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Department of Operations Research and System Theory Report OS-R8718 November

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Edge-Disjoint Homotopic Paths in Straight-Line Planar Graphs

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Abstract. - Let G be a planar graph, embedded without crossings in the euclidean plane $\mathbb{R}^2$, and let $I_1,\ldots,I_p$ be some of its faces (including the unbounded face), considered as open sets. Suppose there exist (straight) line segments $L_1,\ldots,L_t$ in $\mathbb{R}^2$ so that $G \cup I_1 \cup \ldots \cup I_p = L_1 \cup \ldots \cup L_t \cup I_1 \cup \ldots \cup I_p$ and so that each L_i has its end points in $I_1 \cup \ldots \cup I_p$. Let $C_1,\ldots,C_k$ be curves in $\mathbb{R}^2 \setminus (I_1 \cup \ldots \cup I_p)$ with end points in vertices of G. We describe conditions under which there exist pairwise edge-disjoint paths $P_1,\ldots,P_k$ in G so that P_i is homotopic to C_i in $\mathbb{R}^2 \setminus (I_1 \cup \ldots \cup I_p)$, for $i=1,\ldots,k$. This extends results of Kaufmann and Mehlhorn for graphs derived from the rectangular grid.

1980 Mathematics Subject Classification (Revised 1985): Primary: 05C10, 68R10, 90C27. Secondary: 05C38, 57M15.
Key Words & Phrases: Planar, graph, homotopic, edge-disjoint, VLSI, straight-line.

Report OS-R8718
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1. INTRODUCTION AND STATEMENT OF THE THEOREM

Let $G=(V,E)$ be a planar graph, embedded without crossing edges in the euclidean plane $\mathbb{R}^2$. We identify G with its image in $\mathbb{R}^2$. Let $I_1,\ldots,I_p$ be some of its faces, including the unbounded face, called the *black holes*. (We consider faces as *open* sets.) Let moreover paths $C_1,\ldots,C_k$ be given with end points in V, not intersecting any black hole. (That is, for each i, C_i is a continuous function $[0,1]\longrightarrow\mathbb{R}^2\setminus(I_1\cup\ldots\cup I_p)$ with $C(0), C(1)\in V$.)

Motivated by the automatic design of integrated circuits, Kurt Mehlhorn posed the following question:

(1) Under which conditions do there exist pairwise edge-disjoint paths $P_1,\ldots,P_k$ in G so that P_i is homotopic to C_i in the space $\mathbb{R}^2\setminus(I_1\cup\ldots\cup I_p)$ (for $i=1,\ldots,k$)?

Here a *path in* G is a continuous function $P:[0,1]\longrightarrow G$ with $P(0),P(1)\in V$. Paths $P_1,\ldots,P_k$ are *pairwise edge-disjoint* if the following holds: if $P_i(x)=P_j(y)\notin V$ then $x=y$ and $i=j$. (In particular, if $P_1,\ldots,P_k$ are pairwise edge-disjoint, then each P_i does not pass the same edge more than once.) Two paths P,C: $[0,1]\longrightarrow\mathbb{R}^2\setminus(I_1\cup\ldots\cup I_p)$ are *homotopic* (*in* $\mathbb{R}^2\setminus(I_1\cup\ldots\cup I_p)$), denoted by $P\sim C$, if there exists a continuous function $\Phi:[0,1]\times[0,1]\longrightarrow\mathbb{R}^2\setminus(I_1\cup\ldots\cup I_p)$ so that for all $x\in[0,1]$: $\Phi(x,0)=P(x)$, $\Phi(x,1)=C(x)$, $\Phi(0,x)=P(0)$, $\Phi(1,x)=P(1)$. (In particular, $P(0)=C(0)$ and $P(1)=C(1)$.)

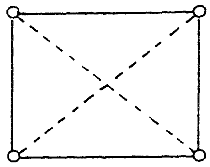
Mehlhorn proposed to study question (1) with the help of the following 'cuts'. A (*homotopic*) *cut* is a continuous function $D:[0,1]\longrightarrow\mathbb{R}^2\setminus(V\cup I_1\cup\ldots\cup I_p)$ so that $D(0)$ and $D(1)$ belong to the boundary of $I_1\cup\ldots\cup I_p$ and so that $|D^{-1}(G)|$ is finite. The *cut condition* (*for* $G;I_1,\ldots,I_p;C_1,\ldots,C_k$) is:

(2) (<u>cut condition</u>) for each cut D: $cr(G,D)\geqslant\sum_{i=1}^{k} mincr(C_i,D)$.

Here we use the following notation for curves $C,D:[0,1]\longrightarrow\mathbb{R}^2\setminus(I_1\cup\ldots\cup I_p)$:

(3)
$$\begin{aligned} cr(G,D) &:= \big|\{y\in[0,1] \mid D(y)\in G\}\big|,\\ cr(C,D) &:= \big|\{(x,y)\in[0,1]\times[0,1] \mid C(x)=D(y)\}\big|,\\ mincr(C,D) &:= \min\{cr(\tilde{C},\tilde{D}) \mid \tilde{C}\sim C,\ \tilde{D}\sim D \text{ in } \mathbb{R}^2\setminus(I_1\cup\ldots\cup I_p)\}. \end{aligned}$$

Clearly, the cut condition is a necessary condition for a positive answer to question (1). It is generally not sufficient, not even for quite simple situations. E.g., take $k=2$, $p=1$, and consider



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where the straight lines stand for edges of G and where the interrupted lines stand for curves C_1 and C_2.

It turned out that one additional condition, the so-called *parity condition*, can be helpful (cf. Section 2 below):

(4) (parity condition) for each cut D: $cr(G,D) \equiv \sum_{i=1}^{k} mincr(C_i, D) \pmod 2$.

Let us now come to stating our theorem. We say that $G; I_1, \ldots, I_p; C_1, \ldots, C_k$ is in the *straight-line case* if:

(5) there are line segment $L_1, \ldots, L_t$ in $\mathbb{R}^2$ so that $G \cup I_1 \cup \ldots \cup I_p = L_1 \cup \ldots \cup L_t \cup I_1 \cup \ldots \cup I_p$ and so that each L_j has its end points in $I_1 \cup \ldots \cup I_p$,

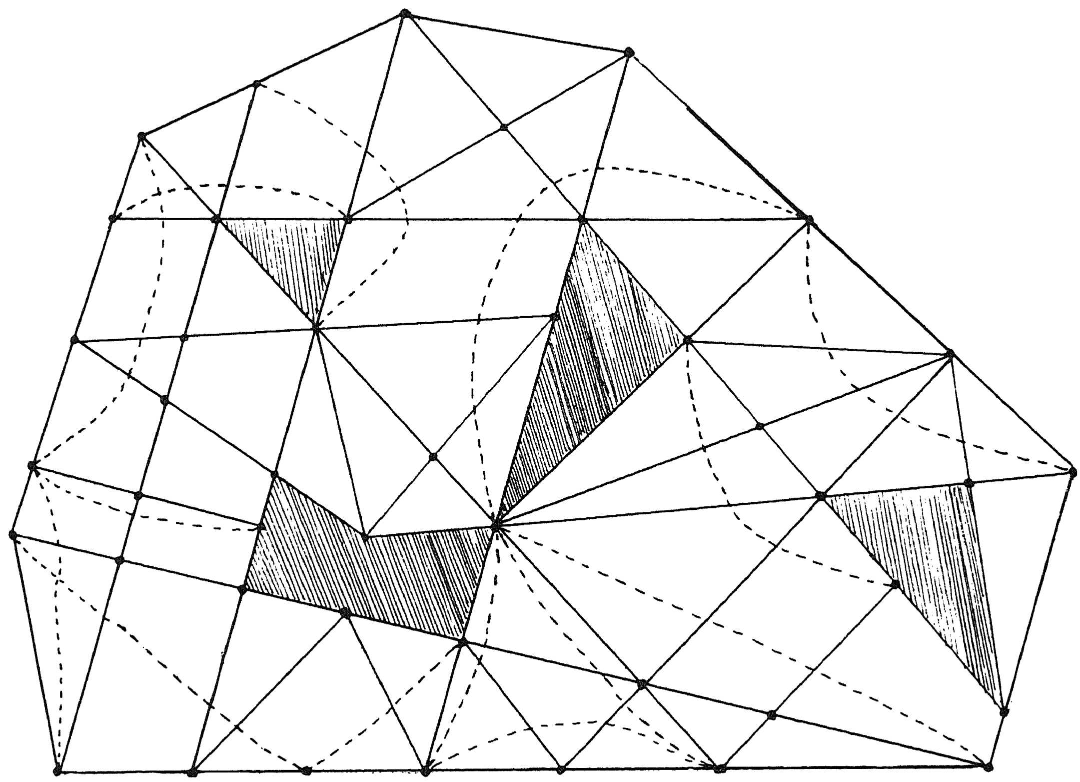
and

(6) if the aperture at vertex v of G is larger than 180°, then the number of times v occurs as end point of the curves C_i is not larger than the number of edges terminating at v.

Here the *aperture* at vertex v of G is the largest angle which can be made at v so that none of the black holes adjacent to v is intersecting the interior of the angle. (More formally, let $\rho > 0$ be so that the circle K of radius ρ and centre v does not contain any other vertex of G in its interior, and does not intersect any edge except for those adjacent to v. Let $K \setminus (I_1 \cup \ldots \cup I_p)$ have components $K_1, \ldots, K_h$, making angles $\varphi_1, \ldots, \varphi_h$. Then the aperture at v is equal to $\max\{\varphi_1, \ldots, \varphi_h\}$.) Edge $e = \{(1-\lambda)u + \lambda v \mid 0 < \lambda < 1\}$ of G is said to *terminate at* v if for some $\mu > 1$ the set $\{(1-\lambda)u + \lambda v \mid 1 < \lambda < \mu\}$ is contained in $I_1 \cup \ldots \cup I_p$.

Theorem. *If we are in the straight-line case and the parity condition holds, then there exist pairwise edge-disjoint paths as in* (1) *if and only if the cut condition holds.*

As an illustration, we give an example of the straight-line case (where the shaded faces, together with the unbounded face, are the black holes, and where the interrupted curves stand for the paths C_i):



The theorem generalizes a result of Kaufmann and Mehlhorn [2] for graphs derived from the rectangular grid in the following way. G is a finite subgraph of the rectangular grid. (That is, V is a finite subset of $\mathbb{Z}^2$, while each edge is a line segment of length 1.) $I_1,\ldots,I_p$ are exactly those faces of G which are not bounded by exactly four edges of G. Moreover, for each vertex v it is required that $\deg(v)+r(v)\leqslant 4$, where $\deg(v)$ denotes the degree of v in G, while $r(v) := \left|\{i=1,\ldots,k \mid C_i(0)=v\}\right| + \left|\{i=1,\ldots,k \mid C_i(1)=v\}\right|$.

<u>Corollary</u> (Kaufmann and Mehlhorn). *If the conditions given in the previous paragraph are satisfied and the parity condition holds, then there exist pairwise edge-disjoint paths as in* (1) *if and only if the cut condition holds.*

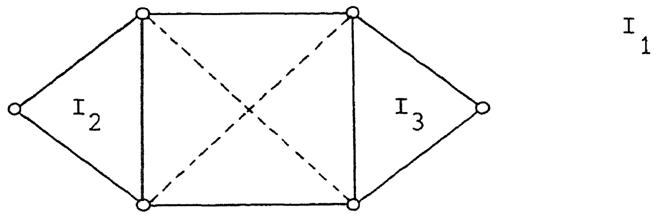
In fact, Kaufmann and Mehlhorn found a linear-time algorithm to find these paths, if they exist.

In Section 4 we give a proof of our theorem. We make use of a Lemma to be proved in Section 3 (showing that in the straight-line case we may restrict the cut condition to (almost) straight cuts (analogous to the idea of "1-bend cuts" in [2]), and of results of [4] to be reviewed in Section 2.

2. REVIEW OF PRELIMINARY RESULTS

In this section we resturn to the general case of a planar graph $G=(V,E)$ embedded without crossing edges in the euclidean plane $\mathbb{R}^2$, with black holes $I_1,\ldots,I_p$ (including the unbounded face) and curves $C_1,\ldots,C_k$. Let each C_i have its end points in vertices on the boundary of $I_1\cup\ldots\cup I_p$.

It was shown by Okamura and Seymour [3] that if $p=1$ the cut condition together with the parity condition imply the existence of paths as in (1). (Note that for $p=1$ two paths P,P' are homotopic if and only if $P(0)=P'(0)$ and $P(1)=P'(1)$.) This was extended by van Hoesel and Schrijver [1] to $p=2$. It cannot be extended to higher p, as is shown for $p=3$ by:



However, it was shown in [4] that, for arbitrary p, the cut condition is equivalent to the existence of a 'fractional' packing of paths as required, i.e., to the existence of paths $P_1^1,\ldots,P_1^{t_1},P_2^1,\ldots\ldots,P_k^1,\ldots,P_k^{t_k}$ and rationals $\lambda_1^1,\ldots,\lambda_1^{t_1},\lambda_2^1,\ldots\ldots,\lambda_k^1,\ldots,\lambda_k^{t_k}>0$ such that:

(7) (i) $P_i^j \sim C_i$ $\qquad (i=1,\ldots,k;\ j=1,\ldots,t_i)$,

(ii) $\sum_{j=1}^{t_i} \lambda_i^j = 1$ $\qquad (i=1,\ldots,k)$,

(iii) $\sum_{i=1}^{k}\sum_{j=1}^{t_i} \lambda_i^j \chi^{P_i^j}(e)\leq 1$ $\qquad (e\in E)$.

Here $\chi^P(e)$ denotes the number of times paths P passes edge e.

Another result from [4] to be used below was derived with the theory of simplicial approximations. Let $C,D:[0,1]\to\mathbb{R}^2\setminus(I_1\cup\ldots\cup I_p)$ be continuous. Let $C(0),C(1),D(0)$ and $D(1)$ be on the boundary of $I_1\cup\ldots\cup I_p$, with $\{C(0),C(1)\}\cap\{D(0),D(1)\}=\emptyset$. Let

(8) $\qquad X := \{(y,z)\in[0,1]\times[0,1] \mid C(y)=D(z)\}$

be finite, where each (y,z) in X gives a crossing of C and D. For $y,y'\in[0,1]$ let $C|_y^{y'}$ denote the path from $C(y)$ to $C(y')$ given by:

(9) $\quad (C\Big|_{y}^{y'})(\lambda) := C((1-\lambda)y+\lambda y') \qquad$ for $\lambda \in [0,1]$.

Similarly for D. Define for $(y,z),(y',z') \in X$:

(10) $\quad (y,z) \approx (y',z') \iff (C\Big|_{y}^{y'}) \sim (D\Big|_{z}^{z'})$ in $\mathbb{R}^2 \setminus (I_1 \cup \ldots \cup I_p)$.

We call the classes of the equivalence relation $\approx$ the *classes of intersections* of C and D. Such a class is called *odd* if it contains an odd number of elements. Let odd(C,D) denote the number of odd classes of X. Then

(11) $\quad \text{mincr}(C,D) = \text{odd}(C,D)$.

3. A LEMMA ON STRAIGHT CUTS

We call a cut $D:[0,1] \longrightarrow \mathbb{R}^2 \setminus (I_1 \cup \ldots \cup I_p)$ a *straight cut* if:

(12) either (i) D is linear,
or (ii) the line segment connecting D(0) and D(1) is contained in G, the functions $D|[0,\frac{1}{2}]$ and $D|[\frac{1}{2},1]$ are linear, there is no vertex of G contained in the interior of the triangle $D(0)D(\frac{1}{2})D(1)$, and no edge is intersected more than once by D.

In (ii) one might think of D as being very close to the line segment connecting D(0) and D(1). So a straight cut is determined by its end points, in case (12)(ii) up to 'slight' homotopic shifts, which however do not change the number of intersections with G.

Lemma. *In the straight-line case, the cut condition holds if and only if* $cr(G,D) \geq \sum_{i=1}^{k} mincr(C_i,D)$ *for each straight cut* D.

Proof. Necessity being trivial, we show sufficiency. Let the cut inequality be satisfied by each straight cut. Suppose there exist a cut $D:[0,1] \longrightarrow \mathbb{R}^2 \setminus (I_1 \cup \ldots \cup I_p)$ so that

(13) $$cr(G,D) < \sum_{i=1}^{k} mincr(C_i,D).$$

We choose D satisfying (13) so that $t:=cr(G,D)$ is as small as possible. The idea of the proof is to straighten out D as much as possible.

First observe that we may assume that, if D(1) is not on the line through the edge containing D(0), then the line segment $\overline{D(0)D(1)}$ does not intersect V (this can be achieved by slightly shifting D(0) over the edge containing D(0)). Moreover, we may assume that there exists an $\varepsilon > 0$ so that

(14) (i) $D|[0,\varepsilon]$ is linear;
(ii) for all $\delta \in (0,\varepsilon]$: $D(\delta)$ does not belong to any line through any pair of vertices of G and not to any line through a pair of a vertex of G and a point of intersection of D and G.

Let $\lambda_1,\ldots,\lambda_t$ be so that $0=\lambda_1<\lambda_2<\ldots<\lambda_{t-1}<\lambda_t=1$, with $D(\lambda_i) \in G$ for all i. Define

(15) $p_1 := D(\varepsilon)$,

$p_i := D(\lambda_i)$ for $i = 2,\ldots,t$.

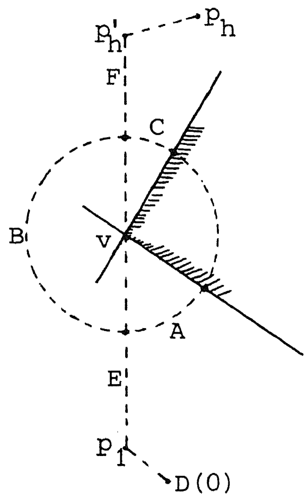
Finally, we may assume that $D|[\varepsilon,\lambda_2]$ and $D|[\lambda_{i-1},\lambda_i]$ are linear functions ($i=3,\ldots,t$) (since in the straight-line case each face not in $\{I_1,\ldots,I_p\}$ is convex).

Let $h(D)$ be the smallest index h with $2\leq h\leq t-1$ so that the angle between $\overline{p_{h-1}p_h}$ and $\overline{p_hp_{h+1}}$ is not $180°$. If no such h exists, let $h(D):=t$. Choose D so that the above applies (fixing $t=cr(G,D)$) and so that $h(D)$ is as large as possible. Let $h:=h(D)$.

First consider the case $h<t$. Choose the highest $\lambda\in[0,1]$ so that the triangle with vertices p_1,p_h and $p_h+\lambda(p_{h+1}-p_h)$ does not intersect $I_1\cup\ldots\cup I_p$. Let $p_h' := p_h+\lambda(p_{h+1}-p_h)$. Let D' be the piecewise linear function obtained from D by replacing parts $\overline{p_1p_h}$ and $\overline{p_hp_h'}$ of D by $\overline{p_1p_h'}$.

If $\lambda=1$, then $p_h'=p_{h+1}$, and hence by (14)(ii) $\overline{p_1p_h'}$ does not intersect any vertex of G. So D' is a cut, with $cr(G,D') = cr(G,D)$ (by the conditions (5) and (6) for the straight-line case) and $D'\sim D$. As $h(D')>h(D)$ this contradicts the fact that we have chosen D so that $h(D)$ is as large as possible.

If $\lambda<1$, then $\overline{p_1p_h'}$ intersects a vertex v of G, on the boundary of $I_1\cup\ldots\cup I_p$. This vertex is unique by (14)(ii), and has aperture larger than 180 . Consider a circle K with center v, not containing any other vertex of G, and not intersect any edge of G except for those adjacent to v. Let



$K\setminus(I_1\cup\ldots\cup I_p)$ have components $K_1,\ldots,K_h$. So each K_i is a cut. We may assume that K_1 intersects D' twice. So K_1 is a circular arc of angle larger than $180°$. Use the notation A,B,C,E,F for the parts of D' and K_1 as indicated in the figure above. Let H denote the part of D from p_h' to p_t. As we have chosen D so that (13) is satisfied with $cr(G,D)$ as small as possible, we have:

(16) $$cr(G,D) = cr(G,EBFH) = cr(G,EA) + cr(G,CFH) + \sum_{j=2}^{h} cr(G,K_j) + \text{(number of edges terminating at v)} \geqslant$$
$$\sum_{i=1}^{k} mincr(C_i,EA) + \sum_{i=1}^{k} mincr(C_i,CFH) + \sum_{j=2}^{h}\sum_{i=1}^{k} mincr(C_i,K_j) +$$
$$+\sum_{i=1}^{k} \text{(number of times v is end point of } C_i) \geqslant \sum_{i=1}^{k} mincr(C_i,D)$$

(using (6)). This contradicts (13).

As $h<t$ leads to a contradiction, we know $h=t$. If the line segment $\overline{D(0)D(1)}$ is not contained in G, then by our assumption this line segment forms a straight cut D', with cr(G,D')=cr(G,D) and $D'\sim D$, whence

(17) $$cr(G,D) = cr(G,D') \geqslant \sum_{i=1}^{k} mincr(C_i,D') = \sum_{i=1}^{k} mincr(C_i,D),$$

contradicting (13). If $\overline{D(0)D(1)}$ is contained in G, then D itself forms a straight cut, contradicting (13). □

4. PROOF OF THE THEOREM

We now prove our theorem:

Theorem. *If we are in the straight-line case and the parity condition holds, then there exist pairwise edge-disjoint paths as in* (1) *if and only if the cut condition holds.*

Proof. By induction on the number of faces not in $\{I_1,\dots,I_p\}$. If each face belongs to $\{I_1,\dots,I_p\}$ then the theorem is trivially true. So assume that not all faces belong to $\{I_1,\dots,I_p\}$.

I. We first consider those situations where the following holds:

(18) G has an edge e_0, connecting vertices u and w, both of degree 2, so that e_0 separates a face in $\{I_1,\dots,I_p\}$ from a face not in $\{I_1,\dots,I_p\}$ and so that one of the curves C_i connects u and w following e_0.

Without loss of generality, e_0 separates face I_1 from face $F \notin \{I_1,\dots,I_p\}$, and C_1 connects u and w following e_0. Moreover, we may assume that none of $C_2,\dots,C_k$ passes e_0 (we can make detours along the other edges of F). By the parity condition, there exist h,j so that C_h has an end point in u and C_j has an end point in w (possibly h=j).

Now let $I_{p+1} := F$. Clearly, $G;I_1,\dots,I_p,I_{p+1};C_1,\dots,C_k$ is again in the straight-line case, in which the parity condition holds. We show:

(19) the cut condition holds for $G;I_1,\dots,I_{p+1};C_1,\dots,C_k$.

As the number of faces not in $\{I_1,\dots,I_{p+1}\}$ is one less than in the original situation, (19) implies by induction that there exist pairwise edge-disjoint paths $P_1 \sim C_1,\dots,P_k \sim C_k$ in $\mathbb{R}^2 \setminus (I_1 \cup \dots \cup I_{p+1})$. This implies $P_1 \sim C_1,\dots,$ $P_k \sim C_k$ in $\mathbb{R}^2 \setminus (I_1 \cup \dots \cup I_p)$ as required.

We prove (19). We will refer to $G;I_1,\dots,I_{p+1};C_1,\dots,C_k$ as the *new structure,* and to $G;I_1,\dots,I_p;C_1,\dots,C_k$ as the *original structure*. For the new structure we use the notation mincr' instead of mincr.

In order to show (19), by the Lemma, it suffices to prove the cut inequality for straight cuts only. Let D be a straight cut in the new structure. If D(0) and D(1) belong to the boundary of $I_1 \cup \dots \cup I_p$, then D is also a cut in the original structure, and the cut inequality follows (as $\mathrm{mincr}'(C_i,D) = \mathrm{mincr}(C_i,D)$ for each i). If both D(0) and D(1) belong to the boundary of $I_{p+1}=F$, then $\mathrm{mincr}'(C_i,D)=0$ for each i (as F is convex), and the cut inequality follows. So we may assume that D(0) belongs to the boundary of $I_1\cup\ldots\cup I_p$ and D(1) belongs to the boundary of F. We can extend D in $\overline{F}$ to a cut D' ending on e_0. Then D' is a cut in the original structure. Thus we have

$$\text{(20)}\qquad \mathrm{cr}(G,D) = \mathrm{cr}(G,D')-1 \geqslant \textstyle\sum_{i=1}^k \mathrm{mincr}(C_i,D') -1 = \sum_{i=1}^k \mathrm{mincr}'(C_i,D),$$

thus showing the cut inequality for D. This proves (19).

II. Now we consider the general case (i.e., we do not assume (18)). As not all faces belong to $\{I_1,\ldots,I_p\}$, there exists an edge, say e_0, separating a face I_h $(1\leqslant h\leqslant p)$ from a face F not in $\{I_1,\ldots,I_p\}$. We may assume h=1. Without loss of generality, no path C_i intersects e_0 or F (we can make detours along the boundary of F). Extend G to a graph G' by adding two new vertices, say u and w, on e_0. Let e_0' be the edge connecting u and w. Let C_{k+1} and C_{k+2} be two curves, each connecting u and w via e_0'. We consider two cases.

<u>Case 1</u>: The cut condition holds for $G';I_1,\ldots,I_p;C_1,\ldots,C_k,C_{k+1},C_{k+2}$. Now we can apply part I of this proof above, and paths $P_1,\ldots,P_k,P_{k+1},P_{k+2}$ as required exist.

<u>Case 2</u>: The cut condition does not hold for $G';I_1,\ldots,I_p;C_1,\ldots,C_k,C_{k+1},C_{k+2}$. Since also in this new situation we are in the straight-line case, by the Lemma there exists a straight cut D so that

$$\text{(21)}\qquad \mathrm{cr}(G',D) < \textstyle\sum_{i=1}^{k+2} \mathrm{mincr}(C_i,D).$$

Since $\mathrm{mincr}(C_{k+1},D)=\mathrm{mincr}(C_{k+2},D)\leqslant 1$ and since the parity condition holds for $G;I_1,\ldots,I_p;C_1,\ldots,C_k$ we know:

$$\text{(22)}\qquad \mathrm{cr}(G,D) = \textstyle\sum_{i=1}^k \mathrm{mincr}(C_i,D),$$

and $\mathrm{mincr}(C_{k+1},D)=\mathrm{mincr}(C_{k+2})=1$. Hence D has one of its end points on e_0'.

As the cut condition holds for $G;I_1,\ldots,I_p;C_1,\ldots,C_k$, there exists a 'fractional' packing of paths $P_1^1,\ldots,P_1^{t_1},\ldots\ldots,P_k^1,\ldots,P_k^{t_k}$, with coefficients $\lambda_1^1,\ldots,\lambda_1^{t_1},\ldots\ldots,\lambda_k^1,\ldots,\lambda_k^{t_k}>0$, satisfying (7). By (22), at least one of the P_i^j, say P_1^1, passes edge e_0. So $P_1^1 = R_1 e_0' R_2$ for certain paths R_1 and R_2.

We now first show the following:

Claim. *For each straight cut* D' *(for* G'*) we have:*

(23) $$\operatorname{mincr}(R_1,D') + \operatorname{mincr}(C_{k+1},D') + \operatorname{mincr}(R_2,D') \leqslant \operatorname{mincr}(C_1,D') + 2.$$

Proof of the Claim. Since

(24) $$cr(G,D) = \sum_{i=1}^{k} \operatorname{mincr}(C_i,D) \leqslant \sum_{i=1}^{k}\sum_{j=1}^{t_i} \lambda_i^j \cdot cr(P_i^j,D) \leqslant cr(G,D),$$

and since $\lambda_1^1 > 0$, we know that $cr(P_1^1,D) = \operatorname{mincr}(C_1,D)$.

Without loss of generality, $\left(P_1^1\Big|_0^{\frac14}\right)$ coincides with path R_1, $\left(P_1^1\Big|_{\frac14}^{\frac34}\right)$ with C_{k+1} and $\left(P_1^1\Big|_{\frac34}^{1}\right)$ with R_2. Moreover, we may assume that $P_1^1(\frac12) = D(0)$.

Let D' be any straight cut. In order to show (23), we may assume that D and D' intersect each other at most once, and that if D' intersects e_0', then D and D' do not intersect.

Let

(25) $$X := \left\{(x,y) \in [0,1]\times[0,1] \;\middle|\; P_1^1(x) = D'(y)\right\}.$$

Let $\approx$ be as in (10). So $\operatorname{mincr}(C_1,D')$ is equal to the number of odd classes of $\approx$. We show:

(26) if $(x,y),(x',y'),(x'',y''),(x''',y''') \in X$ so that $(x,y) \approx (x',y')$, $(x'',y'') \approx (x''',y''')$, $x,x'' \in (0,\frac12)$ and $x',x''' \in (\frac12,1)$, then D and D' intersect and $(x,y) \approx (x'',y'')$.

Indeed, as $(x,y) \approx (x',y')$, we know $\left(P_1^1\Big|_x^{x'}\right) \sim \left(D'\Big|_y^{y'}\right)$. So $\left(P_1^1\Big|_x^{x'}\right)\left(D'\Big|_{y'}^{y}\right)$ forms a homotopically trivial cycle K. Since $\left(P_1^1\Big|_x^{x'}\right)$ passes D(0), D splits K into two homotopically trivial cycles. That is, there is a $\lambda \in (0,1]$ so that:

(27) either (i) $\exists z \in [x,x']$: $\left(P_1^1\Big|_z^{\frac12}\right)\left(D\Big|_0^{\lambda}\right)$ is a homotopically trivial cycle,

or (ii) $\exists z \in (y,y')$: $\left(P_1^1\Big|_x^{\frac12}\right)\left(D\Big|_{\frac12}^{\lambda}\right)\left(D'\Big|_z^{y}\right)$ is a homotopically trivial cycle.

Since $cr(P_1^1,D) = \operatorname{mincr}(P_1^1,D)$, (27)(i) does not occur. So (27)(ii) applies.

Hence

(28) $$(P_1^1\Big|{}^{\frac12}_{x}) \sim (D'\Big|{}^{z}_{y})(D\Big|{}^{\frac12}_{\lambda}).$$

In particular, D and D' intersect, with $D(\lambda) = D'(z)$. One similarly derives from the fact that $(x'',y'') \approx (x''',y''')$ that

(29) $$(P_1^1\Big|{}^{\frac12}_{x''}) \sim (D'\Big|{}^{z}_{y''})(D\Big|{}^{\frac12}_{\lambda}).$$

Therefore,

(30) $$(P_1^1\Big|{}^{x''}_{x}) \sim (P_1^1\Big|{}^{\frac12}_{x})(P_1^1\Big|{}^{x''}_{\frac12}) \sim (D'\Big|{}^{z}_{y})(D\Big|{}^{\frac12}_{\lambda})(D\Big|{}^{\lambda}_{\frac12})(D'\Big|{}^{y''}_{z}) \sim (D'\Big|{}^{y''}_{y}).$$

So $(x,y) \approx (x'',y'')$. This shows (26).

Now $cr(C_{k+1},D') \leq 1$. If $cr(C_{k+1},D') = 0$, then the above implies:

(31) $$odd(P_1^1,D') \geq (odd(R_1,D') - 1) + (odd(R_2,D') - 1),$$

since by (26) all but at most one class of intersections of R_1 and D' is also a class of intersections of P_1^1 and D'. Similarly for R_2. (31) implies (23).

If $cr(C_{k+1},D') = 1$, then D and D' do not intersect, by assumption. Hence, by (26), no class of intersections of P_1^1 and D' contains both (x,y) and (x',y') with $x \in (0,\frac12)$ and $x' \in (\frac12,1)$. Since $cr(C_{k+1},D')=1$, there is only one element (x,y) in X with $x \in (\frac14,\frac34)$. Except for the class of intersections of P_1^1 and D' containing this element, all other classes also form a class of intersections of R_1 and D' or of R_2 and D'. Hence

(32) $$odd(P_1^1,D') \geq odd(R_1,D') + odd(R_2,D') - 1,$$

and (23) follows. <u>End of proof of the Claim</u>.

We next show:

(33) the cut condition holds for $G';I_1,\ldots,I_p;R_1,R_2,C_2,\ldots,C_k,C_{k+1}$.

Suppose not. Since we are again in the straight-line case, by the Lemma there exists a straight cut D' so that

(34) $$mincr(R_1,D') + mincr(R_2,D') + \sum_{i=2}^{k+1} mincr(C_i,D') \geq cr(G,D') + 2,$$

using the fact that the parity condition holds also for $G';I_1,\ldots,I_p;R_1,R_2,$ $C_2,\ldots,C_{k+1}$. Since the cut condition does hold for $G';I_1,\ldots,I_p;C_1,\ldots,C_k$ it follows that

(35) $$\mathrm{mincr}(R_1,D')+\mathrm{mincr}(R_2,D')+\mathrm{mincr}(C_{k+1},D')>\mathrm{mincr}(C_1,D').$$

Hence

(36) $$cr(P_1^1,D') = cr(R_1,D')+cr(R_2,D')+cr(C_{k+1},D')>\mathrm{mincr}(C_1,D').$$

Therefore,

(37) $$cr(G,D')\geqslant \sum_{i=1}^k \sum_{j=1}^{t_i} \lambda_i^j\cdot cr(P_i^j,D')>\sum_{i=1}^k\sum_{j=1}^{t_i} \lambda_i^j\cdot \mathrm{mincr}(C_i,D') =$$
$$=\sum_{i=1}^k \mathrm{mincr}(C_i,D').$$

However, (34) and (37) imply

(38) $$\mathrm{mincr}(R_1,D')+\mathrm{mincr}(R_2,D')+\sum_{i=2}^{k+1} \mathrm{mincr}(C_i,D')\geqslant cr(G,D')+2$$
$$>\sum_{i=1}^k \mathrm{mincr}(C_i,D') +2,$$

contradicting the Claim.

So (33) holds, and hence by part I of this proof there exist pairwise edge-disjoint paths $Q_1'\sim R_1$, $Q_1''\sim R_2$, $Q_2\sim C_2$, $\ldots$, $Q_k\sim C_k$, $Q_{k+1}\sim C_{k+1}$. By sticking Q_1',Q_{k+1},Q_1'' to one path, which is homotopic to C_1, we obtain paths as required. $\square$

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