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Random Upper Semicontinuous Functions and Extremal Processes

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All functions take their values in the extended real line. Spaces of upper semicontinuous (usc) functions on a topological space E are considered and topologized in different ways. Convergence in distribution of random usc functions is characterized for one topology, the sup vague topology. In a canonical way, usc functions correspond to sup measures, union-sup homomorphisms on the open sets of E. Random sup measures are interpreted as extremal processes. By identifying closed subsets of E with their indicator functions we make them a subspace of the usc functions. Consequently, the basics of random closed sets are part of the theory. A function on E is usc iff its hypograph is closed in the product space of domain and range, which establishes another relation between the usc functions on E and the closed subsets of a space, this time different from E. The natural bijections between all these spaces or subsets of them turn out to be lattice isomorphisms, and homeomorphisms if the spaces are provided with the sup vague topology. All spaces are sup vaguely Hausdorff if E is locally quasicompact, but E need not be Hausdorff itself. In fact, it is better to allow E being non-Hausdorff for a smooth theory. At the end of the paper, the developed theory is applied to capacities as a common framework for vague convergence of Radon measures and sup vague convergence of usc functions.

Key Words and Phrases: upper semicontinuous functions, sup measures, spaces of closed sets, hypographs, sup vague topology, sup narrow topology, hypo topology, locally quasicompact spaces, extremal processes, random upper semicontinuous functions, random closed sets, convergence in distribution, vague convergence of capacities, Radon measures as capacities.
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0. Introduction

The original reason for the research leading to the present paper was the necessity of formalizing the notion 'extremal process' in probability theory. What came out of it turned out to be a common framework for random closed sets, parts of optimization theory, theory of hyperspaces in set topology, and extremal processes as intended. Substantial parts of this paper could be classified as set topology, and to a lesser extent as lattice theory, rather than probability theory.

Extremal processes have come up in the probabilistic literature in the following way. Let $(\xi_k)_{k=-\infty}^{\infty}$

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be a sequence of real-valued random variables (for instance independent and identically distributed, but nothing is actually assumed). Set for subsets A of $\mathbb{R}$

$$M_n(A) \;:=\; \left[\bigvee_{k:k/n\in A} X_k - b_n\right] \,/\, a_n\,,$$

where $a_n > 0$ and b_n are 'normalizing constants'. In the older probabilistic literature extremal processes were limits in distribution of $M_n([0,t])$ as $n \to \infty$, regarded as random functions of t (see for instance LAMPERTI (1964), DWASS (1964), RESNICK & RUBINOWITCH (1973)). In the more recent literature the idea gradually broke through that M_n should be regarded as a random set function, for instance on the intervals (PICKANDS (1971), MORI & OODAIRA (1976), MORI (1977), RESNICK (1986, 1987)). However, the full consequence of this idea has not been drawn by these authors, because the special cases considered by them allow a nice and concise description as functionals of point processes in the plane, which aspect attracted the focus of their attention. The point process approach turns out to be too narrow in the study of stationary self-similar extremal processes by O'BRIEN, TORFS & VERVAAT (1988) (for announcements of these results, cf. VERVAAT (1986)), which forced these authors to define extremal processes as random set functions with certain properties. For a related approach, see NORBERG (1987).

It is most convenient to regard an extremal process M as a random $\overline{\mathbb{R}}$-valued function ($\overline{\mathbb{R}} := [-\infty,\infty]$) on the open sets in $\mathbb{R}$ such that M has with probability 1 (wp1) the following property:

$$M(\textstyle\bigcup_{j\in J} G_j) \;=\; \bigvee_{j\in J} M(G_j) \tag{0.1}$$

for each collection $(G_j)_{j\in J}$ of open sets in $\mathbb{R}$. More formally, let E be a topological space, $\mathcal{G} = \mathcal{G}(E)$ the collection of its open sets, and let m be an $\mathbb{I}$-valued function on $\mathcal{G}$. Here $\mathbb{I}$ is a fixed compact subinterval of $\overline{\mathbb{R}}$, $\mathbb{I} = \overline{\mathbb{R}}$ in the previous application, but $\mathbb{I} = [0,1]$ for convenience in most of the present paper. Let us call m a *sup measure* if

$$m(\textstyle\bigcup_{j\in J} G_j) \;=\; \bigvee_{j\in J} m(G_j)$$

for each collection $(G_j)_{j\in J}$ in $\mathcal{G}$. Then an extremal process is just a random sup measure. In order to give this definition sense, it is necessary to make SM, the collection of all sup measures, a measurable space. For the notion of convergence in distribution of extremal processes, SM must be made a topological space, preferably with the measurable structure derived from the topological. It is exactly this what the present paper is about.

The following duality, established in Sections 1 and 2, plays a key role in the theory. If m is a function on $\mathcal{G}(E)$, then its *sup derivative* is the function $d^\vee m$ on E defined by

$$d^\vee m(t) \;:=\; \inf\{m(G)\colon G \in \mathcal{G},\, t \in G\}.$$

If f is a function on E, then its *sup integral* $i^\vee f = f^\vee$ is the function on $\mathcal{G}$ defined by

$$f^\vee(G) \;:=\; \bigvee_{t\in G} f(t) \quad \text{for } G \in \mathcal{G}.$$

It turns out that sup measures correspond one-to-one to upper semicontinuous (usc) functions on E by $d^\vee$ and $i^\vee$. Let $US = US(E)$ be the collection of all usc functions on E (here and in the sequel all functions are assumed to be $\mathbb{I}$-valued unless stated otherwise). We now can topologize SM by topologizing US or vice versa. It turns out that for E locally compact with countable base SM becomes compact (Section 4) and metric (Section 5) with the following notion of *sup vague convergence* (Section 3):

$$m_n \to m \text{ in } SM \;\text{ iff }\; \begin{cases} \limsup m_n(K) \leqslant m(K) \text{ for } K \in \mathcal{K}, \\ \liminf m_n(G) \geqslant m(G) \text{ for } G \in \mathcal{G}, \end{cases} \tag{0.2}$$

where $\mathcal{K} = \mathcal{K}(E)$ is the collection of compact sets in E. In fact, the right hand side need to be required only for subcollections like bases (Section 5).

If E is locally compact with countable base, then the Borel field of SM is the smallest that makes the evaluations $m \mapsto m(A)$ measurable for all $A \in \mathcal{G}$ or all $A \in \mathcal{K}$ (Section 11), and extremal processes (= SM-valued random variables) M_n converge in distribution to M iff all finite-dimensional distributions of the values of M_n at the compact balls in E converge to those of M, where the balls B in E must be restricted to those with $\mathbb{P}[M(B) = M(\operatorname{int} B)] = 1$ (Section 12). The finite-dimensional distributions of the values of the extremal processes on $\mathcal{G}$ can be characterized by requiring (0.1) to hold wp1 separately for each countable collection (G_j) in $\mathcal{G}$ (Section 13). In Section 14 measurability and semicontinuity of the actions of taking suprema and infima in SM are investigated.

Before discussing the connections with the literature, we first indicate some relations with spaces of closed sets. Let $\mathcal{F}(E)$ be the collection of all closed sets in E, and let 1_A for $A \subset E$ be the indicator function of A: $1_A(t) := 1$ if $t \in A$, 0 if $t \in E \setminus A$. Then $F \mapsto 1_F$ maps $\mathcal{F}$ one-to-one into $US(E)$. So we can identify (random) closed sets in E with (random) $\{0, 1\}$-valued usc functions or (random) $\{0, 1\}$-valued sup measures. Moreover, $\mathcal{F}(E)$ is topologized by the relative topology of the sup vague topology on $US(E)$ in its image under $F \mapsto 1_F$. We call this the sup vague topology on $\mathcal{F}(E)$. On the other hand, a function on E is usc iff its *hypograph*

$$\operatorname{hypo} f := \{(t, x) \in E \times (0, 1]: x \leqslant f(t)\}$$

is closed in $E \times (0, 1]$. So 'hypo' maps $US(E)$ one-to-one into $\mathcal{F}(E \times (0, 1])$. Consequently, any topology on $\mathcal{F}(E \times (0, 1])$ determines a relative topology on $US(E)$, and it turns out (Section 7) that the sup vague topology on $\mathcal{F}(E \times (0, 1])$ generates the sup vague topology on $US(E)$ (for this reason often called the *hypo topology*). So it is a matter of taste on which space one wants to define the sup vague topology first. The present paper starts with US (or rather SM), in contrast with most of the literature. The reason is the duality between SM and US, which seems to have been unnoticed so far, but plays a crucial role here.

Actually, the map 'hypo' has even much nicer properties, which become visible only if one is willing to consider non-Hausdorff spaces. Let $(0, 1]\uparrow$ denote the space $(0, 1]$ provided with the *upper topology*, whose nontrivial open sets are $(x, 1]$ for $0 < x < 1$. Then 'hypo' turns out to be an order preserving bijection between $US(E)$ and $\mathcal{F}(E \times (0, 1]\uparrow)$ (Section 1), and a homeomorphism between the spaces provided with the sup vague topology (Section 7). So if $E^* = E \times (0, 1]\uparrow$, then $US(E)$ and $\mathcal{F}(E^*)$ are homeomorphic, whereas in the previous paragraph, with $E^* = E \times (0, 1]$, $US(E)$ is only homeomorphic to a subspace of $\mathcal{F}(E^*)$.

This observation leads us to considering non-Hausdorff spaces E from the beginning. It turns out that $US(E)$ and $\mathcal{F}(E)$ are sup vaguely quasicompact (qcompact) whatever is E (Section 4). Here quasicompactness refers to the finite open subcover property, without Hausdorffness. It turns out that $US(E)$ and $\mathcal{F}(E)$ are in addition Hausdorff (hence compact) in case E is locally qcompact but not necessarily Hausdorff (Section 4). In non-Hausdorff spaces things are not as one is used to (qcompact sets need not be closed, an intersection of two qcompact sets need not be qcompact, lattice-isomorphic topologies need not come from homeomorphic spaces, etc.), and this environment is explored in Sections 8 and 9.

There are related developments in many fields of mathematics. Here we indicate them only globally. More detailed comments are made at the end of each section. Furthermore, we do not discuss the special case that E is compact and metric, in which case $\mathcal{F}(E)$ is equal to the space $\mathcal{K}(E)$ of compact sets, metrized by the Hausdorff distance. This is a classical topic in topology. Random closed sets (= $\mathcal{F}(E)$-valued random variables) are the subject of a monograph by Matheron (1975), which was developed further by Salinetti & Wets (1981, 1986) and Norberg (1984, 1986), with random usc functions appearing in the 1986 papers. Usc functions appear as images with grey levels in Serra (1982), who attributes this idea to Matheron in the early 1970s (personal communication).

Random closed sets appear in the equivalent shape of 'measurable closed multifunction' in the optimization literature (Rockafellar (1976), Castaing & Valadier (1977), Klein & Thompson (1984)). Similarly, random *lower* semicontinuous functions can be identified with 'normal integrands' in the optimization literature (Rockafellar (1976)). The two viewpoints were conciliated by

SALINETTI & WETS (1981, 1986). A whole system of convergence notions ('Γ- and G-limits') was developed for variational analysis by DE GIORGI & FRANZONI (1975), DE GIORGI (1977, 1979) and BUTAZZO (1977). The sup vague convergence notions in $US(E)$ and $\mathcal{F}(E)$ are particular cases of this. Applications in mathematical economics can be found in DEBREU (1966, 1974) and HILDENBRAND (1974).

The following topological literature is relevant for $\mathcal{F}(E)$ and $US(E)$. A convergence concept corresponding to the sup vague topology in $\mathcal{F}(E)$ for locally compact E was studied by CHOQUET (1948) and KURATOWSKI (1966), and actually has its traces in the beginning of this century. The topology itself was studied first by FELL (1962), and later on by DIXMIER (1968) and MATHERON (1975). Spaces of usc functions were already considered by MOSCO (1969) and BUTTAZZO (1977).

Spaces of subsets of a given topological space ('hyperspaces') are a topic of study in set topology. Classical references are MICHAEL (1951) and the more recent monograph by NADLER (1978). However, our sup vague topology does not occur at all in these references. For a possible reason, see our discussion in 4.6. In contrast to this, the recent monograph by KLEIN & THOMPSON (1984) also treats the sup vague topology, motivated by applications in economics and optimization.

The spaces $US(E)$ and $\mathcal{F}(E)$ (with reverse order) play a central role as examples of continuous lattices in GIERZ ET AL. (1980). Actually, a whole chapter in this monograph is devoted to a general and abstract theory of spaces of lower semicontinuous functions, which appear there in the shape of 'Scott continuous functions'. See also MISLOVE (1982) for a fast introduction. Only in the last decade locally qcompact spaces (not necessarily Hausdorff) have been studied, exclusively in the context of lattice theory. See GIERZ ET AL. (1980) and HOFMANN & MISLOVE (1981).

Sup measures are a special case of semilattice homomorphisms in GIERZ ET AL. (1980) and of semigroup-valued measures in SION (1973). The terms 'sup derivative', 'sup integral' and 'sup vague topology' remind us that we are dealing here with the 'minimax analogue' of calculus and analysis of Radon measures, in the sense of CUNINGHAME-GREEN (1979), who developed a matrix calculus and spectral theory with addition $+$ replaced by $\vee$ and multiplication replaced by $+$.

There are two topics in the paper that have not been mentioned yet. Initially, mainly in Sections 3 and 4, a whole class of topologies on SM and US are introduced, by replacing $\mathcal{K}$ in (0.2) by some general class of sets $\mathcal{B}$, and the resulting topologies are called the sup $\mathcal{B}$ topologies. The major reason is that in this way we obtain results as well for the sup $\mathcal{F}$ or 'sup narrow' topology, which is favorite in classical set topology.

Another development is that sup vague convergence of sup measures and vague convergence of (additive) Radon measures can be put into one common framework of vague convergence of *capacities*, monotone set functions on the qcompact sets with certain semicontinuity properties. Section 15 and 16 complement the pioneering paper by NORBERG (1986).

The author has tried to make this paper self-contained, which entails that part of the results is not new. There are several reasons for this. The results elsewhere are often formulated in the context of other fields of mathematics, and therefore not easily understandable for probabilists. And even where the the formulations in the literature are more familiar, the approach in the present paper is rather different. For instance, the basics of random closed sets appear as side results of a more general theory of random usc functions, so that this paper can serve as an alternative to the introduction in MATHERON's (1975) monograph. Furthermore, the generality of non-Hausdorff spaces permeates the paper from the beginning.

There is more in probabilty theory than extremal processes that can benefit from a self-contained and direct introduction to random usc functions. Pointwise ordered pairs of lsc and usc functions $(-f, g \in US, f \leqslant g)$ form a space which is a compactification of the function space $C(E)$. By considering this compactification or related ones the proof of Donsker's theorem can be interpreted in a new way (cf. VERVAAT (1981), LENSTRA (1985) and SALINETTI & WETS (1986)). Furthermore, the most natural context for processes of random closed sets is a generalization of extremal processes, whose values are no longer in $\overline{\mathbb{R}}$, but in a lattice L, for instance $L = \mathcal{F}(E')$ for some other space E'. Lattice-valued usc functions are investigated in GERRITSE (1985), and the corresponding probability theory is being developed by NORBERG. Part of the basic theory has already been dealt with by GIERZ ET AL. (1980).

The prerequisites fot the present paper are the measure theoretical foundations of probability theory, convergence in distribution in Polish spaces and basics of set topology.

0.1. NOTATIONS AND CONVENTIONS

All functions are $\mathbb{I}$-valued, unless stated otherwise; $\mathbb{I}$ is a compact subinterval of $\overline{\mathbb{R}} := [-\infty, \infty]$, for convenience $\mathbb{I} = [0, 1]$ in the present paper, but $\mathbb{I} = \overline{\mathbb{R}}$ is more approptiate for applications.

E is a topological space. No separation axioms are assumed in general. In many places all or part of the regularity conditions show up: E is locally quasicompact with countable base. In particular these are assumed in the probabilistic sections 11, 12 and 13. A subset K of E is *quasicompact* (*qcompact*) if each open cover contains a finite subcover; if K is in addition Hausdorff, then K is *compact*. For $A \subset E$ its *saturation* $\mathrm{sat}A$ is the intersection of all open sets containing A; if $A = \mathrm{sat}A$, then A is *saturated.*

$\mathcal{G}$	$:=$	{open sets};
$\mathcal{F}$	$:=$	{closed sets};
$\mathcal{K}$	$:=$	{qcompact sets};
$\mathcal{Q}$	$:=$	{saturated qcompact sets}
$\mathcal{G}_0$	$:=$	base of open sets;
$\mathcal{K}_0$	$:=$	base-like collection of qcompact sets;
US	$:=$	{upper semicontinuous functions on E};
SM	$:=$	{sup measures on $\mathcal{G}$} (cf. §2);

When E varies and the dependence on E becomes relevant, we write $\mathcal{G}(E)$, $US(E)$, etc. In this paper, F is always a closed set, G an open set, K a qcompact set, Q a saturated qcompact set. In proofs these qualities are not always mentioned. Let $A \subset E$. Then:

1_A is the indicator function of A: $1_A(t) := 1$ for $t \in A$, 0 for $t \in E \setminus A$;
$\mathrm{clos}A$ is the closure of A;
$\mathrm{int}A$ is the interior of A;
$\mathrm{sat}A$ is the saturation of A as defined above (cf. also 1.7);
$\mathrm{sqc}A$ is the smallest qcompact set containing A if it exists (§8);
$A' := E \setminus A$ is the complement of A;

Moreover,

hypo f is the hypograph of f (§1);
$d^{\vee}m$ is the sup derivative of m (§2);
$i^{\vee}f = f^{\vee}$ is the sup integral of f (§2).

Convergence in distribution of random variables is denoted by $\to_d$, equality in distribution by $=_d$.

lsc	=	lower semicontinuous;
usc	=	upper semicontinuous;
rv	=	random variable;
wp1	=	with probability one;
qcompact	=	quasicompact.

0.2. CONTENTS

1. Upper Semicontinuous Functions

In the present section we collect some results about real-valued upper semicontinuous functions, many of them well-known. All functions are defined on a topological space E, without any separation axiom assumed, and take their values in some compact interval $\mathbb{I}$ in the extended real line. In many probabilistic applications $\mathbb{I} = [0, \infty]$ or $\mathbb{I} = [-\infty, \infty]$. For convenience we fix $\mathbb{I} = [0,1]$ in the present paper. We write $\mathbb{I}' := \mathbb{I} \setminus \inf \mathbb{I}$, so $\mathbb{I}' = (0,1]$. For functions $f: E \to \mathbb{I}$ we define the *hypograph* of f by

$$\operatorname{hypo} f \ := \ \{(t,x) \in E \times \mathbb{I}' : x \leqslant f(t)\} .$$

By $[-\infty, \infty]\!\downarrow$ we denote the set $[-\infty, \infty]$ provided with the *lower topology,* whose nontrivial open sets are $[-\infty, x)$ for $x \in (-\infty, \infty]$. A subset A provided with the relative lower topology is denoted $A\!\downarrow$. Similar conventions apply to the *upper topology* on $[-\infty, \infty]$: $[-\infty, \infty]\!\uparrow$ has nontrivial open sets $(x, \infty]$. Observe that a nonempty subset of $A\!\downarrow$ is quasicompact (qcompact) iff it contains its supremum. Quasicompactness refers to the finite open subcover property. A compact set is both quasicompact and Hausdorff.

1.1. Definition. *Let E be a topological space. A function $f: E \to \mathbb{I}$ is* ***upper semicontinuous (usc)*** *at $t \in E$ if*

$$f(t) \ = \ \textstyle\bigwedge_{\text{open } G \ni t} \bigvee_{u \in G} f(u) .$$

A function $f: E \to \mathbb{I}$ is usc if it is usc at all $t \in E$. A function $f: E \to \mathbb{I}$ is ***lower semicontinuous (lsc)*** *if $1 - f$ is usc.*

1.2. Theorem. *The following are equivalent:*
(i) *f is usc;*
(ii) *$\operatorname{hypo} f$ is closed in $E \times \mathbb{I}'$;*
(iii) *$f: E \to \mathbb{I}\!\downarrow$ is continuous, i.e., $f^{\leftarrow}[0,x)$ is open for all $x \in \mathbb{I}'$.*

1.3. Corollaries. The first two corollaries are based on independent observations about functions with closed hypographs that will play a role in the proof of the theorem.
(a) Let $A \subset E$. Then 1_A is usc iff A is closed. (From (ii). Observe that clos hypo 1_A = hypo $1_{\operatorname{clos} A}$.) Similarly, $x\, 1_A + y\, 1_{E \setminus A}$ with $x > y$ is usc iff A is closed.
(b) If $(f_j)_{j \in J}$ is a collection of usc functions, then $\bigwedge_{j \in J} f_j$ is usc. If, moreover, J is finite, then $\bigvee_{j \in J} f_j$ is usc. (From (ii). Observe that hypo $\bigwedge$ = $\bigcap$ hypo, hypo $\bigvee$ = $\bigcup$ hypo, the latter only for finite collections.)

(c) If f is usc and E is qcompact, then f has a maximum. (From (iii). Observe that $f(E)$ in $\mathbb{I}\downarrow$ is qcompact.)

Before proving Theorem 1.2, it is useful to make the following observation. Set

$$f_G \;:=\; (\vee_{u\in G} f(u))\, 1_G + 1_{E\setminus G}$$

for open $G \subset E$, and

$$f^* \;:=\; \wedge_G f_G\,.$$

Then Definition 1.1 tells us that f is usc iff $f = f^*$.

1.4. LEMMA. clos hypo f $=$ hypo f^*.

PROOF. Obviously $f \leqslant f^*$, so hypo $f \subset$ hypo f^*. Furthermore, hypo f^* is closed by the observations in Corollaries 1.3(a,b). So it is sufficient to prove hypo $f^* \subset$ clos hypo f. To this end, consider $(t,x) \notin$ clos hypo f. Then there is an open $G \ni t$ and a real $y < x$ such that $G \times (y, 1]$ does not intersect hypo f. Hence $f(u) < y < x$ for $u \in G$, so $f^*(t) \leqslant y < x$, i.e., $(t,x) \notin$ hypo f^*. □

PROOF OF THEOREM 1.2.
(i) $\Rightarrow$ (iii). If $t \in f^{\leftarrow}[0,x)$, then $\wedge_{\text{open } G\ni t} \vee_{u\in G} f(u) = f(t) < x$, so $\vee_{u\in G} f(u) < x$ for some open $G \ni t$. So $t \in G \subset f^{\leftarrow}[0,x)$, which proves $f^{\leftarrow}[0,x)$ to be open.
(iii) $\Rightarrow$ (ii). We have in general:

$$f \;=\; \wedge_{x\in\mathbb{I}'}(x\, 1_{f^{\leftarrow}[0,x)} + 1_{f^{\leftarrow}[x,1]})\,. \tag{1.1}$$

From (iii) and the independent hypo observations in Corollaries 1.3(a,b) we see that hypo f is closed.
(ii) $\Rightarrow$ (i). By Lemma 1.4 we have $f = f^*$ if hypo f is closed. □

Let $US = US(E)$ be the set of all $\mathbb{I}$-valued usc functions on E. We want to characterize US as a whole. First a notation. By $\mathcal{F} = \mathcal{F}(E)$ we denote the family of closed sets in E.

1.5 THEOREM. (a) *US is the smallest class of $\mathbb{I}$-valued functions on E that contains $x\, 1_F$ for $x \in \mathbb{I}'$, $F \in \mathcal{F}$ and is closed for arbitrary infima and finite suprema.*
(b) *US is the smallest class of $\mathbb{I}$-valued functions on E that contains*

$$x\, 1_F + y\, 1_{E\setminus F} \quad \text{for } x, y \in \mathbb{I},\ x \geqslant y,\ F \in \mathcal{F},$$

and is closed for arbitrary infima.

PROOF. (b) Follows from (1.1), Theorem 1.2 and Corollaries 1.3(a,b).
(a) Follows from (b),

$$x\, 1_F + y\, 1_{E\setminus F} \;=\; x\, 1_F \vee y\, 1_E \quad \text{for } x \geqslant y$$

and Corollary 1.3(b). □

The space $US(E)$ is a complete lattice (all subsets have infima and suprema) with the infimum being pointwise infimum and the supremum of $(f_j)_j$ being $(\vee_j f_j)^*$. The space $\mathcal{F}(E)$ is a complete lattice with the infimum being intersection and the supremum being closure of the union.

1.6. THEOREM. *The map* hypo *is a lattice isomorphism from $US(E)$ onto $\mathcal{F}(E \times \mathbb{I}'\uparrow)$.*

PROOF. The family of closed sets in $E \times \mathbb{I}'\uparrow$ is the smallest class that contains

$$F \times (0,x] \;=\; \text{hypo}\, x\, 1_F \quad \text{for } x \in \mathbb{I}',\ F \in \mathcal{F}(E)$$

and is closed for arbitrary intersections and finite unions. Apply Theorem 1.5(a). □

From Theorem 1.6 we learn that $US(E)$ can be examined by considering $\mathcal{F}(E^*)$ for $E^* = E \times \mathbb{I}'\uparrow$. However, E^* is not Hausdorff. This motivates us to maintain a generality beyond Hausdorffness in the present paper. In non-Hausdorff spaces the following notion will be useful.

1.7. Definition. *The **saturation** of a set $A \subset E$ is the set*

$$\operatorname{sat} A := \bigcap\nolimits_{\text{open } G \supset A} G$$

*If $A = \operatorname{sat} A$, then A is said to be **saturated**.*

All sets in E are saturated iff E is T_1 (cf. §4), in particular if E is Hausdorff. Note that $u \in \operatorname{sat}\{t\} =: \operatorname{sat} t$ iff $t \in \operatorname{clos}\{u\} =: \operatorname{clos} u$. More generally, we have $B \cap \operatorname{sat} A = \varnothing$ iff $\operatorname{clos} B \cap \operatorname{sat} A = \varnothing$ iff $\operatorname{clos} B \cap A = \varnothing$. Applying this for $A = \bigcup_j A_j$ and $B = \{t\}$ we find $\operatorname{sat} \bigcup_j A_j = \bigcup_j \operatorname{sat} A_j$. The intersection of saturated sets is saturated, as in general $\operatorname{sat} \bigcap_j A_j \subset \bigcap_j \operatorname{sat} A_j \subset \operatorname{sat} \bigcap_j \operatorname{sat} A_j$.

1.8. Theorem. *If f is usc and $A \subset E$, then $\bigvee_{t \in A} f(t) = \bigvee_{t \in \operatorname{sat} A} f(t)$.*

Proof. As $\operatorname{sat} A = \bigcup_{t \in A} \operatorname{sat} t$, it is sufficient to prove the theorem for $A = \{a\}$. Since $G \ni a$ implies $G \supset \operatorname{sat} a$ for open G, we have

$$f(a) \leqslant \bigvee\nolimits_{t \in \operatorname{sat} a} f(t) \leqslant \bigwedge\nolimits_{G \ni a} \bigvee\nolimits_{t \in G} f(t) = f(a). \qquad \square$$

1.9. Example. $E = \mathbb{R}\downarrow$. Then $\operatorname{sat} t = (-\infty, t]$ for $t \in E$. The space $US(E)$ consists of all nondecreasing right-continuous $\mathbb{I}$-valued functions on $\mathbb{R}$, and can be identified with the class of all probability distribution functions on the extended real line $[-\infty, \infty]$.

1.10. Literature. Most results are classical knowledge in a perhaps less classical presentation. For a lattice-theoretical approach to *lower* semicontinuous functions, see Chapter II of Gierz & al. (1980). The three characterizations of upper semicontinuity in Theorem 1.2 need no longer be equivalent in case the totally ordered range $\mathbb{I}$ is replaced by a more general lattice or partially ordered space. See Penot & Théra (1982) and Gerritse (1985).

2. Sup Measures

In the present section we introduce the sup measures, which henceforth will be close companions of usc functions. By $\mathcal{G} = \mathcal{G}(E)$ we denote the class of open sets in a topological space E.

2.1. Definition. (a) *The **sup derivative** of a function $m: \mathcal{G} \to \mathbb{I}$ is the function $d^\vee m: E \to \mathbb{I}$ defined by*

$$d^\vee m(t) := \bigwedge\nolimits_{G \ni t} m(G) \quad \text{for } t \in E. \tag{2.1}$$

(b) *The **sup integral** of a function $f: E \to \mathbb{I}$ is the function $f^\vee: \mathcal{G} \to \mathbb{I}$ defined by*

$$f^\vee(G) := \bigvee\nolimits_{t \in G} f(t) \quad \text{for } G \in \mathcal{G},$$

where $\bigvee \varnothing := 0$. Occasionally we will write $i^\vee f$ instead of $f^\vee$.

2.2. Lemma. *Let m and f be as in Definition 2.1. Then*
(a) *$d^\vee m$ is usc,*
(b) $m \geqslant i^\vee d^\vee m$,

(c) $f \leqslant d^{\vee} i^{\vee} f$.

Proof. (a) Note that

$$d^{\vee} m = \wedge_{G \in \mathcal{G}} (m(G) 1_G + 1_{E \setminus G}),$$

so $d^{\vee} m$ is usc by Corollaries 1.3(a,b).
(b,c) Obvious. □

2.3. **Remark.** Note that Definition 1.1 can be rephrased as: f is usc iff $f = d^{\vee} i^{\vee} f =: f^*$. In Lemma 1.4 we recognize f^* as the smallest usc function larger than f, the function with clos hypo$f$ as hypograph.

2.4. **Definition.** *A function* $m: \mathcal{G} \to \mathbb{I}$ *is called a* **sup measure** *if* $m(\varnothing) = 0$ *and for all collections* $(G_j)_{j \in J}$ *of open sets*

$$m(\bigcup_{j \in J} G_j) = \vee_{j \in J} m(G_j). \tag{2.2}$$

Obviously, all sup integrals are sup measures, but different f in Definition 2.1(b) may generate the same sup measure. Example: $E = \mathbb{R}$, $1_{\mathbb{R}}^{\vee} = 1_{\mathbb{Q}}^{\vee} \equiv 1$ on $\mathcal{G} \setminus \{\varnothing\}$. The following theorem shows that all sup measures are sup integrals of usc functions, and that the correspondence is one-to-one.

2.5. **Theorem.** *Let m and f be as in Definition 2.1.*
(a) m is a sup measure iff $m = i^{\vee} d^{\vee} m$.
(b) If m is a sup measure, then $f = d^{\vee} m$ is the largest f and only usc f with $f^{\vee} = m$.
(c) If m is a sup measure, then $\vee_{t \in A} d^{\vee} m(t) = \wedge_{G \supset A} m(G)$ for all sets $A \subset E$.

Proof. (a) The 'if' part is trivial, the 'only if' part a special case of (c) for open A.
(b) Follows from (a), Lemma 2.2(a,c) and Remark 2.3.
(c) For all $t \in A$ we have $d^{\vee} m(t) \leqslant \wedge_{G \supset A} m(G)$, so $\vee_{t \in A} d^{\vee} m(t) \leqslant \wedge_{G \supset A} m(G)$. To prove the reverse inequality, fix $x > \vee_{t \in A} d^{\vee} m(t)$. For each $t \in A$ there is an open $G_t \ni t$ such that $m(G_t) < x$, so $m(\bigcup_{t \in A} G_t) \leqslant x$, implying $\wedge_{G \supset A} m(G) \leqslant x$. □

Let m be an increasing $\mathbb{I}$-valued function on $\mathcal{G}$ and let $\mathcal{G}_0$ be a base of $\mathcal{G}$. Obviously, $d^{\vee} m$ does not change if we restrict G to $\mathcal{G}_0$ in (2.1). Furthermore, if m is a sup measure (hence increasing), then its values on $\mathcal{G}$ are determined by its values on $\mathcal{G}_0$ and (2.2). The following theorem characterizes which functions on $\mathcal{G}_0$ can be extended to sup measures on $\mathcal{G}$.

2.6. **Theorem.** (Extension Theorem). *Let $\mathcal{G}_0$ be a base of $\mathcal{G}$. If m is an $\mathbb{I}$-valued function on $\mathcal{G}_0$ such that $m(\varnothing) = 0$ and (2.2) holds whenever $G_j \in \mathcal{G}_0$ for $j \in J$ and $\bigcup_{j \in J} G_j \in \mathcal{G}_0$, then m can be extended to a unique sup measure on $\mathcal{G}$ by (2.2).*

Proof. By rephrasing Definition 2.1(a) and the proofs of Lemma 2.2(a) and Theorem 2.5(a) with $\mathcal{G}_0$ instead of $\mathcal{G}$, we obtain that $d^{\vee} m$ (in its new definition) is usc, and that $m = i^{\vee} d^{\vee} m$ on $\mathcal{G}_0$. Hence the unique extension of m to $\mathcal{G}$ is $i^{\vee} d^{\vee} m$. □

If the topology $\mathcal{G}$ of E has a countable base, then (2.2) is equivalent to its restriction to countable J, whether or not restricted further to $\mathcal{G}_0$. In particular this is the case if $E = \mathbb{R}$ and $\mathcal{G}_0$ is the collection of open intervals in $\mathbb{R}$.

2.7. **Example.** Let $E = \mathbb{R}\!\downarrow$ as in Example 1.9. Then the sup measures m can be identified via $m(-\infty, \cdot)$ with the nondecreasing left-continuous $\mathbb{I}$-valued functions on $\mathbb{R}$, and $d^{\vee} m$ turns out to be the right-continuous version of $m(-\infty,\cdot)$.

2.8. LITERATURE. SHILKRET (1970) investigated sup measures with emphasis on analogues of integration theorems in measure theory. This research was continued in the wider context of capacities by NORBERG (1986). Sup measures are a special case of semigroup-valued measures as studied by SION (1973). Sup measures are semilattice homomorphisms between $(\mathcal{G}, \cup)$ and $(\mathbb{I}, \vee)$, which are studied in categorical generality by GIERZ & AL. (1980). With some effort the duality between sup measures and usc functions can be related to a Galois connection (cf. GIERZ & AL. (1980, §0.3), the dual of m being $x \mapsto \operatorname{int} f^{\leftarrow}[0,x]$ with $f = d^{\vee} m$). Lemma 2.2(a) and part of Theorem 2.5 have been proved previously by BUTTAZZO (1977), Lemmas (1.5) and (1.6)) and GRAF (1980, Proposition 6.1). The terminology 'sup measure', 'sup derivative', 'sup integral' indicates that we are dealing here with 'minimax' analogues of measure theory and calculus, in the sense of CUNINGHAME-GREENE (1979) (replace + by $\vee$). Theorem 2.5(a) can be seen as the analogue of the Main Theorem of integral calculus, which identifies the indefinite integral as an antiderivative.

3. THE SUP TOPOLOGIES

In the present section we introduce a class of topologies on $SM = SM(E)$, the lattice of sup measures on $\mathcal{G}(E)$. By Theorem 2.5 we may identify SM with US via the bijections

$$SM(E) \underset{i^{\vee}}{\overset{d^{\vee}}{\rightleftarrows}} US(E),$$

so all topologies on SM carry over to US by declaring $d^{\vee}$ and $i^{\vee}$ homeomorphisms. The map 'ind': $\mathcal{F} \ni F \mapsto 1_F$ injects $\mathcal{F}$ into US, and each topology on US induces in this way a relative topology on $\mathcal{F}$.

Recall that the sup measures as defined in Section 2 have the open sets $\mathcal{G}$ as their domain. However, by Theorem 2.5(c) there is a canonical extension to all subsets A of E by

$$m(A) := \bigvee_{t\in A} d^{\vee} m(t) = \bigwedge_{G\supset A} m(G). \tag{3.1}$$

The right-hand side depends only on A via $\operatorname{sat} A$, so

$$m(A) = m(\operatorname{sat} A) \quad \text{for } A \subset E, \tag{3.2}$$

which result is equivalent to Theorem 1.8 by Theorem 2.5. Two classes of subsets of E will determine the topology on SM, the open sets $\mathcal{G}$ and another class $\mathcal{B}$, the *bounding class* of the topology. For a bounding class we require only that it contains $\varnothing$ (this condition does not matter here, but will be convenient later on when we consider $\mathcal{F}(E)$). Examples of bounding classes are:

$$\begin{array}{lllllll}
\mathcal{B} & = & \{\varnothing\}, & & & & \\
\mathcal{B} & = & \mathcal{J} & = & \mathcal{J}(E) & := & \{\text{finite subsets of } E\}, \\
\mathcal{B} & = & \mathcal{K} & = & \mathcal{K}(E) & := & \{\text{qcompact subsets of } E\}, \\
\mathcal{B} & = & \mathcal{F} & = & \mathcal{F}(E) & := & \{\text{closed subsets of } E\} \text{ as defined before}, \\
\mathcal{B} & = & \mathcal{G} & = & \mathcal{G}(E), & & \\
\mathcal{B} & = & \mathcal{F}_d & & & := & \{d\text{-bounded closed subsets of } E\},
\end{array}$$

where d is a metric that metrizes the topology of E.

3.1. DEFINITION. *The **sup topology** on $SM(E)$ with **bounding class** $\mathcal{B}$, or the **sup** $\mathcal{B}$ **topology** on $SM(E)$, is the smallest topology that makes the evaluations*

$$m \mapsto m(A) \quad \textit{usc for } A \in \mathcal{B}, \ \textit{lsc for } A \in \mathcal{G}. \tag{3.3}$$

3.2. REMARKS. (a) The sets

$$\{m \in SM: m(B) < x\},\ \{m \in SM: m(G) > x\} \quad \text{for } B \in \mathfrak{B},\ G \in \mathcal{G},\ x \in \mathbb{I} \tag{3.4}$$

form a subbase of the sup $\mathfrak{B}$ topology in SM.
(b) A net (m_n) converges sup $\mathfrak{B}$ to m in SM iff

$$\begin{aligned} \text{limsup}_n m_n(B) &\leqslant m(B) \text{ for } B \in \mathfrak{B}, \\ \text{liminf}_n m_n(G) &\geqslant m(G) \text{ for } G \in \mathcal{G}. \end{aligned} \tag{3.5}$$

For *additive* Radon measures and $\mathfrak{B} = \mathcal{K}$ (3.5) is known to characterize vague convergence. Similarly, (3.5) with $\mathfrak{B} = \mathcal{F}$ characterizes weak (or narrow) convergence for *additive* bounded measures, in particular probability measures. Therefore we call sup $\mathcal{K}$ convergence also *sup vague* convergence, and sup $\mathcal{F}$ convergence also *sup weak* (or *sup narrow*) convergence.
(c) If $m_n \to m$ in the sup $\{\varnothing\}$ topology, then also $m_n \to m'$ for each $m' \leqslant m$. So the sup $\mathfrak{B}$ topology is not Hausdorff for $\mathfrak{B} = \{\varnothing\}$. This may happen also for other $\mathfrak{B}$.
(d) The sup $\mathcal{G}$ topology on SM is relative to the product topology on $\mathbb{I}^{\mathcal{G}}$.
(e) The sup $\mathfrak{B}$-topology does not change if
(e1) $\mathfrak{B}$ is enlarged to be closed for finite unions,
(e2) $\mathfrak{B}$ is replaced by $\mathfrak{B}^{\text{sat}} := \{\text{sat}\, B: B \in \mathfrak{B}\}$.
For (e1), note that

$$\begin{aligned} \text{limsup}_n m_n(B_1 \cup B_2) &= \text{limsup}_n (m_n(B_1) \vee m_n(B_2)) \\ &\leqslant \text{limsup}_n m_n(B_1) \vee \text{limsup}_n m_n(B_2) \leqslant m(B_1) \vee m(B_2) = m(B_1 \cup B_2) \end{aligned}$$

if $m_n \to m$ and $B_1, B_2 \in \mathfrak{B}$; (e2) follows from (3.2).

3.3. Definition. *The sup $\mathfrak{B}$ topology on $US(E)$ is the topology that makes the bijection $d^\vee$ between $SM(E)$ and $US(E)$ a homeomorphism. The sup $\mathfrak{B}$ topology on $\mathcal{F}(E)$ is the topology that makes the injection 'ind' from $\mathcal{F}(E)$ into $US(E)$ a homeomorphism.*

3.4. Properties. (a) The sup $\mathfrak{B}$ topology on $US(E)$ is the smallest that makes the sup evaluations

$$f \mapsto f^\vee(A) \quad \text{usc for } A \in \mathfrak{B},\ \text{lsc for } A \in \mathcal{G}, \tag{3.6}$$

so $f_n \to f$ in $US(E)$ iff

$$\begin{aligned} \text{limsup}_n f_n^\vee(B) &\leqslant f^\vee(B) \text{ for } B \in \mathfrak{B}, \\ \text{liminf}_n f_n^\vee(G) &\geqslant f^\vee(G) \text{ for } G \in \mathcal{G}. \end{aligned} \tag{3.7}$$

(b) The sets

$$\{F \in \mathcal{F}: F \cap B = \varnothing\} \text{ for } B \in \mathfrak{B}, \quad \{F \in \mathcal{F}: F \cap G \neq \varnothing\} \text{ for } G \in \mathcal{G} \tag{3.8}$$

form a subbase for the sup $\mathfrak{B}$ topology on $\mathcal{F}(E)$. Note that $\mathcal{F}$ itself belongs to it, because $\varnothing \in \mathfrak{B}$. A net (F_n) converges sup $\mathfrak{B}$ to F in $\mathcal{F}(E)$ iff the following implications hold:

$$\begin{aligned} F \cap B = \varnothing &\Rightarrow F_n \cap B = \varnothing \text{ for all sufficiently large } n \ (B \in \mathfrak{B}), \\ F \cap G \neq \varnothing &\Rightarrow F_n \cap G \neq \varnothing \text{ for all sufficiently large } n \ (G \in \mathcal{G}). \end{aligned} \tag{3.9}$$

In set topology one usually preferred to consider $\mathcal{F}(E)$ with the sup $\mathcal{F}$ topology. In probability and optimization one considered $\mathcal{F}(E)$, and more recently also $US(E)$, with the sup $\mathcal{K}$ topology, for locally compact E. When it comes to probability in the present paper, we will restrict ourselves to the sup $\mathcal{K}$ topology on $US(E)$ for locally qcompact E (not necessarily Hausdorff).

The properties of the sup $\mathfrak{B}$ topologies depend strongly on the separation axioms assumed for the topology $\mathcal{G}$ on E and the interaction between $\mathcal{G}$ and the bounding class $\mathfrak{B}$. Here we list the separation axioms and interaction hypotheses that occur in this paper.

3.5. Separation axioms for $\mathcal{G}$. *The space E (or rather its topology $\mathcal{G}$) is*

(a) T_0 *if for each* $\{t,u\} \subset E$ *there is a* $G \in \mathcal{G}$ *such that* $\#(G \cap \{t,u\}) = 1$,
(b) T_1 *if for each* $\{t,u\} \subset E$ *there are* $G_1, G_2 \in \mathcal{G}$ *such that* $G_1 \cap \{t,u\} = \{t\}$, $G_2 \cap \{t,u\} = \{u\}$,
(c) T_2 *(Hausdorff) if* G_1 *and* G_2 *in (b) can be chosen disjoint,*
(d) T_3 *if for each* $t \in E$ *and* $F \in \mathcal{F}$ *with* $t \notin F$ *there are disjoint* $G_1, G_2 \in \mathcal{G}$ *such that* $t \in G_1$, $F \subset G_2$.

3.6. LOCAL AXIOMS FOR $\mathcal{B}$. *The space E (or rather its topology* $\mathcal{G}$*) is*
(a) ***locally*** $\mathcal{B}$ *if for each* $t \in E$ *and each open* $G \ni t$ *there is a* $B \in \mathcal{B}$ *such that* $t \in \operatorname{int} B \subset B \subset G$,
(b) ***internally*** $\mathcal{B}$ *if for each* $t \in E$ *there is a* $B \in \mathcal{B}$ *such that* $t \in \operatorname{int} B$,
(c) ***fragmentally*** $\mathcal{B}$ *if for each* $t \in E$ *and each open* $G \ni t$ *there is a* $B \in \mathcal{B}$ *such that* $t \in B \subset G$.
Synonyms for locally, internally and fragmentally $\mathcal{K}$ are locally, internally and fragmentally qcompact (compact if E is T_2).

3.7. PROPERTIES. (a) If E is locally $\mathcal{B}$, then E is internally $\mathcal{B}$ and fragmentally $\mathcal{B}$.
(b) The space E is locally $\mathcal{B}$ iff $\mathcal{B}$ contains a neighborhood base at t for each $t \in E$ iff for each open G there is a collection $\{B_j\}_{j\in J} \subset \mathcal{B}$ such that

$$G = \bigcup_{j\in J} \operatorname{int} B_j = \bigcup_{j\in J} B_j .$$

(c) The space E is internally $\mathcal{B}$ iff $E = \bigcup_{B\in B^*} \operatorname{int} B$, in particular if $E \in \mathcal{B}$.
(d) If E is T_2, then E is locally compact iff E is internally compact. A similar equivalence does not hold for local qcompactness in absence of T_2. In particular, a qcompact E is internally qcompact by (c), but need not be locally qcompact. For an example of the latter, consider the one-point qcompactification E' of a Hausdorff space E, obtained by adding one point ∞ to E and making the complements in E' of compact sets in E to its open neighborhoods. Then E' is qcompact, and E' is locally qcompact iff E' is Hausdorff iff E is locally compact.
(e) The space E is locally $\mathcal{G}$ and internally $\mathcal{F}$. The space E is locally $\mathcal{F}$ iff E is T_3.
(f) If $\mathcal{J} \subset \mathcal{B}$, then E is fragmentally $\mathcal{B}$; $\mathcal{J} \subset \mathcal{K}$; $\mathcal{J} \subset \mathcal{F}$ iff E is T_1.
(g) $\mathcal{F} \subset \mathcal{K}$ iff $E \in \mathcal{K}$. If E is T_2, then $\mathcal{K} \subset \mathcal{F}$.

3.8. EXAMPLE. Let $E = \mathbb{R}\!\downarrow$ as in Examples 1.9 and 2.7. Then

$$\begin{aligned}
&\mathcal{G} = \{\varnothing, \mathbb{R}, (-\infty,t): t \in \mathbb{R}\},\\
&\mathcal{F} = \{\varnothing, \mathbb{R}, [t,\infty): t \in \mathbb{R}\},\\
&\mathcal{K} = \{\varnothing, A \subset \mathbb{R}: \sup A \in A\},\\
&\mathcal{F}^{\text{sat}} = \{\varnothing, \mathbb{R}\},\\
&\mathcal{K}^{\text{sat}} = \{\varnothing, (-\infty,t]: t \in \mathbb{R}\}.
\end{aligned}$$

The space $\mathbb{R}\!\downarrow$ is T_0, but not T_1, T_2, T_3 (= locally $\mathcal{F}$) or fragmentally $\mathcal{F}$. It is locally qcompact with countable base $\{\varnothing, (-\infty,t): t \in \mathbb{Q}\}$. The spaces US and SM have been described in Examples 1.9 and 2.7. Recall that US can be identified with the probability distribution functions on $[-\infty, \infty]$. The following characterizes $f_n \to f$ sup $\mathcal{B}$ in US for different $\mathcal{B}$:
(a) $\mathcal{B} = \{\varnothing\}$: $f(t-) \leqslant \liminf_n f_n(t-)$ for $t \in \mathbb{R}$,
(b) $\mathcal{B} = \mathcal{F}$ or $\{\varnothing, \mathbb{R}\}$: (a) and $\limsup_n f_n(\infty-) \leqslant f(\infty-)$,
(c) $\mathcal{B} = \mathcal{G}$: $\lim_n f_n(t) = f(t)$ for $t \in \mathbb{R}$,
(d) $\mathcal{B} = \mathcal{K}$ or $\mathcal{J}$:
$f(t-) \leqslant \liminf_n f_n(t-) \leqslant \limsup_n f_n(t) \leqslant f(t)$ for $t \in \mathbb{R}$
$\Leftrightarrow \lim_n f_n(t) = f(t)$ for all t where $f(t-) = f(t)$.
So US is not sup $\mathcal{B}$ Hausdorff for $\mathcal{B} = \{\varnothing\}$ or $\mathcal{F}$. It is Hausdorff, even compact for $\mathcal{B} = \mathcal{K}$ or $\mathcal{G}$, and metrizable for $\mathcal{B} = \mathcal{K}$ (by a Lévy-type distance) but not for $\mathcal{B} = \mathcal{G}$. So $US(\mathbb{R}\!\downarrow)$ is sup vaguely compact and metrizable, even though $\mathbb{R}\!\downarrow$ is not Hausdorff. In the next sections we will identify the relevant properties of $\mathbb{R}\!\downarrow$ as local qcompactness with countable base.

3.9. LITERATURE. Sup $\mathfrak{B}$ topologies for various $\mathfrak{B}$ were considered first by MROWKA (1970). The sup $\mathfrak{F}$ or sup narrow topology in $\mathfrak{F}$ has been a major topic in set topology (e.g. MICHAEL (1951) and NADLER (1978)). More common names are 'Vietoris' or 'finite' topology. A convergence concept in $\mathfrak{F}$ that is topologized by the sup $\mathfrak{K}$ or sup vague topology in case E is locally qcompact is known since long, in fact as a combination of notions of upper and lower limits in $\mathfrak{F}$ (cf. CHOQUET (1948), KURATOWSKI (1966, §27) and BERGE (1963, §I.9)). The first discussion about the sup vague topology in $\mathfrak{F}$ as a topology is given by FELL (1962), followed by DIXMIER (1968), and MATHERON (1975) for E locally compact with countable base. Both the sup narrow and sup vague topologies are treated by KLEIN & THOMPSON (1984).

If E is metric and metrized by d, then $\mathfrak{K}$ is metrized by the *Hausdorff metric*

$$\rho(K_1, K_2) \;:=\; \bigvee_{t \in K_1} d(t, K_2) \vee \bigvee_{t \in K_2} d(t, K_1)$$

(establishing distance ∞ between $\varnothing$ and nonempty sets). If E is compact, then $\mathfrak{F}$ and $\mathfrak{K}$ coincide, and the sup vague and sup narrow topologies are the same and generated by the Hausdorff distance. The space $\mathfrak{K}$ with the Hausdorff distance is classical and will not be discussed here.

The sup vague or 'hypo' (cf. Section 7) topology on US has been considered by several authors. Some of them start with the convergence concept: $f_n \to f$ in US iff for each $t \in E$ we have $\limsup_n f_n(t_n) \leqslant f(t)$ for all sequences $(t_n) \to t$ in E, and $\lim_n f_n(t_n) = f(t)$ for some sequence $(t_n) \to t$ in E (E locally compact with countable base). This is the case with DE GIORGI & FRANZONI (1975), BUTTAZZO (1977), who in fact study a more general collection of upper and lower limits in topology ('Γ- and G-limits'), which is also considered by DE GIORGI (1977, 1979). Other authors start with the embedding 'hypo': $US(E) \to \mathfrak{F}(E \times \mathbb{I}')$, like MOSCO (1969) for convex functions, BEER (1982) for compact E and SALINETTI & WETS (1986). Characterizations (3.6) and (3.7) of the sup vague topology in US also occur in SALINETTI & WETS (1986) and NORBERG (1986). Only NORBERG (1986) and the present paper take it as starting point.

The sup vague topologies in US and $\mathfrak{F}$ are a special case of the Lawson topology in continuous lattices (with reverse order), cf. Th.II.4.7 and Ch.III of GIERZ ET AL. (1980).

4. GENERAL PROPERTIES OF THE SUP TOPOLOGIES

We assume $SM(E)$, $US(E)$ and $\mathfrak{F}(E)$ provided with a sup $\mathfrak{B}$ topology for some bounding class $\mathfrak{B}$. Note that $\mathfrak{F}$ is a subspace of US after identification with its image under 'ind'. We start with examining this subspace.

4.1. THEOREM. *The range* ind($\mathfrak{F}$) *is sup* $\mathfrak{B}$ *closed in US iff E is locally* $\mathfrak{B}$.

PROOF. If $f \in US \setminus \text{ind}(\mathfrak{F})$, then $f(t) \in (0,1)$ for some $t \in E$. Select x, y such that $0 < x < f(t) < y < 1$. Since $f \in US$, there is a $G_0 \in \mathcal{G}$ such that $t \in G_0$ and $f^\vee(G_0) < y$.

If E is locally $\mathfrak{B}$, then select $B \in \mathfrak{B}$ with $t \in \text{int} B \subset B \subset G_0$, so the basic open set

$$\{g \in US : g^\vee(B) < y,\ g^\vee(\text{int} B) > x\}$$

contains f but does not intersect ind($\mathfrak{F}$). This proves that $US \setminus \text{ind}(\mathfrak{F})$ is open.

Conversely, if f is an accumulation point of ind($\mathfrak{F}$), then for all open G with $t \in G \subset G_0$ and all $B \in \mathfrak{B}$ with $t \in B \subset G_0$ there is an $F \in \mathfrak{F}$ with

$$1_F \;\in\; \{g \in US : g^\vee(B) < y,\ g^\vee(G) > x\},$$

so with $F \cap B = \varnothing, F \cap G \neq \varnothing$. It follows that $G \subset B$ is impossible for all such B and G, so E is not locally $\mathfrak{B}$. □

4.2. THEOREM. *The following are equivalent:*

(i) *SM and US are qcompact;*

(ii) *$\mathcal{F}$ is qcompact;*
(iii) $\mathcal{B} \subset \mathcal{K}$.

4.3. THEOREM. (a) *SM, US and $\mathcal{F}$ are T_0.*
(b) *If E is fragmentally $\mathcal{B}$, then SM, US and $\mathcal{F}$ are T_1.*
(c) *If E is locally $\mathcal{B}$, then SM, US and $\mathcal{F}$ are T_2.*
(d) *If $\mathcal{F}$ is T_2, then E is internally $\mathcal{B}$.*

4.4. COROLLARIES. (a) The spaces *SM*, *US* and $\mathcal{F}$ are sup vaguely (= sup $\mathcal{K}$) qcompact for general E, and moreover compact if E is locally qcompact. If E is T_2 and *SM*, *US* or $\mathcal{F}$ is sup vaguely compact, then E is internally compact, so locally compact by Property 3.7(d). So if E is T_2, then each of *SM*, *US* and $\mathcal{F}$ is compact iff E is locally compact.
(b) The spaces *SM*, *US* and $\mathcal{F}$ are sup weakly (= sup $\mathcal{F}$) qcompact iff E is qcompact, and sup weakly compact if E is qcompact and T_3 (= locally $\mathcal{F}$).

PROOF OF THEOREM 4.2. By (3.4) each closed set in *US* is an intersection of finite unions of

$$\{f : f^\vee(B) \geqslant x\}, \ \{f : f^\vee(G) \leqslant y\} \ \text{ for } B \in \mathcal{B},\ G \in \mathcal{G},\ x, y \in \mathbb{I}.$$

By Alexander's subbase theorem (KELLEY (1955, p.139)) *US* is qcompact iff for each instance of

$$\bigcap_{i\in I}\{f : f^\vee(B_i) \geqslant x_i\} \cap \bigcap_{j\in J}\{f : f^\vee(G_j) \leqslant y_j\} =: \bigcap_{i\in I} F_{1,i} \cap \bigcap_{j\in J} F_{2,j} = \varnothing \tag{4.1}$$

the same holds true with I and J replaced by finite subsets. Set

$$g := \wedge_{j\in J}(y_j 1_{G_j} + 1_{E\setminus G_j}).$$

Then g is usc by Corollaries 1.3(a,b), and $\bigcap_{j\in J} F_{2,j} = \{f : f \leqslant g\}$. Furthermore, if $f_1, f_2 \in US$, $f_1 \leqslant f_2$ and $\bigcap_{i\in I} F_{1,i}$ contains f_1, then also f_2. So (4.1) holds iff

$$g^\vee(B_i) < x_i \quad \text{for some } i \in I. \tag{4.2}$$

(iii) $\Rightarrow$ (i). Assume $\mathcal{B} \subset \mathcal{K}$. Assume further that $E = \bigcup_{j\in J} G_j$ (this is no restriction: if necessary, add a j with $G_j = E$ and $y_j = 1$ in (4.1)). Suppose that (4.1) holds and fix an i that realizes (4.2). Let $J_i \subset J$ be the collection of j such that $B_i \cap G_j \neq \varnothing$ and $y_j < x_i$. Then $B_i \subset \bigcup_{j\in J_i} G_j$. As B_i is qcompact, we have $B_i \subset \bigcup_{j\in J_\#} G_j$ for some finite $J_\# \subset J_i$. Defining $g_\#$ by reducing J to $J_\#$ in the definition of g we find

$$g_\#^\vee(B_i) \leqslant \vee_{j\in J_\#} g_\#^\vee(G_j) \leqslant \vee_{j\in J_\#} y_j < x_i,$$

so (4.1) already holds with $\{i\}$ instead of I and $J_\#$ instead of J. We have proved that *US* is qcompact.
(i) $\Rightarrow$ (iii). Conversely, if *US* is qcompact, consider (4.1) with only one i, $B_i = B \in \mathcal{B}$, $x_i := 1$, $y_j := 0$ for $j \in J$. Then (4.1) is equivalent to (4.2), thus to

$$B \subset \bigcup_{j\in J} G_j. \tag{4.3}$$

Reduction to finite J, being possible as *US* is qcompact, is equivalent to the same reduction in (4.3). Hence B is qcompact, which proves $\mathcal{B} \subset \mathcal{K}$.
(iii) $\Leftrightarrow$ (ii). Repeat the proof of (iii) $\Leftrightarrow$ (i) with ind($\mathcal{F}$) instead of *US*, $\{0,1\}$-valued f, $x_i := 1$, $y_j := 0$. □

PROOF OF THEOREM 4.3. (a,b,c). It is sufficient to prove the statements for *US*, as *SM* is homeomorphic and $\mathcal{F}$ is a subspace. So suppose $g, h \in US$ and $g \neq h$. Then $g(t) \neq h(t)$ for some $t \in E$, say $g(t) < h(t)$. Let $g(t) < x < h(t)$. Then $g^\vee(G) < x$ for some open $G \ni t$, so the basic open set $\{f \in US : f^{\vee}(G) > x\}$ contains h, but not g. This proves US to be T_0.

If, moreover, E is fragmentally $\mathfrak{B}$, then we can find $B \in \mathfrak{B}$ with $t \in B \subset G$, and the basic open set $\{f \in US : f^{\vee}(B) < x\}$ contains g, but not h. This proves US to be T_1.

Finally, if E is locally $\mathfrak{B}$, then select $B \in \mathfrak{B}$ with $t \in \operatorname{int} B \subset B \subset G$. Then $\{f \in US : f^{\vee}(B) < x\}$ and $\{f \in US : f^{\vee}(\operatorname{int} B) > x\}$ are disjoint neighborhoods of g and h, which proves US to be T_2. □

Before proving Theorem 4.3(d), we examine first the basic open sets in $\mathfrak{F}$, intersections of finitely many sets in the subbase (3.8). If $\mathfrak{B}$ is closed for finite unions, then they have the form:

$$\{F \in \mathfrak{F} : F \cap B = \varnothing, F \cap G_j \neq \varnothing \text{ for } j = 1, 2, ..., n\} \tag{4.4}$$

with $n \in \mathbb{N}_0$, $B \in \mathfrak{B}$ and $G_j \in \mathfrak{G}$ for $j = 1, 2, ..., n$. In particular, we must know when a set as in (4.4) is not empty.

4.5. Lemma. *The set in (4.4) is empty iff $G_j \subset \operatorname{sat} B$ for some j.*

Proof. The set in (4.4) is empty iff for each open $H \supset B$ ($H := E \setminus F$), there is a G_j with $H \supset G_j$. The latter holds if $G_j \subset \operatorname{sat} B$ for some j. If, on the other hand, $G_j \subset \operatorname{sat} B$ for no j, then there is for each j an open $H_j \supset B$ such that $H_j \not\supset G_j$. Consequently, $H := \bigcap_{j=1}^{n} H_j \supset B$ and $H \supset G_j$ for no j. □

Proof of Theorem 4.3(d). It is no restriction to assume $\mathfrak{B}$ closed for finite unions. For if $\mathfrak{C}$ is the collection of finite unions in $\mathfrak{B}$, then $\mathfrak{C}^{\text{sat}}$ is the collection of finite unions in $\mathfrak{B}^{\text{sat}}$, and if E is internally $\mathfrak{C}^{\text{sat}}$, then E is also internally $\mathfrak{B}^{\text{sat}}$. So let $\mathfrak{B}$ be closed for finite unions and let $\mathfrak{F}$ be T_2. Fix $t \in E$. Then $\varnothing$ and $\operatorname{clos} t$ have disjoint neighborhoods in $\mathfrak{F}$. By (4.4) their form is

$$\begin{aligned} U &= \{F \in \mathfrak{F} : F \cap B_1 = \varnothing\}, \\ V &= \{F \in \mathfrak{F} : F \cap B_2 = \varnothing, F \cap G_j \neq \varnothing \text{ for } j = 1, 2, ..., n\} \end{aligned}$$

with $t \in G_j$ for $j = 1, 2, ..., n$ (note that $G_j \cap \operatorname{clos} t \neq \varnothing$ iff $t \in G_j$). By Lemma 4.5 we have $U \cap V = \varnothing$ iff $G_j \subset \operatorname{sat}(B_1 \cup B_2)$ for some j. So there is a j with $t \in G_j \subset \operatorname{sat}(B_1 \cup B_2)$, which proves E to be internally B^{sat}. □

4.6. Literature. Special cases of Theorem 4.1 occur in Klein & Thompson (1984). The sup narrow topology in $\mathfrak{F}$ is more 'hereditary' in its properties (cf. Property 4.4(b)). This could explain why this topology received almost exclusive attention from set topologists. The fact that $\mathfrak{F}$ is sup vaguely qcompact has been proved by many authors, e.g. Choquet (1948), Fell (1962), Klein & Thompson (1984), and Matheron (1975) for E locally compact with countable base. Sup vague qcompactness of US has been proved by Buttazzo (1977) for E having countable base and Salinetti & Wets (1981) for $E = \mathbb{R}^d$. It also follows from Th.II.4.7 and Th.III.1.10 of Gierz et al. (1980). The last part of Corollary 4.4(a) has been proved also by Dixmier (1968) and Gierz et al. (1980). The latter reference contains also an extension of the equivalence to non-Hausdorff spaces: If E is *sober* (cf. Section 9), then US and $\mathfrak{F}$ are sup vaguely Hausdorff iff E is locally qcompact.

5. Thinning the Convergence Criteria to Bases

The object of the present section is to thin out characterization (3.3) of the sup $\mathfrak{B}$ topology on SM (and thus also on US and $\mathfrak{F}$) to equivalent characterizations with $\mathfrak{B}$ and $\mathfrak{G}$ replaced by subclasses $\mathfrak{B}_0$ and $\mathfrak{G}_0$. The $\mathfrak{G}$ part is easy.

5.1. Lemma. *Let $\mathfrak{G}_0$ be a base of $\mathfrak{G}$. If the evaluation*

$$SM \ni m \mapsto m(G) \in \mathbb{I}$$

is lsc for $G \in \mathcal{G}_0$, then also for $G \in \mathcal{G}$.

PROOF. If $G \in \mathcal{G}$, then $G = \bigcup_{j \in J} G_j$ for some collection $\{G_j\}_{j \in J} \subset \mathcal{G}_0$, so $m \mapsto m(G) = \bigvee_{j \in J} m(G_j)$ is lsc as supremum of lsc functions. □

The $\mathcal{B}$ part is more demanding, and will in fact be handled only in case $\mathcal{B} = \mathcal{K}$.

5.2. LEMMA. *Let E be locally $\mathcal{K}_0$ with $\mathcal{K}_0 \subset \mathcal{K}$. If*

$$SM \ni m \mapsto m(K) \in \mathbb{I}$$

is usc for $K \in \mathcal{K}_0$, then also for $K \in \mathcal{K}$.

PROOF. We may and will assume $\mathcal{K}_0$ to be closed for finite unions (cf. Remark 3.2(e1)). We will show that

$$m(K) = \bigwedge_{Q \in \mathcal{K}_0 : Q \supset K} m(Q) \quad \text{for } K \in \mathcal{K}, \tag{5.1}$$

from which it follows that $m \mapsto m(K)$ is usc as infimum of usc functions. Trivially we have $\leqslant$ instead of $=$ in (5.1). To prove the reverse inequality, let $m(K) < x$. We will show that $m(Q) \leqslant x$ for some $Q \in \mathcal{K}_0$ with $Q \supset K$. We have $K \subset G := (d^\vee m)^\leftarrow[0,x)$, which is open by Lemma 2.2 and Theorem 1.2. By Property 3.7(b) we have $G = \bigcup_{j \in J} \operatorname{int} K_j = \bigcup_{j \in J} K_j$ for some collection $\{K_j\}_{j \in J} \subset \mathcal{K}_0$. As $K \in \mathcal{K}$ and $K \subset G$, there is a finite subset $J_\#$ of J such that

$$K \subset \bigcup_{j \in J_\#} \operatorname{int} K_j \subset \bigcup_{j \in J_\#} K_j =: Q \subset G.$$

Now $Q \in \mathcal{K}_0$, $Q \supset K$ and $m(Q) \leqslant m(G) \leqslant x$. □

5.3. THEOREM. *If $\mathcal{G}_0$ is a base of $\mathcal{G}$, $\mathcal{K}_0 \subset \mathcal{K}$ and E is locally $\mathcal{K}_0$, then the sup vague (= sup $\mathcal{K}$) topology on SM is the smallest that makes the evaluations*

$$m \mapsto m(A) \quad \textit{usc for } A \in \mathcal{K}_0, \quad \textit{lsc for } A \in \mathcal{G}_0.$$

PROOF. Combine Lemmas 5.1 and 5.2. □

5.4. LEMMA. *If $\mathcal{G}_0$ is a countable base of $\mathcal{G}$ and E is locally $\mathcal{K}$, then E is locally $\mathcal{K}_0$ for some countable $\mathcal{K}_0 \subset \mathcal{K}$.*

PROOF. For all $G_1, G_2 \in \mathcal{G}_0$ such that there is at least one $K \in \mathcal{K}$ with $G_1 \subset \operatorname{int} K \subset K \subset G_2$, select one $K(G_1, G_2) \in \mathcal{K}$. Set $\mathcal{K}_0 := \{K(G_1, G_2)\}$. □

5.5. THEOREM. *If E is locally qcompact, then SM, US or $\mathcal{F}$ are sup vaguely metrizable iff E has a countable base.*

5.6. REMARK. If E is locally qcompact, then SM, US and $\mathcal{F}$ are sup vaguely compact by Corollary 4.4(a), so metrizable iff they have a countable base.

PROOF OF THEOREM 5.5. By Remark 5.6 we must show that SM, US or $\mathcal{F}$ has a countable base iff E has one. If E has a countable base $\mathcal{G}_0$, then E is locally $\mathcal{K}_0$ for some countable $\mathcal{K}_0 \subset \mathcal{K}$ by Lemma 5.4, and a countable subbase of the sup vague topology on SM is given by the subbase in (3.4) with $B \in \mathcal{K}_0$, $G \in \mathcal{G}_0$ and $x \in \mathbb{I} \cap \mathbb{Q}$. So SM has a countable base, as does US and its subspace $\mathcal{F}$.

If $\mathcal{F}$ has a countable base $\mathcal{U}$ of the sup vague topology, then it has also a countable base $\mathcal{V}$

consisting of finite intersections of subbase sets as in (3.8) (select one such set between each pair of $U_1 \subset U_2$ in $\mathcal{U}$). Let τ be the coarser topology in $\mathcal{F}$ with subbase consisting of $\varnothing$ and $\{F \in \mathcal{F}: F \cap G \neq \varnothing\}$ for $G \in \mathcal{G}$. Then each $\{F \in \mathcal{F}: F \cap G \neq \varnothing\}$ is union of elements of $\mathcal{V}$, but does not have any $\{F \in \mathcal{F}: F \cap K = \varnothing\}$ as subset, since it does not contain $\varnothing \in \{F \in \mathcal{F}: F \cap K = \varnothing\}$. So $\mathcal{W}$ consisting of all elements of $\mathcal{V}$ that are finite intersections of sets $\{F \in \mathcal{F}: F \cap G \neq \varnothing\}$ is a countable base for τ. Let c be the map

$$E \ni t \mapsto \operatorname{clos} t \in \mathcal{F}.$$

Then $c(t) = c(u)$ iff no open set in E separates t and u. Identifying such points we make E a T_0 space, and c an injection. Furthermore, c is bicontinuous if $\mathcal{F}$ is provided with the topology τ:

$$\{t \in E: c(t) \cap G \neq \varnothing\} = G$$

for $G \in \mathcal{G}$. So E is a subspace of $(\mathcal{F}, \tau)$ after identification via c. As $(\mathcal{F}, \tau)$ has a countable base, E does. □

We now investigate how we can select the subclasses $\mathcal{K}_0$ of $\mathcal{K}$ as in Theorem 5.3 or Lemma 5.4 under more specific assumptions. Note that $\operatorname{sat} K$ is qcompact if K is.

5.7 Example. If E is locally compact (thus Hausdorff), then with a base $\mathcal{G}_0$ we can choose $\mathcal{K}_0 := \{\operatorname{clos} G: G \in \mathcal{G}_0\} \cap \mathcal{K}$.

5.8 Example. If E is locally compact with countable base, then E is metrizable, say by d. Set for $t \in E$ and $r \in (0, \infty)$:

$$\begin{aligned} B(t,r) &:= \{u \in E: d(t,u) < r\}, \\ B(t,r+) &:= \{u \in E: d(t,u) \leq r\}. \end{aligned} \tag{5.2}$$

Let D be a countable dense subset of E. Then a countable base $\mathcal{G}_0$ and a countable $\mathcal{K}_0 \subset \mathcal{K}$ such that E is locally $\mathcal{K}_0$ are given by

$$\begin{aligned} \mathcal{G}_0 &:= \{B(t,r): t \in D,\ r \in \mathbb{Q} \cap (0,\infty),\ B(t,r+) \in \mathcal{K}\}, \\ \mathcal{K}_0 &:= \{B(t,r+): t \in D,\ r \in \mathbb{Q} \cap (0,\infty),\ B(t,r+) \in \mathcal{K}\}. \end{aligned} \tag{5.3}$$

Note that for fixed t we have $B(t,r+)$ compact for all sufficiently small r (not for all r: consider $E = (0,1)$ with the usual metric and topology). One can metrize the same topology in such a way that all $B(t,r+)$ are compact (cf. Vaughan (1937)). Our present choice of $\mathcal{K}_0$ with $\mathcal{G}_0$ does *not* follow the recipe of Example 5.7, as $\operatorname{clos} B(t,r) = B(t,r+)$ need not hold in general (consider $r = 1$ in a discrete E with $d(t,u) = 1$ for $t \neq u$).

5.9. Literature. The combination of Theorem 5.3 with Example 5.8 for $E = \mathbb{R}^d$ has been proved by Salinetti & Wets (1981). For a completely different approach (cf. lines following Example 10.2), see Norberg (1984, 1986). The 'if' part of Theorem 5.5 has been proved by Dixmier (1968) and Matheron (1975).

6. Examples and Further Properties

The following examples exhibit some properties of the sup $\mathcal{K}$ and $\mathcal{F}$ topologies in $US(\mathbb{R})$.

6.1. Examples. $E = \mathbb{R}$, $n = 1, 2, \ldots$, $f_n \in US(\mathbb{R})$.
(a) $f_n := 1_{\{n\}}$. Then $f_n \to 0_{\mathbb{R}}$ sup $\mathcal{K}$, but (f_n) does not converge sup $\mathcal{F}$.
(b) $f_n(t) := ½ + ½\cdot(-1)^n \cos n^{½} t$. Then $f_n \to 1_{\mathbb{R}}$ sup $\mathcal{K}$ and $\mathcal{F}$, whereas $(f_n(t))$ does not converge in $\mathbb{I}$ for any t.

(c) $f_n := 1_{\{1/n\}}$. Then $f_n \to 1_{\{0\}}$ sup $\mathcal{K}$ and $\mathcal{F}$, whereas $f_n \to 0_{\mathbf{R}}$ pointwise.
(d) $f_n := 1_{\{1/n\}}$ for even n, $1_{\{1-1/n\}}$ for odd n. Then (f_n) does not converge sup $\mathcal{K}$ or $\mathcal{F}$, whereas $f_n \to 0_{\mathbf{R}}$ pointwise.
(e) $f_n := 1_{(-\infty,1/n]} + 1_{[2/n,\infty)}$. Then $f_n \to 1_{\mathbf{R}}$ sup $\mathcal{K}$, sup $\mathcal{F}$ and pointwise.

We now show that in many instances monotone nets in SM and US converge.

6.2. **Theorem.** *(a) If (m_n) is an increasing net in SM and $m(G) := \bigvee_n m_n(G)$ for $G \in \mathcal{G}$, then $m \in SM$ and $m_n \to m$ sup $\mathcal{B}$ for any bounding class $\mathcal{B}$.*
(b) If (f_n) is a decreasing net in US with pointwise infimum f, then $f \in US$, $f_n \to f$ sup $\mathcal{K}$ and $f_n^\vee(K) \to f^\vee(K)$ for $K \in \mathcal{K}$.

Proof. (a) Obviously, SM is closed for arbitrary suprema by (2.2), and $\liminf_n m_n(G) = \lim_n m_n(G) = m(G)$ for $G \in \mathcal{G}$. Furthermore, $m_n(B) \leqslant m(B)$, so $\limsup_n m_n(B) \leqslant m(B)$ for $B \in \mathcal{B}$.
(b) US is closed for arbitrary infima (Corollary 1.3(b)), so $f \in US$. Let $K \in \mathcal{K}$. Since $f_n^\vee(K)$ is nonincreasing in n and $f_n^\vee(K) \geqslant f^\vee(K)$, we have $\lim_n f_n^\vee(K) \geqslant f^\vee(K)$. If $\lim_n f_n^\vee(K) > x > f^\vee(K)$ for some $x \in \mathbb{I}$, then the nonempty qcompact sets $K \cap f_n^{\leftarrow}[x, 1]$ would decrease to the empty set $K \cap f^{\leftarrow}[x, 1]$, which is impossible. So $f_n^\vee(K) \to f^\vee(K)$. Trivially, $\lim_n f_n^\vee(G) \geqslant f^\vee(G)$ for $G \in \mathcal{G}$, as $f_n^\vee(G) \geqslant f^\vee(G)$, so $f_n \to f$ sup $\mathcal{K}$. □

6.3. **Corollary.** If $K_n \downarrow K$ in $\mathcal{K} \cap \mathcal{F}$ and $m \in SM$, then $m(K_n) \downarrow m(K)$.

Proof. Apply Theorem 6.2(b) to $f_n := 1_{K_n} d^\vee m$ with K_1 instead of the K in Theorem 6.2(b). □

Even for nonmonotone nets the convergence $f_n^\vee(K) \to f^\vee(K)$ in Theorem 6.2(b) is interesting.

6.4. **Theorem.** *Let $f_n, f \in US$. Then the following statements are equivalent.*
(i) $f_n \to f$ sup $\mathcal{K}$ and pointwise;
(ii) $f_n^\vee(K) \to f^\vee(K)$ for $K \in \mathcal{K}$.

Proof. (i) $\Rightarrow$ (ii). Let $K \in \mathcal{K}$. By (3.7) and Corollary 1.3(c) we have for some $t_K \in K$:

$$\limsup_n f_n^\vee(K) \leqslant f^\vee(K) = f(t_K) = \lim_n f_n(t_K) \leqslant \liminf_n f_n^\vee(K).$$

(ii) $\Rightarrow$ (i). Choosing $K = \{t\}$ we obtain pointwise convergence. For $G \in \mathcal{G}$ we have

$$\begin{aligned}\liminf_n f_n^\vee(G) &= \liminf_n \bigvee_{t\in G} f_n(t) \geqslant \bigvee_{t\in G} \liminf_n f_n(t) \\ &= \bigvee_{t\in G} \lim_n f_n(t) = \bigvee_{t\in G} f(t) = f^\vee(G).\end{aligned}$$

Together with the hypothesis this implies $f_n \to f$ sup $\mathcal{K}$ by (3.7). □

6.5. **Remark.** One can prove that $f_n \to f$ sup $\mathcal{K}$ and pointwise iff $f_n \to f$ locally uniformly in the semimetric $d(x,y) := (x-y)^+$ on $\mathbb{I}\!\uparrow$, i.e., iff $d(f(t), f_n(t)) \to 0$ uniformly on qcompact sets in E. See Section 8 for the definition of 'semimetric'.

6.6. **Literature.** The results in Remark 6.5 and related relative compactness criteria have been obtained by Salinetti & Wets (1979), Dolecki et al. (1983) and Beer (1982).

7. Hypo Topologies

Here is a diagram of one-to-one maps that we have found in Sections 1, 2 and 3. Horizontal arrows denote surjections, vertical arrows injections; $SM = SM(E)$ is the lattice of sup measures on $\mathcal{G}(E)$, $US = US(E)$ the lattice of usc functions on E, 'ind' the indicator map $\mathcal{F}(E) \ni F \mapsto 1_F$ and 'id' is the identity map. All maps are in fact lattice isomorphisms, since they are order preserving.

$$\begin{tikzcd}
SM(E) \arrow[r, shift left, "d^\vee"] & US(E) \arrow[l, shift left, "i^\vee"] \arrow[r, shift left, "\mathrm{hypo}"] & \mathcal{F}(E \times \mathbb{I}'\!\uparrow) \arrow[l, shift left] \arrow[d, "\mathrm{id}"] \\
 & \mathcal{F}(E) \arrow[u, "\mathrm{ind}"] & \mathcal{F}(E \times \mathbb{I}')
\end{tikzcd}$$

In Sections 3 and 4 we considered topologies on the different spaces in relation with the maps $d^\vee$, $i^\vee$ and 'ind'. In the present section we will concentrate on relations with the maps 'hypo' and 'id'.

Set $E^* := E \times \mathbb{I}'\!\uparrow$. Each class $\mathfrak{B}^*$ of subsets of E^* determines a sup $\mathfrak{B}^*$ topology on $\mathcal{F}(E)$, by Definitions 3.1 and 3.3. We carry this over to a topology on $US(E)$ by 'hypo'.

7.1. Definition. *The* ***hypo*** $\mathfrak{B}^*$ ***topology*** *is the topology on* $US(E)$ *that makes* $US(E)$ *homeomorphic to* $\mathcal{F}(E \times \mathbb{I}'\!\uparrow)$ *with the sup* $\mathfrak{B}^*$ *topology via* hypo.

Set $\mathcal{K}^*(E) := \mathcal{K}(E^*)$. Then the sup vague topology on $\mathcal{F}(E^*)$ is the sup $\mathcal{K}^*$ topology (cf. Remark 3.2(b)). Let us call the hypo $\mathcal{K}^*$ topology on $US(E)$ the *hypo vague topology*. The following is a very convenient property that justifies our preference for vague topologies. Note that there is no condition at all on the underlying topological space E.

7.2. Theorem. *The sup vague and hypo vague topologies on* $US(E)$ *coincide.*

Proof. Recall that all elements of $\mathcal{F}(E^*)$ have the form hypo$f$ for some $f \in US(E)$, by Theorem 1.6. For $G \in \mathcal{G}(E)$, $x \in [0, 1)$ and $f : E \to \mathbb{I}$ we have

$$f^\vee(G) > x \;\Leftrightarrow\; \mathrm{hypo} f \cap (G \times (x, 1]) \neq \varnothing. \tag{7.1a}$$

For $K \in \mathcal{K}(E)$, $x \in (0, 1]$ and $f \in US(E)$ we have by Corollary 1.3(a)

$$f^\vee(K) < x \;\Leftrightarrow\; \mathrm{hypo} f \cap (K \times [x, 1]) = \varnothing. \tag{7.1b}$$

The $f \in US(E)$ satisfying the left-hand sides of (7.1) form a subbase of the sup vague topology on $US(E)$. Note that $G \times (x, 1] \in \mathcal{G}^* := \mathcal{G}(E^*)$ and that $K \times [x, 1] \in \mathcal{K}^*$ (cf. lines preceding Definition 1.1 with $A\!\uparrow$ instead of $A\!\downarrow$). Consequently, the sets hypo$f \in \mathcal{F}(E^*)$ satisfying the right-hand sides of (7.1) are open as subbase sets of the sup vague topology on $\mathcal{F}(E^*)$. We have shown that hypo$^\leftarrow$ is continuous.

The remainder of this proof serves to show that also 'hypo' is continuous. So we must show that

$$\{f \in US(E) \colon \mathrm{hypo} f \cap G^* \neq \varnothing\} \quad \text{for } G^* \in \mathcal{G}^* \tag{7.2a}$$

and

$$\{f \in US(E) \colon \mathrm{hypo} f \cap K^* = \varnothing\} \quad \text{for } K^* \in \mathcal{K}^* \tag{7.2b}$$

are open subsets of $US(E)$. By Lemma 5.1 applied to $\mathcal{F}(E^*)$ as subspace of $US(E^*) \simeq SM(E^*)$ we need to show (7.2a) to be open only for G^* varying through a base $\mathcal{G}_0^*$ of $\mathcal{G}^*$. Such a base are the open rectangles $G \times (x, 1]$ as in (7.1a), and (7.1a) gives us what we need.

We now consider (7.2b). Let $\pi_1 \colon (t, x) \mapsto t$ be the projection in E^* onto the first component. Set for $n = 1, 2, \ldots$

$$K_n^* \;:=\; \bigcup_{k=1}^{2^n} \left[\pi_1(K^* \cap (E \times (0, k2^{-n}])) \times [(k-1)2^{-n}, 1]\right].$$

Then $K_n^* \supset K^*$, so $\operatorname{hypo} f \cap K^* = \varnothing$ if $\operatorname{hypo} f \cap K_n^* = \varnothing$ for some n. Conversely, if $\operatorname{hypo} f \cap K_n^* = \varnothing$ for all n, then there are $t_n \in \pi_1 K^*$ and $(t_n, x_n) \in K^*$ such that $x_n - f(t_n) \leqslant 2^{-n}$. Since $\pi_1 K^*$ and K^* are qcompact, we arrive after passing to subsequences at the situation $t_n \to t_0$ in $\pi_1 K^*$ and $(t_n, x_n) \to (t_0, x_0)$ in K^*. The latter convergence implies $x_n \to x_0$ in $\mathbb{I}'\!\uparrow$, i.e., $\liminf x_n \geqslant x_0$ in $\mathbb{I}'$. Since f is usc, it follows that

$$f(t_0) \geqslant \limsup f(t_n) \geqslant \limsup (x_n - 2^{-n}) \geqslant x_0,$$

while $(t_0, x_0) \in K^*$. So $\operatorname{hypo} f \cap K^* \neq \varnothing$. We have proved

$$\{f \in US(E)\colon \operatorname{hypo} f \cap K^* = \varnothing\} \;=\; \bigcup_{n=1}^{\infty} \{f \in US(E)\colon \operatorname{hypo} f \cap K_n^* = \varnothing\}$$

$$=\; \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{2^n} \{f \in US(E)\colon f^\vee(\pi_1(K^* \cap (E \times (0, k2^{-n}])) < (k-1)2^{-n}\}.$$

The right-hand side is a union of open sets, since $f^\vee$ is applied to qcompact sets. For the last observation, note that a continuous function π_1 is applied to a qcompact set, an intersection of the qcompact set K^* with the closed set $E \times (0, k2^{-n}]$ in E^*. □

We now consider the last (vertical) arrow 'id' in the diagram at the beginning of this section. From the next theorem it follows that it is a homeomorphism if the spaces on its both sides are provided with the sup vague topology.

7.3. Theorem. (a) *$\mathcal{F}(E \times \mathbb{I}'\!\uparrow)$ with the sup vague topology is a subspace of $\mathcal{F}(E \times \mathbb{I}')$ with the sup vague topology.*
(b) *If E is locally qcompact, then this subspace is closed.*

Before proving the theorem we introduce some convenient notation and a lemma. For $x \in \mathbb{I}'$, set $\uparrow\! x := [x, 1]$. For $C \subset \mathbb{I}'$, set $\uparrow\! C := \bigcup_{x \in C} \uparrow\! x$. For $A \subset E \times \mathbb{I}'$, set $\uparrow\! A := \bigcup_{(t,x) \in A} \{t\} \times \uparrow\! x$. Note that $\uparrow\! A \subset \operatorname{sat} A$ in $E \times \mathbb{I}'\!\uparrow$, so that for $F^* \in \mathcal{F}(E \times \mathbb{I}'\!\uparrow)$ we have

$$F^* \cap A = \varnothing \quad\Leftrightarrow\quad F^* \cap \uparrow\! A = \varnothing \quad\Leftrightarrow\quad F^* \cap \operatorname{sat} A = \varnothing \tag{7.3}$$

(cf. lines following Definition 1.7). In general, saturations of qcompact sets are qcompact. Consequently, $\uparrow\! K^* \in \mathcal{K}(E \times \mathbb{I}'\!\uparrow)$ if $K^* \in \mathcal{K}(E \times \mathbb{I}'\!\uparrow)$, but we can say more.

7.4. Lemma. *If $K^* \in \mathcal{K}(E \times \mathbb{I}'\!\uparrow)$, then $\uparrow\! K^* \in \mathcal{K}(E \times \mathbb{I}')$.*

Proof. Let $\mathcal{H}$ be the base of $\mathcal{G}(E \times \mathbb{I}')$ consisting of rectangles $H = G \times I$ with $G \in \mathcal{G}(E)$ and I an open interval in $\mathbb{I}'$. Then $\uparrow\! H = G \times \uparrow\! I \in \mathcal{G}(E \times \mathbb{I}'\!\uparrow)$. Suppose $\uparrow\! K^* \subset \bigcup_{j \in J} H_j$ with $H_j \in \mathcal{H}$. Then also $\uparrow\! K^* \subset \bigcup_{j \in J} \uparrow\! H_j$. Since $\uparrow\! K^* \in \mathcal{K}(E \times \mathbb{I}'\!\uparrow)$, there is a finite $J_\# \subset J$ such that $\uparrow\! K^* \subset \bigcup_{j \in J_*} \uparrow\! H_j$. Let π_1 and π_2 be the projections on the first and second component of $E \times \mathbb{I}'$, and define $\downarrow\! H$ starting from $\downarrow\! x := (0, x]$ for $x \in \mathbb{I}'$. Then $\pi_1 \uparrow\! K^* = \pi_1 K^* \in \mathcal{K}(E)$, and $\pi_2 \uparrow\! K^*$ has the form $[x, 1]$, so belongs to $\mathcal{K}(\mathbb{I}')$. Consequently, $\pi_1 \uparrow\! K^* \times \pi_2 \uparrow\! K^*$ belongs to $\mathcal{K}(E \times \mathbb{I}')$, and so does

$$(\pi_1 \uparrow\! K^* \times \pi_2 \uparrow\! K^*) \setminus \bigcup_{j \in J_*} \downarrow\! H_j \;=\; \uparrow\! K^* \setminus \bigcup_{j \in J_*} H_j.$$

Consequently, a finite subcollection of $(H_j)_{j \in J}$ covers the right-hand side, so a finite subcollection covers $\uparrow\! K^*$. □

Proof of Theorem 7.3 (a) First of all, the topology of $E \times \mathbb{I}'\!\uparrow$ is coarser than that of $E \times \mathbb{I}'$, so $\mathcal{F}(E \times \mathbb{I}'\!\uparrow) \subset \mathcal{F}(E \times \mathbb{I}')$. The subbase open sets of $\mathcal{F}(E \times \mathbb{I}'\!\uparrow)$ are $\{F^*\colon F^* \cap G^* \neq \varnothing\}$ and $\{F^*;\, F^* \cap K^* = \varnothing\}$ for $G^* \in \mathcal{G}(E \times \mathbb{I}'\!\uparrow)$ and $K^* \in \mathcal{K}(E \times \mathbb{I}'\!\uparrow)$. Recalling that $\mathcal{G}(E \times \mathbb{I}'\!\uparrow) \subset$

$\mathcal{G}(E \times \mathbb{I}')$ we identify $\{F^*: F^* \cap G^* \neq \varnothing\}$ as the trace in $\mathcal{F}(E \times \mathbb{I}'\uparrow)$ of a subbase open subset of $\mathcal{F}(E \times \mathbb{I}')$. By (7.3) we have

$$\{F^*: F^* \cap K^* = \varnothing\} \quad = \quad \{F^*: F^* \cap \uparrow K^* = \varnothing\},$$

and by Lemma 7.4 we can identify the right hand side as the trace in $\mathcal{F}(E \times \mathbb{I}'\uparrow)$ of a subbase open subset of $\mathcal{F}(E \times \mathbb{I}')$. We have proved (a).
(b) For general E, $\mathcal{F}(E \times \mathbb{I}'\uparrow)$ is sup vaguely qcompact by Corollary 4.4(a). If E is locally qcompact, then so is $E \times \mathbb{I}'$, so $\mathcal{F}(E \times \mathbb{I}')$ is Hausdorff by Theorem 4.3(b). In this case the qcompact subset $\mathcal{F}(E \times \mathbb{I}'\uparrow)$ is closed. □

7.5 LITERATURE. BUTTAZZO (1977, Prop. (1.12)) proved that $US(E)$ is sup vaguely homeomorphic to a subset of $\mathcal{F}(E \times \mathbb{I}')$. Theorem 7.2 is a consequence of a general representation theorem in GIERZ ET AL. (1980) that identifies certain continuous lattices as $\mathcal{F}(E^*)$ with E^* the set of primes of the lattice in question.

8. NON-HAUSDORFF LOCALLY QCOMPACT SPACES

The next two sections can be skipped by readers who are not interested in non-Hausdorff spaces. In the present section all material is concentrated that may be relevant for readers who want to restrict their considerations of non-Hausdorffness to $E \times \mathbb{I}'\uparrow$ with E Hausdorff.

Let us consider the diagram at the beginning of Section 7. From the theorems in Sections 4 and 7 we know that things are particularly nice if E is locally qcompact, but not necessarily Hausdorff, and all spaces are endowed with the sup vague topology. Then all spaces are compact, and all arrows are homeomorphisms (into when vertical). If, in addition, E is Hausdorff, or more specially, metric, then Examples 5.7 and 5.8 indicate convenient choices of subcollections $\mathcal{K}_0$ of $\mathcal{K}$ for defining smaller subbases of the sup vague topologies on the spaces $SM(E)$, $US(E)$ and $\mathcal{F}(E)$ (cf. Theorem 5.3).

In this section we explore what remains of this when E is not Hausdorff. This is useful, because we want to be able to consider also $E^* := E \times \mathbb{I}'\uparrow$, which is not Hausdorff, but is locally qcompact if E is. The following examples are instructive.

8.1 EXAMPLE. Let $E = \mathbb{R}\downarrow$ as in Example 1.9. Nonempty $A \subset \mathbb{R}\downarrow$ are qcompact iff $\sup A \in A$. Thus $K_n := (-\infty, 0) \cup \{n\}$ is qcompact for $n = 1, 2$, but $K_1 \cap K_2 = (-\infty, 0)$ is not. We see that $\mathcal{K}(\mathbb{R}\downarrow)$ is not closed for finite intersections. Let G be open and nontrivial, so $G = (-\infty, x)$ for some $x \in \mathbb{R}$. Then G is relatively qcompact, i.e., contained in some qcompact set. There is even a smallest qcompact set K containing G, viz. $K = (-\infty, x]$. We cannot obtain K by taking closures as in Example 5.7, since $\operatorname{clos} G = \mathbb{R}$. In fact, the only closed qcompact set is $\varnothing$.

8.2 EXAMPLE. Let $E = \mathbb{Q}\downarrow$ with the relative lower topology from $\mathbb{R}\downarrow$. Again, nonempty $A \subset \mathbb{Q}\downarrow$ are qcompact iff $\sup A \in A$. The generic open set is $(-\infty, x) \cap \mathbb{Q}$ with $x \in \mathbb{R}$. Now $G := (-\infty, \pi) \cap \mathbb{Q}$ is relatively qcompact since $G \subset (-\infty, q] \cap \mathbb{Q}$ for $q > \pi$, $q \in \mathbb{Q}$. However, there is no smallest qcompact set containing G.

The first step to overcome these problems is considering *saturated* qcompact sets rather than qcompact sets. We write $\mathcal{Q} = \mathcal{Q}(E)$ for the collection of saturated qcompact sets, with generic element Q. It is immediate that $\operatorname{sat} K \in \mathcal{Q}$ if $K \in \mathcal{K}$.

We have $\mathcal{Q}(\mathbb{R}\downarrow) = \{\varnothing, (-\infty, x]: x \in \mathbb{R}\}$ and $\mathcal{Q}(\mathbb{Q}\downarrow) = \{\varnothing, (-\infty, q] \cap \mathbb{Q}: q \in \mathbb{Q}\}$. Note that in both cases $\mathcal{Q}$ is closed for finite intersections, but that only $\mathcal{Q}(\mathbb{R}\downarrow)$ is closed for arbitrary intersections. There are E for which $\mathcal{Q}(E)$ is not even closed for finite intersections (cf. Example 9.7(b)).

These observations lead us to the following regularity condition that we will impose on E.

8.3 DEFINITION. *The topological space is a Q_δ space if the collection $\mathcal{Q}$ of its saturated qcompact sets is* *closed for arbitrary intersections.*

Hausdorff spaces are Q_δ, and so are $\mathbb{R}\downarrow$ and $\mathbb{I}'\uparrow$, but $\mathbb{Q}\downarrow$ is not. If $A \subset E$ is relatively qcompact and E is Q_δ, then the intersection of all saturated qcompact sets containing A is the smallest such set. We will denote it by $\operatorname{sqc} A$, the *saturated compactification* of A. For Hausdorff E we have $\operatorname{sqc} A = \operatorname{clos} A$ for relatively compact subsets A. For non-Hausdorff spaces which are Q_δ, 'sqc' takes over the role of 'clos'. We now can generalize Example 5.7 to

8.4. Example. If E is locally qcompact and Q_δ, then with a base $\mathcal{G}_0$ we can choose

$$\mathcal{K}_0 := \{\operatorname{sqc} G \colon G \in \mathcal{G}_0,\ G \text{ relatively qcompact}\}$$

in Theorem 5.3

It would be nice to generalize Example 5.8 as well to non-Hausdorff spaces. The only way to do this is by generalizing the notion of 'metric', since all metric spaces are Hausdorff. Here are some partial results.

8.5 Definition. *A **semimetric** on E is a map $d\colon E \times E \to [0,\infty)$ such that $d(t,t) = 0$ for $t \in E$ and satisfying the **triangle inequality***

$$d(t,v) \leqslant d(t,u) + d(u,v) \quad \text{for } t,\ u,\ v \in E\,.$$

Note that we do not require $d(t,u) = d(u,t)$. We define the balls $B(t,r)$ and $B(t,r+)$ for semimetrics as in (5.2). As for metrics, one proves that the balls $B(t,r)$ form a base of a topology, by definition the topology *generated* or *semimetrized* by d. For example, $\mathbb{R}\downarrow$ (Example 8.1) is semimetrized by $d(t,u) := (u-t)^+$, and more generally, $(\mathbb{R}\downarrow)^n$ by $d(t,u) := \bigvee_{k=1}^n (u_k - t_k)^+$.

In general we have for semimetric E

(a) $\operatorname{sat} t = \{u \in E \colon d(t,u) = 0\}$,

(b) $\operatorname{clos} u = \{t \in E \colon d(t,u) = 0\}$,

(c) the net $(t_n)_n$ converges to t in E iff $d(t,t_n) \to 0$.

Note that (a) and (b) express the more general equivalence $u \in \operatorname{sat} t \Leftrightarrow t \in \operatorname{clos} u$.

8.6. Theorem. *If E has a countable base, then E is semimetrizable.*

Proof. Let $G_1, G_2, \ldots$ be a base for E. Define for $t, t \in E$

$$d(t,u) := \textstyle\sum_{n=1}^\infty 1_{G_n}(t)\, 1_{G_n^c}(u)\, 2^{-n}.$$

One easily checks that d is a semimetric. We now show that d generates the same topology as $G_1, G_2, \ldots$. If $t \in G_n$, then $\{s \colon d(t,s) < 2^{-n}\} \subset G_n$, so G_n is d-open. On the other hand, with N such that $\sum_{n>N} 2^{-n} < \epsilon$ we find

$$t \in \bigcap_{\{n \leqslant N : t \in G_n\}} G_n \subset \{u \colon d(t,u) < \epsilon\}. \qquad \square$$

Recall that the balls $B(t,r+)$ are defined as in (5.2) for semimetrics d. In general, the balls $B(t,r+)$ need not be closed. If E is locally compact (thus also Hausdorff) and is metrized by d, then for fixed t the (then closed) balls $B(t,r+)$ are compact for all small r (cf. Example 5.7). If E is locally qcompact and semimetrized by d, then the balls $B(t,r+)$ are saturated (as intersection of the open sets $B(t,s)$ for $s > r$), but not necessarily qcompact, even for small r. However, in the Q_δ space $\mathbb{R}\downarrow$ of Example 8.1 all balls $B(t,r+) = (-\infty, t+r]$ are qcompact.

8.7 Definition. *If E is semimetrized by d, then d is said to be $\mathcal{Q}$ **compatible**, and E is said to be* *semimetrized $\mathcal{Q}$ **compatibly** by d if for each $t \in E$ we have $B(t, r+) \in \mathcal{Q}$ for all small r.*

8.8 Corollary. If E is locally qcompact and $\mathcal{Q}$ compatibly semimetrized by d, then $\mathcal{K}_0$ as in (5.3) can be substituted in Theorem 5.3.

8.9 Examples. (a) All metrics on locally compact spaces are $\mathcal{Q}$ compatible.
(b) The semimetric $d(t, u) = (u-t)^+$ on $\mathbb{R}\downarrow$ is $\mathcal{Q}$ compatible.
(c) The semimetric $d(t, u) = (u-t)^+$ on $\mathbb{Q}\downarrow$ is not $\mathcal{Q}$ compatible: $B(0, \pi+) = (-\infty, \pi) \cap \mathbb{Q}$ is not qcompact. However, there is another semimetric d' that generates the same topology and is $\mathcal{Q}$ compatible: $d'(t, u) := \varphi((u-t)^+)$, where $\varphi(0) := 0$, $\varphi(t) := 2^{-n}$ for $t \in [2^{-(n+1)}, 2^{-n})$, $n \in \mathbb{Z}$.

We do not know whether all semimetrizable locally qcompact spaces can be semimetrized $\mathcal{Q}$ compatibly. It is even hard to verify if specific spaces are $\mathcal{Q}$-compatibly semimetrizable, as for instance $(([-\infty,0) \cup [1,\infty))\downarrow)^2$. However, in many specific cases it is easy to find $\mathcal{Q}$ compatible semimetrics, and the number of such cases is extended by

8.10. Lemma. *If $E^{(1)}$ and $E^{(2)}$ are locally qcompact and $\mathcal{Q}$ compatibly semimetrizable, then so is $E := E^{(1)} \times E^{(2)}$.*

Proof. First of all, E is locally $\mathcal{K}_0$ with $\mathcal{K}_0$ the qcompact rectangles, so E is locally qcompact. If $E^{(n)}$ is $\mathcal{Q}$ compatibly semimetrized by the semimetric $d^{(n)}$ for $n = 1, 2$, then E is by the semimetric $d(t, u) := \bigvee_{n=1}^{2} d^{(n)}(t^{(n)}, u^{(n)})$. □

8.11. Corollary. If E is locally qcompact and $\mathcal{Q}$ compatibly semimetrizable, then so are $E \times \mathbb{I}'\uparrow$ and $E \times \mathbb{I}'$. In particular, the conclusion holds true if E is locally compact and metrizable.

8.12. Literature. A bit different notion of 'semimetric' occurs in Nachbin (1965).

9. More about Non-Hausdorff Spaces

First we make a fundamental observation about the spaces in the diagram of lattice isomorphisms in the beginning of Section 7. If we are given a topological space $(E, \mathcal{G})$, then $US(E)$ depends only on this space via $\mathcal{G}$. More specifically, if $(E_1, \mathcal{G}_1)$ and $(E_2, \mathcal{G}_2)$ are two topological spaces such that $\mathcal{G}_1$ and $\mathcal{G}_2$ are lattice isomorphic, then $US(E_1)$ and $US(E_2)$ are lattice isomorphic. This is obvious in the diagram on the left side since $SM(E)$ is a space of functions on $\mathcal{G}$, and on the right side since $\mathcal{F}(E \times \mathbb{I}'\uparrow)$ depends on $(E, \mathcal{G})$ only via $\mathcal{G}$. If, moreover, the bounding class $\mathcal{B}$ in the sup $\mathcal{B}$ topology on $US(E)$ depends on $(E, \mathcal{G})$ only via $\mathcal{G}$ (which is the case for $\mathcal{B} = \mathcal{F}$ but not for $\mathcal{B} = \mathcal{K}$), then $US(E)$ as a topological space depends on $(E, \mathcal{G})$ only via $\mathcal{G}$.

This makes it useful to study which topological spaces E have lattice isomorphic topologies $\mathcal{G}$. First we must get rid of a trivial complication. If two points in E are not separated by any open set, then we can identify them without affecting the lattice of open sets. By identifying all nonseparated points we make E a T_0 space. Therefore we will often assume that E is T_0.

We now start with an example. As in Example 8.2, let $\mathbb{Q}\downarrow$ be the rationals provided with the lower topology, the relative topology from $\mathbb{R}\downarrow$. Then its nontrivial open sets are given by $(-\infty, x) \cap \mathbb{Q}$ for $x \in \mathbb{R}$. We see that the topology of $\mathbb{Q}\downarrow$ is lattice isomorphic to that of $\mathbb{R}\downarrow$. Intuitively we may feel that $\mathbb{R}$ as a total space fits better in the topology than $\mathbb{Q}$. We now provide theoretical support for this feeling. At this point it is more convenient to see a topology determined by the closed sets $\mathcal{F}$ rather than the open sets $\mathcal{G}$.

9.1. Definition. *A set $F \in \mathcal{F}$ is called **prime** if $F \neq \varnothing$ and $F = F_1 \cup F_2$ with $F_1, F_2 \in \mathcal{F}$ implies $F = F_1$ or F_2.*

9.2. Remark. From the definition it follows that $\operatorname{clos} A$ $(A \subset E)$ is prime iff $A \neq \emptyset$ and $A \cap G_n \neq \emptyset$ for open G_n $(n = 1, 2)$ implies $A \cap G_1 \cap G_2 \neq \emptyset$. In particular singleton closures are prime. Moreover, in a Hausdorff space a prime closed set cannot contain two points, so the prime closed sets are just the singletons. The characterization in the first clause of this remark remains valid with G_1 and G_2 coming from a base $\mathcal{G}_0$ of $\mathcal{G}$.

9.3. Example. In $\mathbb{Q}\!\downarrow$ the prime closed sets $\mathbb{Q}$ and $[x, \infty) \cap \mathbb{Q}$ with $x \in \mathbb{R} \setminus \mathbb{Q}$ are no singleton closures. In $\mathbb{R}\!\downarrow$ the total set $\mathbb{R}$ is prime closed and no singleton closure.

The observations in the example suggest us what to do. If F is a prime closed set that is not a singleton closure, than add a new point x to E that by definition is contained in each open set that intersects F, to obtain $F = \operatorname{clos} x$. Formally one performs this by making a new topological space whose points are the primes in $\mathcal{F}(E)$. See Section 1 of Hofmann & Mislove (1981), from which we borrow the following definition and result.

9.4. Definition. *A topological space E is **sober** if it is T_0 and each prime closed set in E is a singleton closure.*

9.5. Theorem. *For each space E there is a sober space* $\operatorname{sob} E$, *unique up to homeomorphism, such that $\mathcal{G}(\operatorname{sob} E)$ is lattice isomorphic to $\mathcal{G}(E)$.*

9.6. Example. $\operatorname{sob} \mathbb{Q}\!\downarrow \simeq \operatorname{sob} \mathbb{R}\!\downarrow \simeq [-\infty, \infty)\!\downarrow$.

We call $\operatorname{sob} E$ the *sobrification* of E. The term is not very suggestive, as $\operatorname{sob} E$ is a kind of completion of E. It is the largest T_0 space with topology lattice isomorphic to $\mathcal{G}(E)$. We make a T_0 space E a topological subspace of $\operatorname{sob} E$ by identifying points whose closure complements are mapped on each other by the lattice isomorphism between the topologies. We already noticed that $US(E)$ and $US(\operatorname{sob} E)$ are lattice isomorphic, and homeomorphic with the sup weak topologies but not necessarily with the sup vague topologies.

It is hard to find examples of the latter, but Hofmann & Lawson (1978, §7) exhibit one in which every qcompact set in E has empty interior, whereas $\operatorname{sob} E$ is locally qcompact. Consequently, $US(\operatorname{sob} E)$ is sup vaguely Hausdorff by Theorem 4.3(c), whereas $US(E)$ is not, by Theorem 4.3(d).

In Examples 8.1 and 8.2 we observed that in general $\mathcal{K}$ is not closed for intersections, but that $\mathcal{Q} := \{\operatorname{sat} K : K \in \mathcal{K}\}$ is closed for intersections in some of the cases where $\mathcal{K}$ fails to be so. We called E a Q_δ space if $\mathcal{Q}$ is closed for arbitrary intersections, and found that $\mathbb{R}\!\downarrow$ is Q_δ. The following list of examples is instructive.

9.7. Examples. (a) E is countable, the open sets are empty or cofinite. Then E is T_1 but not T_2, and not sober as the total set E is prime closed. All subsets are qcompact and saturated, so E is locally $\mathcal{Q}$ and Q_δ. The sobrification of E is obtained by adding a point ∞ to each nonempty open set.

(b) $E = \mathbb{N} \cup \{\infty_1, \infty_2\}$ with as open sets all subsets of $\mathbb{N}$ and all cofinite subsets of E that intersect $\{\infty_1, \infty_2\}$. Then E is T_1 (so all subsets are saturated) but not T_2; E is sober; $A \subset E$ is qcompact iff A is finite or A intersects $\{\infty_1, \infty_2\}$. E is locally $\mathcal{Q}$, but $\mathcal{Q}$ is not closed for finite intersections: consider $Q_n := \mathbb{N} \cup \{\infty_n\}$ for $n = 1, 2$. However, $\mathcal{Q}$ is closed for intersections of decreasing nets in $\mathcal{Q}$.

(c) E is Hausdorff, but not necessarily locally compact. Then E is sober and Q_δ.

From Hofmann & Mislove (1981) and Gierz & al. (1980) we quote the following definition and results.

In a sober space, $\mathcal{Q}$ is closed for intersections of decreasing nets in $\mathcal{Q}$ (Hofmann & Mislove (1981, Prop. 2.19)). Consequently, a sober space is Q_δ iff $\mathcal{Q}$ is closed for finite intersections. A space E is called *supersober* if the set of limit points of each ultrafilter on E is either empty or a singleton closure. A T_1 space which is not T_2, is not supersober. If E is supersober, then E is sober and Q_δ (GIERZ & AL. (1980, VII-1.11)). If E is sober, Q_δ and locally $\mathcal{Q}$, then E is supersober (HOFMANN & MISLOVE (1981, Th.4.8)).

We will not prove or use these results here, but rather content ourselves with obtaining directly a collection of weaker results which serves our needs.

9.8. LEMMA. *Let E be locally $\mathcal{K}_0$ with $\mathcal{K}_0 \subset \mathcal{K}$ and such that $\mathcal{K}_0$ is closed for finite intersections.*
(a) *If $(t_\alpha)_\alpha$ is a convergent net in E and* $\operatorname{Lim} t_\alpha$ *its set of limits, then* $\operatorname{Lim} t_\alpha$ *is prime closed.*
(b) *If in addition E is sober, then E is Q_δ.*

PROOF. (a) In general, the set $\operatorname{Lim} t_\alpha$ is closed. Let G_1, G_2 be two open sets intersecting $\operatorname{Lim} t_\alpha$. We must prove that $G_1 \cap G_2 \cap \operatorname{Lim} t_\alpha \neq \varnothing$. Select $u_n \in G_n \cap \operatorname{Lim} t_\alpha$ and $K_n \in \mathcal{K}_0$ such that $u_n \in \operatorname{int} K_n \subset K_n \subset G_n$ for $n = 1, 2$. Since $u_n = \lim t_\alpha$, we have that $t_\alpha \in \operatorname{int} K_n \subset K_n$ for all sufficiently large α, so $t_\alpha \in K_1 \cap K_2$ for all sufficiently large α. Since $K_1 \cap K_2$ is qcompact, there is a $u \in K_1 \cap K_2$ with $u = \lim t_\alpha$. So $u \in K_1 \cap K_2 \cap \operatorname{Lim} t_\alpha \subset G_1 \cap G_2 \cap \operatorname{Lim} t_\alpha$.
(b) Let $Q_j \in \mathcal{Q}$ for $j \in J$ and set $Q := \bigcap_{j \in J} Q_j$. Then Q is saturated as intersection of saturated sets. It remains to show that Q is qcompact. Let (t_α) be a net in Q. Then (t_α) is a net in Q_j (for some fixed j) and Q_j is qcompact, so there is a convergent subnet with at least one of its limits in Q_j. Think (t_α) replaced by this convergent subnet. By (a) and the sobriety of E there is a $u \in E$ such that $\operatorname{Lim} t_\alpha = \operatorname{clos} u$. For all j we have that t_α is a convergent net in the qcompact set Q_j, so $Q_j \cap \operatorname{clos} u \neq \varnothing$. As Q_j is saturated, it follows that $u \in Q_j$ for all j, so $u \in Q$. We have proved that $Q \cap \operatorname{Lim} t_\alpha \neq \varnothing$, so Q is qcompact. □

We now turn to product spaces.

9.9. LEMMA. *Let $E := E^{(1)} \times E^{(2)}$ with the product topology. Then the prime closed sets in E are the rectangles with prime closed sides in $E^{(1)}$ and $E^{(2)}$.*

PROOF. Let π^n for $n = 1, 2$ be the projection in E on the nth component $E^{(n)}$. The key observations are that for open $G^{(1)}$ in $E^{(1)}$ and closed F in E we have

$$G^{(1)} \cap \pi^1 F \neq \varnothing \quad \Leftrightarrow \quad (G^{(1)} \times E^{(2)}) \cap F \neq \varnothing , \tag{9.1}$$

and that the open sets for testing primality of F as in Remark 9.2 may be the open rectangles

$$G^{(1)} \times G^{(2)} \;=\; (G^{(1)} \times E^{(2)}) \cap (E^{(1)} \times G^{(2)}) . \tag{9.2}$$

Considering (9.1) for two open sets $G^{(1)}$ we see that $\operatorname{clos} \pi^1 F$ (and similarly $\operatorname{clos} \pi^2 F$) is prime, if F is prime. If $F^{(n)}$ is prime closed in $E^{(n)}$ for $n = 1, 2$, then $F^{(1)} \times F^{(2)}$ is in E, which one verifies by intersecting $F^{(1)} \times F^{(2)}$ with open rectangles as in (9.2).

It remains to prove that $F = \operatorname{clos} \pi^1 F \times \operatorname{clos} \pi^2 F =: F^{(1)} \times F^{(2)}$ for each prime closed F in E. So suppose there is a $t \in (F^{(1)} \times F^{(2)}) \setminus F$. As F is closed, there is an open rectangle $G^{(1)} \times G^{(2)}$ containing t but not intersecting F, which contradicts the primality of F and $F \cap (G^{(1)} \times E^{(2)}) \neq \varnothing$, $F \cap (E^{(1)} \times G^{(2)}) \neq \varnothing$ (note that $\pi^1 t \in G^{(1)}$ and $\pi^1 t \in F^{(1)} = \operatorname{clos} \pi^1 F$, so $G^{(1)} \cap \pi^1 F \neq \varnothing$). □

9.10. THEOREM. *Let $E = E^{(1)} \times E^{(2)}$ with the product topology.*
(a) *If $E^{(1)}$ and $E^{(2)}$ are locally qcompact, then so is E.*
(b) *If $E^{(1)}$ and $E^{(2)}$ are sober, then so is E.*
(c) *If $E^{(1)}$ and $E^{(2)}$ are sober, locally qcompact and Q_δ, then so is E.*

PROOF. (a) E is locally $\mathcal{K}_0$ with $\mathcal{K}_0$ the qcompact rectangles.

(b) Follows from Lemma 9.9 and $\mathrm{clos}(t,u) = (\mathrm{clos}\, t) \times (\mathit{clos}\, u)$.
(c) Let $\mathcal{K}_0$ be the compact rectangles in E. Then E is locally $\mathcal{K}_0$ and $\mathcal{K}_0$ is closed for finite (even arbitrary) intersections because $E^{(1)}$ and $E^{(2)}$ are Q_δ. By (b), E is sober. So E satisfies all assumptions of Lemma 9.8(b), which proves E to be Q_δ. □

9.11. COROLLARY. If E is sober, locally qcompact and Q_δ, then so are $E \times \mathbb{I}'\uparrow$ and $E \times \mathbb{I}'$.

9.12. LITERATURE. Most results of this section can be found in HOFMANN & MISLOVE (1981) and GIERZ ET AL. (1980).

10. OTHER CRITERIA FOR CONVERGENCE

Let E be locally qcompact and $\mathcal{Q}$ compatibly semimetrized by d, which is in particular the case if E is locally compact and metrized by d. Let the balls $B(t,r)$ and $B(t,r+)$ be defined by (5.2). If $f_n \to f$ sup $\mathcal{K}$ in US, then

$$f^\vee(B(t,r)) \leqslant \liminf_n f_n^\vee(B(t,r)) \leqslant \limsup_n f_n^\vee(B(t,r+)) \leqslant f^\vee(B(t,r+)) \tag{10.1}$$

for all $t \in E$ and $r > 0$ such that $B(t,r+)$ is qcompact (which is the case for all sufficiently small r, depending on t). If

$$f^\vee(B(t,r)) = f^\vee(B(t,r+)) \tag{10.2}$$

for some t and r, then

$$f^\vee(B(t,r)) = \lim_n f_n^\vee(B(t,r)) . \tag{10.3}$$

As the function $r \mapsto f^\vee(B(t,r))$ is monotone, we have (10.2) for fixed t violated for at most countably many r. Consequently, if

$$\begin{aligned} \mathcal{G}_f &:= \{B(t,r): f^\vee(B(t,r)) = f^\vee(B(t,r+)),\ B(t,r+) \in \mathcal{K}\} , \\ \mathcal{K}_f &:= \{B(t,r+): f^\vee(B(t,r)) = f^\vee(B(t,r+)),\ B(t,r+) \in \mathcal{K}\} , \end{aligned} \tag{10.4}$$

then $\mathcal{G}_f$ is a base of $\mathcal{G}$ and E is locally $\mathcal{K}_f$. So (10.1) restricted to balls in $\mathcal{G}_f$ or $\mathcal{K}_f$ (which is (10.3) with the same restriction) implies $f_n \to f$ sup $\mathcal{K}$ by Theorem 5.3. We conclude:

10.1. THEOREM. *We have $f_n \to f$ sup vaguely in US iff $f_n^\vee(B) \to f^\vee(B)$ for all $B \in \mathcal{G}_f$ or all $B \in \mathcal{K}_f$ defined in (10.4).*

10.2. EXAMPLE. $E = \mathbb{R}$: $f_n \to f$ sup vaguely in $US(\mathbb{R})$ iff $\lim_n f_n^\vee(B) = f^\vee(B)$ for all open bounded intervals B such that $f^\vee(B) = f^\vee(\mathrm{clos}\, B)$.

A unifying approach to some of the preceding results is based on semiseparating classes as considered by NORBERG (1984, 1986). First, let E be locally compact (thus Hausdorff) with countable base. A class $\mathcal{A}$ of subsets of E is called *separating* if for all open G and compact K with $G \subset K$ there is an $A \in \mathcal{A}$ such that $G \subset A \subset K$. A class $\mathcal{A}$ is *semiseparating* if the class of finite unions of elements in $\mathcal{A}$ is separating. Examples of semiseparating classes are $\mathcal{A} = \mathcal{G}_f$ and $\mathcal{A} = \mathcal{K}_f$. NORBERG (1986) related sup vague convergence of sup measures to the limiting behavior of their values on semiseparating classes. We refer to his work for the results, and confine ourselves to indicating some connections and a possible generalization to non-Hausdorff E.

A sup measure is *inner continuous* on $\mathcal{G}$ in the sense that

$$m(G_n) \uparrow m(G) \quad \text{if } G_n \uparrow G,\ \ G_n, G \in \mathcal{G}$$

(cf. (2.2)). An inner continuous set function $m: \mathcal{G} \to \mathbb{I}$ is a sup measure iff

$$m(G_1 \cup G_2) = m(G_1) \vee m(G_2) \quad \text{for } G_1, G_2 \in \mathcal{G}.$$

A sup measure is *outer continuous* on $\mathcal{K} \cap \mathcal{F}$, i.e., Corollary6.33 holds true. This suffices for the case of Hausdorff E considered by NORBERG.

Generalization to the non-Hausdorff case is possible for locally qcompact sober Q_δ E. In this case it is necessary to consider only semiseparating classes of saturated sets that separate open and saturated qcompact sets. The role of compact closure of relatively compact sets is taken over by the sqc operation in Example 8.4. The following lemma shows that sup measures are outer continuous on the saturated qcompact sets $\mathcal{Q}$.

10.3. LEMMA. *If E is locally qcompact, sober and Q_δ, m is a sup measure and $(Q_n)_n$ is a decreasing net in $\mathcal{Q}$ with intersection Q, then $m(Q_n) \downarrow m(Q)$.*

PROOF. Obviously, $\lim_n m(Q_n) \geqslant m(Q)$. By Theorem 2.5 and Corollary 1.3(c) there is a $t_n \in Q_n$ such that $d^\vee m(t_n) = m(Q_n)$. Since the Q_n's are qcompact, there is a convergent subnet $(t_{n'})$. By Lemma 9.8(a) the set of its limits is prime closed., so has the form $\operatorname{clos} u$ for a $u \in E$, as E is sober. Since the Q_n's are saturated and $Q_n \cap \operatorname{clos} u \neq \varnothing$, we have $u \in Q_n$ for all n, so $u \in Q$. As $u = \lim_{n'} t_{n'}$, we have

$$\begin{aligned} m(Q) &\geqslant d^\vee m(u) = \textstyle\bigwedge_{G \ni u} m(G) \geqslant \limsup_{n'} d^\vee m(t_{n'}) \\ &\geqslant \limsup_{n'} m(Q_{n'}) = \lim_n m(Q_n). \end{aligned}$$

This combined with the first observation proves the lemma. □

NORBERG (1984, 1986) assumed the sets in the semiseparating classes to be Borel measurable, which becomes necessary in the context of SM- or US-valued random variables.

10.4. LITERATURE. Theorem 10.1 has been proved also by SALINETTI & WETS (1981, 1986) and NORBERG (1986).

11. MEASURABILITY, RANDOM VARIABLES AND EXTREMAL PROCESSES

Let in general $\operatorname{Bor} E$ denote the Borel field of a topological space E, the σ-field generated by $\mathcal{G}(E)$. We begin with investigating $\operatorname{Bor} SM$ and $\operatorname{Bor} \mathcal{F}$, where throughout this section SM and $\mathcal{F}$ are endowed with the sup vague topology. In general it is hard to characterize $\operatorname{Bor} SM$ further, but if SM has a countable base, then $\operatorname{Bor} SM$ is already generated by its subbase (3.4), as now each open set in SM is countable union of finite intersections of subbase elements. Now $\operatorname{Bor} SM$ can be characterized succinctly.

11.1. THEOREM. *If $\mathcal{G}(E)$ has a countable base, $\mathcal{G}_0$ is a base of $\mathcal{G}$ and E is locally $\mathcal{K}_0$ with $\mathcal{K}_0 \subset \mathcal{K}$, then $\operatorname{Bor} SM$ is the smallest σ-field that makes the evaluations $m \mapsto m(A)$ measurable for all $A \in \mathcal{G}_0$ or all $A \in \mathcal{K}_0$.*

PROOF. SM has a countable base by Theorem 5.5 and Remark 5.6. In the proofs of Lemmas 5.1 and 5.2 all J can be taken or made countable, which shows measurability of $A \mapsto m(A)$ for $A \in \mathcal{G}_0$ (or $\mathcal{K}_0$) to be equivalent to that for $A \in \mathcal{G}$ (or $\mathcal{K}$). Measurability for all $A \in \mathcal{G}$ or $\mathcal{K}$ implies measurability for all $A \in \mathcal{G} \cup \mathcal{K}$ by (3.1) with $G_n \downarrow A \in \mathcal{K}$ and Property 3.7(b) with J made countable.

11.2. DEFINITION. *An **extremal process** is an SM-valued random variable (rv). A **random usc function** is a US-valued rv. A **random closed set** is an $\mathcal{F}$-valued rv.*

11.3. Corollary. In the situation of Theorem 11.1 an extremal process is a mapping M from some probability space into SM such that $M(A)$ is an $\mathbb{I}$-valued rv for each $A \in \mathcal{G}_0$ or each $A \in \mathcal{K}_0$.

11.4. Remark. Let $\mathcal{C}$ be the smallest σ-field in SM that makes all evaluations $m \mapsto m(\{t\}) = d^{\vee} m(t)$ measurable. Then $\mathcal{C} \subset \mathrm{Bor}\, SM$, but $\mathcal{C}$ is in general strictly smaller than $\mathrm{Bor}\, SM$. To see this, set $E := [0,1]$ and let the rv ξ have a uniform distribution in E. Set $M_1 :\equiv 0$, $M_2 := 1_{\{\xi\}}$. Then M_1 and M_2 are extremal processes with different distributions on $\mathrm{Bor}\, SM$: $M_1(E) = 0$ wp1, $M_2(E) = 1$ wp1, but equal distributions on $\mathcal{C}$: $M_1(\{t\}) = M_2(\{t\}) = 0$ wp1 for each $t \in E$.

11.5. Theorem. *Let E, $\mathcal{G}_0$ and $\mathcal{K}_0$ be as in Theorem 11.1, and let M be an extremal process. Then the probability distribution of M is determined by the finite-dimensional distributions of* $(M(G))_{G \in \mathcal{G}_0}$ *or* $(M(K))_{K \in \mathcal{K}_0}$.

Proof. The family of sets $\bigcap_{j \in J} \{m \colon m(G_j) \leq x_j\}$ for finite subcollections $(G_j)_{j \in J}$ of $\mathcal{G}_0$ generates $\mathrm{Bor}\, SM$ by Theorem 11.1, and is closed for finite intersections. Apply Theorem 10.3 of Billingsley (1979). The proof for $\mathcal{K}_0$ is similar.

11.6. Remark. If M is an extremal process, then $d^{\vee} M$ is a random usc function. If X is a random usc function, then $X^{\vee}$ is an extremal process.

Let M be an extremal process. So far we have seen that $M(A)$ is a rv in $\mathbb{I}$ for $A \in \mathcal{G} \cup \mathcal{K}$. Although $M(A)$ need not be a rv for all $A \subset E$, even not for all $A \in \mathrm{Bor}\, E$ in case the σ-field in the underlying probability space does not contain all $\mathbb{P}$ nullsets, we can extend $\mathcal{G} \cup \mathcal{K}$ a bit further. Obviously, $M(\bigcup_{n=1}^{\infty} A_n) = \bigvee_{n=1}^{\infty} M(A_n)$ is a rv if each $M(A_n)$ is. So $M(A)$ is a rv for each $A \in (\mathcal{G} \cup \mathcal{K})^{\sigma}$, the family of countable unions of elements of $\mathcal{G} \cup \mathcal{K}$. If E is locally qcompact with countable base, then $\mathcal{G} \subset \mathcal{K}^{\sigma}$, so $(\mathcal{G} \cup \mathcal{K})^{\sigma} = \mathcal{K}^{\sigma}$. We have found

11.7. Theorem. *If E is locally qcompact with countable base and M is an extremal process on E, then* $M(A)$ *is a rv for each* $A \in \mathcal{K}^{\sigma}$ *(the σ-qcompact sets), in particular for open A.*

11.8. Remark. By (3.2) we have $M(A) = M(\mathrm{sat} A)$ wp1 for all $A \in \mathcal{K}^{\sigma}$ simultaneously. So we do not lose anything by restricting $\mathcal{K}^{\sigma}$ to the saturated sets in $\mathcal{K}^{\sigma}$. As $\bigcap \mathrm{sat} = \mathrm{sat} \bigcap$ (cf. §1), we have $\{A \in \mathcal{K}^{\sigma} \colon A = \mathrm{sat} A\} = \mathcal{Q}^{\sigma}$, the class of countable unions of saturated qcompact sets. In the next section it will be convenient to restrict $\mathcal{Q}^{\sigma}$ a bit further to

$$\mathcal{D} := \{A \in \mathcal{Q}^{\sigma} \colon A \subset Q \text{ for some } Q \in \mathcal{Q}\}. \tag{11.1}$$

We call $\mathcal{D}$ the *natural domain* of extremal processes.

We now turn to random closed sets (cf. Definition 11.2). They can be regarded as $\{0,1\}$-valued extremal processes or $\{0,1\}$-valued random usc functions. The previous theorems specialize to the following result.

11.9. Theorem. *Let $\mathcal{G}$ have a countable base, $\mathcal{G}_0$ be a base of $\mathcal{G}$ and E be locally $\mathcal{K}_0$ with $\mathcal{K}_0 \subset \mathcal{K}$. Then the following holds.*
(a) $\mathrm{Bor}\, \mathcal{F}$ *is the smallest σ-field that contains* $\{F \in \mathcal{F} \colon F \cap A \neq \varnothing\}$ *for all* $A \in \mathcal{G}_0$ *or all* $A \in \mathcal{K}_0$.
(b) *A random closed set is a mapping X from some probability space into $\mathcal{F}$ such that* $[X \cap A \neq \varnothing]$ *is an event for all* $A \in \mathcal{G}_0$ *or all* $A \in \mathcal{K}_0$.
(c) *If in addition $\mathcal{G}_0$ (or $\mathcal{K}_0$) is closed for finite unions, then the probability distribution of a random closed set X is determined by* $T(A) := \mathbb{P}[X \cap A \neq \varnothing]$ *for* $A \in \mathcal{G}_0$ *(or $\mathcal{K}_0$); T is called the* ***distribution function*** *of X.*

(d) *If X is a random closed set, then $[X \cap A \neq \varnothing]$ is an event for each $A \in \mathcal{K}^o$.*

PROOF. (a,b,d) Straightforward from Theorem 11.1 and Corollary 11.3.
(c) In the first instance, Theorem 11.5 translates into the distribution of X being determined by the finite-dimensional distributions of

$$\left[1_{[X \cap A \neq \varnothing]}\right]_{A \in \mathcal{G}_0} \quad \text{or} \quad \left[1_{[X \cap A \neq \varnothing]}\right]_{A \in \mathcal{K}_0}, \tag{11.2}$$

where $\mathcal{G}_0$ and $\mathcal{K}_0$ need not yet be closed for finite unions. In general, the finite-dimensional distributions of a collection of $\{0,1\}$-valued rv's $(\epsilon_j)_{j \in J}$ determine and are determined by $\mathbb{P}[\epsilon_i = 0 \text{ for } i \in I]$ for all finite $I \subset J$. So $\mathbb{P}[X \cap A = \varnothing] = 1 - \mathbb{P}[X \cap A \neq \varnothing]$ with A varying through the finite unions in $\mathcal{G}_0$ (or $\mathcal{K}_0$) determines the finite-dimensional distributions of (11.2). The relevant direction of determination can be read from

$$\begin{aligned}
&\mathbb{P}[\epsilon_i = 0 \text{ for } i \in K,\ \epsilon_i = 1 \text{ for } i \in L \setminus K] \\
&= \textstyle\sum_{I: K \subset I \subset L} (-1)^{\#(I \setminus K)} \mathbb{P}[\epsilon_i = 0 \text{ for } i \in I]
\end{aligned}$$

for finite K, L with $K \subset L \subset J$. □

11.10. REMARK. It is possible to characterize those $T: \mathcal{K} \mapsto [0,1]$ such that T is the distribution function of a random closed set X. See MATHERON (1975, §2.2), SALINETTI & WETS (1986) and ROSS (1986) for Hausdorff E, and REVUZ (1955) and HONEYCUT (1971) for more general E.

11.11. LITERATURE. Random closed sets ($= \mathcal{F}(E)$-valued rv's) are the subject of the monograph by MATHERON (1975). They appear in the shape of 'measurable closed multifunctions' in the optimization literature (ROCKAFELLAR (1976), CASTAING & VALADIER (1977)). SALINETTI & WETS (1981) conciliate the two points of view. Random lower semicontinuous functions appear in the shape of 'normal integrands' in the optimization literature (ROCKAFELLAR (1976)). SALINETTI & WETS (1986) conciliate the two points of view. See also NORBERG (1984) for random closed sets and NORBERG (1986) for random usc functions.

12. CONVERGENCE IN DISTRIBUTION

As in the previous section, E is locally qcompact with countable base, and SM and $\mathcal{F}$ are provided with the sup vague topology. By Corollary 4.4(a) and Theorem 5.5, SM and $\mathcal{F}$ are metrizable and compact. So the general theory about convergence in distribution as treated in BILLINGSLEY (1968) applies immediately to extremal processes and random closed sets, with the pleasant circumstance that the collection of all probability distributions on BorSM or Bor$\mathcal{F}$ is narrowly (= weakly) compact, so we need not worry about tightness conditions. Since the distribution of an extremal process M is determined by that of $(M(G))_{G \in \mathcal{G}_0}$ with $\mathcal{G}_0$ a base of $\mathcal{G}$, we may expect that convergence in distribution of M_n to M in SM is determined by something like convergence in distribution of $(M_n(G))_{G \in \mathcal{G}_0}$ to $(M(G))_{G \in \mathcal{G}_0}$ in $\mathbb{I}^{\mathcal{G}_0}$. We are going to make this precise.

As in the classical theory of convergence in distribution, we must be careful with sets at which the limit M is discontinuous with positive probability. Recall the definition of the natural domain $\mathfrak{D}$ of extremal processes in (11.1), the definition of Q_δ in Definition 8.3 and the definition of sqc after Definition 8.3, that each Hausdorff space E is Q_δ and that in Hausdorff spaces the sqc operation is the same as taking closure for relatively compact sets.

12.1 DEFINITION. *Let E be locally qcompact and Q_δ with countable base.*
(a) *Let M be an extremal process on E. A set $A \in \mathfrak{D}$ is called a* **continuity set** *of M if $M(\text{int}A) = M(\text{sqc}A)$ wp1. The family of all continuity sets of M is denoted by $\mathcal{C}(M)$.*

(b) *A class $\mathcal{A} \subset \mathcal{Q}^{\sigma}$ is* ***probability determining*** *if the distributions of extremal processes M are determined by the finite-dimensional distributions of $(M(A))_{A \in \mathcal{A}}$.*
(c) *A class $\mathcal{A} \subset \mathcal{Q}^{\sigma}$ is* ***convergence determining*** *if for each two extremal processes M_1 and M_2 the class $\mathcal{A} \cap \mathcal{C}(M_1) \cap \mathcal{C}(M_2)$ is probability determining.*

For the next theorem, recall Definition 8.7 of $\mathcal{Q}$ compatible semimetric and note that all metrizable locally compact (Hausdorff) E are $\mathcal{Q}$ compatibly semimetrized by their metrics.

12.2. Theorem. *Let E be $\mathcal{Q}$ compatibly semimetrized by semimetric d and have a countable base. Let D be a dense subset of E. Then the classes of balls*

$$\mathcal{G}_0 := \{B(t,r):\, t \in D,\ r>0,\ B(t,r+) \in \mathcal{K}\},$$
$$\mathcal{K}_0 := \{B(t,r+):\, t \in D,\ r>0,\ B(t,r+) \in \mathcal{K}\}$$

both are convergence determining.

Proof. Since $\mathcal{G}_0$ is a base of $\mathcal{G}$ and E is locally $\mathcal{K}_0$ (cf. Example 5.8 and Corollary 8.8), $\mathcal{G}_0$ and $\mathcal{K}_0$ are probability determining by Theorem 11.5. It is obvious that $\mathcal{G}_0$ and $\mathcal{K}_0$ keep these properties if r is allowed to vary only through a dense subset of $(0,\infty)$ for each $t \in D$. So we are done if we prove that $B(t,r), B(t,r+) \in \mathcal{C}(M)$ for all but countably many $r \in (0,\infty)$, where M is an extremal proces.

Let $t \in D$. Then $r \mapsto M(B(t,r))$ is a nondecreasing left-continuous function, whereas $M(B(t,r+)) = \lim_{s \downarrow r} M(B(t,s))$ (wp1). In this situation we have $M(B(t,r)) = M(B(t,r+))$ wp1 iff $M(B(t,s)) \to_d M(B(t,r))$ in $\mathbb{I}$ as $s \downarrow r$. So it is sufficient to show that the map $r \mapsto \operatorname{law} M(B(t,r))$ has only countably many discontinuities. The countable collection of bounded continuous nondecreasing functions

$$\Phi := \{0 \vee (ax+b) \wedge 1:\, a, b \in \mathbb{Q},\ a>0\}$$

determines convergence in distribution in $\mathbb{I}$:

$$X_n \to_d X \text{ in } \mathbb{I} \quad \text{iff} \quad \mathbb{E}\varphi(X_n) \to \mathbb{E}\varphi(X) \text{ for } \varphi \in \Phi.$$

Furthermore, $r \mapsto \mathbb{E}\varphi(M(B(t,r)))$ is nondecreasing for $\varphi \in \Phi$ as $r \mapsto \varphi(M(B(t,r)))$ is nondecreasing wp1. So there are only countably many r at which $r \mapsto \mathbb{E}\varphi(M(B(t,r)))$ is discontinuous for at least one $\varphi \in \Phi$. Only at these points $r \mapsto \operatorname{law} M(B(t,r))$ can be discontinuous, so only at these points we may have $M(B(t,r)) \neq M(B(t,r+))$ with positive probability. □

The next theorem clarifies the term 'continuity set' in Definition 12.1(a). Note that we can also speak about continuity sets of deterministic sup measures m, as they can be regarded as degenerate extremal processes. Consequently, $\mathcal{C}(m) = \{A \in \mathcal{D}:\, m(\operatorname{int} A) = m(\operatorname{sqc} A)\}$.

12.3. Theorem. *Let E be locally qcompact and Q_δ with countable base.*
(a) *If $m_0 \in SM$ and $A \in \mathcal{C}(m_0)$, then the map $SM \ni m \mapsto m(A) \in \mathbb{I}$ is continuous at m_0.*
(b) *Let $\mathcal{A}$ be a convergence determining class and M_n, M be extremal processes. Then $M_n \to_d M$ in SM iff*

$$(M_n(A))_{A \in \mathcal{C}(M)} \to_d (M(A))_{A \in \mathcal{C}(M)} \quad \text{in } \mathbb{I}^{\mathcal{C}(M)} \tag{12.1}$$

(i.e., the finite-dimensional distributions of the left-hand side converge to those of the right-hand side).

Proof. (a) Suppose $m_n \to m_0$ in SM. Note that $A \subset \operatorname{sqc} A \in \mathcal{Q} \subset \mathcal{K}$. By (3.5) we have

$$\begin{aligned} m_0(A) = m_0(\operatorname{int} A) &\leqslant \liminf m_n(\operatorname{int} A) \leqslant \liminf m_n(A) \\ &\leqslant \limsup m_n(A) \leqslant \limsup m_n(\operatorname{sqc} A) \leqslant m_0(\operatorname{sqc} A) = m_0(A). \end{aligned}$$

(b) If $M_n \to_d M_0$ in SM and $A_1, A_2, ..., A_k \in \mathcal{C}(M_0)$, then $SM \ni m \mapsto (m(A_i))_{i=1}^k \in \mathbb{I}^k$ is wpl continuous at M_0, so $(M_n(A_i))_{i=1}^k \to_d (M_0(A_i))_{i=1}^k$ in $\mathbb{I}^k$ by the Continuous Mapping Theorem (BILLINGSLEY (1968), §5). Conversely, if (12.1) holds, then each M_0 to which some subsequence of (M_n) converges in distribution must have the same finite-dimensional distributions as M for $A \in \mathcal{Q} \cap \mathcal{C}(M) \cap \mathcal{C}(M_0)$, so $M_0 =_d M$. Since SM is compact, (M_n) is relatively compact for convergence in distribution, so $M_n \to_d M$ in SM. □

12.4. REMARK. One can prove that $A \in \mathcal{C}(m_0)$ is also necessary for continuity of $m \mapsto m(A)$ at m_0 in case $A \in \mathfrak{D}$, $\operatorname{int}\operatorname{sqc}\operatorname{int} A = \operatorname{int} A$ and $\operatorname{sqc}\operatorname{int}\operatorname{sqc} A = \operatorname{sqc} A$.

Identifying random closed sets X with the associated $\{0,1\}$-valued extremal processes $M := 1_X^\vee$ we can translate Definition 12.1(a) into the following.

12.5. DEFINITION. *Let X be a random closed set in E with distribution function $T := \mathbb{P}[X \cap \cdot \neq \varnothing]$ considered on $\mathcal{G} \cup \mathcal{K}$. Then $A \in \mathfrak{D}$ is called a continuity set of X if $T(\operatorname{int} A) = T(\operatorname{sqc} A)$, and $\mathcal{C}(X)$ is the class of all such A.*

12.6. THEOREM. *Let E be locally qcompact and Q_δ with countable base, and let $\mathcal{Q}$ be a convergence determining class which is closed for finite unions. Let X_n, X be random closed sets in E with distribution functions T_n, T. Then $X_n \to_d X$ in $\mathcal{F}$ iff $T_n(A) \to T(A)$ for all $A \in \mathcal{Q} \cap \mathcal{C}(X)$.*

PROOF. Similar to the proof of Theorem 12.3(b). Use Theorem 11.9(c) for the uniqueness of limit points of convergent subsequences and note that $\mathcal{C}(X)$ is closed for finite unions. For the latter, note that $\operatorname{sqc}(A \cup B) = \operatorname{sqc} A \cup \operatorname{sqc} B$ and $\operatorname{int}(A \cup B) \supset \operatorname{int} A \cup \operatorname{int} B$, so $(\operatorname{sqc}(A \cup B)) \setminus \operatorname{int}(A \cup B) \subset ((\operatorname{sqc} A) \setminus \operatorname{int} A) \cup ((\operatorname{sqc} B) \setminus \operatorname{int} B)$. □

12.7. APPLICATIONS. (a) If M_n, M are extremal processes on an interval $E \subset \mathbb{R}$, then $M_n \to_d M$ iff

$$(M_n(J_i))_{i=1}^k \to_d (M(J_i))_{i=1}^k \quad \text{in } \mathbb{I}^k$$

for each finite sequence $(J_i)_{i=1}^k$ of open intervals which are relatively compact in E and such that $M(J_i) = M(\operatorname{clos} J_i)$ wpl for $1 = 1, 2, ..., k$.
(b) If X_n, X are random closed sets in $\mathbb{R}^d$ with distribution functions T_n, T, then $X_n \to_d X$ iff $T_n(A) \to T(A)$ for all finite unions A of blocks in $\mathbb{R}^d$ such that $T(\operatorname{int} A) = T(\operatorname{clos} A)$.

12.8. LITERATURE. Convergence in distribution for random closed sets is studied by SALINETTI & WETS (1981) for $E = \mathbb{R}^d$ and by NORBERG (1984). Convergence in distribution for random usc functions is studied by SALINETTI & WETS (1986) for $E = \mathbb{R}^d$ and by NORBERG (1986). For convergence in probability, see SALINETTI, VERVAAT & WETS (1986). Convergence of probability measures on semi-lattices is studied by NORBERG (1986a).

13. THE EXISTENCE THEOREM FOR EXTREMAL PROCESSES

As in the previous sections we assume that E is locally qcompact with countable base. We need the following lemma, which will be proved in Section 14 (cf. Remark 14.15(a)).

13.1 LEMMA. *Let J be countable. Then the mapping $US^J \ni (f_j)_{j \in J} \to \bigwedge_{j \in J} f_j \in US$ is measurable, so $\bigwedge_{j \in J} X_j$ is a US-valued rv if all X_j are.*

Let M be an extremal process and $\mathcal{G}_0$ a base of $\mathcal{G}$. By Theorem 11.5 the probability distribution of M is determined by the distribution of the $\mathbb{I}^{\mathcal{G}_0}$-valued rv $(M(G))_{G \in \mathcal{G}_0}$. However, if we do not assume an extremal process to be given, but start only with an $\mathbb{I}^{\mathcal{G}_0}$-valued rv $(N(G))_{G \in \mathcal{G}_0}$, then it need not be true that there is an extremal process M such that $M(G) = N(G)$ wp1 for each $G \in \mathcal{G}_0$ (separately). Obviously, a necessary condition for the existence of such an M is

$$N(\bigcup_{j=1}^{\infty} G_j) = \vee_{j=1}^{\infty} N(G_j) \text{ wp1} \tag{13.1}$$

for each separate sequence $(G_j)_{j=1}^{\infty}$ in $\mathcal{G}_0$ with $\bigcup_{j=1}^{\infty} G_j \in \mathcal{G}_0$. The next theorem tells us that this condition is also sufficient.

13.2 THEOREM. (Existence Theorem for extremal processes). *Let E be locally qcompact and Q_δ with countable base, and let $\mathcal{G}_0$ be a base of $\mathcal{G}$ that does not contain $\varnothing$. Let $(N(G))_{G\in\mathcal{G}_0}$ be an $\mathbb{I}^{\mathcal{G}_0}$-valued rv such that (13.1) holds wp1 for each separate sequence $(G_j)_{j=1}^{\infty}$ in $\mathcal{G}_0$ with $\bigcup_{j=1}^{\infty} G_j \in \mathcal{G}_0$. Then there is an extremal process M such that $M(G) = N(G)$ wp1 for each $G \in \mathcal{G}_0$ separately.*

13.3 REMARKS. Note that in the theorem the exceptional event of probability 0 that (13.1) does not hold may depend on the sequence $(G_j)_{j=1}^{\infty}$. The stronger condition that (13.13.1or all sequences $(G_j)_{j=1}^{\infty}$ simultaneously reduces Theorem 13.2 to a trivial consequence of the Extension Theorem 2.6. If $\mathcal{G}_0$ is countable, then it follows that $M = N$ wp1 on $\mathcal{G}_0$, so N is wp1 the restriction of the extremal process M (again by Theorem 2.6). If $\mathcal{G}_0$ is uncountable (for instance if $\mathcal{G}_0 = \mathcal{G}$), this need not be true, as shows the following example.

13.4 EXAMPLE. $E = \mathbb{R}$, $\mathcal{G}_0 = \{\text{open intervals}\}$, ξ is a rv with a uniform distribution in (0,1), $N(G) := 1_{[\xi\in\partial G]}$ for $G \in \mathcal{G}_0$, where ∂G is the boundary of G. Then N is wp1 not the restriction to $\mathcal{G}_0$ of an extremal process, but $M \equiv 0$ makes the theorem work.

13.5 REMARK. The complication in Example 13.4 is avoided by assuming N to be monotone on $\mathcal{G}_0$. Then the conclusion of Theorem 13.2 can be strengthened to $M(G) = N(G)$ wp1 for all $G \in \mathcal{G}_0$ simultaneously.

PROOF OF THEOREM 13.2. Let $\mathcal{G}_1 \subset \mathcal{G}_0$ be a countable base of $\mathcal{G}$ consisting of relatively qcompact sets, and let

$$X := \wedge_{G\in\mathcal{G}_1}(N(G)1_G \vee 1_{G^c}). \tag{13.2}$$

Then X is a US-valued rv by Lemma 13.1, so $M := X^\vee$ is an extremal process. It is obvious that

$$M(G) \leqslant N(G) \text{ wp1 for all } G \in \mathcal{G}_1. \tag{13.3}$$

Let $(G_k)_{k=1}^{\infty}$ be an enumeration of $\mathcal{G}_1$ and set

$$X_n := \wedge_{k=1}^{n}(N(G_k)1_{G_k} \vee 1_{G_k^c}).$$

We are going to prove

$$N(G_k) \leqslant X_n^\vee(G_k) \text{ for } n \geqslant k. \tag{13.4}$$

Let Δ_n be the collection of atoms of the field generated by $G_1, G_2, \ldots, G_n$. Then X_n is constant at each $D \in \Delta_n$ with value $\wedge_{j\leqslant n,\, G_j\supset D} N(G_j)$. Hence

$$X_n^\vee(G_k) = \vee_{D\in\Delta_n,\, D\subset G_k} \wedge_{j\leqslant n,\, G_j\supset D} N(G_j) \text{ for } k \leqslant n. \tag{13.5}$$

If (13.4) would not hold for some fixed $n \geqslant k$, then for each atom $D \subset G_k$ as in (13.5) there is a $G_{j(D)}$ with $j(D) \leqslant n$ and $G_{j(D)} \supset D$ such that $N(G_{j(D)}) < N(G_k)$. Hence

$$N(G_k) > \vee_{D\subset G_k} N(G_{j(D)}) = N(\bigcup_{D\subset G_k} G_{j(D)}) \text{ wp1},$$

contradicting (13.1), since $\bigcup_{D\subset G_k} G_{j(D)} \supset G_k$. This proves (13.4). As $X_n \downarrow X$ pointwise, we have by

(13.3), (13.4) and Theorems 6.2(b) and 6.4

$$N(G_k) = \lim_{n\to\infty} X_n^\vee(G_k) \leqslant \lim_{n\to\infty} X_n^\vee(\operatorname{sqc} G_k) = M(\operatorname{sqc} G_k)\,.$$

Combining this result with (13.3) we find

$$M(G) \leqslant N(G) \leqslant M(\operatorname{sqc} G) \ \text{ wp1 for all } G \in \mathcal{G}_1\,. \tag{13.5}$$

Now take $G_0 \in \mathcal{G}_0$. Then there is a (countable) subcollection $\mathcal{G}_2$ of $\mathcal{G}_1$ such that $G_0 = \bigcup_{G\in\mathcal{G}_2} G = \bigcup_{G\in\mathcal{G}_2} \operatorname{sqc} G$. By (13.1) and (13.5) we have wp1

$$M(G_0) = \vee_{G\in\mathcal{G}_2} M(G) \leqslant \vee_{G\in\mathcal{G}_2} N(G) = N(G_0) \leqslant \vee_{G\in\mathcal{G}_2} M(\operatorname{sqc} G) = M(G_0)\,,$$

so $M(G_0) = N(G_0)$ wp1 for each separate $G_0 \in \mathcal{G}_0$. □

13.6 LITERATURE. For existence theorems for random closed sets based on their probability distribution functions, see REVUZ (1955), MATHERON (1975), BERG ET AL. (1984, Th.4.6.18) and SALINETTI & WETS (1986). Where in NORBERG (1984, 1986) theorems are claimed to generalize Theorem 13.2, it is ignored that 'wp1' in Theorem 13.2 refers to each separate sequence (G_j). The exceptional null event may vary with it. For existence theorems for probability measures on semi-lattices, see NORBERG (1986a).

14. SEMICONTINUITY OF THE LATTICE OPERATIONS

In the present section we first return to the generality of a topological space E without further assumptions, and the sup $\mathfrak{B}$ topologies on SM, US and $\mathcal{F}$. The spaces SM, US and $\mathcal{F}$ are lattices with as partial orders the pointwise order of functions on $\mathcal{G}$ for SM, the pointwise order of functions on E for US, and set inclusion for $\mathcal{F}$. The lattices SM and US are isomorphic via $d^\vee$ and $i^\vee$, and $F \mapsto 1_F$ maps $\mathcal{F}$ isomorphically onto a sublattice of US, in which $\wedge$ and $\vee$ give the same result as in US.

We first investigate when the above partial orders are closed. Recall that a partial order $\leqslant$ on a topological space T is closed if

$$\operatorname{graph} \leqslant \ := \ \{(x,y) \in T^2;\, x \leqslant y\}$$

is closed in T^2, or equivalently, if for all limits x and y of convergent nets (x_α) and (y_α) in T with $x_\alpha \leqslant y_\alpha$ for all α we have $x \leqslant y$. Note that the order in the subspace $\mathcal{F}$ is closed if the order in US is.

14.1 THEOREM. (a) *If E is locally $\mathfrak{B}$, then the orders in SM, US and $\mathcal{F}$ are sup $\mathfrak{B}$ closed.*
(b) *If the order in $\mathcal{F}$ is sup $\mathfrak{B}$ closed, then E is internally $\mathfrak{B}$.*

14.2 COROLLARY. If E is Hausdorff, then the orders in SM, US and $\mathcal{F}$ are sup $\mathfrak{B}$ closed iff E is locally compact (cf. Property 3.7(d)).

PROOF OF THEOREM 14.1. (a) We give the proof for US. Let (h, g) be outside the graph of $\leqslant$, so $g(t) < h(t)$ for some $t \in E$. The construction in the proof of Theorem 4.3(c) gives the sides of an open rectangle around (h, g) that does not intersect graph$\leqslant$.
(b) It follows by Proposition I.2 of NACHBIN (1965) that $\mathcal{F}$ is Hausdorff. Apply Theorem 4.3(d). □

We now turn to the lattice operations. We write $\vee^{SM}$ and $\wedge^{SM}$ for the lattice operations in SM, and $\vee^{US}$ and $\wedge^{US}$ for the lattice operations in US. Note that $\vee^{SM}$ is the same as taking pointwise suprema of functions on $\mathcal{G}$, but that $\wedge^{SM}$ is more complicated:

$$\wedge_j^{SM} m_j \;=\; i^\vee d^\vee \wedge_j m_j \;=\; i^\vee \wedge_j d^\vee m_j\,. \tag{14.1}$$

Proof. The first identity follows from Lemma 2.2(b) and Theorem 2.5(a). The second identity with $\leqslant$ instead follows from the monotonicity of $d^\vee$ and $i^\vee$ implying subsequently

$$\begin{aligned} &\wedge_j m_j \leqslant m_k, \\ &d^\vee \wedge_j m_j \leqslant d^\vee m_k, \\ &d^\vee \wedge_j m_j \leqslant \wedge_j d^\vee m_j, \\ &i^\vee d^\vee \wedge_j m_j \leqslant i^\vee \wedge_j d^\vee m_j. \end{aligned}$$

On the other hand we have $m_k = i^\vee d^\vee m_k \geqslant i^\vee \wedge_j d^\vee m_j$ with a sup measure on the right-hand side, so

$$\wedge_j^{SM} m_j \geqslant i^\vee \wedge_j d^\vee m_j. \qquad \square$$

Analogously, $\wedge^{US}$ is the same as taking pointwise infima of functions on E, but now $\vee^{US}$ is more complicated:

$$\vee_j^{US} f_j \;=\; d^\vee i^\vee \vee_j f_j \;=\; d^\vee \vee_j i^\vee f_j. \tag{14.2}$$

Consequently, we prefer considering $\vee$ in SM and $\wedge$ in US (often without writing the upper indices).

We now want to investigate the topological properties of the lattice operations. The following concepts will be useful.

14.3. **Definition.** (a) (cf. (3.3)). *The **upper topology** on SM is the topology with subbase consisting of $\{m: m(G) > x\}$ for $G \in \mathcal{G}$ and $x \in [0, 1)$. The $\mathfrak{B}$ **lower topology** on SM is the topology with subbase consisting of $\{m: m(B) < x\}$ for $B \in \mathfrak{B}$ and $x \in (0, 1]$. The lower and upper topologies are defined on US and $\mathcal{F}$ by declaring $d^\vee$ and 'ind' homeomorphisms (so have for subbases the corresponding halves of (3.7) and (3.8)). We write $SM\!\uparrow$, $US\!\uparrow$ and $\mathcal{F}\!\uparrow$ for the spaces with the upper topologies, and $SM\!\downarrow$ or $SM\!\downarrow_{\mathfrak{B}}$, etc. for the spaces with the $\mathfrak{B}$ lower topologies.*

(b) *Let T be a topological space. A mapping $\varphi: T \to SM$, US or $\mathcal{F}$ is called **lower semicontinuous (lsc)** if $\varphi: T \to SM\!\uparrow$, $US\!\uparrow$ or $\mathcal{F}\!\uparrow$ is continuous, and φ is called $\mathfrak{B}$ **upper semicontinuous (usc)** if $\varphi: T \to SM\!\downarrow_{\mathfrak{B}}$, $US\!\downarrow_{\mathfrak{B}}$ or $\mathcal{F}\!\downarrow_{\mathfrak{B}}$ is continuous. If T has the form $T = (SM\!\uparrow)^J$, $(US\!\uparrow)^J$ or $(\mathcal{F}\!\uparrow)^J$ with the product topology, then lsc functions on T are called **lower continuous**. If T has the form $T = (SM\!\downarrow_{\mathfrak{B}})^J$, $(US\!\downarrow_{\mathfrak{B}})^J$ or $(\mathcal{F}\!\downarrow_{\mathfrak{B}})^J$ with the product topology, then usc functions on T are called $\mathfrak{B}$ **upper continuous**.*

14.4. **Corollary.** In the situation

$$S \xrightarrow{\psi} T \xrightarrow{\varphi} SM,\ US \text{ or } \mathcal{F},$$

$\psi \circ \varphi$ is lsc ($\mathfrak{B}$ usc) if ψ is continuous and φ is lsc ($\mathfrak{B}$ usc), and $\psi \circ \varphi$ is lower ($\mathfrak{B}$ upper) continuous for appropriate S and T if both ψ and φ are.

14.5. **Remark.** If E is locally qcompact with countable base, then lsc and $\mathfrak{B}$ usc functions are Borel measurable, by Theorem 11.1.

14.6. **Theorem.** *Let J be an arbitrary index set. Then*

$$SM^J \ni (m_j)_{j \in J} \mapsto \vee_{j \in J} m_j \in SM \tag{14.3}$$

is lower continuous, and $\mathfrak{B}$ upper continuous (so $\mathfrak{B}$ continuous) if J is finite.

14.7. **Corollary.** The mapping $\mathcal{F}^J \ni (F_j)_{j \in J} \mapsto \operatorname{clos} \bigcup_{j \in J} F_j \in \mathcal{F}$ is lower continuous, and $\mathfrak{B}$ upper continuous (so $\mathfrak{B}$ continuous) if J is finite.

PROOF OF THEOREM 14.6. (a) Lower continuity follows from

$$\{(m_j)\colon \bigvee_j m_j(G) > x\} \;=\; \bigcup_k \{(m_j)\colon m_k(G) > x\} \;\in\; \mathcal{G}((SM\!\uparrow)^J).$$

It remains to proof that the mapping is $\mathcal{B}$ usc in case J is finite. If so, then $\bigvee_j^{US}$ corresponds to taking pointwise supremum of usc functions, and by Theorem 2.5(c) we have for $B \in \mathcal{B}$

$$\bigvee_j^{SM} m_j(B) \;=\; \bigvee_{t\in B}\left[\bigvee_j d^\vee m_j\right](t) \;=\; \bigvee_j \bigvee_{t\in B} d^\vee m_j(t) \;=\; \bigvee_j m_j(B).$$

Consequently,

$$\{(m_j)\colon \bigvee_j^{SM} m_j(B) < x\} \;=\; \bigcap_k \{(m_j)\colon m_k(B) < x\} \;\in\; \mathcal{G}((SM\!\downarrow_{\mathcal{B}})^J). \qquad \square$$

14.8. REMARK. If J is infinite, then the mapping (14.3) need not be continuous. We exhibit this in $\mathcal{F}$ rather than SM. Let E be a separable metric space, $(t_j)_{j=1}^\infty$ a dense sequence in E, $\mathcal{B} = \mathcal{K}$ and $F_{j,n} := \varnothing$ for $j < n$, $\{t_j\}$ for $j \geqslant n$. Then we have $F_{j,n} \to \varnothing =: F_j$ as $n \to \infty$, so that

$$\operatorname{clos} \bigcup_{j=1}^\infty F_{j,n} \;=\; \operatorname{clos}\{t_n, t_{n+1}, \ldots\} \;\to\; E,$$

whereas $\bigcup_{j=1}^\infty F_j = \varnothing$.

We now are going to study the semicontinuity of $\wedge$. We will restrict our attention to $\mathcal{B} = \mathcal{K}$. The following assumption will be crucial.

14.9. ASSUMPTION. *For $K \in \mathcal{K}$ and $G_1, G_2 \in \mathcal{G}$ such that $K \subset G_1 \cup G_2$ there are $K_1, K_2 \in \mathcal{K}$ such that $K_1 \subset G_1$, $K_2 \subset G_2$ and $K \subset K_1 \cup K_2$.*

14.10. LEMMA. *Sufficient conditions for Assumption 14.9 to hold are that E is locally qcompact or that E is Hausdorff.*

There are spaces E which satisfy Assumption 14.9 but do not satisfy the condition of Lemma 14.10. We do not know to what extent Assumption 14.9 holds.

PROOF OF LEMMA 14.10. First suppose that E is locally qcompact. Let $K \subset G_1 \cup G_2$. Then we can select for each $t \in K$ a qcompact $K(t)$ such that $t \in \operatorname{int} K(t)$ and $K((t) \subset G_1$ if $t \in G_1 \setminus G_2$, $K(t) \subset G_2$ if $t \in G_2 \setminus G_1$, and $K(t) \subset G_1 G_2$ if $t \in G_1 G_2$. First select finite subsets J_1 of $G_1 \setminus G_2$ and J_2 of $G_2 \setminus G_1$ such that $K \setminus G_2 \subset \bigcup_{t\in J_1} \operatorname{int} K(t)$ and $K \setminus G_1 \subset \bigcup_{t\in J_2} \operatorname{int} K(t)$. After this, select a finite subset J_3 of $G_1 G_2$ such that

$$\left[\bigcup_{t\in J_1} \operatorname{int} K(t)\right]^c \;\subset\; \bigcup_{t\in J_2 \cup J_3} \operatorname{int} K(t),$$

$$\left[\bigcup_{t\in J_2} \operatorname{int} K(t)\right]^c \;\subset\; \bigcup_{t\in J_1 \cup J_3} \operatorname{int} K(t)$$

(note that the left-hand sides are closed, hence qcompact). Set $K_n := \bigcup_{t\in J_n \cup J_3} K(t)$ for $n = 1, 2$. Then K_1 and K_2 have the properties claimed in Assumption 14.9.

Next suppose that E is Hausdorff. Then the trace topology on a compact set K is locally compact, so by the first part of the lemma there exist compact sets $K_1, K_2 \subset K$ such that $K_1 \subset KG_1$, $K_2 \subset KG_2$ and $K_1 \cup K_2 = K$. $\square$

14.11. THEOREM. *Let J be an arbitrary index set. Then*

$$\mathcal{F}^J \ni (F_j)_{j\in J} \mapsto \bigcap_{j\in J} F_j \in \mathcal{F} \tag{14.4}$$

is $\mathcal{K}$ upper continuous if Assumption 14.9 holds. The mapping is not $\mathcal{K}$ continuous, even if J is finite.

PROOF. First an example showing that (14.4) is not $\mathcal{K}$ continuous if $\#J = 2$. Let $E := \mathbb{R}$, $F_{\pm n} := \{\pm 1/n\}$ for $n = 1, 2, \ldots$. Then $F_{jn} \to \{0\} =: F_j$ for $j = +, -$, whereas $F_{+n}F_{-n} = \varnothing$, $F_+F_- = \{0\}$.

We now prove $\mathcal{K}$ upper continuity of (14.4) in steps, for increasing size of J. First $\#J = 2$. We must prove that for $K \in \mathcal{K}$

$$U := \{(F_1, F_2) \in \mathcal{F}^2 \colon F_1F_2K = \varnothing\}$$

is open in $\mathcal{F}^2$. If $(F_{10}, F_{20}) \in \mathcal{F}^2$, then select K_1 and K_2 according to Assumption 14.9 for $G_n = F_{n0}^c$, $n = 1, 2$. Then the set

$$\{(F_1, F_2) \colon F_1K_2 = \varnothing\} \cap \{(F_1, F_2) \colon F_2K_1 = \varnothing\}$$

is open in $\mathcal{F}^2$, contains (F_{10}, F_{20}) and is contained in U. So U is open.

If $\mathcal{K}$ upper cuntinuity of (14.4) has been proved for $\#J = n$, then it follows for $\#J = n+1$ by considering the composition

$$(F_1, \ldots, F_n, F_{n+1}) \mapsto \left(\bigcap_{j=1}^n F_j, F_{n+1}\right) \mapsto \bigcap_{j=1}^n F_j \cap F_{n+1}$$

in view of Corollary 14.4. So $\mathcal{K}$ upper continuity follows by induction for finite J. For infinite J, note that for qcompact K

$$\{(F_j)_{j\in J} \colon K \cap \bigcap_{j\in J} F_j = \varnothing\} = \bigcup_{\text{finite } J_* \subset J} \{(F_j)_{j\in J} \colon K \cap \bigcap_{j\in J_*} F_j = \varnothing\}.$$

The set on the right-hand side is open because of our previous result for finite J. □

14.12. THEOREM. *Let J be an arbitrary index set. Then*

$$US^J \ni (f_j)_{j\in J} \mapsto \bigwedge_{j\in J} f_j \in US \tag{14.5}$$

is $\mathcal{K}$ upper continuous if Assumption 14.9 holds. The mapping is not $\mathcal{K}$ continuous, even if J is finite.

PROOF. By Theorem 7.2 the spaces $US(E)$ and $\mathcal{F}(E \times \mathbb{I}'\uparrow)$ are sup $\mathcal{K}$ homeomorphic. So Theorem 14.12 follows from Theorem 14.11 if we show that Assumption 14.9 also holds for $E^* := E \times \mathbb{I}'\uparrow$. To this end, suppose that $F_1^*, F_2^* \in \mathcal{F}(E^*)$ and $K^* \in \mathcal{K}(E^*)$ and that $F_1^*F_2^*K^* = \varnothing$. Adopt the notations after Theorem 7.3. Then $F_1^* \cap F_2^* \cap \uparrow K^* = \varnothing$ by (7.3), and also $\uparrow K^* \in \mathcal{K}(E^*)$. Because F_1^* and F_2^* are hypographs, we have $\pi_1(F_1^* \cap \uparrow K^*) \cap \pi_1(F_2^* \cap \uparrow K^*) = \varnothing$. Now $\pi_1(\uparrow K^*)$ is qcompact since π_1 is continuous, and $\pi_1(F_n^* \cap \uparrow K^*)$ is closed in $\pi_1(\uparrow K^*)$ for $n = 1, 2$, since π_1 is a closed mapping when restricted to the qcompact domain $\pi_1 \uparrow K^* \times \pi_2 \uparrow K^*$. By Assumption 14.9 holding for E we can find K_1 and $K_2 \in \mathcal{K}(E)$ such that

$$K_1 \cap \pi_1(F_2^* \cap \uparrow K^*) = K_2 \cap \pi_1(F_1^* \cap \uparrow K^*) = \varnothing$$

and $\pi_1 \uparrow K^* \subset K_1 \cup K_2$. Then with $K_n^* := K_n \times \pi_2 \uparrow K^*$ we have $K_1^*F_2^* = K_2^*F_1^* = \varnothing$ and $K^* \subset K_1^* \cup K_2^*$. □

14.13. REMARKS. (a) If J is countable, then $\operatorname{Bor} SM^J$ is the J-fold product σ-field of $\operatorname{Bor} SM$. So if $(M_j)_{j\in J}$ is a countable collection of extremal processes, then $\bigvee_{j\in J} M_j$ and $\bigwedge_{j\in J}^{SM} M_j$ are extremal processes. Considering the sup derivatives of M_j we obtain Lemma 13.1.
(b) If J is uncountable, then $\operatorname{Bor} SM^J$ is strictly larger than the J-fold product σ-field of $\operatorname{Bor} SM$ (cf. NELSON (1959), Theorem 2.1 and Corollary 2.1), so $\bigvee_j M_j$ and $\bigwedge_j^{SM} M_j$ need no longer be extremal processes if all M_j are. However, for each system of extremal processes there is a 'version' (i.e., another system of extremal processes with the same joint distributions for each finite subsystem of extremal processes) which is $\operatorname{Bor} SM^J$ measurable. Its distribution over $\operatorname{Bor} SM^J$ is unique if we require in addition that it is regular. All this is an immediate application of Theorem 1.1 of NELSON (1959).

14.14. LITERATURE. For special cases of Theorems 14.6 and 14.11, see BERGE (1963), KURATOWSKI (1968, §43) and MATHERON (1975).

15. CAPACITIES

15.1. DEFINITION. *A **precapacity** is a function* $c: \mathcal{K}(E) \to [0, \infty] =: \mathbb{J}$ *such that* $c(\varnothing) = 0$ *and c is increasing:* $c(K_1) \leqslant c(K_2)$ *if* $K_1 \subset K_2$.

Examples of precapacities are obtained by restricting countably additive measures μ on $\operatorname{Bor} E$ to $\mathcal{K}(E)$. In this case we have $c(K_1 \cup K_2) = c(K_1) + c(K_2)$ for disjoint $K_1, K_2 \in \mathcal{K}(E)$, or more generally,

$$c(K_1 \cup K_2) + c(K_1 \cap K_2) = c(K_1) + c(K_2) \quad \text{in case also } K_1 \cap K_2 \in \mathcal{K}(E). \tag{15.1}$$

Other examples of precapacities are the canonical extensions of sup measures m on $\mathcal{G}(E)$ restricted to $\mathcal{K}(E)$: $c(K) := \bigwedge_{G \supset K} m(G)$ (cf. Theorem 2.5(c)). In this case we have

$$c(K_1 \cup K_2) = c(K_1) \vee c(K_2). \tag{15.2}$$

Equivalently, if $f \in US(E)$, then $c(K) := f^\vee(K)$ defines a precapacity with the same properties. Finally, if $\mathcal{F} \in \mathcal{F}(E)$, then the same procedure for $f = 1_F$ gives a $\{0, 1\}$-valued precapacity c satisfying (15.2).

Precapacities can be extended to all subsets of E by

$$c(A) := \bigvee_{K \subset A} c(K) \quad \text{for } A \subset E. \tag{15.3}$$

In particular the extension to $\mathcal{G}(E)$ is important.

15.2. DEFINITION. *A precapacity is **upper semicontinuous (usc)**, if*

$$c(K) = \bigwedge_{G \supset K} c(G) \quad \textit{for } K \in \mathcal{K}(E). \tag{15.4}$$

Obviously, $c(K) = c(\operatorname{sat} K)$ for usc precapacities, so we can restrict their domain to the saturated qcompact sets $\mathcal{Q}(E)$.

15.3. DEFINITION. *A **capacity** is an usc precapacity with domain restricted to* $\mathcal{Q}(E)$.

In the literature one sees often the following 'upper continuity' condition, which reads in a generalization to the non-Hausdorff case:

$$c(K_n) \downarrow c\left(\bigcap K_n\right) \text{ for all decreasing nets } (K_n) \text{ in } \mathcal{Q}(E) \text{ with } \bigcap K_n \in \mathcal{Q}(E). \tag{15.5}$$

15.4. THEOREM. (a) *Capacities c satisfy (15.5).*
(b) *If a precapacity c satisfies (15.5) and E is locally qcompact and* $\mathcal{Q}(E)$ *is closed for finite intersections (in particular if E is locally compact), then (15.4) holds, so c is a capacity.*

PROOF. (a) If $K_n \downarrow K := \bigcap K_n$ in $\mathcal{Q}$, then $c(K) \leqslant \lim c(K_n)$ since $K \subset K_n$ for all n. Conversely, we have for each $G \supset K$ that $K_n G^c \downarrow \varnothing$, so $K_n G^c = \varnothing$ for large n, i.e., $K_n \subset G$ for large n. Hence $\lim c(K_n) \leqslant c(K)$.
(b) Let $K \in \mathcal{Q}$. By applying Property 3.7(b) we find for each instance of $K \subset G$ a $K' \in \mathcal{Q}(E)$ such that $K \subset \operatorname{int} K' \subset K' \subset G$. We have $K = \bigcap_{G \supset K} G$ because K is saturated. Selecting with each such G a K' as above and applying (15.5) to the net of finite intersections of such K' we find $c(K) \geqslant \bigwedge_{G \supset K} c(G)$. The reverse inequality is obvious. □

15.5. EXAMPLES. (a) Let $E = \mathbb{N} \cup \{\infty\}$ be the Appert-Varadarajan space, i.e., all sets $\{n\} \subset \mathbb{N}$ are open and a subset $G \subset E$ containing ∞ is open iff $\lim n^{-1} \# (G \cap \{1, ..., n\}) = 1$. Then E is Hausdorff and $\mathcal{K}$ consists of its finite subsets. If $c = \#$, then c is the extension of a finite precapacity on $\mathcal{K}$ to all subsets of E. Obviously, c catisfies (15.5). However, c is not usc since $c(G) = \infty$ for nonempty G.
(b) Here is an example of a precapacity c with different limits $\lim c(K_n)$ for different decreasing sequences (K_n) with the same intersection. Let $E := \{(0,0)\} \cup (0,1]^2$ with the trace topology and trace distance d from $\mathbb{R}^2$. Set $V := \{0\} \times \mathbb{R}$ and $o := (0,0)$. Then

$$c(K) := \frac{d(V,K) + d(o,K)}{d(V,K)} \quad \text{for } K \in \mathcal{K}(E \setminus \{o\})$$

defines a capacity on $E \setminus \{o\}$. Let c' be the precapacity on E defined by $c'(K) := c(K \setminus \{o\})$. If (K_n) is a decreasing sequence of line segments starting at o, then $\lim c(K_n)$ depends on the slope of these segments.

For the moment, we return to precapacities. Consider a precapacities c which are restrictions to $\mathcal{K}(E)$ of Radon measures μ on Bor E (i.e., μ is finite on $\mathcal{K}(E)$ and $\mu(A) = \bigvee_{K \subset A} \mu(K)$ for $A \in$ Bor E). The well-known vague topology on spaces of Radon measures (cf. BERG ET AL. (1984, §2.4)) suggests us the *vague topology* on spaces of precapacities, with subbase

$$\{c: c(G) > x\}, \quad \{c: c(K) < x\} \quad \text{for } G \in \mathcal{G},\ K \in \mathcal{K} \text{ and } x \in \mathbb{J}. \tag{15.6}$$

Note that the trace topology on the space of the c arising from sup measures, usc functions or closed sets coincides with what we called the sup vague topologies on *SM*, *US* and $\mathcal{F}$. Similarly, the case of bounded measures μ on Bor E suggests us to extend the notion of *narrow* (= *weak*) *topology* to spaces of precapacities, with subbase

$$\{c: C(G) > x\}, \quad \{c: c(F) < x\} \quad \text{for } G \in \mathcal{G},\ F \in \mathcal{F} \text{ and } x \in \mathbb{J}. \tag{15.7}$$

Again, the trace topology on the precapacities coming from *SM*, *US* or $\mathcal{F}$ corresponds to the sup narrow topology. We will study these topologies, in the case of the vague topology including the relations with spaces of Radon measures and spaces of sup measures. The latter aspect for the narrow topology is more complicated, and will be dealt with in another paper.

In the previous sections we have assumed that sup measures and usc functions have their values in $\mathbb{I} = [0,1]$. By obvious transformations we may replace $\mathbb{I}$ with any compact interval in $[-\infty, \infty]$, in particular by $\mathbb{J} = [0, \infty]$, the range of capacities. In the present section we will think $\mathbb{I}$ replaced by $\mathbb{J}$.

We now take the following point of view. We consider $\mathfrak{Q}(E)$ as space on its own, with as points the saturated qcompact subsets K of E, and want to regard (15.6) and (15.7) as special cases of sup topologies on $US(\mathfrak{Q}(E))$. In particular, this implies that we provide $\mathfrak{Q}(E)$ with a (non-Hausdorff) topology $\mathcal{G}^{\mathfrak{Q}}$ with base $\mathcal{G}_0^{\mathfrak{Q}}$ consisting of

$$\mathfrak{Q}(G) = \{K \in \mathfrak{Q}(E): K \subset G\} \quad \text{for } G \in \mathcal{G}(E). \tag{15.8}$$

Then c determines a sup measure $c^\vee$ on $\mathcal{G}^{\mathfrak{Q}}$ by

$$c^\vee(G^{\mathfrak{Q}}) = \bigvee_{K \in G^{\cdot}} c(K) \quad \text{for } G^{\mathfrak{Q}} \in \mathcal{G}^{\mathfrak{Q}}.$$

For $G^{\mathfrak{Q}} \in \mathcal{G}_0^{\mathfrak{Q}}$ this specializes to

$$c^\vee(\mathfrak{Q}(G)) = c(G) \text{ as defined in (15.3).}$$

Our new upper semicontinuity assumption about c is that $c \in US(\mathfrak{Q}(E))$, so $c = d^\vee i^\vee c$, which may be written as (cf. proof of Theorem 2.6)

$$c(K) = \bigwedge_{G^{\cdot} \in \mathcal{G}^{\cdot}: G^{\cdot} \ni K} c^\vee(G^{\mathfrak{Q}}) = \bigwedge_{G^{\cdot} \in \mathcal{G}_0^{\cdot}: G^{\cdot} \ni K} c^\vee(G^{\mathfrak{Q}}) = \bigwedge_{G \in \mathcal{G}(E): G \supset K} c^\vee(\mathfrak{Q}(G))$$

$$= \bigwedge_{G \in \mathcal{G}(E): G \supset K} c(G),$$

which is (15.4). Consequently,

15.6. THEOREM. *A precapacity c is a capacity, i.e., (15.4) holds, iff* $c \in US(\mathcal{Q}(E))$.

We write $CAP = CAP(E)$ for the family of all capacities on E (or rather $\mathcal{Q}(E)$). Recall that $CAP(E) = US(\mathcal{Q}(E))$, where $\mathcal{Q}(E)$ is provided with the topology with subbase $\mathcal{G}_0^{\mathcal{Q}}$ consisting of the sets in (15.8). Let $\mathfrak{B}$ be a class of subsets of E. Then

$$\mathfrak{B}^{\mathcal{Q}} := \{\mathcal{Q}(B): B \in \mathfrak{B}\}$$

is a class of subsets of $\mathcal{Q}(E)$. In view of (15.6) and (15.7) we define the $\mathfrak{B}$ topology on CAP as topology with subbase

$$\{c: c(G) > x\}, \ \{c: c(B) < x\} \quad \text{for } G \in \mathcal{G},\ B \in \mathfrak{B} \text{ and } x \in \mathbb{J}.$$

Since $c(B) = c^{\vee}(\mathcal{Q}(B))$, we see immediately that the $\mathfrak{B}$ topology on $CAP(E)$ is the same as the sup $\mathfrak{B}^{\mathcal{Q}}$ topology on $US(\mathcal{Q}(E))$.

15.7. THEOREM. *If E is locally* $\mathfrak{B}$ *and* $\mathfrak{B}$ *is closed for finite unions, then* $\mathcal{Q}(E)$ *is locally* $\mathfrak{B}^{\mathcal{Q}}$*, and CAP is* $\mathfrak{B}$ *Hausdorff.*

PROOF. The generic element of the subbase $\mathcal{G}_0^{\mathcal{Q}}$ is in (15.8). Let $K_0 \in \mathcal{Q}(G)$, so $K_0 \subset G$. For each $t \in K_0$, select a $B(t) \in \mathfrak{B}$ such that $t \in \operatorname{int} B(t) \subset B \subset G$. Then $K_0 \subset \bigcup_{t \in K_0} \operatorname{int} B(t)$. Select a finite subset $K_\#$ of K_0 such that $K_0 \subset \bigcup_{t \in K_\#} \operatorname{int} B(t)$, and set $B := \bigcup_{t \in K_\#} B(t)$. Then $B \in \mathfrak{B}$ and $K_0 \subset \operatorname{int} B \subset B \subset G$. So

$$K_0 \in \mathcal{Q}(\operatorname{int} B) \subset \mathcal{Q}(B) \subset \mathcal{Q}(G),$$

where $\mathcal{Q}(\operatorname{int} B) \in \mathcal{G}_0^{\mathcal{Q}}$ and $\mathcal{Q}(B) \in \mathfrak{B}^{\mathcal{Q}}$. So $\mathcal{Q}(E)$ is locally $\mathfrak{B}^{\mathcal{Q}}$. Then $US(\mathcal{Q}(E))$ is sup $\mathfrak{B}$ Hausdorff by Theorem 4.3(c), so CAP is $\mathfrak{B}$ Hausdorff. □

15.8. COROLLARY. If E is locally closed ($= T_3$, cf. Property 3.7(e)), then CAP is narrowly Hausdorff.

15.9. THEOREM. (a) *The space CAP is vaguely qcompact.*
(b) *If E is locally qcompact, then CAP is vaguely compact.*

PROOF. (a) Follows from Corollary 4.4(a).
(b) Combine (a) and Theorem 15.7 for $\mathfrak{B} = \mathcal{Q}$. □

15.10. LITERATURE. The present section complements and generalizes aspects of NORBERG (1986). For similar results on narrow convergence of capacities, see SALINETTI & WETS (1987), VERVAAT (1988) and future work by the author. DAL MASO (1980) has a similar approach to capacities based on topologies in spaces of increasing functions (DAL MASO (1979)).

16. SUP AND RADON MEASURES AS SPECIAL CAPACITIES

In the beginning of the previous section we observed that restrictions of sup and Radon measures to $\mathcal{K}$ are precapacities with specific behavior for unions in $\mathcal{K}$ (cf. (15.2) and (15.1)). In the present section we are going to characterize the spaces of these restrictions a subspaces of the (pre)capacities. The presentation is self-contained for sup measures. The corresponding results for Radon measures demand much more theory and are quoted from the literature.

We start with some generalities about precapacities.

16.1. LEMMA. *If c is a precapacity and (G_n) is an increasing net in $\mathcal{G}$ with union $G := \bigcup_n G_n$, then $c(G_n) \uparrow c(G)$.*

PROOF. If $x < c(G) = \bigvee_{K \subset G} c(K)$, then there is a $K \subset G$ such that $x < c(K)$. Since K is qcompact, there is an n such that $K \subset G_n$, so $x < c(G_n)$. Hence $x < \lim c(G_n)$, which proves $c(G) \leqslant \lim c(G_n)$. The reverse inequality is trivial. □

16.2. COROLLARY. Let c be a precapacity.
(a) If

$$c(G_1 \cup G_2) \;=\; c(G_1) \vee c(G_2) \quad \text{for } G_1, G_2 \in \mathcal{G}, \tag{16.1}$$

then $c(\bigcup_j G_j) = \bigvee_j c(G_j)$ for arbitrary collections in $\mathcal{G}$ (apply the lemma to the net of finite unions).
(b) If $c(G_1 \cup G_2) = c(G_1) + c(G_2)$ for disjoint G_1, G_2 in $\mathcal{G}$, then $c(\bigcup_j G_j) = \sum_j c(G_j)$ for arbitrary collections of disjoint sets in $\mathcal{G}$ (idem).

16.3. LEMMA. *If Assumption 14.9 holds and*

$$c(K_1 \cup K_2) \;=\; c(K_1) \vee c(K_2) \quad \text{for } K_1, K_2 \in \mathcal{K}, \tag{16.2}$$

then (16.1) holds.

PROOF. If $x < c(G_1 \cup G_2)$, then there is a qcompact $K \subset G_1 \cup G_2$ such that $x < c(K)$. By Assumption 14.9 there are $K_1 \subset G_1$ and $K_2 \subset G_2$ such that $K \subset K_1 \cup K_2$. Hence

$$x \;<\; c(K) \;<\; c(K_1 \cup K_2) \;=\; c(K_1) \vee c(K_2) \;\leqslant\; c(G_1) \vee c(G_2).$$

We have proved $c(G_1 \cup G_2) \leqslant c(G_1) \vee c(G_2)$. The reverse inequality is trivial. □

16.4. THEOREM. *If Assumption 14.9 holds, the a capacity c is the restriction to $\mathcal{K}$ of the extension of a sup measure on $\mathcal{G}$ iff (16.2) holds.*

PROOF. By Lemma 16.3 and Corollary 16.2(a) we see that c is a sup measure on $\mathcal{G}$. Upper semicontinuity of c guarantees that $c(K) = \bigwedge_{G \ni K} c(G)$, in accordance with Theorem 2.5(c). □

16.5. COROLLARY. If Assumption 14.9 holds, then there is for each capacity c satisfying (16.2) a unique $f \in US$ such that $c(K) = f^{\vee}(K)$ for $K \in \mathcal{Q}$.

We now turn to Radon measures and henceforth assume that E is Hausdorff. There are two different definitions of Radon measures in the literature. Following BERG ET AL. (1984) we say that a countably additive measure μ on $\operatorname{Bor} E$ is *Radon* if μ is finite on $\mathcal{K}$ and $\mu(A) = \bigvee_{K \subset A} \mu(K)$ for $A \in \operatorname{Bor} A$. Most other authors, starting with BOURBAKI (1965), require in addition that μ is *locally finite*: for each $t \in E$ there is an open $G \ni t$ such that $\mu(G) < \infty$. It is not hard to see that a Radon measure is locally finite iff its restriction to $\mathcal{K}$ is usc as a precapacity, so is a capacity.

Here is a list of plausible characterizations of finite additivity of precapacities on $\mathcal{K}$. Each line is implied by the next.

$$c(K_1 \cup K_2) \;=\; c(K_1) + c(K_2) \quad \text{if } K_1 K_2 = \varnothing\,; \tag{16.4a}$$

$$c(K_1 \cup K_2) \;\leqslant\; c(K_1) + c(K_2) \quad \&\, (16.4a)\,; \tag{16.4b}$$

$$c(K_1 \cup K_2) + c(K_1 \cap K_2) \;=\; c(K_1) + c(K_2)\,; \tag{16.4c}$$

$$c(K_1) = c(K_1 \setminus K_2) + c(K_2) \quad \text{if } K_1 \supset K_2. \tag{16.4d}$$

16.6. THEOREM. *A precapacity c is the restriction of a Radon measure to $\mathcal{K}$ iff c is finite-valued on $\mathcal{K}$ and (16.4d) holds.*

PROOF. BERG ET AL. (1984, Th.2.1.4). □

16.7. THEOREM. *A capacity c is the restriction of a (necessarily locally finite) Radon measure to $\mathcal{K}$ iff c is finite-valued on $\mathcal{K}$ and (16.4b) holds.*

PROOF. BOURBAKI (1965, Th.IX.3.1 + Remark 1). □

16.8. EXAMPLE. Let $E = \mathbb{R}$ and let $c([a, b]) := e^{b-a} - 1$ for compact intervals $[a, b]$. Extend c to finite disjoint unions of such intervals by (16.4a), and subsequently to all of $\mathcal{K}$ by (15.5). Then c is a capacity by Theorem 15.4(b). Furthermore, c satisfies (16.4a), but is not the restriction of a Radon measure, since it does not satisfy (16.4b).

16.9 LITERATURE. For related problems in partially ordered sets, see NORBERG (1987a).

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