., $D_j(z)=\pi(v)$ iff $z=1$);

(iii) (65) splits D into 'equal' parts; that is, for $j=0,\dots,n-1$: $D(z) = D_j(z^n)$ if $2\pi j/n \leq \arg z \leq 2\pi(j+1)/n$;

(iv) D_0 starts via $\pi(F)$.

Now let D' be the lifting of D to S' by 1. For $j\in\mathbb{Z}$, let $D'_j:[0,1]\to S'$ be the j-th part of D'; that is,

$$D'_j(x) = D'(\frac{j+x}{n}) \qquad \text{for } x\in[0,1]. \tag{68}$$

So D'_j is a path in S' from

$$(p,\mathrm{hom}(D^{2\pi ijy/n})_{y\in[0,1]}) \text{ to } (p,\mathrm{hom}(D^{2\pi i(j+1)y/n})_{y\in[0,1]}). \tag{69}$$

Let for $j\in\mathbb{Z}$, $\theta_j:S'\to S'$ denote the shift of S' over the homotopy class $\mathrm{hom}(D(e^{2\pi ijy/n})_{y\in[0,1]})\in \mathrm{Hom}(p,p)$; that is:

$$\theta_j(q,\lambda) := (q,\mathrm{hom}(D(e^{2\pi ijy/n})_{y\in[0,1]})\cdot\lambda). \tag{70}$$

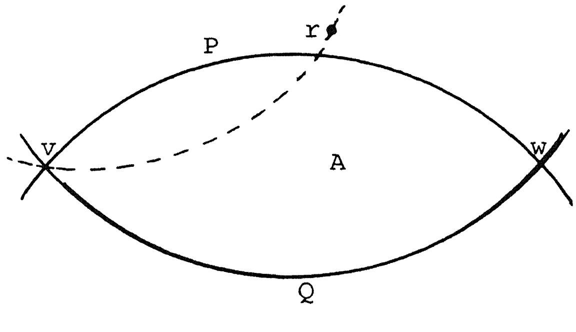
As is well-known, θ_j is an orientation-preserving homeomorphism of S'. It brings G' to G' and $\mathfrak{D}'$ to $\mathfrak{D}'$. Give the following names to copies of objects in (60):

$$A_j := \theta_j(A),\ P_j := \theta_j(P),\ Q_j := \theta_j(Q),$$
$$v_j := \theta_j(v),\ w_j := \theta_j(w),\ J_j := \theta_j(J),\ F_j := \theta_j(F). \tag{71}$$

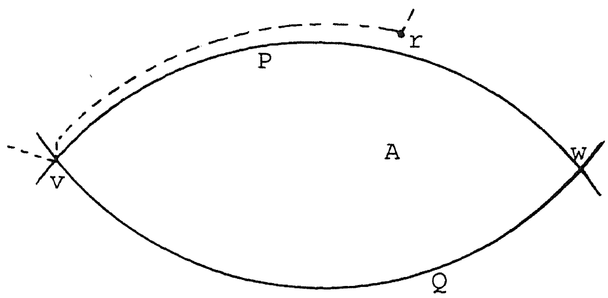
Claim 3. *If D_j starts via $\pi(F)$, then the image of D'_j is contained in $\overline{A_j}$.*

Proof of Claim 3. Without loss of generality, j=0. Note that $A_0 = A$. If D_0 starts via $\pi(F)$, then D'_0 starts via F (i.e., $D'_0[(0,\varepsilon)]\subseteq F$ for some $\varepsilon>0$). If the image of D'_0 is not contained in $\overline{A}$, the path D'_0 should leave $\overline{A}$ somewhere for the first time. First suppose this is by crossing $\overline{P}$:

(72)



where r is a point on D_0' after it has crossed P, but before crossing any other element of G'. Now there is a curve from v to r having at least two fewer crossings with G' than part v-r of D_0' has (by Claim 2):

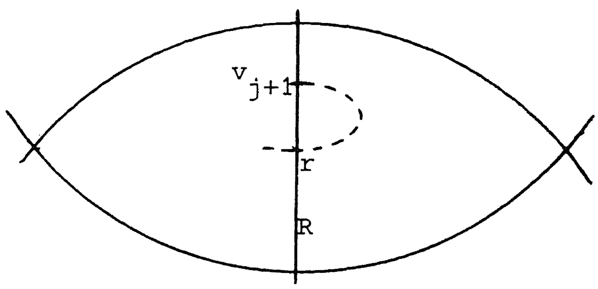
(73) 

We can thus redefine part v-r of D_0' , and redefine part D_0 of D accordingly (note that any homotopic shift of paths in S' gives by the projection $\pi: S' \longrightarrow S$ a homotopic shift of the projections of these paths in S). Thus we modify D to $\tilde{D} \sim D$. However, for $\tilde{D}$ we have $cr(G,\tilde{D}) \leq cr(G,D) - 2$, while $\tau(\tilde{D}) = \tau(D) - 1$, contradicting (64).

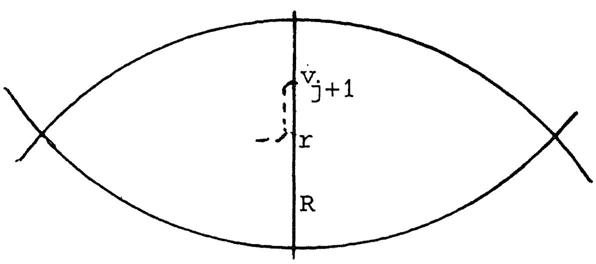
The case that D_0' leaves $\bar{A}$ by crossing $\bar{Q}$ is treated similarly. □

laim 4. *If D_j starts via $\pi(F)$, and v_{j+1} belongs to chord $\bar{R}$ of A_j, then* $)_j'$ *has no other point in common with $\bar{R}$.*

roof of Claim 4. Suppose D_j' intersects chord $\bar{R}$ more than once. Then, p to symmetry, we have:

(74) 

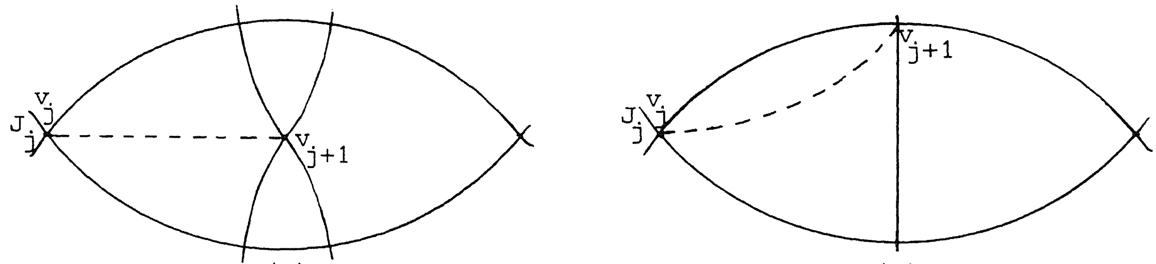
or some point r on $\bar{R}$ (v_{j+1} can be one of the end points of $\bar{R}$). Replacing part f D_j' as follows:

(75) 

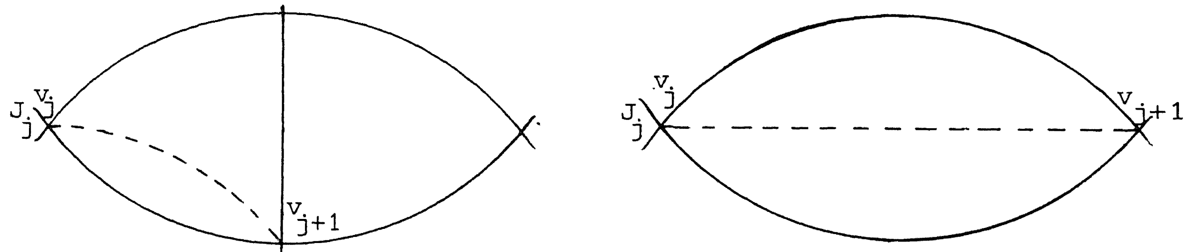
we obtain a path $\widetilde{D_j'}$ homotopic to D_j' in S', such that if we replace part D_j of D by $\pi\circ\widetilde{D_j'}$ we obtain a path $\tilde{D}$ homotopic to D in S with $cr(G,\tilde{D}) \leq cr(G,D)-2$ and $\tau(\tilde{D}) = \tau(D)-1$, contradicting (64). $\square$

By Claim 3, if D_j starts via $\pi(F)$, then v_{j+1} belongs to $\overline{A_j}$. There are four possibilities: (a) $v_{j+1} \in A_j$, (b) $v_{j+1} \in P_j$, (c) $v_{j+1} \in Q_j$, (d) $v_{j+1} = w_j$. By Claim 4, this corresponds to the following figures, where the interrupted curve stands for D_j':

(76)



(a) (b)

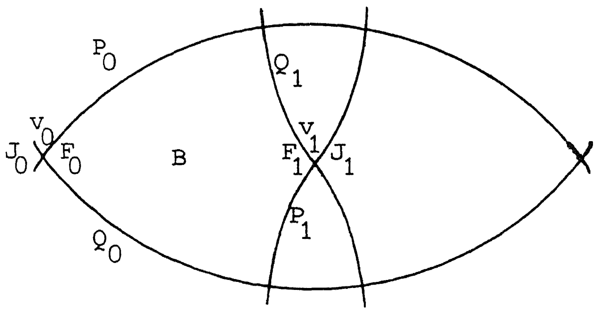


(c) (d)

<u>Claim 5</u>. *If D_j starts via $\pi(F)$, then D_j ends via $\pi(J)$.*

<u>Proof of Claim 5</u>. Suppose to the contrary that D_j both starts and ends via $\pi(F)$. Without loss of generality j=0. Suppose first we are in case (76)(a), leading to the following configuration:

(77)



where B is the open region indicated, bounded by parts of $\overline{P_0}$, $\overline{Q_1}$, $\overline{P_1}$ and $\overline{Q_0}$. So $B \subseteq A_0 \cap A_1$ (by Claim 2 applied to A_1).

Clearly, there exists a continuous function $\phi:\overline{A_0} \to \overline{B}$ with the following properties:

(78) (i) $\phi|_{\overline{B}} = \mathrm{id}|_{\overline{B}}$ (the identical function on $\overline{B}$);
(ii) $\phi[\overline{A_0}\setminus\overline{B}] \subseteq \overline{P_1} \cup \overline{Q_1}$;
(iii) $\phi[\overline{P_0}\setminus\overline{B}] \subseteq \overline{Q_1}$;
(iv) $\phi[\overline{Q_0}\setminus\overline{B}] \subseteq \overline{P_1}$.

The function $\theta_1^{-1}: S' \to S'$ is a continuous function, without fixed points (as D_0 is not null-homotopic, by (67)(i)), so that $\theta_1^{-1}[\overline{A_1}] = \overline{A_0}$. Hence $\phi\circ\theta_1^{-1}$ is a continuous function with $\phi\circ\theta_1^{-1}[\overline{B}] \subseteq \overline{B}$. As $\overline{B}$ is homeomorphic to a compact convex set, by Brouwer's fixed point theorem there exists an $r \in \overline{B}$ with $\phi(\theta_1^{-1}(r)) = r$.

If $\theta_1^{-1}(r) \in \overline{B}$, then $r = \phi(\theta_1^{-1}(r)) = \theta_1^{-1}(r)$, contradicting the fact that θ_1^{-1} has no fixed points.

If $\theta_1^{-1}(r) \in \overline{A_0} \setminus \overline{B}$, then $r = \phi(\theta_1^{-1}(r)) \in \overline{P_1} \cup \overline{Q_1}$. If $r \in \overline{P_1}$ then $\theta_1^{-1}(r) \in \overline{P_0}\setminus\overline{B}$ and hence $r = \phi(\theta_1^{-1}(r)) \in \overline{Q_1} \cap \overline{B}$, implying $r = v_1$, whence $v_1 = \phi(\theta_1^{-1}(v_1)) = \phi(v_0) = v_0$ – a contradiction. If $r \in \overline{Q_1}$ a contradiction follows similarly.

The cases (76)(b),(c) and (d) lead similarly (in fact, more simply) to contradictions. □

As a direct consequence we have:

Claim 6. *Each* D_j *starts via* $\pi(F)$ *and ends via* $\pi(J)$.

Proof of Claim 6. By (67)(iv) we know that D_0 starts via $\pi(F)$. Hence, by (66) and Claim 5, it follows inductively that each D_j starts via $\pi(F)$ and ends via $\pi(J)$. □

We next study the subset

(79) $$Y := \bigcup_{j\in\mathbb{Z}} \overline{A_j}$$

of S', and curves in $\mathfrak{D}'$ entering Y. Consider any curve C in $\mathfrak{D}'$ entering Y. That is, we can write

(80) $$C = (\ldots.,e_0,u_0,e_1,u_1,e_2,u_2,e_3,u_3,\ldots)$$

where e_i is an edge connecting vertices u_{i-1} and u_i ($i \in \mathbb{Z}$), so that $e_0 \not\subseteq Y$, while $e_1 \subseteq Y$ (we consider edges as open curves). We will show that C will leave Y again, and before leaving Y it intersects the curve D' in a point which is not a vertex of G'.

<u>Claim 7</u>. *There exists a number* $h \geq 1$ *so that* $e_1,\dots,e_h \subseteq Y$ *and* $e_{h+1} \nsubseteq Y$, *and so that the 'subcurve'*

(81) $$I := (u_0, e_1, u_1, e_2, u_2, \dots, e_h, u_h)$$

of C *intersects* D' *in a point which is not a vertex of* G'.

<u>Proof of Claim 7</u>. I. Let $H := \{h \geq 1 \mid e_1,\dots,e_h \subseteq Y\}$. We first show that only a finite number of sets $\overline{A_j}$ intersects $\bigcup_{h\in H} e_h$, which implies that H is finite (since each $\overline{A_j}$ contains only a finite number of edges, and since all $e_1, e_2, e_3, \dots$ are distinct edges of G'), and hence $h \in H$ with $h+1 \notin H$ does exist, as required.

If $\overline{A_j}$ would intersect $\bigcup_{h\in H} e_h$ for infinitely many j, there exists a j with $0 \leq j < n$ such that $\overline{A_{mn+j}}$ intersects $\bigcup_{h\in H} e_h$ for infinitely many $m \in \mathbb{Z}$. Since $\overline{A_{mn+j}} = \theta_{mn+j}(\overline{A})$, and since $\overline{A}$ contains only finitely many edges, there exists an edge e of G' contained in $\overline{A}$ so that $e_{h'} = \theta_{m'n+j}(e)$ and $e_{h''} = \theta_{m''n+j}(e)$ for some $h', h'' \in H$ and $m', m'' \in \mathbb{Z}$, with $m' \neq m''$. Without loss of generality, $h' \leq h''$. If $h' = h''$, we would have

(82) $$e = \theta^{-1}_{m''n+j}(\theta_{m'n+j}(e)) = \theta_{(m'-m'')n}(e),$$

a contradiction. If $h' < h''$, then

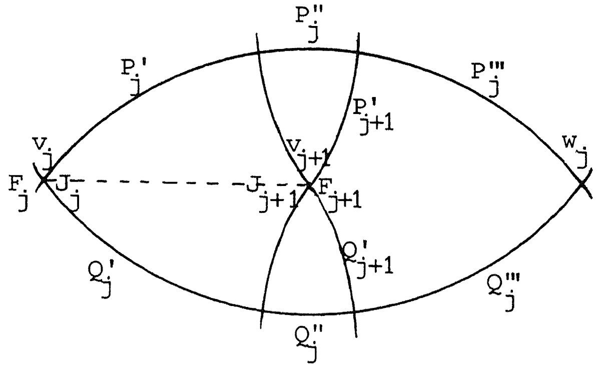
(83) $$e_{h''} = \theta_{m''n+j}(\theta^{-1}_{m'n+j}(e_{h'})) = \theta_{(m''-m')n}(e_{h'}),$$

whence, by symmetry, $e_{h''-h'} = \theta_{(m''-m')n}(e_0)$. This contradicts the facts that $e_0 \nsubseteq Y$, $e_{h''-h'} \subseteq Y$ and $\theta^{-1}_{(m''-m')n}(Y) = Y$.

II. So h as required exists. We next show that I intersects D' in a point which is not a vertex of G'. Let j be the largest integer so that I intersects $\overline{A_j}$ (j exists by the first part of this proof). We consider four cases.

<u>Case A</u>: For j we are in situation (76)(a):

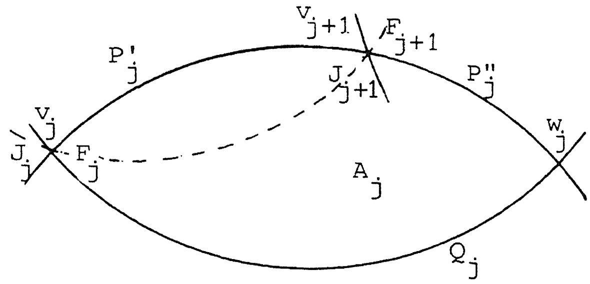
(84)



Note that parts P_j''' of P_j and Q_j''' of Q_j are fully contained in $\overline{A_{j+1}}$, by Claim 2, applied to A_{j+1}. Since I does not intersect $\overline{A_{j+1}}$, it follows that I does not intersect $\overline{P'_{j+1}} \cup \overline{P'''_j} \cup \overline{Q'''_j} \cup \overline{Q'_{j+1}}$. Hence I contains one of the chords of A_j connecting $\overline{P'_j} \cup \overline{P''_j}$ and $\overline{Q'_j} \cup \overline{Q''_j}$ as a subcurve. This implies that I intersects D' in a point which is not a vertex of G' (here we use (67)(ii)).

<u>Case B</u>: For j we are in situation (76)(b):

(85)



Note that part $\overline{P''_j}$ of $\overline{P_j}$ fully is contained in $\overline{Q_{j+1}}$, since $\overline{P_j}$ and $\overline{Q_j}$ contain the same number of vertices of G' (by Claim 2), implying that $\overline{P_j}$ and $\overline{Q_{j+1}}$ contain the same number of vertices of G' (as Q_{j+1} is a shift of Q_j).

In particular, $\overline{P''_j} \subseteq \overline{A_{j+1}}$. Since I does not intersect $\overline{A_{j+1}}$, it follows that I does not intersect $\overline{P''_j}$. Hence I contains one of the chords of A_j connecting P'_j and Q_j as a subset. This implies that I intersects D' in a point which is not a vertex of G' (here we use (67) (ii)).

<u>Case C</u>: For j we are in situation (76)(c). Similar to Case B.

<u>Case D</u>: For j we are in situation (76)(d). Since I does not intersect $\overline{A_{j+1}}$, we know that I cannot contain $\overline{P_j}$ or $\overline{Q_j}$ as a subcurve. Therefore, I contains one of the chords of A_j connecting P_j and Q_j , and hence it intersects D' in a point which is not a vertex of G' (again by (67)(ii)). $\square$

Consider now the covering surface S" of S generated by D (cf. Section 2). Let $D'':S_1 \rightarrow S''$ be the lifting of D to S". So D" is homotopic to the unit circle under the homeomorphism of S" and $\mathbb{C} \setminus \{0\}$. Let again $\psi: S' \rightarrow S''$ denote the quotient map (i.e., $\psi(q,\lambda) := (q, \langle\lambda\rangle)$), and let $\pi': S'' \rightarrow S$ denote the projection of S" on S (i.e., $\pi'(q, \langle\lambda\rangle) := q$). Let $G'' := (\pi')^{-1}[G] = \psi[G']$ be the 'lifting' of G to S". Again, the edges of G" can be decomposed into cycles and infinite paths ('curves') so that in each vertex two opposite edges are consecutively in the same curve. Let $\mathfrak{D}''$ denote this partition. So $\mathfrak{D}''$ corresponds to the projection of $\mathfrak{D}'$ on S".

Let $Y' := \psi[Y] = \bigcup_{j\in\mathbb{Z}}\psi[\overline{A_j}]$. Then $Y' = \bigcup_{j=0}^{n-1}\psi[\overline{A_j}]$, since $\psi[\overline{A_j}] = \psi[\overline{A_{n+j}}]$ for all $j\in\mathbb{Z}$.

Let E_0 be the collection of edges e of G" such that $e \not\subseteq Y'$ and $\bar{e}\cap Y' \neq \emptyset$. Note that in any vertex of G' (on S'), none or at least two of the adjacent faces are contained in Y. Hence, also in any vertex of G" (on S"), none or at least two of the adjacent faces are contained in Y'. Therefore, for any edge in E_0, if any end point belongs to Y', the opposite edge is contained in Y'.

Since Y' is compact, and contains the curve D", there are curves D_1 and D_2 on S" along the outer and inner boundaries, respectively, of Y', but disjoint from Y', intersecting only edges in E_0, and each of them at most once. Note that, by Proposition 2, both D_1 and D_2 are homotopic to D", and hence, $\pi'\circ D_1$ and $\pi'\circ D_2$ are homotopic to $\pi'\circ D" = D$.

<u>Claim 8</u>. $cr(G",D_1) + cr(G",D_2) \leqslant 2(cr(G",D") - 2n)$.

<u>Proof of Claim 8</u>. I. First observe the following. Consider any edge e in E_0. Then there is a curve C' in $\mathfrak{D}"$ with

(86) $$C' = (\ldots,e_0',u_0',e_1',u_1',\ldots)$$

where $e_0' = e$ and $e_1' \subseteq Y'$. Hence there is a curve

(87) $$C = (\ldots,e_0,u_0,e_1,u_1,\ldots)$$

in $\mathfrak{D}'$ so that $\psi(u_j)=u_j'$ and $\psi[e_j]=e_j'$ for all j. By Claim 7, there is an $h\in\mathbb{Z}$ so that $e_1,\ldots,e_h \subseteq Y$ and so that e_h intersects D' in a point $x \notin V'$. Hence, $e_1',\ldots,e_h' \subseteq Y'$ and e_h' intersects D" in a point $x' \notin V"$.

II. Now let α denote the number of subcurves

(88) $$(e_0',u_1',e_1',\ldots,e_m',u_m',e_{m+1}')$$

of curves in $\mathfrak{D}"$ so that $e_0',e_{m+1}' \not\subseteq Y'$ while $e_1',\ldots,e_m' \subseteq Y'$ (identifying (88) with its reverse). Then $cr(G",D_1)+cr(G",D_2) \leqslant 2\alpha$. Moreover, $\alpha \leqslant cr(G,D)-2n$, by I. above. Hence the claim follows. □

Without loss of generality, $cr(G",D_1) \leqslant cr(G,D)-2n$. Then $cr(G,\pi'\circ D_1) = cr(G",D_1) \leqslant cr(G,D)-2n < mincr(G,D)$, by (64). As $\pi'\circ D_1 \sim D$, this is a contradiction. □□□

5. PROOF OF THEOREM 1.

We now prove:

THEOREM 1. *Let* G *be an eulerian graph embedded on a compact orientable surface* S. *Then the edges of* G *can be decomposed into cycles* $C_1,\ldots,C_t$ *in such a way that for each closed curve* D *on* S:

$$\text{(89)} \qquad \text{mincr}(G,D) = \sum_{i=1}^{t} \text{mincr}(C_i,D).$$

PROOF. First note that the inequality $\geqslant$ in (89) trivially holds for any decomposition of the edges of G into cycles $C_1,\ldots,C_t$, since if $\check{D}\sim D$ is so that $\text{cr}(G,\check{D}) = \text{mincr}(G,D)$, then

$$\text{(90)} \qquad \text{mincr}(G,D) = \text{cr}(G,\check{D}) = \sum_{i=1}^{t} \text{cr}(C_i,\check{D}) \geqslant \sum_{i=1}^{t} \text{mincr}(C_i,D).$$

To see the reverse inequality, suppose $G=(V,E)$ forms a counterexample with

$$\text{(91)} \qquad \sum_{v\in V} 2^{\deg(v)}$$

as small as possible (where deg(v) denotes the degree of v in G).

Then G is minimally connected: If G would have a component which is a null-homotopic circuit, we could remove this circuit without changing mincr(G,D) for any closed curve D. Since the sum (91) does decrease, the edges of the smaller graph can be decomposed into cycles satisfying (89), and hence also for the original graph such a decomposition would exist.

Similarly, suppose the connectivity would be preserved by opening a vertex v from face F' to face F". Let G' be the graph after this opening. As for G' the sum (91) is smaller than for G, we can split the edges of G' into cycles satisfying (89), implying that the same holds for the original graph G.

So G is minimally connected, and therefore, by the Main Lemma, the straight decomposition forms a minimally crossing collection of primitive closed curves $C_1,\ldots,C_t$. By Proposition 11, for any closed curve D there exists $\tilde{D}\sim D$ so that $\text{cr}(C_i,\tilde{D}) = \text{mincr}(C_i,D)$ for $i=1,\ldots,t$. Hence

$$\text{(92)} \qquad \text{mincr}(G,D) \leqslant \text{cr}(G,\tilde{D}) = \sum_{i=1}^{t} \text{cr}(C_i,\tilde{D}) = \sum_{i=1}^{t} \text{mincr}(C_i,D). \qquad \square$$

6. PROOF OF THEOREM 2.

Using the duality relation of graphs embedded on a surface we derive from Theorem 1:

THEOREM 2. *Let* $G=(V,E)$ *be a bipartite graph embedded on a compact orientable surface* S*, and let* $C_1,\ldots,C_k$ *be cycles in* G*. Then there exist closed curves* $D_1,\ldots,D_t : S_1 \to S$ *so that (i) no* D_j *intersects* V*, (ii) each edge of* G *is intersected by exactly one* D_j *and by that* D_j *only once, (iii) for each* $i=1,\ldots,k$*:*

$$(93) \qquad \text{minlength}_G(C_i) = \sum_{j=1}^{t} \text{mincr}(C_i,D_j).$$

Proof. We can extend (the embedded) G to a bipartite graph L embedded on S, containing G as a subgraph, so that each face of L (i.e., component of $S\setminus L$) is simply connected (i.e., homeomorphic to $\mathbb{C}$). Let $d:=\max\{\text{minlength}_G(C_i) \mid i=1,\ldots,k\}$. By inserting d new vertices on each edge of L not occurring in G, we obtain a bipartite graph H satisfying

$$(94) \qquad \text{minlength}_G(C_i) = \text{minlength}_H(C_i)$$

for $i=1,\ldots,k$.

Consider a dual graph H^* of H on S. Since H is bipartite, H^* is eulerian. Hence by the theorem, the edges of H^* can be decomposed into cycles $D_1,\ldots,D_t$ so that for any closed curve C on S:

$$(95) \qquad \text{mincr}(H^*,C) = \sum_{j=1}^{t} \text{mincr}(D_j,C).$$

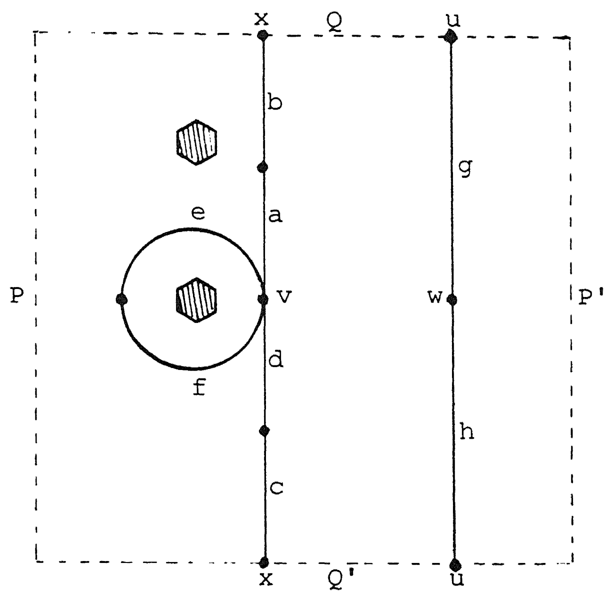
Now for each $i=1,\ldots,k$, $\text{mincr}(H^*,C_i) = \text{minlength}_H(C_i) = \text{minlength}_G(C_i)$, and (93) follows. □

Remark. The proof of Theorem 2 also gives the following. Let $G=(V,E)$ be a bipartite graph embedded on a compact orientable surface S, such that each face is simply connected. Then there exist closed curves $D_1,\ldots,D_t$: $S_1 \to S$ so that (i) no D_j intersects V, (ii) each edge of G is intersected by exactly one D_j and by that D_j only once, (iii) for each cycle C in G:

$$(96) \qquad \text{minlength}_G(C) = \sum_{j=1}^{t} \text{mincr}(C,D_j).$$

In this statement we cannot delete the condition that each face be simply connected, as is shown by the following example:

(97)



The surface is obtained from the square by identifying P and P' and identifying Q and Q' (thus obtaining a torus), and next deleting the interiors of the two hexagons and identifying their boundaries (in such a way that the surface obtained is orientable). For i=0,1,2,..., let C_i be the cycle in G which, starting in v, first follows e and f once, and next follows the cycle a,b,c, d i times. Then $\text{minlength}_G(C_i) = 4i+2$. Suppose now that $D_1,\dots,D_t$ are closed curves as above. Choose an arbitrary path $P:S_1 \rightarrow S$ from v to w. Then C_i is homotopic to the closed curve $\tilde{C}_i$ obtained by, starting from v, first following e and f, next following P, then following the cycle g,h i times, and finally following P back from w to v. Hence, for each i,

(98)
$$4i+2 = \text{minlength}_G(C_i) = \sum_{j=1}^{t} \text{mincr}(C_i,D_j) \leq \sum_{j=1}^{t} \text{cr}(\tilde{C}_i,D_j) =$$
$$\sum_{j=1}^{t}\left(\text{cr}(C_0,D_j) + 2\cdot\text{cr}(P,D_j) + i\cdot\text{cr}((w,g,u,h,w),D_j)\right) =$$
$$\sum_{j=1}^{t}\left(\text{cr}(C_0,D_j) + 2\cdot\text{cr}(P,D_j)\right) + 2i.$$

As the first term in this last sum is independent of i, this is a contradiction.

6. PROOF OF THEOREM 3.

Using the polarity relation of convex cones in euclidean space we derive from Theorem 2 the following 'homotopic circulation theorem':

THEOREM 3. *Let* $G=(V,E)$ *be a graph embedded on a compact orientable surface* S, *and let* $c:E\rightarrow\mathbb{Q}_+$ *('capacity' function). Let* $C_1,\ldots,C_k$ *be cycles in* G, *pairwise not-homotopic, and let* $d_1,\ldots,d_k\in\mathbb{Q}_+$ *('demands'). Then there exist cycles* $\Gamma_1,\ldots,\Gamma_u$ *in* G *and* $\lambda_1,\ldots,\lambda_u\geq 0$ *so that:*

$$\text{(99)}\qquad \text{(i)}\quad \sum_{j=1,\ \Gamma_j\sim C_i}^{u}\lambda_j=d_i \qquad (i=1,\ldots,k),$$

$$\text{(ii)}\quad \sum_{j=1}^{u}\lambda_j\cdot\chi^{\Gamma_j}(e)\leq c(e) \qquad (e\in E),$$

if and only if for each closed curve D *on* S *not intersecting* V *we have:*

$$\text{(100)}\qquad \sum_{i=1}^{k}d_i\cdot\text{mincr}(C_i,D)\leq\sum_{e\in E}c(e)\cdot\chi^{D}(e).$$

PROOF. Necessity. Suppose there exist $\Gamma_1,\ldots,\Gamma_u,\lambda_1,\ldots,\lambda_u$ as required, and let D be a closed curve on S not intersecting V. Then:

$$\text{(101)}\qquad \sum_{e\in E}c(e)\cdot\chi^{D}(e)\geq\sum_{e\in E}\chi^{D}(e)\sum_{j=1}^{u}\lambda_j\cdot\chi^{\Gamma_j}(e)=$$

$$\sum_{j=1}^{u}\lambda_j\sum_{e\in E}\chi^{D}(e)\cdot\chi^{\Gamma_j}(e)=\sum_{j=1}^{u}\lambda_j\text{cr}(\Gamma_j,D)\geq\sum_{j=1}^{u}\lambda_j\text{mincr}(\Gamma_j,D)\geq$$

$$\sum_{i=1}^{k}\text{mincr}(C_i,D)\sum_{\substack{j=1\\ \Gamma_j\sim C_i}}^{u}\lambda_j=\sum_{i=1}^{k}d_i\text{mincr}(C_i,D).$$

Sufficiency. Suppose (100) is satisfied for each closed curve D not intersecting V. Let K be the convex cone in $\mathbb{R}^k\times\mathbb{R}^E$ generated by the vectors:

$$\text{(102)}\qquad (\varepsilon_i;\chi^{\Gamma}) \qquad (i=1,\ldots,k;\ \Gamma\text{ cycle in }G\text{ with }\Gamma\sim C_i);$$

$$(\underline{0};\varepsilon_e) \qquad (e\in E).$$

Here ε_i denotes the i-th unit bases vector in $\mathbb{R}^k$. Similarly, ε_e denotes the e-th unit basis vector in $\mathbb{R}^E$. $\underline{0}$ denotes the origin in $\mathbb{R}^k$.

Although (102) gives infinitely many vectors, K is finitely generated. This can be seen as follows. For each fixed i, call a cycle $\Gamma\sim C_i$ *minimal* if there is no cycle $\Gamma'\sim C_i$ with $\chi^{\Gamma'}(e)\leq\chi^{\Gamma}(e)$ for each edge e, and with strict inequality for at least one edge e. So the set $\{\chi^{\Gamma}\mid\Gamma\text{ minimal cycle with }\Gamma\sim C_i\}$

forms an antichain in $\mathbb{Z}_+^E$ and is therefore finite. Since we can restrict, for each i=1,...,k, the χ^Γ in (102) to those with Γ minimal, K is finitely generated.

What we must show is that the vector (d;c) belongs to K, where d:= $(d_1,\ldots,d_k)$. By Farkas' lemma, it suffices to show that for each vector (p;b) $\in \mathbb{Q}^k \times \mathbb{Q}^E$ with nonnegative inner product with each of the vectors (102), also the inner product with (d;c) is nonnegative. So let (p;b) have nonnegative inner product with each of (102). This is equivalent to:

$$\text{(103)}\qquad \begin{array}{lll} \text{(i)} & p_i + \sum_{e\in E} b(e)\,\chi^\Gamma(e) \geq 0 & (i=1,\ldots,k;\ \Gamma \text{ cycle in G with } \Gamma \sim C_i); \\ \text{(ii)} & b(e) \geq 0 & (e \in E). \end{array}$$

Without loss of generality, each entry in (p;b) is an even integer. Let G' be the graph arising from G by replacing each edge e by a path of length b(e) (that is, b(e)-1 new vertices are inserted on e, if b(e) $\geq$ 1; e is contracted if b(e)=0). Each cycle C_i directly gives a cycle C_i' in G'. Then by (103)(i):

$$\text{(104)}\qquad -p_i \leq \text{minlength}_{G'}(C_i') \qquad \text{for } i=1,\ldots,k.$$

Since G' is bipartite, by Corollary 1, there exist closed curves $D_1,\ldots,D_t$ on S so that (i) each D_j intersects G' only in edges of G', (ii) each edge of G' is intersected by exactly one D_j and only once by that D_j, and (iii) for each i=1,...,k:

$$\text{(105)}\qquad \text{minlength}_{G'}(C_i') = \sum_{j=1}^t \text{mincr}(C_i',D_j).$$

Note that (ii) is equivalent to:

$$\text{(106)}\qquad b(e) = \sum_{j=1}^t \chi^{D_j}(e).$$

Therefore, using (100), (104), (105) and (106):

$$\text{(107)}\qquad \sum_{e\in E} b(e)c(e) = \sum_{j=1}^t \sum_{e\in E} \chi^{D_j}(e)c(e) =$$

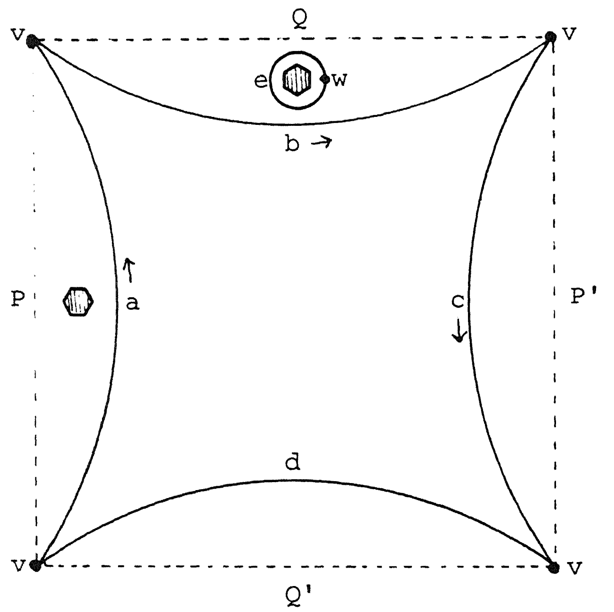
$$\sum_{j=1}^t \sum_{i=1}^k d_i\cdot\text{mincr}(C_i,D_j) = \sum_{i=1}^k d_i\cdot\sum_{j=1}^t \text{mincr}(C_i',D_j) =$$

$$\sum_{i=1}^k d_i\cdot\text{minlength}_{G'}(C_i') \geq -\sum_{i=1}^k p_i d_i.$$

So $(p;b)\cdot(d;c)^T \geq 0$. ∎

Remark. In Note 3 in Section 1 we mentioned some special cases where we can take the λ_i in Theorem 3 integral. Here we give an example that in general it is not enough to require that all capacities and demands are integral so that the difference in (100) is even for each D:

(108)



The surface is obtained from the square by identifying P and P' and identifying Q and Q' (thus obtaining a torus) and next deleting the interiors of the two hexagons and identifying their boundaries (in such a way that the surface obtained is orientable). The graph has two vertices, v and w, and four loops (a,b,c,d) at v and one loop (e) at w. Curve C_1 follows the edges a and b, and curve C_2 follows the edges b and c - in the directions indicated. Taking all capacities and demands equal to 1, we see that for each closed curve D not intersecting v and w, the difference in (100) is an even nonnegative integer. However, no edge-disjoint cycles homotopic to C_1 and C_2, respectively, exist in G.

8. PROOF OF THEOREM 4.

As a consequence of Theorem 3 we derive a 'homotopic flow-cut theorem':

THEOREM 4. *Let* $G=(V,E)$ *be a planar graph embedded in the complex plane* $\mathbb{C}$. *Let* $I_1,\ldots,I_p$ *be (the interiors of) some of the faces of* G, *including the unbounded face. Let* $P_1,\ldots,P_k$ *be paths in* G *with end points on the boundary of* $I_1\cup\ldots\cup I_p$. *Then there exist paths* $P_1^1,\ldots,P_1^{t_1},P_2^1,\ldots,P_2^{t_2},\ldots,P_k^1,\ldots,P_k^{t_k}$ *in* G *and rationals* $\lambda_1^1,\ldots,\lambda_1^{t_1},\lambda_2^1,\ldots,\lambda_2^{t_2},\ldots,\lambda_k^1,\ldots,\lambda_k^{t_k}\geqslant 0$ *so that:*

$$\text{(109)}\quad \begin{array}{lll} \text{(i)} & P_i^j\sim P_i \text{ in } \mathbb{C}\setminus(I_1\cup\ldots\cup I_p) & (i=1,\ldots,k;j=1,\ldots,t_i),\\ \text{(ii)} & \sum_{j=1}^{t_i}\lambda_i^j=1 & (i=1,\ldots,k),\\ \text{(iii)} & \sum_{i=1}^{k}\sum_{j=1}^{t_i}\lambda_i^j\,\chi^{P_i^j}(e)\leqslant 1 & (e\in E), \end{array}$$

if and only if for each path $D:[0,1]\rightarrow\mathbb{C}\setminus(I_1\cup\ldots\cup I_p)$, *connecting two points on the boundary of* $I_1\cup\ldots\cup I_p$, *not intersecting* V, *and intersecting* G *only a finite number of times, we have:*

$$\text{(110)}\quad \sum_{i=1}^{k}\text{mincr}(P_i,D)\leqslant\text{cr}(G,D).$$

PROOF. Necessity. Similar to the proof of necessity in Theorem 3.

Sufficiency. Suppose the condition mentioned holds. We may assume that G is embedded on the 2-dimensional sphere S_2, and that $I_1,\ldots,I_p$ are faces on S_2. For each pair of vertices v,v' of G on the boundary of $I_1\cup\ldots\cup I_p$, say on the boundaries of I_j and $I_{j'}$, respectively, make a handle $H_{v,v'}$ connecting I_j and $I_{j'}$. This yields the compact orientable surface S. Next, for each path P_i from, say, v to v', we extend G by an edge, say e_i, connecting v and v' over the handle $H_{v,v'}$. It is easy to see that we can do this in such a way that the new edges do not intersect each other and do not intersect the 'old' edges of G. Let G' denote the extended graph. Each P_i now has been 'closed' to a cycle, say C_i, in G'. We apply Theorem 3, with $d_i:=1$ for $i=1,\ldots,k$, and $c(e):=1$ for each edge e. Then the 'circulation' described by (99) yields a 'multi-commodity flow' described by (109). So it suffices to check condition (100), i.e., that for any closed curve D on S not intersecting V we have:

$$\text{(111)}\quad \sum_{i=1}^{k}\text{mincr}(C_i,D)\leqslant\text{cr}(G',D).$$

We distinguish three cases.

Case 1: D is contained in $S_2 \setminus (I_1 \cup \ldots \cup I_p)$. Let y be some point on D, let z be some point on the boundary of $I_1 \cup \ldots \cup I_p$ (so that $z \notin V$), let R be some path in $S_2 \setminus (I_1 \cup \ldots \cup I_p)$ connecting z and y (so that R does not intersect V and does intersect G only a finite number of times), and let for $n \in \mathbb{N}$, Q_n be the path from z to z which first follows R from z to y, then follows closed curve D n times, and next returns to z over R. Let r be the number of edges intersected by R. Let D^n denote the closed curve with $D^n(z) := D(z^n)$ for $z \in S_1$. Then for all $n \in \mathbb{N}$:

(112) $$n \cdot \sum_{i=1}^k \operatorname{mincr}(C_i, D) = \sum_{i=1}^k \operatorname{mincr}(C_i, D^n) \leqslant \sum_{i=1}^k \operatorname{mincr}(P_i, Q_n) \leqslant \operatorname{cr}(G, Q_n) = 2r + n \cdot \operatorname{cr}(G', D).$$

[The first equality follows from Proposition 5 in Section 3. The first inequality follows from the fact that for each i, if $Q \sim Q_n$ is a path in $S_2 \setminus (I_1 \cup \ldots \cup I_p)$ attaining $\operatorname{mincr}(P_i, Q_n)$, then, as a closed curve, Q is (freely) homotopic to D^n, and hence $\operatorname{mincr}(C_i, D^n) \leqslant \operatorname{cr}(C_i, Q) = \operatorname{cr}(P_i, Q) = \operatorname{mincr}(P_i, Q_n)$. The second inequality follows from (110). The last equality follows from the definition of path Q_n.] Since (112) holds for each n, while r is fixed, (111) follows.

Case 2: D does not intersect $S_2 \setminus (I_1 \cup \ldots \cup I_p)$. Now:

(113) $$\operatorname{cr}(G', D) = \sum_{i=1}^k \operatorname{cr}(C_i, D) \geqslant \sum_{i=1}^k \operatorname{mincr}(C_i, D).$$

Case 3: D intersects both $S_2 \setminus (I_1 \cup \ldots \cup I_p)$ and its complement in S, say H. Then we can split D into paths $D_1, D_2, \ldots, D_{2u}$ so that for odd i, D_i is contained in $S_2 \setminus (I_1 \cup \ldots \cup I_p)$ and connects two points on the boundary of $I_1 \cup \ldots \cup I_p$, while for even i, D_i is contained in H, except for its end points. Then we have:

(114) $$\operatorname{cr}(G', D) = \sum_{j=1}^u \operatorname{cr}(G, D_{2j-1}) + \sum_{j=1}^u \sum_{i=1}^k \chi^{D_{2j}}(e_i) \geqslant \sum_{j=1}^u \sum_{i=1}^k \operatorname{mincr}(P_i, D_{2j-1}) + \sum_{j=1}^u \sum_{i=1}^k \chi^{D_{2j}}(e_i) = \sum_{i=1}^k \sum_{j=1}^u \left(\operatorname{mincr}(P_i, D_{2j-1}) + \chi^{D_{2j}}(e_i)\right) \geqslant \sum_{i=1}^k \operatorname{mincr}(C_i, D),$$

thus proving (111). The first inequality here follows from (110). The second inequality can be seen as follows. Define for any two paths $P, Q: [0,1] \rightarrow S_2 \setminus (I_1 \cup \ldots \cup I_p)$ with $X := \{(x,y) \in [0,1] \times [0,1] \mid P(x) = Q(y)\}$ finite, the following relation $\approx$ on X:

(115) $(x,y) \approx (x',y')$ if and only if the path $P(\lambda x'+(1-\lambda)x)_{\lambda\in[0,1]}$ is homotopic to the path $Q(\lambda y'+(1-\lambda)y)_{\lambda\in[0,1]}$

(note that both paths connect $P(x)=Q(y)$ with $P(x')=Q(y')$). This defines an equivalence relation on X. Call a class *odd* if it contains an odd number of elements. Next define:

(116) $odd(P,Q) :=$ number of odd classes of $\approx$.

Clearly, $cr(P,Q) \geqslant odd(P,Q)$. It is not difficult to see that, if both P and Q have their end points on the boundary of $I_1\cup\ldots\cup I_p$, and if $\tilde{P}\sim P$, $\tilde{Q}\sim Q$ in $S_2\setminus(I_1\cup\ldots\cup I_p)$, so that $\tilde{P}$ and $\tilde{Q}$ have a finite number of intersections, then $odd(\tilde{P},\tilde{Q})=odd(P,Q)$. [This follows with the theory of simplicial approximation (cf. Seifert and Threlfall [22:§44]): Make triangulations Γ and Δ of $S_2\setminus(I_1\cup\ldots\cup I_p)$ so that P and $\tilde{P}$ follow edges of Γ and Q and $\tilde{Q}$ follow edges of Δ, and so that the vertices and edges of Γ have only a finite number of intersections with those of Δ. Then $\tilde{P}$ arises from P by a series of reroutings along triangles of Γ; each such rerouting does not change $odd(P,Q)$. Similarly for Q.]

Hence

(117) $mincr(P,Q) \geqslant odd(P,Q)$.

Therefore, for each fixed $i=1,\ldots,k$:

(118)
$$\sum_{j=1}^{u}\left(mincr(P_i,D_{2j-1}) + \chi^{D_{2j}}(e_i)\right) \geqslant$$
$$\sum_{j=1}^{u}\left(odd(P_i,D_{2j-1}) + \chi^{D_{2j}}(e_i)\right) \geqslant odd(C_i,D) = mincr(C_i,D).$$

For the definition of $odd(C,D)$ for closed curves C,D we refer to Section 3. The second inequality in (118) follows from the fact that if two intersections of P_i and D_{2j-1} are equivalent according to (115), then they are equivalent intersections of C_i and D according to (45). Hence each odd class of intersections of C_i and D includes at least one odd class of intersections of P_i and D_{2j-1} for some $j=1,\ldots,u$, or contains at least one intersection of D_{2j} with e_i. □

9. PROOF OF THEOREM 5.

We finally show that the condition given in the 'Main Lemma' is necessary and sufficient, which is the content of Theorem 5. As a preparation, we show the following 'cross-counting' lemma, based on an elementary exchange of summation.

LEMMA. *Let* $G=(V,E)$ *be an eulerian graph embedded on the compact orientable surface* S. *Let* $C_1,\ldots,C_s$ *be cycles in* G *and let* $\lambda_1,\ldots,\lambda_s>0$ *so that for each* $e\in E$ *we have* $\sum_{i=1}^{s}\lambda_i\chi^{C_i}(e)\leq 1$. *Then for each* $i=1,\ldots,s$:

$$\text{(119)}\qquad \sum_{v\in V}\mu_i(v)(\tfrac{1}{2}\deg(v)-1)\geq\sum_{j=1}^{s}\lambda_j\cdot\text{mincr}(C_i,C_j),$$

where $\mu_i(v)$ *denotes the number of times* C_i *passes vertex* v. *Moreover, equality in* (119) *implies that* C_i *belongs to the straight decomposition of* G.

PROOF. To avoid notational problems we assume that G has no parallel edges (by inserting extra vertices on edges we can accomplish this). Let for any $v\in V$, $\delta(v)$ be the set of edges incident to v. Denote for each $i=1,\ldots,s$ and each choice of $v\in V$, $e,e'\in\delta(v)$:

(120) $\mu_i(e,e')$:= the number of times cycle C_i passes v by going from e to e' or from e' to e.

So $\mu_i(e,e')=0$ if $e=e'$. Define, for $v\in V$, $e,e',e'',e'''\in\delta(v)$:

$$\text{(121)}\qquad \tau(e,e',e'',e''') := \begin{cases} 1, & \text{if } e,e' \text{ crosses } e'',e''';\\ \tfrac{1}{2}, & \text{if } e,e' \text{ semi-crosses } e'',e''';\\ 0, & \text{otherwise.}\end{cases}$$

Here we say that e,e' *crosses* e'',e''' if e,e',e'',e''' are all distinct and if in the cyclic order of edges incident to v, we have e,e'',e',e''' or e,e''',e',e''. Moreover, e,e' *semi-crosses* e'',e''' if $|\{e,e',e'',e'''\}|=3$ and $|\{e,e'\}\cap\{e'',e'''\}|=1$.

One easily checks that for $i,j=1,\ldots,s$:

$$\text{(122)}\qquad \text{mincr}(C_i,C_j)\leq\frac{1}{4}\sum_{v\in V}\sum_{\substack{e,e',e'',\\ e'''\in\delta(v)}}\tau(e,e',e'',e''')\,\mu_i(e,e')\mu_j(e'',e''').$$

Now for each $v\in V$ and $e,e'\in\delta(v)$ with $e\neq e'$:

$$(123)\quad \sum_{j=1}^{s}\lambda_j \sum_{e'',e'''\in\delta(v)} \tau(e,e',e'',e''')\mu_j(e'',e''') = \sum_{\substack{e'',e'''\in\delta(v),\\ e'',e'''\text{ crosses}\\ e,e'}} \sum_{j=1}^{s}\lambda_j\cdot\mu_j(e'',e''') + \tfrac{1}{2}\sum_{\substack{e'',e'''\in\delta(v)\\ e'',e'''\text{ semi-crosses}\\ e,e'}} \sum_{j=1}^{s}\lambda_j\cdot\mu_j(e'',e''')$$

$$= \sum_{\substack{e''\in\delta(v)\\ e''\neq e,e'}} \sum_{j=1}^{s}\lambda_j\Bigg(\sum_{\substack{e'''\in\delta(v)\\ e'',e'''\text{ crosses}\\ e,e'}}\mu_j(e'',e''') + \tfrac{1}{2}\sum_{e'''\in\{e,e'\}}\mu_j(e'',e''')\Bigg) +$$

$$+ \sum_{e''\in\{e,e'\}} \sum_{j=1}^{s}\lambda_j\,\tfrac{1}{2}\Bigg(\sum_{\substack{e'''\in\delta(v)\\ e'''\neq e,e'}}\mu_j(e'',e''')\Bigg) \leq$$

$$\sum_{\substack{e''\in\delta(v)\\ e''\neq e,e'}} \sum_{j=1}^{s}\lambda_j \sum_{e'''\in\delta(v)}\mu_j(e'',e''') = \sum_{\substack{e''\in\delta(v)\\ e''\neq e,e'}} \sum_{j=1}^{s}\lambda_j\cdot\chi^{C_j}(e'') \leq \deg(v)-2.$$

Hence, by (122) and (123),

$$(124)\quad \sum_{v\in V}\mu_i(v)(\deg(v)-2) = \tfrac{1}{2}\sum_{v\in V}\sum_{e,e'\in\delta(v)}\mu_i(e,e')(\deg(v)-2) \geq$$

$$\tfrac{1}{2}\cdot\sum_{v\in V}\sum_{e,e'\in\delta(v)}\mu_i(e,e')\cdot\sum_{j=1}^{s}\lambda_j\sum_{e'',e'''\in\delta(v)}\tau(e,e',e'',e''')\mu_j(e'',e''') \geq$$

$$2\cdot\sum_{j=1}^{s}\lambda_j\cdot\mathrm{mincr}(C_i,C_j).$$

This shows (119).

Next suppose we have equality in (119). To show that C_i belongs to the straight decomposition, suppose $v\in V$, $e,e'\in\delta(v)$, $e\neq e'$, such that e and e' are not opposite edges and $\mu_i(e,e')\geq 1$. Equality in (119) implies equality throughout in (124). Let $e_1=e$,$e_2,\ldots,e_p,e_{p+1}=e',e_{p+2},\ldots,e_d$ be the edges incident to v in cyclic order (so $d=\deg(v)$). Since e and e' are not opposite, we may assume $p\leq\frac{1}{2}d-1$. Then

$$(125)\quad \sum_{\substack{e'',e'''\in\delta(v)\\ e'',e'''\text{ crosses}\\ e,e'}} \sum_{j=1}^{s}\lambda_j\cdot\mu_j(e'',e''') = 2\cdot\sum_{h=2}^{p}\sum_{g=p+2}^{d}\lambda_j\cdot\mu_j(e_h,e_g) =$$

$$= 2\cdot\sum_{h=}^{p}\sum_{j=1}^{s}\lambda_j\sum_{g=p+2}^{d}\mu_j(e_h,e_g) \leq 2\cdot\sum_{h=}^{p}\sum_{j=1}^{s}\lambda_j\cdot\chi^{C_j}(e_h) \leq 2p-2,$$

and

$$(126)\qquad \tfrac{1}{2}\cdot \sum_{\substack{e'',e'''\in\delta(v)\\ e'',e''' \text{ semi-crosses}\\ e,e'}} \sum_{j=1}^{s} \lambda_j\cdot\mu_j(e'',e''') <$$

$$\tfrac{1}{2}\cdot \sum_{\substack{e'',e'''\in\delta(v)\\ e'',e''' \text{ semi-crosses}\\ e,e'}} \sum_{j=1}^{s} \lambda_j\cdot\mu_j(e'',e''') + \sum_{j=1}^{s} \lambda_j\cdot\mu_j(e,e') =$$

$$\tfrac{1}{2} \sum_{\substack{e''\in\delta(v)\\ e''\neq e,e'}} \sum_{j=1}^{s} \lambda_j(\mu_j(e,e'')+\mu_j(e',e'')) +$$

$$\tfrac{1}{2} \sum_{\substack{e'''\in\delta(v)\\ e'''\neq e,e'}} \sum_{j=1}^{s} \lambda_j(\mu_j(e,e''')+\mu_j(e',e''')) + \sum_{j=1}^{s} \lambda_j\cdot\mu_j(e,e') =$$

$$\sum_{e''\in\delta(v)} \sum_{j=1}^{s} \lambda_j\cdot(\mu_j(e,e'')+\mu_j(e',e'')) = \sum_{j=1}^{s} \lambda_j(\chi^{C_j}(e)+\chi^{C_j}(e')) \leq 2.$$

This implies strict inequality in (123) - a contradiction. □

This lemma is used in proving:

<u>THEOREM 5</u>. *Let* G = (V,E) *be an eulerian graph embedded on the compact orientable surface* S. *Then* G *is minimally connected if and only if the straight decomposition of* G *forms a minimally crossing collection of primitive closed curves.*

<u>PROOF</u>. The content of the 'Main Lemma' being the 'only if' part, we show here the 'if' part. So let the straight decomposition $\mathcal{D}=(C_1,\ldots,C_k)$ of G be a minimally crossing collection of primitive closed curves. We show that (27)(i) and (ii) are satisfied.

Condition (27)(ii) is trivial, since any component of G being a nullhomotopic circuit would give a curve in $\mathcal{D}$ which is not primitive.

Now to show (27)(i), suppose that the connectivity is preserved by opening vertex w from face F' to face F", where F' and F" are opposite in w. Let G' be the graph obtained by this 'opening' (cf. (26)). Since for each closed curve D on S:

$$(127)\qquad \sum_{i=1}^{k} \text{mincr}(C_i,D) \leq \text{mincr}(G,D) = \text{mincr}(G',D),$$

we know by Theorem 3 that in G' there exist cycles $C_1^1, \ldots, C_1^{t_1}, \ldots, C_k^1, \ldots, C_k^{t_k}$ and rationals $\lambda_1^1, \ldots, \lambda_1^{t_1}, \ldots, \lambda_k^1, \ldots, \lambda_k^{t_k} > 0$ so that:

(128) $$C_i^j \sim C_i \qquad (i=1,\ldots,k; j=1,\ldots,t_i),$$
$$\sum_{j=1}^{t_i} \lambda_i^j = 1 \qquad (i=1,\ldots,k),$$
$$\sum_{i=1}^{k} \sum_{j=1}^{t_i} \lambda_i^j \cdot \chi^{C_i^j}$$

By identifying the two new w' and w" we obtain cycles in G satisfying the condition of the Lemma. Call the cycle in G obtained from C_i^j again C_i^j. Then

(129) $$\sum_i \sum_{i'} \text{mincr}(C_i, C_{i'}) = \sum_{i,j} \sum_{i',j'} \lambda_i^j \lambda_{i'}^{j'} \text{mincr}(C_i^j, C_{i'}^{j'}) \leq$$
$$\sum_{i,j} \lambda_i^j \sum_{v \in V} \mu_i^j(v)(\tfrac{1}{2}\deg(v)-1) = \sum_{v \in V} (\tfrac{1}{2}\deg(v)-1) \sum_{i,j} \lambda_i^j \mu_i^j(v) \leq$$
$$\sum_{v \in V} (\tfrac{1}{2}\deg(v)-1)(\tfrac{1}{2}\deg(v)) = \sum_{\substack{i,i' \\ i \neq i'}} \text{cr}(C_i, C_{i'}) + 2\sum_i \text{cr}(C_i) =$$
$$\sum_i \sum_{i'} \text{mincr}(C_i, C_{i'})$$

(cf. Proposition 10 in Section 3), where i and i' range over 1,...,k, j over $1,\ldots,t_i$ and j' over $1,\ldots,t_{i'}$, and where $\mu_i^j(v)$ denotes the number of times cycle C_i^j passes v.

So we have equality throughout. By the Lemma it implies that each C_i^j belongs to the straight decomposition of G. Hence, as the C_i^j arose from cycles in G', vertex w cannot be used more than $\tfrac{1}{2}\deg(v)-1$ times, i.e.,

(130) $$\sum_{i,j}{}' \lambda_i^j \cdot \mu_i^j(w) \leq \tfrac{1}{2}\deg(w)-1.$$

This contradicts the equality throughout in (129). □

10. FURTHER RESULTS ON CURVES ON SURFACES

We finally derive from our results some further properties of curves on compact orientable surfaces. First we notice:

Lemma 1. *Let* C *be a closed curve on the compact orientable surface* S *such that* $\mathrm{mincr}(C,D)=0$ *for each closed curve* D. *Then* C *is null-homotopic.*

Proof. Let S have genus g. That is, S is homeomorphic to the 2-dimensional sphere added with g 'handles'. If g=0, C is null-homotopic. If $g \geqslant 1$, there exists a minimally crossing collection of primitive closed curves $E_1, E_2, \dots, E_{2g-1}, E_{2g}$ on S so that $S \setminus \bigcup_{i=1}^{2g} E_i[S_1]$ is homeomorphic to $\mathbb{C}$. As $\mathrm{mincr}(C,E_i)=0$ for $i=1,\dots,2g$, by Proposition 12 there exists $\tilde{C} \sim C$ such that $\mathrm{cr}(\tilde{C},E_i)=0$ for $i=1,\dots,2g$. Therefore, $\tilde{C}$ is a curve in $S\setminus\bigcup_{i=1}^{2g} E_i[S_1] \simeq \mathbb{C}$, and hence C is null-homotopic. □

A second auxiliary result is derived from Theorem 3 (the 'homotopic circulation theorem') and the Lemma in Section 9:

Lemma 2. *Let* B *and* C *be primitive closed curves on a compact orientable surface* S *and* $m,n \in \mathbb{N}$ *so that* $\mathrm{mincr}(B^m,D)=\mathrm{mincr}(C^n,D)$ *for each closed curve* D. *Then* $B \sim C$ *or* $B \sim C^{-1}$, *and* m=n.

Proof. By symmetry we may assume:

$$(131)\qquad m^2\cdot\mathrm{mincr}(B) \leqslant n^2\cdot\mathrm{mincr}(C).$$

Moreover, we may assume that $\mathrm{cr}(B)=\mathrm{mincr}(B)$. Consider the graph G made up by B, where the vertex set V of G is exactly the set of points of self-crossings of B. Then for each closed curve D:

$$(132)\qquad \mathrm{cr}(G,D)=\mathrm{cr}(B,D)\geqslant \mathrm{mincr}(B,D)=\tfrac{1}{m}\mathrm{mincr}(B^m,D)=\tfrac{1}{m}\mathrm{mincr}(C^n,D)=\tfrac{n}{m}\mathrm{mincr}(C,D).$$

Hence, by Theorem 3, there exist closed curves $C_1,\dots,C_s \sim C$ and $\lambda_1,\dots,\lambda_s > 0$ so that:

$$(133)\qquad \text{(i)}\ \lambda_1+\dots+\lambda_s = \tfrac{n}{m},$$
$$\text{(ii)}\ \textstyle\sum_{i=1}^{s} \lambda_i \chi^{C_i}(e) \leqslant 1 \ \text{ for every edge } e \text{ of } G.$$

Denote by $\mu_i(v)$ the number of times C_i passes v. Then we derive from the Lemma in Section 9 (using Proposition 10):

(134) $$2\frac{n^2}{m^2}\cdot \mathrm{mincr}(C) = \sum_{i=1}^{s} \lambda_i \sum_{j=1}^{s} \lambda_j \cdot \mathrm{mincr}(C_i,C_j) \leqslant$$
$$\sum_{i=1}^{s} \lambda_i \cdot \sum_{v\in V} \mu_i(v)(\tfrac{1}{2}\deg(v)-1) = \sum_{v\in V}(\tfrac{1}{2}\deg(v)-1)\sum_{i=1}^{s}\mu_i(v)\lambda_i =$$
$$\sum_{v\in V}(\tfrac{1}{2}\deg(v)-1)\tfrac{1}{2}\sum_{e\in\delta(v)}\sum_{i=1}^{s}\lambda_i \chi^{C_i}(e) \leqslant$$
$$\sum_{v\in V}(\tfrac{1}{2}\deg(v)-1)(\tfrac{1}{2}\deg(v)) = 2\mathrm{mincr}(B).$$

(Here $\delta(v)$ denotes the set of edges incident to v.) By (131) we have equality throughout in (134). Hence for each i we have equality in (119). By the Lemma, this implies that C_i belongs to the straight decomposition of G. Hence $C_i \sim B$ or $C_i \sim B^{-1}$, implying $B \sim C$ or $B \sim C^{-1}$.

Moreover, by Lemma 1, $\mathrm{mincr}(B,D) > 0$ for at least one closed curve D. So m=n follows from: $m\cdot\mathrm{mincr}(B,D) = \mathrm{mincr}(B^m,D) = \mathrm{mincr}(C^n,D) = n\cdot\mathrm{mincr}(C,D) = n\cdot\mathrm{mincr}(B,D)$. □

We derive the following 'unique factorization theorem' for closed curves, generalizing a result of Marden, Richards and Rodin [11]:

<u>Proposition 13</u>. *For each not-null-homotopic closed curve* E *on a compact orientable surface* S, *there exists a primitive closed curve* C, *unique up to homotopy, and a unique* $n \in \mathbb{N}$ *such that* $E \sim C^n$.

<u>Proof</u>. We first show that $E \sim C^n$ for at least one primitive closed curve C and $n \in \mathbb{N}$. By Lemma 1, there exists a closed curve D with $\mathrm{mincr}(E,D) > 0$. So if $E \sim C^n$ for some closed curve C and $n \in \mathbb{N}$, then (by Proposition 5):

(135) $$0 < \mathrm{mincr}(E,D) = \mathrm{mincr}(C^n,D) = n\cdot\mathrm{mincr}(C,D),$$

implying that $n \leqslant \mathrm{mincr}(E,D)$. Hence there exists a largest $n \in \mathbb{N}$ for which there exists a closed curve C with $E \sim C^n$. It follows that C is primitive (otherwise, $C \sim B^m$ for some $m \geqslant 2$, and hence $E \sim B^{mn}$ with $mn > n$).

Second we show that C and n are unique (up to homotopy). Suppose $C^n \sim B^m$ where C and B are primitive and $m,n \in \mathbb{N}$. In particular, $\mathrm{mincr}(B^m,D) = \mathrm{mincr}(C^n,D)$ for each D. Hence by Lemma 2, $B \sim C$ or $B \sim C^{-1}$, and m=n. Suppose $B \sim C^{-1}$. Then C^{2n} is null-homotopic, and hence $\mathrm{mincr}(C,D) = \mathrm{mincr}(C^{2n},D)/2n = 0$

for each closed curve D. By Lemma 1 this implies that C is null-homotopic, contradicting the primitivity of C. □

Finally we derive that any curve homotopy class can be identified by the function 'mincr' (up to inverting the curve):

Proposition 14. *Let* B *and* C *be closed curves on a compact orientable surface* S, *such that* mincr(B,D) = mincr(C,D) *for each closed curve* D. *Then* $B \sim C$ *or* $B \sim C^{-1}$.

Proof. Directly from Lemma 2 and Proposition 13. □

Acknowledgements. I am grateful to Crispin St.J.A. Nash-Williams for asking me, in Oberwolfach, the question whether Theorem 4 above could be true, which has led me to the present research. I also thank Stan van Hoesel, Laci Lovász, Paul Seymour and Éva Tardos for stimulating discussions.

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