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On Single Lane Roads

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A road which narrows at a bottleneck from an ∞-lane road to a one-lane road is studied with the aid of two stochastic processes. Special attention is given to headways and gaps. At the bottleneck an equilibrium headway can be viewed as the maximum of a shifted exponential random variable and a minimum headway. After the bottleneck the situation becomes far more complicated. However, limiting results are obtained for headways and gaps at a large distance from the bottleneck. The asymptotic behavior of headways and gaps is largely determined by the behavior of the desired speed distribution at the lower extreme of its support.

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1. INTRODUCTION.

In the past, the problem of describing traffic flow has been approached from various angles. For instance, LIGHTHILL & WHITHAM (1955) and PAYNE (1976) use hydrodynamic theory, while PRIGOGINE & HERMAN (1971), PAVERI-FONTANA (1975) and PHILLIPS (1979) use kinetic theory. However, in the light of the discrete nature of traffic, description with the aid of point processes seems to be the most appropriate.

Up to now, there is no point process model which gives a satisfactory general description of traffic flow. The work of BRILL (1971), MILLER (1962), MORSE & YAFFE (1971), NEWELL (1966), RØRBECH (1976), and especially RÉNYI (1964) give the strong impression that one cannot do without unrealistic simplifications in order to keep the resulting model tractable. Hence, the conclusion of BREIMAN (1969) that working along these lines is, at present, virtually useless, still seems to be true.

The problems involved in constructing general point process models for describing traffic flow almost always concern the modelling of overtaking behavior. Thus, it might be fruitful to restrict attention in the first instance to those situations in which overtaking behavior does not give rise to problems.

When the traffic volume on a multilane road is low, overtaking is always possible, so one may assume that each car drives at its desired speed, which remains constant over time. By slightly modifying a result of BREIMAN (1963), a major result was obtained by THEDÉEN (1964): under rather weak additional assumptions, the distribution of cars along the road at a given time t tends to Poisson point process, as t tends to infinity. Later, this result was put into a more general context by KALLENBERG (1978).

In the opposite situation, when overtaking is never possible, it is assumed that each car drives at its desired speed, unless its headway (the time distance between a car and its predecessor, measured at a fixed point along the road) threatens to become less than its minimum headway, the minimal value the car driver is willing to accept. Then, in order to prevent this, the speed is adjusted accordingly.

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The practical value of the no overtaking models is not restricted to the single lane only: in dense traffic the possibility of changing lanes is nearly absent, and in that case a n-lane road can be expected to behave approximately as n single lane roads.

The early work of MILLER (1965) and HODGSON (1968) on no overtaking lacks realism, with the assumption of minimum headways of length zero (leading to an infinite road capacity) and Poisson or deterministic arrivals at the beginning of the no overtaking zone. This assumption is also made by LÁNSKÝ (1978).

The arrival processes used by COWAN (1971, 1975, 1980) are more realistic because they include non-zero minimum headways with allowance for bunching of vehicles. COWAN (1975) deals with a number of stochastic and statistical problems concerning headways, but in the context of single lane studies he deals with two issues:

(i) the passage of vehicles wishing to travel at various speeds along a single-lane road when arrivals are a renewal process (with a proportion of vehicles having headway equal to the minimum headway);

(ii) the headways created when an infinite-lane road merges into a single-lane road (here we shall refer to such a point as a bottleneck).

COWAN (1975) deals with two cases, random and deterministic minimum headways. In this paper we show that his conclusion regarding the headway distribution at the bottleneck when minimum headways are random is false. We correct this result.

As Cowan notes, the headway process created at such bottlenecks is not a renewal process, so he does not consider the combined bottleneck/passage problem in his 1975 paper, even though the headway distribution asserted to arise in situation (ii) agrees with the forms used as arrival processes in the passage problem. In doing so we make the simplifying assumption that our arrivals are a renewal process (when the output of the bottleneck is not strictly so). We prove new theorems regarding the limiting behavior of the headway distribution at large distances from the start of the single lane. Of special interest are results where we show how the limiting distribution of gaps between traffic bunches depends in subtle ways upon the lower tail of the desired speed distribution.

COWAN (1980) considers non-renewal arrivals at a single-lane road, mostly in the context where the minimum headways are deterministic. Within this context he studies the joint bottleneck/passage problem, rigorously treating the non-renewal aspects of the bottleneck process which feeds the single lane. In discussing the limiting distribution for the inter-bunch gaps, however, he only deals with the case where there is a slowest class of vehicles. Our limit theory is more general with regard to the lower tail of the speed distribution, though less general in other respects due to our renewal assumption. We anticipate that the limit theory developed in this paper will have application in the situation described in COWAN (1980).

Also, because of our new headway distribution at the bottleneck, our approach to the problem of random minimum headways in the passage problem differs from COWAN (1975, 1980).

2. THE BOTTLENECK.

Enumerate the cars in the order by which they pass the origin. Whenever this enumeration is in some way essential, we shall denote a process as a sequence of random variables, otherwise as a set of random variables. Assign to car n two random variables: S_n, the minimum headway of car n, and V_n, the desired speed of car n. $(S_n)_{n=1}^{\infty}$ and $(V_n)_{n=1}^{\infty}$ are assumed to be two independent sequences of i.i.d. random variables. Denote the cumulative distribution function of S_n by G and of V_n by K. Assume that V_n has a probability density k satisfying

$$\int_0^{\infty} \frac{k(u)}{u}\, du < \infty. \tag{2.1}$$

In reality, there exists a positive lower limit to the desired speeds, i.e. a positive number v_0 such that $K(v_0)=0$; this ensures that (2.1) holds.

Define $A_n^{(r)}$ as the time instance at which car n passes point r. For convenience suppose that

$A_0^{(0)}=0$. In this section we shall study the process $(A_n^{(0)})_{n=1}^{\infty}$. Concerning this process we assume that there exists a Poisson point process $(D_n^{(0)})_{n=1}^{\infty}$ such that for $n \geqslant 1$

$$A_n^{(0)} = \max(D_n^{(0)}, A_{n-1}^{(0)} + S_n). \tag{2.2}$$

To make this assumption plausible, think of $D_n^{(0)}$ as the time instance at which car n would pass the origin if the road did not narrow at that point. Having in mind the results of BREIMAN (1963) and THEDÉEN (1964), we may assume that for a fixed time t the set of positions of cars along such a road constitutes a Poisson point process, and hence by a theorem of RYLL-NARDZEWSKI (1954) - here (2.1) is needed - $\{D_n^{(0)} : n=1,2,...\}$ is a Poisson point process. Assume arrivals at the origin on the road which does not narrow are in the same order as arrivals at the origin on the road which does narrow, then $(D_n^{(0)})_{n=1}^{\infty}$ is also a Poisson point process. Now car n will pass the origin at time $D_n^{(0)}$, unless its headway becomes less than S_n: in that case it will follow its predecessor at a time distance S_n. Consequently, we have (2.2).

THEOREM 2.1. *Let $(A_n^{(0)})_{n=1}^{\infty}$ be a sequence of random variables such that there exists a Poisson point process $(D_n^{(0)})_{n=1}^{\infty}$ with intensity λ, and a sequence $(S_n)_{n=1}^{\infty}$ of i.i.d. nonnegative random variables, independent of $(D_n^{(0)})_{n=1}^{\infty}$, satisfying equation (2.2). If $\lambda ES_n<1$, then the random variable defined by*

$$Y_n^{(0)} = A_n^{(0)} - A_{n-1}^{(0)} \tag{2.3}$$

has an equilibrium distribution as n tends to infinity, with cumulative distribution function

$$F(y) = (1-e^{-\lambda(y-\theta)})\, G(y), \tag{2.4}$$

where

$$\theta = \frac{1}{\lambda}\left(\ln(1-\lambda ES_n) - \ln E\exp\{-\lambda S_n\}\right). \tag{2.5}$$

PROOF. Let W_n, X_n and U_n be random variables defined by

$$W_n = A_n^{(0)} - D_n^{(0)}, \tag{2.6}$$

$$X_n = D_n^{(0)} - D_{n-1}^{(0)}, \tag{2.7}$$

and

$$U_n = S_n - X_n. \tag{2.8}$$

Equation (2.2) is equivalent to

$$W_n = \max(0, W_{n-1} + U_n), \tag{2.9}$$

and both $(X_n)_{n=1}^{\infty}$ and $(U_n)_{n=1}^{\infty}$ are sequences of i.i.d. random variables. Note that U_n is independent of W_{n-1}. Hence $(W_n)_{n=1}^{\infty}$ is the queueing process induced by the sequence $(U_n)_{n=1}^{\infty}$ (FELLER (1971), p. 194). Actually, the same queueing process is generated by the M/G/1 queue in which the nth customer arrives at $D_n^{(0)}$ and has service time S_{n+1}. Thus, we may apply standard queueing theory techniques to $(W_n)_{n=1}^{\infty}$. If $\lambda ES_n<1$, then $EU_n<0$, and by a theorem of LINDLEY (1952) it follows that W_n has a stationary distribution. In equilibrium W_n satisfies the Pollaczek-Khintchine formula (COOPER (1981), p. 217), and in particular

$$E\exp\{-\lambda W_n\} = \frac{1-\lambda ES_n}{E\exp\{-\lambda S_n\}}. \tag{2.10}$$

Substituting (2.2) and (2.6)-(2.8) into (2.3) gives

$$Y_n^{(0)} = \max(X_n - W_{n-1}, S_n). \tag{2.11}$$

Hence, if W_n has a stationary distribution $Y_n^{(0)}$ also has a stationary distribution. The random variable W_{n-1} is independent of X_n, because it only depends on the sequence $(X_k)_{k<n}$. Thus, for $t \geq 0$ the cumulative distribution function of $X_n - W_{n-1}$ is equal to

$$\int_0^\infty (1-e^{-\lambda(t+w)})\, dP(W_{n-1} \leq w) = 1 - e^{-\lambda t}\, E\exp\{-\lambda W_n\}$$

$$= 1 - e^{-\lambda(t-\theta)}$$

with

$$\theta = \frac{1}{\lambda} \ln E\exp\{-\lambda W_n\}. \tag{2.12}$$

The result follows because $X_n - W_{n-1}$ is independent of S_n.

COROLLARY 2.1. *One may interpret a headway at the bottleneck as the maximum of a shifted exponential random variable and a minimum headway. To be precise: there exists a random variable T_n, which is independent of S_n, and has an exponential distribution, such that $Y_n^{(0)}$ equals the maximum of $T_n+\theta$ and S_n with probability 1.*

An important observation is made in the proof of theorem 2.1: the bottleneck generates the same queueing process as the M/G/1 queue in which the nth customer arrives at $D_n^{(0)}$ and has service time S_{n+1}. Since $(S_n)_{n=1}^\infty$ is a sequence of i.i.d. random variables, the results for the standard M/G/1 queue (in which the nth customer arrives at $D_n^{(0)}$ and has service time S_n) also apply for the bottleneck, though there is no exact analogy between the two.

For instance, let n_k be the kth index such that $Y_n^{(0)} > S_n$. Then $n_k - n_{k-1}$ is equivalent to the number of customers served during the kth busy period in the M/G/1 queue. Therefore its probability generating function obeys

$$\Pi(z) = z \int_0^\infty e^{-\lambda(1-\Pi(z))t} dG(t). \tag{2.13}$$

In general, as COWAN (1975) notes, Π will not be equal to the probability generating function belonging to the geometric distribution

$$\Pi_g(z) = \frac{(1-\rho)z}{1-\rho z},$$

implying that in general $(A_n^{(0)})_{n=1}^\infty$ is not a renewal process.

Another thing to be learned from the M/G/1 queue is the important role of the quantity λES_n, the probability that an arriving customer finds the server occupied. At the bottleneck λES_n, where λES_n may be interpreted as the probability that the headway of car n equals its minimum headway, plays an equally important role.

Theorem 2.1 may be of some interest to queueing theorists: it follows that the distribution of the time between successive entries to service in the M/G/1 queue is also given by (2.4). To the author's knowledge the process of entries to service has never been studied before.

The output process of the bottleneck should not be confused with the output process of the M/G/1 queue. The distribution of the time between successive exits from service is given by:

$$H(y) = G(y) - (1-\lambda ES_n)\int_0^y e^{-\lambda(y-s)} dG(s) \tag{2.14}$$

(KLEINROCK (1975), p.148, p.238). The output process of a M/G/1 queue is of interest when studying tandem queues. Surveys of results on departure processes and tandem queues are given by BURKE (1972) and DALEY (1976). A classical result was obtained by BURKE (1956): the output process of a M/M/1 queue is a Poisson process with the same intensity as the arrival process. One can easily check by using (2.4) that the output process of the corresponding bottleneck is certainly not Poisson.

Let us consider θ more closely. From (2.12) we see that $\lambda\theta$ is the logarithm of the Laplace-transform of W_n evaluated at a point $\lambda>0$. Therefore the value of θ must be negative. Furthermore, from (2.4) we have for the expectation of $Y_n^{(0)}$:

$$EY_n^{(0)} = ES_n + \frac{e^{\lambda\theta}}{\lambda} E\exp\{-\lambda S_n\} = \frac{1}{\lambda}. \tag{2.15}$$

Thus, the value of θ given in (2.5) is precisely that value which makes $EY_n^{(0)}$ equal $1/\lambda$, or, in other words, makes the intensity of the process $(A_n^{(0)})_{n=1}^{\infty}$ equal the intensity of the process $(D_n^{(0)})_{n=1}^{\infty}$.

Now draw a distinction between headways which are equal to the minimum headways *(following* headways) and headways which exceed the minimum headway *(non-following* or *leading* headways). By Corollary 2.1 the distribution of leading headways in the equilibrium situation is equal to the distribution of $(T_n+\theta \mid T_n+\theta>S_n)$:

$$F_L(y) = \left[\int_0^{\infty} e^{-\lambda t} dG(t)\right]^{-1} \int_0^{y} \lambda e^{-\lambda t} G(t)dt, \tag{2.16}$$

and the distribution of following headways is in that case equal to the distribution of $(S_n \mid S_n>T_n+\theta)$:

$$F_F(y) = \frac{G(y) - \int_0^{y} e^{-\lambda(t-\theta)} dG(t)}{1 - \int_0^{\infty} e^{-\lambda(t-\theta)} dG(t)} = \frac{G(y) - \int_0^{y} e^{-\lambda(t-\theta)} dG(t)}{\lambda \int_0^{\infty} t \, dG(t)}. \tag{2.17}$$

Note that, though all following headways are minimum headways, $G(y)$ is in general not equal to $F_F(y)$. Large minimum headways have an increased chance to be following headways. Also, $G(y)$ is in general not equal to the distribution function of leading minimum headways $(S_n \mid S_n<T_n+\theta)$:

$$G_L(y) = \left[\int_0^{\infty} e^{-\lambda t} dG(t)\right]^{-1} \int_0^{y} e^{-\lambda t} dG(t). \tag{2.18}$$

The distribution of following minimum headways is, of course, equal to the distribution of following headways.

Formula (2.4) can now be rewritten as

$$F(y) = \rho F_F(y) + (1-\rho)F_L(y), \tag{2.19}$$

with

$$\rho = \lambda \int_0^{\infty} t \, dG(t) \tag{2.20}$$

as fraction following headways.

When the variables S_n take a value $\tau>0$ with probability 1, our model becomes:

$$F_T(y) = \begin{cases} 0 & \text{if } y < \tau \\ 1-(1-\rho)e^{-\lambda(y-\tau)} & \text{if } y \geqslant \tau \end{cases} \tag{2.21}$$

with

$$\rho = \lambda\tau \tag{2.22}$$

which is identical to a model proposed by TANNER (1961) in another context. Cowan's (1975) distribution for bottleneck headways agrees with (2.21) and (2.22) in this special case.

Some traffic theorists have proposed general traffic models which can be viewed as generalizations of the Tanner model, obtained by simply plugging in a distribution for τ, and lifting the restriction on ρ. We shall denote a fraction of followers which does not necessarily equal λES_n by the symbol p. The way the distribution for τ is plugged in depends on the probabilistic interpretation of the term $e^{-\lambda(y-\tau)}$ in the Tanner model. If one interprets this term as $P(T_n>y \mid T_n>\tau)$, then one obtains the *Semi Poisson* model (BUCKLEY (1968)), given by:

$$F_{SP}(y) = pG(y) + (1-p)\left[\int_0^\infty e^{-\lambda t}dG(t)\right]^{-1}\int_0^y \lambda e^{-\lambda t}G(t)dt. \tag{2.23}$$

An interpretation of $e^{-\lambda(y-\tau)}$ as $P(T_n+\tau>y)$ leads to the *M4* model given by COWAN (1975):

$$F_{M4}(y) = pG(y) + (1-p)\int_0^y G(y-t)\lambda e^{-\lambda t}dt. \tag{2.24}$$

This model also appeared under the name *Generalized Queueing* model in BRANSTON (1976).

Clearly, Cowan's (1975) claim that his M4 model arises at a bottleneck is false unless minimum headways are deterministic. We have shown that the correct distribution of headways at the bottleneck is given by (2.4). In the next sections we shall link the output of the bottleneck with the input of the single-lane passage problem using (2.4) and renewal assumptions.

3. BEYOND THE BOTTLENECK

3.1. Journey times

In this section we shall study the process $(A_n^{(r)})_{n=1}^\infty$ for $r>0$ (i.e. at a point downstream of the bottleneck) by basically the same methods by which we have studied $(A_n^{(0)})_{n=1}^\infty$. First, we shall assume that $A_0^{(r)}=0$, and that $(A_n^{(r)})_{n=1}^\infty$ satisfies

$$A_n^{(r)} = \max(D_n^{(r)}, A_{n-1}^{(r)}+S_n) \tag{3.1}$$

for some process $(D_n^{(0)})_{n=1}^\infty$, suitably chosen. Next, we shall use equation (3.1) to find the counterpart of $(W_n)_{n=1}^\infty$ in section 2, which is used to derive the properties of $(A_n^{(r)})_{n=1}^\infty$.

Unless car n is impeded by its predecessor, it will pass point r at a time exactly r / V_n after passing the bottleneck, where V_n is the desired speed of car n. By the same line of reasoning as in section 2, it is reasonable to assume that equation (3.1) holds for the process $(D_n^{(r)})_{n=1}^\infty$ given by

$$D_n^{(r)} = A_n^{(0)}+r / V_n. \tag{3.2}$$

Denoting the journey time $A_n^{(r)}-A_n^{(0)}$, the time car n needs to get from the bottleneck to point r, by $Z_n^{(r)}$, and, as before, $A_n^{(0)}-A_{n-1}^{(0)}$ by $Y_n^{(0)}$, we have by (3.1) and (3.2)

$$Z_n^{(r)} = \max(r / V_n, Z_{n-1}^{(r)} - Y_n^{(0)}+S_n). \tag{3.3}$$

In this formula both $Y_n^{(0)}$ and $Z_{n-1}^{(r)}$ appear. Unfortunately, both variables depend on $Y_{n-1}^{(0)}$, and therefore will in general not be independent. This difficulty is presumably the reason why Cowan did not formally link the bottleneck problem with the passage problem in his 1975 paper (though the different approach of his 1980 paper adresses the link in a restricted context). To overcome this difficulty in our current context we assume that the arrivals at the single lane road form a renewal process with (2.4) as a headway distribution.

THEOREM 3.1. *Let $(V_n)_{n=1}^\infty$ and $(S_n)_{n=1}^\infty$ be two independent sequences of i.i.d. random variables, and $(Y_n^{(0)})_{n=1}^\infty$ a sequence of i.i.d. random variables satisfying*

(i) $P(Y_n^{(0)} \geqslant S_n) = 1$

(ii) *For each* $y>0$*:*

$$P(Y_n^{(0)}-S_n>y) = (1-\rho)e^{-\lambda y}, \tag{3.4}$$

where $0<\rho<1$, *and* $\lambda>0$.
(iii) $(Y_n^{(0)}-S_n \mid Y_n^{(0)}>S_n)$ *is independent of* V_n *and* S_n.
Then, if $EV_n^{-1}<\infty$, *the random variable* $Z_n^{(r)}$ *defined by (3.3) has an equilibrium distribution with cumulative distribution function*

$$\Omega^{(r)}(z) = \Psi^{(r)}(z)\exp\left\{-\lambda\int_z^\infty(1-\Psi^{(r)}(t))dt\right\} \tag{3.5}$$

where

$$\Psi^{(r)}(z) = \frac{(1-\rho)\Phi^{(r)}(z)}{1-\rho\Phi^{(r)}(z)}, \tag{3.6}$$

$$\Phi^{(r)}(z) = 1-K(r/z), \tag{3.7}$$

and K *is the distribution function of* V_1.

PROOF. The existence of the equilibrium distribution follows from LOYNES (1962) or HELLAND & NILSEN (1976). Now let $Z_{n-1}^{(r)}$ be a random variable with distribution function $\Omega^{(r)}$, which is independent of r/V_n, $Y_n^{(0)}$, and S_n. Then $\max(r/V_n,\ Z_{n-1}^{(r)}-Y_n^{(0)}+S_n)$ must also have distribution function $\Omega^{(r)}$, from which we have

$$\Omega^{(r)}(z) = \Phi^{(r)}(z)\left\{\Omega^{(r)}(z) + (1-\rho)\int_z^\infty e^{-\lambda(u-z)}d\Omega^{(r)}(u)\right\}$$

where $\Phi^{(r)}(z) = 1-K(r/z)$ is the cumulative distribution function of r/V_n. Rearranging this equation gives

$$\Omega^{(r)}(z) = (1-\rho)\frac{\Phi^{(r)}(z)}{1-\Phi^{(r)}(z)}\int_z^\infty e^{-\lambda(u-z)}d\Omega^{(r)}(u). \tag{3.8}$$

Differentiating with respect to z leads to a differential equation with (3.5) as solution under the boundary condition $\Omega^{(r)}(\infty)=1$.

COWAN (1975) obtained equation (3.5) in the case where $Y_n^{(0)}$ obeys the M4 model. Theorem 3.1 shows that also other choices of $Y_n^{(0)}$ lead to (3.5), as was already suggested in COWAN (1975). E.g. we may choose $Y_n^{(0)}$ according to (2.4), with the modification that $(Y_n^{(0)})_{n=1}^\infty$ now is a sequence of i.i.d. random variables, satisfying condition (iii) of theorem 3.1. We may expect that this choice leads to a close approximation of the process derived in the previous section, when the latter is a sequence of nearly independent random variables. Hence, under these circumstances (3.5) will hold approximately.

Some probabilistic insight in equation (3.5) is gained by computing $P(Z_{n-1}^{(r)}-Y_n^{(0)}+S_n\leq z \mid Y_n^{(0)}>S_n)$.

$$\begin{aligned} P(Z_{n-1}^{(r)}-Y_n^{(0)}+S_n\leq z \mid Y_n^{(0)}>S_n) &= \Omega^{(r)}(z)+\int_z^\infty e^{-\lambda(u-z)}d\Omega^{(r)}(u) \\ &= \Omega^{(r)}(z)+\frac{1-\Phi^{(r)}(z)}{(1-\rho)\Phi^{(r)}(z)}\Omega^{(r)}(z) \\ &= \exp\left\{-\lambda\int_z^\infty(1-\Psi^{(r)}(t))dt\right\} \end{aligned} \tag{3.9}$$

where the second and third line follow from (3.8) and (3.5) respectively. Now remark that $Z_{n-1}^{(r)}$ and $Y_n^{(0)}>S_n$ are independent. Hence

$$P(Z_{n-1}^{(r)} \leqslant z \mid Z_{n-1}^{(r)} - Y_n^{(0)} + S_n \leqslant z,\ Y_n^{(0)} > S_n) = \frac{P(Z_{n-1}^{(r)} \leqslant z)}{P(Z_{n-1}^{(r)} - Y_n^{(0)} + S_n \leqslant z \mid Y_n^{(0)} > S_n)}$$

$$= \Psi^{(r)}(z) \tag{3.10}$$

which yields an interpretation of $\Psi^{(r)}(z)$ as a conditional probability.

3.2. Headways at a large distance from the bottleneck
We have for $Y_n^{(r)} = A_n^{(r)} - A_{n-1}^{(r)}$:

$$Y_n^{(r)} = \max(Y_n^{(0)} - Z_{n-1}^{(r)} + r / V_n,\ S_n). \tag{3.11}$$

Substituting $\max(T_n + \theta,\ S_n)$ for $Y_n^{(0)}$ we arrive at

$$Y_n^{(r)} = \max(T_n + \theta - Z_{n-1}^{(r)} + r / V_n,\ S_n - Z_{n-1}^{(r)} + r / V_n,\ S_n). \tag{3.12}$$

THEOREM 3.2. *Let $(T_n)_{n=1}^{\infty}$, $(S_n)_{n=1}^{\infty}$, and $(V_n)_{n=1}^{\infty}$ be three independent sequences of i.i.d. random variables, let $Y_n^{(0)}$ equal $\max(T_n + \theta,\ S_n)$ and $(Z_n^{(r)})_{n=1}^{\infty}$ satisfy (3.3), and let $(Y_n^{(r)})_{n=1}^{\infty}$ be defined by (3.12). Furthermore, let T_1 be an exponential random variable with expectation $1/\lambda$, let θ and ρ be given by (2.5) and (2.20) respectively, and let K be the cumulative distribution function of V_n. If $\rho < 1$, then for every v such that $K(v) > 0$:*

$$\lim_{r\to\infty} \frac{G(y) - P(Y_n^{(r)} \leqslant y \mid V_n = v)}{\exp\{-\lambda r \tau(v)\}} = \exp\{-\gamma(v)(y-\theta)\} G(y), \tag{3.13}$$

where

$$\gamma(v) = \lambda \frac{K(v)}{1 - \rho + \rho K(v)} \tag{3.14}$$

and

$$\tau(v) = \int_0^v \frac{1}{s^2} \frac{K(s)}{1 - \rho + \rho K(s)} ds. \tag{3.15}$$

PROOF. The distribution of $Y_n^{(r)}$, being the maximum of three dependent variables, can be quite complicated. Therefore we introduce an easier to handle approximation to $Y_n^{(r)}$:

$$\tilde{Y}_n^{(r)} = \max(T_n + \theta - Z_{n-1}^{(r)} + r / V_n,\ S_n) \tag{3.16}$$

which is the maximum of two independent variables. To evaluate the quality of $\tilde{Y}_n^{(r)}$ as an approximation to $Y_n^{(r)}$ let us compute the probability that $\tilde{Y}_n^{(r)}$ differs from $Y_n^{(r)}$, given that $V_n = v$. First note that $\tilde{Y}_n^{(r)} \neq Y_n^{(r)}$ if and only if the following two independent events:

$$\{T_n + \theta < S_n\} \text{ and } \{r / V_n > Z_{n-1}^{(r)}\}$$

occur simultaneously. The probability of the first event, which is independent of $\{V_n = v\}$, is simply equal to ρ, the probability of being a follower at the bottleneck. Conditional on the event $\{V_n = v\}$, the probability of the second event is:

$$P(Z_{n-1}^{(r)} < r / v) = \Omega^{(r)}(r / v)$$

Combining the two probabilities, and expressing $\Omega^{(r)}$ in K, the desired speed distribution function, we have

$$P(\tilde{Y}_n^{(r)} \neq Y_n^{(r)} \mid V_n = v) = \rho(1-\rho) \frac{1 - K(v)}{1 - \rho + \rho K(v)} \exp\left\{ -\lambda r \int_0^v \frac{1}{s^2} \frac{K(s)}{1 - \rho + \rho K(s)} ds \right\},$$

and finally by letting r tend to infinity

$$\lim_{r\to\infty} P(\tilde{Y}_n^{(r)} \neq Y_n^{(r)} \mid V_n = v) = 0. \tag{3.17}$$

We will now determine the asymptotic behavior of the distribution of $(\tilde{Y}_n^{(r)} \mid V_n = v)$, and hence also of $(Y_n^{(r)} \mid V_n = v)$, as r tends to infinity. From (3.16) we obtain:

$$\begin{aligned} P(\tilde{Y}_n^{(r)} \leq y \mid V_n = v) &= P(T_n + \theta - Z_{n-1}^{(r)} + r/v \leq y)\, G(y) \\ &= (1 - P(Z_{n-1}^{(r)} - T_n \leq r/v + \theta - y))\, G(y). \end{aligned}$$

By (3.10) we can write:

$$\begin{aligned} P(Z_{n-1}^{(r)} - T_n \leq r/v + \theta - y) &= \exp\left\{ -\lambda \int_{r/v+\theta-y}^{\infty} \frac{1-\Phi^{(r)}(t)}{1-\rho\Phi^{(r)}(t)}\, dt \right\} \\ &= \exp\left\{ -\lambda r \int_0^{\frac{v}{1-v(y-\theta)/r}} \frac{1}{s^2} \frac{K(s)}{1-\rho+\rho K(s)}\, ds \right\}. \end{aligned}$$

Using the right-continuity of the integrand we obtain the Taylor expansion

$$\int_0^{\frac{v}{1-vz}} \frac{1}{s^2} \frac{K(s)}{1-\rho+\rho K(s)} ds = \int_0^{v} \frac{1}{s^2} \frac{K(s)}{1-\rho+\rho K(s)} ds + \frac{K(v)}{1-\rho+\rho K(v)} z + o(z)$$

for $z \downarrow 0$, and get for fixed v:

$$\begin{aligned} \int_0^{\frac{v}{1-v(y-\theta)/r}} \frac{1}{s^2} \frac{K(s)}{1-\rho+\rho K(s)} ds &= \int_0^{v} \frac{1}{s^2} \frac{K(s)}{1-\rho+\rho K(s)} ds + \\ &+ \frac{K(v)}{1-\rho+\rho K(v)} \frac{y-\theta}{r} + o(1/r) \end{aligned} \tag{3.18}$$

for $r \to \infty$. Thus

$$\lim_{r\to\infty} \frac{G(y) - P(\tilde{Y}_n^{(r)} \leq y \mid V_n = v)}{\exp\{-\lambda r \tau(v)\}} = \exp\{-\gamma(v)(y-\theta)\} G(y),$$

and hence by (3.17) the theorem follows.

COROLLARY 3.1. *For large r a headway, conditional upon the desired speed of the next car, is approximately the maximum of a shifted exponential random variable and a minimum headway. Both the location and the scale of the shifted exponential random variable depend on the desired speed. We note that the scale parameter $\gamma(v)$ does not depend on r, and that the location parameter $\theta - r\tau(v)/\gamma(v)$ tends to $-\infty$, as r tends to ∞, if $\tau(v) > 0$.*

The following corollary is a consequence of the remainder term in (3.18) being non-negative.

COROLLARY 3.2. *Under the conditions of theorem 3.2 it follows that for every $r > 0$,*

$$0 \leq \frac{G(y) - P(Y_n^{(r)} \leq y)}{\int_0^{\infty} \exp\{-\gamma(v)(y-\theta) - \lambda r \tau(v)\} dK(v)} \leq G(y). \tag{3.19}$$

As it will turn out, the behavior of headways at a large distance from the bottleneck is largely determined by the behavior of K at the point v_0 defined by:

$$v_0 = \inf\{v: K(v)>0\}. \tag{3.20}$$

Of crucial importance is the existence of a lower speed class, i.e. whether or not $K(v_0)$ is positive. This is already indicated by (3.19), since the denominator in the middle part tends to zero if and only if $K(v_0)>0$. Corollary 3.3 summarizes the asymptotic behavior of headways.

COROLLARY 3.3. *Let π denote $K(v_0)$. As r tends to infinity, the following hold:*
(i) *If $v>v_0$, then $P(Y_n^{(r)} \leqslant y \mid V_n = v)$ tends to $G(y)$.*
(ii) *If $\pi>0$, then $P(Y_n^{(r)} \leqslant y \mid V_n = v_0)$ converges to $(1-\exp\{-\lambda\frac{\pi}{1-\rho+\rho\pi}(y-\theta)\})G(y)$.*
(iii) *$P(Y_n^{(r)} \leqslant y)$ converges to $(1-\pi\exp\{-\lambda\frac{\pi}{1-\rho+\rho\pi}(y-\theta)\})G(y)$.*

3.3. Gaps at a large distance from the bottleneck
Related to headways are the so-called 'gaps', which we can define mathematically as $(Y_n^{(r)}-S_n \mid Y_n^{(r)}>S_n)$. They can be thought of as the free time remaining between the passing of consecutive bunches of cars. Note that the gaps at the bottleneck have an exponential distribution. Now we shall study gaps at a point downstream of the bottleneck.

COROLLARY 3.4. *If $\pi>0$, then $P(Y_n^{(r)}-S_n>y \mid Y_n^{(0)}>S_n)$ converges to $\exp\{-\lambda\pi y/(1-\rho+\rho\pi)\}$.*

Thus if $\pi>0$, the gaps eventually become exponentially distributed again. COWAN (1980) proves this within the context of his bunch-gap arrival process. Corollary 3.4 is therefore a special case of his result. He has a general distribution of bunch sizes at the entry point, whereas under our renewal assumption initial bunches are geometrically distributed.

We now present important results for $\pi=0$ not covered by Cowan's 1980 theory. Indeed, they are divergent from his unproven remarks in 1975 and 1980 concerning limiting gap distributions.

THEOREM 3.3. *Let the sequence $(Z_n^{(r)})_{n=1}^{\infty}$ defined by (3.3) have an equilibrium cumulative distribution function $\Omega^{(r)}(z)$ given by (3.5). Let $(V_n)_{n=1}^{\infty}$ be a sequence of i.i.d. random variables with cumulative distribution function K, such that V_n is independent of $Z_{n-1}^{(r)}-Y_n^{(0)}+S_n$ for each $n \geqslant 0$. Then*

$$\lim_{r\to\infty} P(Y_n^{(r)}-S_n>y \mid Y_n^{(r)}>S_n,\ V_n=v)=\exp\{-\gamma(v)y\} \tag{3.21}$$

for every v such that $0<K(v)<1$, with $\gamma(v)$ as given in (3.14).

PROOF. We have from (3.11)

$$Y_n^{(r)}-S_n = \max(Y_n^{(0)}-S_n+r/V_n-Z_{n-1}^{(r)},\ 0).$$

Hence for $y \geqslant 0$:

$$P(Y_n^{(r)}-S_n>y \mid V_n=v) = P(Z_{n-1}^{(r)}-Y_n^{(0)}+S_n<r/v-y).$$

Choose r large enough such that $K(\frac{v}{1-vy/r})<1$. Then, from (3.3) and (3.5), it follows that

$$P(Y_n^{(r)}-S_n>y \mid V_n=v) = \frac{1-\rho}{1-\rho\Phi^{(r)}(r/v-y)}\exp\left\{-\lambda\int_{r/v-y}^{\infty}(1-\Psi^{(r)}(t))dt\right\}.$$

$$= \frac{1-\rho}{1-\rho+\rho K(\frac{v}{1-vy/r})}\exp\left\{-\lambda r\int_0^{\frac{v}{1-vy/r}}\frac{1}{s^2}\frac{K(s)}{1-\rho+\rho K(s)}ds\right\}. \tag{3.22}$$

Thus

$$P(Y_n^{(r)}-S_n>y \mid Y_n^{(r)}>S_n,\ V_n=v) =$$

$$= \frac{1-\rho+\rho K(v)}{1-\rho+\rho K(\frac{v}{1-vy/r})}\exp\left\{-\lambda r \int_v^{\frac{v}{1-vy/r}} \frac{1}{s^2}\frac{K(s)}{1-\rho+\rho K(s)}ds\right\}.$$

The theorem follows by using essentially the same expansion as (3.18).

COROLLARY 3.5. *For large r a gap, conditional upon desired speed of the next car, is approximately exponentially distributed, with scale depending on desired speed.*

THEOREM 3.4. *Under the conditions of Theorem 3.3, let v_0 defined by (3.20) be positive. If k, the derivative of K, exists on some neighborhood of v_0 and satisfies:*

$$\lim_{v\downarrow v_0}(v-v_0)^{2-\alpha}k(v) = \beta \tag{3.23}$$

for given values $\alpha \geqslant 2$ and $0<\beta<\infty$, then

$$\lim_{r\to\infty} P(Y_n^{(r)}-S_n>zr^{1-1/\alpha} \mid Y_n^{(r)}>S_n) = \frac{\int_0^\infty \exp\{-C(w+v_0^2 z)^\alpha\}w^{\alpha-2}dw}{\int_0^\infty \exp\{-Cw^\alpha\}w^{\alpha-2}dw} \tag{3.24}$$

where

$$C = \frac{\beta\lambda}{\alpha(\alpha-1)(1-\rho)v_0^2} \tag{3.25}$$

PROOF. We have

$$r^{1/\alpha}P(Y_n^{(r)}-S_n>zr^{1-1/\alpha}) =$$

$$= \int_0^\infty P(Y_n^{(r)}-S_n>zr^{1-1/\alpha} \mid V_n=v_0+wr^{-1/\alpha})r^{1/\alpha}dK(v_0+wr^{-1/\alpha}), \tag{3.26}$$

where we recognize the left-hand side of (3.22) as part of the integrand. Since by (3.23)

$$\lim_{v\downarrow v_0}(v-v_0)^{-\alpha}\lambda\int_0^v \frac{1}{s^2}\frac{K(s)}{1-\rho+\rho K(s)}ds = C, \tag{3.27}$$

it follows that there is a $\delta>0$ such that for $v_0<v<v_0+\delta$

$$k(v)<2\beta(v-v_0)^{\alpha-2},$$

and

$$\lambda\int_0^v \frac{1}{s^2}\frac{K(s)}{1-\rho+\rho K(s)}ds > \frac{C}{2}(v-v_0)^\alpha.$$

Split the integrating region in (3.26) in two parts: $\{w : 0\leqslant w\leqslant \delta r^{1/\alpha}\}$, and $\{w : w>\delta r^{1/\alpha}\}$. The integral over the second part can be bounded by

$$r^{1/\alpha}\exp\{-\lambda r \int_{v_0}^{v_0+\delta} \frac{1}{s^2}\frac{K(s)}{1-\rho+\rho K(s)}ds\}$$

which tends to zero as r tends to infinity. Since

$$\frac{v_0+wr^{-1/\alpha}}{1-(v_0+wr^{-1/\alpha})zr^{-1/\alpha}} = v_0+(w+v_0^2z)r^{-1/\alpha}+o(r^{-1/\alpha})$$

for $r\rightarrow\infty$, and $w>0$, we may rewrite the integral over the first part as

$$\int_0^{\delta r^{1/\alpha}} c(w,r)\exp\left\{-\lambda r\int_0^{v_0+(w+v_0^2z)r^{-1/\alpha}}\frac{1}{s^2}\frac{K(s)}{1-\rho+\rho K(s)}ds\right\}k(v_0+wr^{-1/\alpha})\,dw,$$

where $c(w,r)$ is bounded, and tends to 1 for every w as r tends to infinity. Since we may bound the integrand by $2\beta e^{-\frac{1}{2}Cw^\alpha}w^\alpha$, it follows by dominated convergence and by equation (3.27) that this integral converges to

$$\beta\int_0^\infty \exp\{-C(w+v_0^2z)^\alpha\}w^{\alpha-2}dw.$$

Now (3.24) follows directly from

$$P(Y_n^{(r)}-S_n>zr^{1-1/\alpha}\mid Y_n^{(r)}>S_n) = \lim_{r\rightarrow\infty}\frac{r^{1/\alpha}P(Y_n^{(r)}-S_n>zr^{1-1/\alpha})}{r^{1/\alpha}P(Y_n^{(r)}-S_n>0)}.$$

This result is in line with the previous results: the local behavior of the desired speed distribution at the point v_0 determines for a large part the asymptotic behavior of headways and gaps at a large distance from the bottleneck. There is one familiar special case of the distribution given in the right-hand side of (3.24).

COROLLARY 3.6. *If $k(v_0)>0$, then gaps are asymptotically half-normal.*

The half-normal distribution is better known as the distribution of the absolute value of a normal variable, or as the distribution of the supremum of a Brownian motion over a fixed interval.

4. DISCUSSION

Under some circumstances (especially when ρ is small) the situation of section 3 can be considered as an approximation to the situation after the bottleneck under the model of section 2. In KONING (1985) Monte Carlo experiments were performed to assess the quality of this approximation. Even for values of ρ close to 1, the approximation seemed to work well.

Only the work of COWAN (1971, 1975, 1980) is comparable to this study. However, his results differ from those presented here. At the bottleneck he claims that the M4 model (2.24) should arise, and for large r he finds, while assuming minimum headways which take a value τ with probability 1, that $P(Y_n^{(r)}-S_n>y\mid Y_n^{(r)}>S_n)$ is closely approximated by

$$\exp\left\{-\frac{\xi\lambda y}{1-\rho}\right\},$$

where ξ is the probability that car n is unimpeded at point r. It is not hard to show that ξ tends to zero as r tends to infinity. Hence, his result implies that there exists no limiting distribution for $P(Y_n^{(r)}-S_n>y\mid Y_n^{(r)}-S_n)$.

In COWAN (1975) it is remarked that the exponential distribution of gaps at the bottleneck is preserved in a certain sense: as r tends to infinity this distribution becomes exponential again. In COWAN (1980) this remark is proved for the case in which there exists a 'slowest' class of vehicles.

Our results show that the remark is not true for the case in which the speed distribution has a density. Thus, there is a marked difference between these two cases.

Fundamental in the treatment of the single lane road is equation (3.3), which can be rewritten as:

$$Z_n = \max(Q_n, Z_{n-1} - U_n), \tag{4.1}$$

and can be thought of as a generalization of the much studied equation:

$$W_n = \max(0, W_{n-1} - U_{n-1}) \tag{4.2}$$

which arises in the study of the single server queue with independent interarrival and service times. Equation (4.1) deserves the same amount of attention: it not only arises here, but also in the random exchange model (HELLAND & NILSEN (1976)), and in the study of the single server queue with weakly dependent inputs (KINGMAN (1965)).

The type of road studied here is of little practical value. One could enhance the practical value e.g. by considering an ∞-lane road which narrows at a bottleneck to an n-lane road, with no lane changing after the bottleneck. However, in making this small step towards a true general model one can expect to encounter the same problems as involved in the step from the M/G/1 queue to the M/G/n queue. Up to now no exact general solutions are known for the M/G/n queue (cf. COHEN (1982)), only approximate general solutions (KÖLLERSTRÖM (1974)).

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