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Distance Regular Antipodal Covers of Johnson and Hamming Graphs

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We determine distance regular antipodal covers of Johnson and Hamming graphs and some graphs related to them.

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0. Introduction

In this paper we give non-existence proofs for distance regular antipodal covers of many Johnson and Hamming graphs, and related graphs.
There is a partial overlap with result of Ivanov [3], on distance transitive graphs, our proofs do not use the assumption of distance transitivity.

1. Basic theory

Lemma 1.1. *Let Γ be a distance regular graph of diameter m which contains a quadrangle and let $\tilde{\Gamma}$ be a distance regular antipodal r-cover of Γ of diameter $d \geqslant 5$. Then $m \leqslant \frac{k}{\lambda+2} - \frac{1}{2}\lfloor \frac{d}{2m+1} \rfloor$.*

Proof. It is straightforward to check that $d \geqslant 5$ implies that $\tilde{\Gamma}$ also contains a quadrangle $k = |\Gamma(x)| = |\tilde{\Gamma}(\tilde{x})|$ and $\lambda = |\Gamma_1(x) \cap \Gamma_1(y)| = |\tilde{\Gamma}_1(\tilde{x}) \cap \tilde{\Gamma}_1(\tilde{y})|$ for all $\tilde{x},\tilde{y} \in \tilde{\Gamma}$, $x,y \in \Gamma$ with $d(\tilde{x},\tilde{y}) = d(x,y) = 1$. Applying Terwilliger's diameter bound to $\tilde{\Gamma}$ yields $d \leqslant \frac{2k}{\lambda+2}$. As $d \in \{2m, 2m+1\}$ the Lemma follows. □

Lemma 1.2. *If Γ is a strongly regular graph and $\tilde{\Gamma}$ a distance regular antipodal r-cover of Γ of diameter 4. Then the following holds*
(i) *the μ-graph of Γ is a disjoint union of r graphs of the same cardinality.*
(ii) *either $\lambda = 0$ or $\lambda^2 + 4k$ is a square.*

Proof. Straightforward. □

Remark. The same idea of proof leads to the following more general result.

Lemma. *If Γ is a distance regular antipodal graph of diameter $2m$ with folded graph $\overline{\Gamma}$ and $x,y \in \overline{\Gamma}$ with $d(x,y) = m$ then for all $i \neq 0,m$ $\overline{\Gamma}_i(x) \cap \overline{\Gamma}_{m-i}(y)$ is the disjoint union of t graphs of the same cardinality where $t = |\Gamma_{2m}(x)| + 1$.*

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2. Graphs related to Johnson schemes

Lemma 2.1. *Let $\tilde{\Gamma}$ be a distance regular antipodal covering of the odd graph O_m. Then one of the following holds:*
(i) *$\tilde{\Gamma}$ is the standard double covering of O_m*
(ii) *$m = 3$, $\tilde{\Gamma}$ is the dodecahedral graph.*

Proof. In view of [3] we only have to prove that there exists no distance regular graph with intersectionary

$$\iota(\tilde{\Gamma}) = \{3,2,2,1,1;1,1,1,2,3\}$$

or

$$\iota(\tilde{\Gamma}) = \{4,3,3,1,1,1;1,1,1,3,3,4\}.$$

In the first case we get

$$1+1 = b_3+c_3 > a_3 + \frac{a_4 b_3}{a_3} + \frac{a_2 c_3}{a_3} = 1$$

contradicting proposition 3.3 of [4].
The non existence of the second graph follows directly from D.H. Smith [5] by imitating the proof for 18 adjacent to 49, this yields again a contradiction. □

Lemma 2.2.
(i) *The Johnson graph has no distance regular antipodal covers.*
(ii) *The even graph has a unique distance regular antipodal cover, namely $J(2m,m)$.*

Proof. Actually Moon proved Lemma 5.4 from Ivanov [3] without the assumption of distance transitivity whence the Lemma.
Note that the non existence also follows from Lemma 1.1 and 1.2. □

Lemma 2.3. *Let Γ' be the complement graph of Γ.*
(i) *There exists no distance regular antipodal cover of E'_4 and E'_5.*
(ii) *If $\tilde{\Gamma}$ is a distance regular antipodal cover of $\Gamma_n = J(n,2)'$ $n \geq 5$. Then $n \leq 7$ and*
  a) *$n = 5$, $\tilde{\Gamma}$ is the dodecahedral graph or the Desargues graph.*
  b) *$n = 6$, $\Gamma_6 = GQ(2,2)$ and $\tilde{\Gamma} =$ 3-cover of $GQ(2,2) = \frac{1}{2}$ foster graph.*
  c) *$n = 7$, $\tilde{\Gamma}$ is the unique graph with*

$$\iota(\Gamma) = \{10,6,4,1:1,2,6,10\}.$$

Proof. (i) By Ivanov [3] diam $(\tilde{\Gamma}) = 4$ and non existence follows from Lemma 1.2(ii).
(ii) If $n = 5$ then $\Gamma_5 = O_3$ and (a) follows from Lemma 2.1. If $n \geq 6$ then Lemma 1.1 yields diam $(\tilde{\Gamma}) = 4$ and as μ-graph $(\Gamma_n) \cong \Gamma_{n-3}$, Lemma 1.2(i) yields $n \leq 7$.
If $n = 7$ then Γ_7 is a locally petersen graph and (c) follows from Hall [1]. If $n = 6$ then $\Gamma_6 = GQ(2,2)$ and (b) follows from parameter restrictions. □

3. Graphs related to Hamming schemes

Lemma 3.1. *If $\tilde{\Gamma}$ is a distance regular antipodal cover of $H(q,d)$, $d \geqslant 2$ then $q = d = 2$ and $\tilde{\Gamma}$ is the 8-gon.*

Proof. Lemma 1.1 yields $\text{diam}(\tilde{\Gamma}) = 4$ i.e. $d = 2$ and Lemma 1.2(ii) yields $q = 2$ whence the Lemma. □

Lemma 3.2. *There exists no distance regular antipodal cover of the complement graph of $H(q,2)$ $q \geqslant 3$.*

Proof. One easily checks that $k = (q-1)^2$ and $\lambda = (q-2)^2$ and that there exists a quadrangle Lemma 1.1. and Lemma 2.1.(ii) prove the non existence. □

Lemma 3.3. *Let $\tilde{\Gamma}$ be a distance regular antipodal cover of the folded n-cube $n \geqslant 4$. Then*
(i) *$\tilde{\Gamma}$ is the n-cube*
(ii) *$n = 5$ and $\tilde{\Gamma}$ is the unique graph with $\iota(\Gamma) = \{5,4,1,1;1,1,4,5\}$*
(iii) *$n = 4$ and $\tilde{\Gamma} \cong 4.k_{4,4}$*

Proof. Let Γ be the n-cube
Case 1. *n is odd, $2m+1$ say.* $\iota(\Gamma) = \{n, n-1, ..., m+2; 1, 2, ..., m\}$.
a) cover of odd diameter.
Then there is a θ with $m+2 \geqslant (r-1)\theta \geqslant \theta \geqslant m$.
As $0 = \lambda \leqslant \mu \neq 1$ we must have $c_{i+1} > c_i$.
Hence $r = 2$, $\theta = m+1$ and $\iota(\Gamma) = \iota(H(2,m))$.
Thus Γ is the n-cube.
b) cover of even diameter.
Then $m+2 \geqslant \frac{r-1}{r} m \geqslant m$ and $m-1 \leqslant \frac{m}{r}$.
Hence $r = m = 2$ and $\iota(\tilde{\Gamma}) = \{5,4,1,1;1,1,4,5\}$, i.e. $\tilde{\Gamma}$ is Well's graph.

Case 2. *n is even, $2m$ say.* $\iota(\Gamma) = \{n, ..., m+1; 1, ..., m-1, 2m\}$.
a) cover of odd diameter.
Then $m+1 \geqslant 2m$ contradiction.
b) cover of even diameter.
Then $m+1 \leqslant \frac{r-1}{r} 2m \leqslant m-1$ and $m-1 \leqslant \frac{2m}{r} \leqslant m+1$.
If $m \geqslant 3$ then $0 = \lambda \leqslant \mu \neq 1$ hence

$$c_{i+1} > c_i, r = 2 \text{ and } \Gamma \cong H(2,m).$$

If $m = 2$ then $\iota(\tilde{\Gamma}) = \{4,3,\frac{r-1}{r}4,1;1,\frac{4}{r},3,4\}$ and if $r = 2$ then $\tilde{\Gamma}$ is the cube if $r = 4$ then $\tilde{\Gamma}$ is $4.k_{4,4}$. □

The only folded n-cube $n \in \{4,5\}$ whose complement graph is connected is the 5-cube.

Lemma 3.4. *There exists no distance regular antipodal cover of the complement of the folded 5-cube.*

Proof. Let Γ be the complement of the folded 5-cube. Then $\iota(\Gamma) = \{10,3;1,6\}$ unimodilarity yields $\text{diam}(\tilde{\Gamma}) = 4$ but 76 is not a square. □

Remark. In general we have the following situation. If Γ is a distance regular bipartite and antipodal graph of diameter 5 then $\Gamma^{\pm}$ is isomorphic with the complement of the folded graph of Γ.

Final conclusion.
Together with HEMMETER [2] and the fact that the reduction to a primitive graph takes at most two steps (with exceptions for cliques) one can easily show that the above Lemmi imply that all imprimitive graphs associated with Johnson and Hamming schemes are known.

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