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Reliability of stochastically forced systems (extended version)

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Reliability of Stochastically Forced Systems
(extended version)

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The reliability of a mechanical system is related to a first passage problem in the theory of stochastic dynamical systems. The phenomenon of fatigue is modelled as well, by allowing a slow change in time of the system parameters. The present methods yield expressions for the lifetime on the basis of the dynamics of the mechanical system. For that reason it is expected that better predictions of the lifetime can be made, e.g. in case of accelerated life testing, than by pure statistical methods.

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Note: This paper is an extended version of the identically titled original paper, written together with J. Grasman (Department of Mathematics, State University Utrecht, Utrecht, The Netherlands) for the proceedings of the workshop "Road-vehicle Systems and Related Mathematics", held at Turin, Italy, 1987. The sections 0-3.2 contain the original paper with some minor additions and changes, the sections 3.3-4.1 are new and illustrate computational aspects. This paper will not be submitted for publication.

0. Introduction

The lifetime of a mechanical system subjected to external perturbations is at one hand directly influenced by the frequency and amplitude of these perturbations. At the other hand, due to slow changes in the properties of the physical system (fatigue), it will enter a domain in parameter space, where even very small external perturbations may cause a break down of the system. The usual approach for analysing the reliability of systems is a pure statistical one. In this paper a different stand point is taken. We model the response of a system to stochastic perturbations and include the slow change of parameters. Usually, these problems are treated separately. However,

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in our model of wearout, it is seen that the interaction of the fast state variation of the system with the slow change of a parameter may be the key to the understanding of certain problems in mechanical reliability, like crack growth and other problems of fatigue.

In this paper, lifetime (or time to first failure) is related to the the first passage time in the theory of stochastic dynamical systems. Using asymptotic methods, we are able to derive expressions for the statistical moments of the first passage time in simple systems, which mimic the response of more complex mechanical systems to a stochastic input. In section 1 this point is explained in more detail. In section 2 we analyse such a simple system: the Ornstein-Uhlenbeck process. In order to incorporate the possibility of parameter change due to aging or wearout, we slightly modify the process. In section 3 we deal with systems with an oscillatory response. We work out a mechanical system proposed by Katz and Schuss [10], for which they introduced the concept of reliability index.

1. The response of a system to a stochastic input

We consider physical systems described by n state variables

$$X(t) = (X_1(t), \ldots, X_n(t)), \tag{1.1}$$

satisfying a nonlinear differential equation of the form

$$\frac{dX}{dt} = f(X) + \sqrt{\epsilon}\, \sigma(X)\xi(t), \tag{1.2}$$

where $\sigma(X)$ is an $n \times n$-matrix and

$$\xi(t) = (\xi_1(t), \ldots, \xi_n(t)) \tag{1.3}$$

an input vector. For stochastic inputs this vector can indeed be taken n-dimensional without loss of generality. Assuming that the vector function $f(X)$ is known, and that the stochastic input is an n-dimensional white noise process, we analyse properties of the system, like the frequency of large deviations from a stable deterministic equilibrium at, say, $X = 0$. Let in the phase space R^n a domain Ω exist, containing $X = 0$, in which the physical system operates without failure. The stochastic variable $T(x)$ is the time of first passage of the boundary $\partial\Omega$, starting at $x \in \Omega$. Its statistics are a measure for the failure time of the system. Let $g(x,t)$ be the probability density of $T(x)$. The statistical moments

$$T_m(x) = \int_0^\infty t^m g(x,t)\, dt, \qquad m = 1, 2, \ldots \tag{1.4}$$

satisfy a recurrent system of elliptic boundary value problems:

$$\frac{1}{2}\epsilon \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x) \frac{\partial^2 T_m}{\partial x_i \partial x_j} + \sum_{i=1}^{n} f_i(x) \frac{\partial T_m}{\partial x_i} = -m T_{m-1}(x), \qquad \text{in } \Omega \tag{1.5a}$$

$$T_m(x) = 0, \qquad \text{at } \partial\Omega \tag{1.5b}$$

with

$$a_{ij}(x) = \sum_{k=1}^{n} \sigma_{ik}(x)\sigma_{kj}(x), \tag{1.5c}$$

and $T_0(x) = 1$, see Gardiner [6]. In Grasman [8], it has been shown that for this class of problems an asymptotic solution, valid for $0 < \epsilon \ll 1$, can be constructed by singular perturbation techniques. The first two statistical moments of $T(0)$ give an indication of the density of the failure time, which we can approximate by a normal density, or, more in line with our first passage time approach, we can fit the inverse Gaussian density to the two statistical moments. With an eye at the reality of reliability analysis, this approach would be too academic for several reasons. First, there may be the problem that the equations of the system are not completely known, or that at forehand, it is not clear how the domain Ω is shaped. Secondly, in practice the physical system suffers from aging and wearout, meaning that parameters slowly change and may leave the domain in parameter space, where the system operates safely. Thirdly, dealing with systems with a high reliability, it is noted that the stress parameter (the equivalent of ϵ) is very small under normal conditions, which makes it difficult to obtain the failure time density from normal use. In order to get some insight, one increases the stress parameter in a test, which reduces the lifetime of the system. Then one tries to extrapolate from these test results the lifetime under normal conditions. The Weibull density, inverse Gaussian density, and Gamma density, with stress dependent parameters given as acceleration functions, are used for that purpose, see Viertl [17]. The question arises whether we can extract information from our stochastic model, which may increase the predictive power of this so-called accelerated life testing.

In order to meet the above objections, we analyse in our present approach two simple dynamical systems, representing the two basic responses of a dynamical system when brought out of equilibrium by perturbations. It is observed that by the restoring force a system returns to its stable rest state either exponentially fast or by means of a damped oscillation. This behaviour is governed by the eigenvalue(s) nearest to the imaginary axis. Other modes of the system tend to damp out more rapidly. The reciproke of the absolute value of the real part of the eigenvalue(s) nearest to the imaginary axis is the relaxation time of the full system. Thus, if we wish to study the response of a system to external perturbations, we may as well consider its dynamics in the eigenspace that corresponds to the eigenvalue(s) mentioned above. Generically, there are two possibilities:

a) One real eigenvalue is nearest to the imaginary axis. This leads us to the scalar stochastic differential equation

$$\frac{dX}{dt} = -bX + \sqrt{\epsilon}\,\sigma(X)\xi(t), \tag{1.6}$$

which will be studied in relation with reliability in the next section.

b) For two complex conjugate eigenvalues the dynamics are governed by a damped oscillation. In section 3, we will study a particular mechanical system of this type: a nonflexable rod, held in its position by a hinge, subjected to a load with a stochastic component. For this problem, Katz and Schuss [10] introduced the notion of reliability index.

2. The Ornstein-Uhlenbeck process and some extensions

In this section we analyse in more detail equation (1.6) with the following modifications. The diffusion coefficient $\sigma(X)$ is taken identically one to avoid unnecessary complications in the calculations. The parameter b will, in a later stage, depend on time and also on the state, in order to account for aging and wearout of the system, which expresses itself in a change of the restoring mechanism. Thus, the stochastic system

$$\frac{dX}{dt} = -bX + \sqrt{\epsilon}\,\xi(t), \qquad X(0) = x, \tag{2.1}$$

known as the Ornstein-Uhlenbeck process, is taken as a starting point. We consider the problem of stochastic exit from the domain $|X| < L$, assuming that for $|X| > L$ the deflection is so large that it damages the system. Appropriate scaling changes the domain into $|X| < 1$ and the parameter range into $0 < \epsilon \ll 1$. The first passage time of arriving at $|X| = 1$, starting at $X(0) = x$, with $|x| < 1$, has a density $f(x,t)$ for which the Laplace transform with respect to t can be expressed in terms of parabolic cylinder functions, see e.g. Capocelli and Ricciardi [4]. However, employing the smallness of ϵ, we arrive at much simpler asymptotic expressions for the first two statistical moments of $f(x,t)$. The moments satisfy the boundary value problems:

$$\frac{\epsilon}{2}\frac{d^2T_1}{dx^2} - bx\frac{dT_1}{dx} = -1, \qquad T_1(\pm 1) = 0, \tag{2.2}$$

and

$$\frac{\epsilon}{2}\frac{d^2T_2}{dx^2} - bx\frac{dT_2}{dx} = -2T_1(x), \qquad T_2(\pm 1) = 0, \tag{2.3}$$

which yield:

$$T_1(x) = \frac{-2}{\epsilon}\int_{-1}^{x}\int_{0}^{y} \exp\left\{\frac{b(y^2 - z^2)}{\epsilon}\right\}\, dz\; dy, \tag{2.4}$$

and

$$T_2(x) = \frac{-4}{\epsilon}\int_{-1}^{x}\int_{0}^{y} T_1(z)\exp\left\{\frac{b(y^2 - z^2)}{\epsilon}\right\}\, dz\; dy, \tag{2.5}$$

so that for $0 < \epsilon \ll 1$:

$$T_1(0) \approx \frac{1}{2b}\sqrt{\frac{\pi\epsilon}{b}}\exp\left(\frac{b}{\epsilon}\right), \qquad T_2(0) \approx 2T_1^2(0) - \frac{2}{b^2}\sqrt{\frac{\pi\epsilon}{b}}\exp\left(\frac{b}{\epsilon}\right). \tag{2.6}$$

The first expression is the expectation of the exit time if starting at $x = 0$, the variance is equal to $T_2(0) - T_1^2(0)$. On the basis of this model, we make a prediction of the lifetime density under normal conditions, given the density under heavy conditions. Since we can compute the relaxation parameter b and the proportionality factor between the physical stress parameter s and the parameter ϵ from the failure time density under heavy conditions, we just have to substitute the ϵ value for normal conditions.

2.1 Aging and reliability

We include in our model the aging of the physical system. It is assumed to change the restoring force in the following manner:

$$\frac{dX}{dt} = -b(t)X + \sqrt{\epsilon}\,\xi(t), \tag{2.7a}$$

$$\frac{db}{dt} = -\gamma h(b), \qquad \gamma, b, h > 0, \tag{2.7b}$$

where $h(b)$ is a function fitting the experimental data. We think of a positive $h(b)$, expressing the weakening of the restoring force due to aging, for example by crack growth. The deterministic system corresponding to (2.7) with $h \equiv 1$ has been depicted in Figure 1. Let b_c be a critical value for b, below which the mechanical system does not function properly. Starting from $b > b_c$, it takes the time

$$T = -\frac{1}{\gamma}\int_b^{b_c} \frac{ds}{h(s)} \tag{2.8}$$

to reach the value b_c. During this time, stochastic fluctuations work upon the system, by (2.7a). As a consequence, the point of exit at $b = b_c$ is described by a probability density function, which is derived as follows. The stationary backward equation corresponding to (2.7) is:

$$0 = -bx\frac{\partial u}{\partial x} + \frac{\epsilon}{2}\frac{\partial^2 u}{\partial x^2} - \gamma h\frac{\partial u}{\partial b}. \tag{2.9a}$$

We study exit from the region $\Omega_b = \{(X, b')|b_c \le b' \le b\}$ at the boundary $b = b_c$, which gives rise to the conditions:

$$u(x, b_c) = g(x), \tag{2.9b}$$

$$\lim_{x\to\pm\infty} u(x, b') = 0, \qquad b_c \le b' \le b. \tag{2.9c}$$

With $g(x) = \delta(x - x^*)$, the function $u(x, b)$ is the probability of exit at (x^*, b_c), starting at (x, b). By the transformation

$$y = \frac{x}{\sqrt{f(b)}}, \tag{2.10}$$

equation (2.9a) becomes:

$$0 = y\frac{\partial u}{\partial y}F(b) + \frac{\epsilon}{2}\frac{\partial^2 u}{\partial y^2} - \gamma h f\frac{\partial u}{\partial b}, \tag{2.11a}$$

with:

$$F(b) = -bf + \frac{\gamma}{2}hf'. \tag{2.11b}$$

We choose f as the positive solution of

$$F(b) \equiv -1, \tag{2.12}$$

by which equation (2.11a) takes the form

$$0 = \frac{\epsilon}{2}\frac{\partial^2 u}{\partial y^2} - y\frac{\partial u}{\partial y} - \gamma hf\frac{\partial u}{\partial b}, \tag{2.13}$$

which can be solved by separation of variables to give:

$$u(x,b) = \sum_{\lambda} e^{\frac{x^2}{2\epsilon f} - \frac{\lambda}{\gamma}\int^b \frac{ds}{h(s)f(s)}} \left[c_\lambda D_\lambda\left(\sqrt{\frac{2}{\epsilon f}}x\right) + \tilde{c}_\lambda D_{-\lambda-1}\left(i\sqrt{\frac{2}{\epsilon f}}x\right)\right], \tag{2.14}$$

where λ is the separation constant and D_ν are parabolic cylinder functions. In order to satisfy the boundary conditions (2.9c), $\lambda = n = 0, 1, 2, \ldots$ for which the parabolic cylinder functions are Hermite polynomials, and, for all λ, $\tilde{c}_\lambda = 0$. The unknown coefficients c_n are determined by the condition (2.9b). We find:

$$u(x,b) = \int_{-\infty}^{\infty} g(y)\frac{e^{-\frac{y^2}{\epsilon f(b_c)}}}{\sqrt{\pi\epsilon f(b_c)}} H(x,y,b)\, dy, \tag{2.15a}$$

with

$$H(x,y,b) = \sum_{n=0}^{\infty} \frac{1}{2^n n!} H_n\left(\frac{x}{\sqrt{\epsilon f(b)}}\right) H_n\left(\frac{y}{\sqrt{\epsilon f(b_c)}}\right) e^{-\frac{n}{\gamma}\int_{b_c}^{b}\frac{ds}{h(s)f(s)}} \tag{2.15b}$$

The sum (2.15b) can be evaluated using the following generating function formula, [11, p.252]:

$$(1-z^2)^{-\frac{1}{2}} \exp\left[y^2 - \frac{(y-zx)^2}{1-z^2}\right] = \sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)}{2^n n!} z^n. \tag{2.16}$$

The function in the integrand of (2.15a), apart from $g(y)$, is the probability density of exit at $b = b_c$, starting from (x, b). The result (2.15) is valid for all $\epsilon, \gamma > 0$. In applications, ϵ is often small and γ very small. For $\gamma \ll 1$ and $b_c \gg O(\sqrt{\gamma})$, equation (2.11b) with (2.12) is solved by

$$f(b) = \frac{1}{b}, \tag{2.17}$$

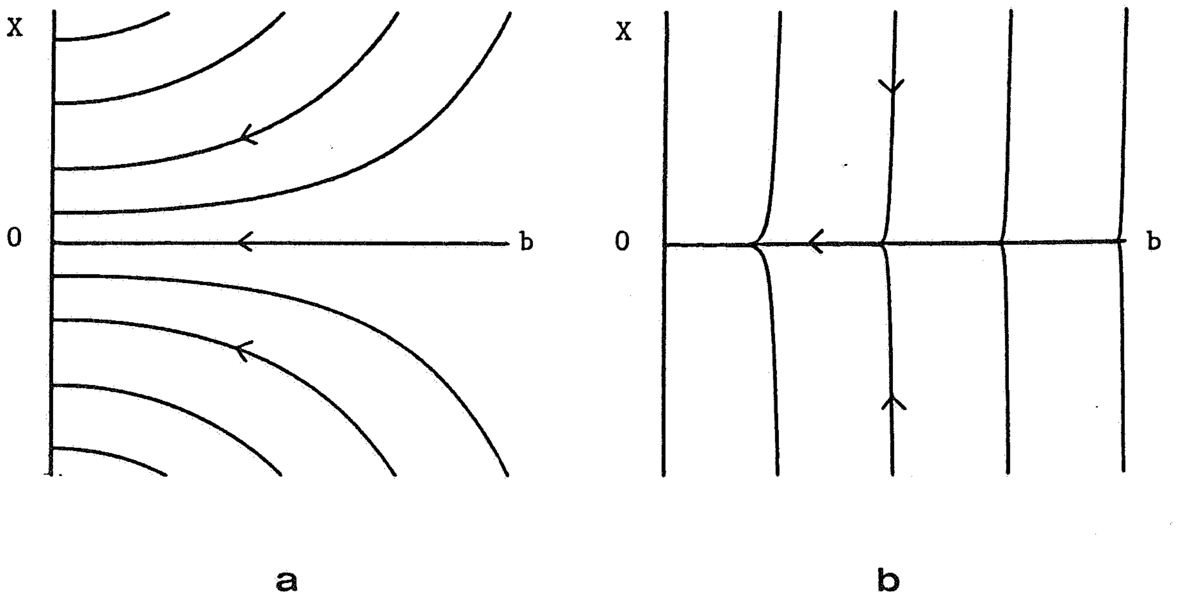


Fig 1. The deterministic system corresponding to the aging model with $h(b) \equiv 1$, and a) $\gamma = 1$, b) $\gamma = .01$. In practice γ is a very small parameter, the behaviour is qualitatively as shown in b), with X approaching the value zero quickly after starting.

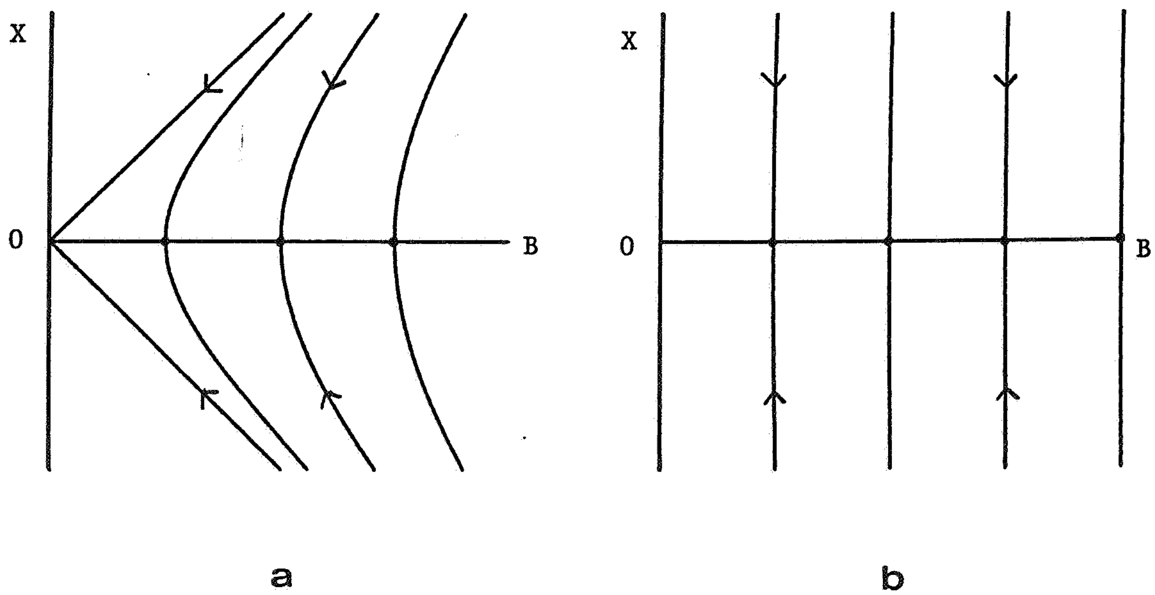


Fig 2. The deterministic system corresponding to the wearout model with $h(X)$ defined by (2.22) and a) $\gamma = 1$, b) $\gamma = .01$. In practice X is a fast variable compared to B, as in b).

and, assuming that $b - b_c$ is larger than $O(\gamma)$, so that $H \approx 1$, the exit density becomes:

$$\sqrt{\frac{b_c}{\epsilon\pi}} \exp\left(-\frac{y^2 b_c}{\epsilon}\right). \tag{2.18}$$

Note that (2.18) solves the stationary forward equation corresponding to (2.7a) with $b = b_c$. This is in accordance with intuition. The results (2.15, 2.18) can be expressed in terms of T instead of b_c, by the relation (2.8). As an example, we take $h(b) = 1$, that is, the restoring force decreases linearly with time. The density at time T, starting at (x, b) on $T = 0$ is given by:

$$\sqrt{\frac{b-\gamma T}{\pi\epsilon}}\, e^{-\frac{y^2(b-\gamma T)}{\epsilon}} \sum_{n=0}^{\infty} \frac{1}{2^n n!} H_n\left(\sqrt{\frac{b}{\epsilon}}x\right) H_n\left(\sqrt{\frac{b-\gamma T}{\epsilon}}y\right) e^{-nT(b-\frac{\gamma}{2}T)}. \tag{2.19}$$

After the relatively short initial time period of order $T \sim O(\gamma^0)$, (2.19) becomes:

$$\sqrt{\frac{b-\gamma T}{\pi\epsilon}}\, e^{-\frac{y^2(b-\gamma T)}{\epsilon}}, \tag{2.20}$$

which is valid on the time scale $T \sim O(\gamma^{-1})$. From the densities (2.18, 2.20) we can compute the probability that the deflection X of the system is above a certain unsafe value and set out a standard for b_c, or the replacement time of the mechanical system.

2.2 Wearout and reliability

We now consider the situation that the restoring mechanism weakens due to previous deflections. Then the system obeys:

$$\frac{dX}{dt} = -BX + \sqrt{\epsilon}\,\xi(t), \tag{2.21a}$$

$$\frac{dB}{dt} = -\gamma h(X), \qquad \gamma, h > 0. \tag{2.21b}$$

The function h determines how wearout depends on the deflections X. We shall continue with the function

$$h(X) = X^2, \tag{2.22}$$

expressing an increasing wearout for larger deflections. Note that in this model wearout has a feedback mechanism that speeds itself up: large deflections cause weakening of the restoring force, which in turn allows for larger deflections. In Figure 2 the deterministic system corresponding to (2.21) has been depicted. We will derive an approximate expression for the decay of B with time. By assumption, γ is a very small parameter, so that X is a fast variable compared to B. As we have seen in the discussion of the aging model (2.7), the density of X for a given value b of B is then approximately the stationary density corresponding to (2.21a), i.e.:

$$v(x|B=b) = \sqrt{\frac{b}{\epsilon\pi}}\, e^{-\frac{x^2 b}{\epsilon}}. \tag{2.23}$$

The function (2.22) is approximated by its expectation value with respect to the density (2.23):

$$E(X^2|B=b) = \int_{-\infty}^{\infty} x^2 \sqrt{\frac{b}{\epsilon\pi}}\, e^{-\frac{x^2 b}{\epsilon}}\, dx = \frac{\epsilon}{2b}. \tag{2.24}$$

Using this expression for h, equation (2.21b) leads to

$$\frac{db}{dt} = -\frac{\gamma\epsilon}{2b}, \tag{2.25}$$

which is solved by:

$$b = \sqrt{b_0^2 - \gamma\epsilon t}, \tag{2.26}$$

where b and b_0 are the values of B at the times t and 0, respectively. Starting from $B = b_0$, it takes the time

$$t = \frac{b_0^2 - b_c^2}{\gamma\epsilon} \tag{2.27}$$

to reach a critical value b_c. Combining the results (2.23, 2.26), the probability density of the system (2.21) is approximately:

$$v(x,t) \approx \frac{(b_0^2 - \gamma\epsilon t)^{\frac{1}{4}}}{\sqrt{\epsilon\pi}}\, e^{-\frac{x^2}{\epsilon}\sqrt{b_0^2 - \gamma\epsilon t}}. \tag{2.28}$$

The way we treated the X-variable is an example of adiabatic elimination of the fast variable, see for example Gardiner [6].

3. Stochastic damped oscillation

In this section we will analyse in detail the following model problem. An unloaded stiff rod of length l, with mass m at distance l' from the hinge O, and spring constant μ at O, carries out small oscillations around the equilibrium position $\varphi = 0$. Next a load P_d is applied to the rod, acting under the small angle φ_d, as indicated in Figure 3a. As a consequence, a new equilibrium position

$$\varphi_e = \frac{-\varphi_d P_d}{\mu l - P_d} \tag{3.1}$$

is established at the place of minimal potential energy (= 0 at φ_e, by definition). The Lagrange equation of motion reads:

$$m\frac{l'^2}{l^2}\frac{\partial^2 z}{\partial t^2} + \mu z - \left(\frac{z}{l} - \varphi_d\right) P_d = 0, \qquad z = l\varphi. \tag{3.2}$$

To describe a realistic system, the model is extended with a stochastic load component and damping:

$$m\frac{l'^2}{l^2}\frac{\partial^2 z}{\partial t^2} + \alpha\frac{\partial z}{\partial t} + \mu z - \left(\frac{z}{l} - \varphi_d\right) P_d = \left|\frac{z}{l} - \varphi_s\right| \ \xi(\gamma t) P_s, \tag{3.3}$$

where ξ is a Gaussian white noise process with intensity one; the square of the factor in front of ξ is the intensity ($= 0$ for $\varphi = \varphi_s$, which is a small angle) and γ^{-1} the time scale of the stochastic process; P_s is a load; α is the damping constant. Suppose the rod is part of a mechanical structure, that functions well as long as the energy of the rod is below a critical energy R^2. By

$$\tilde{m} = m\frac{l'^2}{l^2}, \tag{3.4a}$$

$$\tilde{\mu} = \mu - \frac{P_d}{l}, \qquad \text{(we assume the no 'buckling' condition } \tilde{\mu} > 0\text{)}, \tag{3.4b}$$

the transformations

$$z^* = z\frac{\sqrt{\tilde{\mu}}}{R}, \qquad t^* = t\sqrt{\frac{\tilde{\mu}}{\tilde{m}}}, \qquad P_d^* = \frac{P_d}{R\sqrt{\tilde{\mu}}},$$
$$P_s^* = \frac{P_s}{R\sqrt{\tilde{\mu}}}, \qquad \gamma^* = \gamma\sqrt{\frac{\tilde{m}}{\tilde{\mu}}}, \tag{3.4c}$$

and the white noise scaling property

$$\xi(\gamma t) = \xi(\gamma^* t^*) = \frac{1}{\sqrt{\gamma^*}}\ \xi(t^*), \tag{3.4d}$$

we obtain the equation of motion in dimensionless variables (denoted by *):

$$\frac{\partial^2 z^*}{\partial t^{*2}} + \frac{\alpha}{\sqrt{\tilde{m}\tilde{\mu}}}\frac{\partial z^*}{\partial t^*} + z^* + \varphi_d P_d^* = \frac{P_s^*}{\sqrt{\gamma^*}}\left|\frac{R}{l\sqrt{\tilde{\mu}}}z^* - \varphi_s\right| \ \xi(t^*). \tag{3.5}$$

We make the following assumptions on the magnitude of the various terms:

$$\frac{P_s^*}{\sqrt{\gamma^*}} = \sqrt{\epsilon}, \qquad \frac{R}{l\sqrt{\tilde{\mu}}} = k_1, \qquad \frac{\alpha}{\sqrt{\tilde{m}\tilde{\mu}}} = \epsilon k, \tag{3.6}$$

in which $0 < \epsilon \ll 1$ and k, k_1 are order ϵ^0 constants. Equation (3.5) becomes:

$$\frac{\partial^2 z^*}{\partial t^{*2}} + \epsilon k\frac{\partial z^*}{\partial t^*} + z^* + \varphi_d P_d^* = \sqrt{\epsilon}\,|k_1 z^* - \varphi_s| \ \xi(t^*), \tag{3.7}$$

describing an $O(\epsilon)$ damped oscillation of the stiff rod from a load consisting of a deterministic part of $O(\epsilon^0)$ and a stochastic part with intensity of $O(\epsilon)$. Introduction of the variable

$$\eta = z^* + \varphi_d P_d^*, \tag{3.8}$$

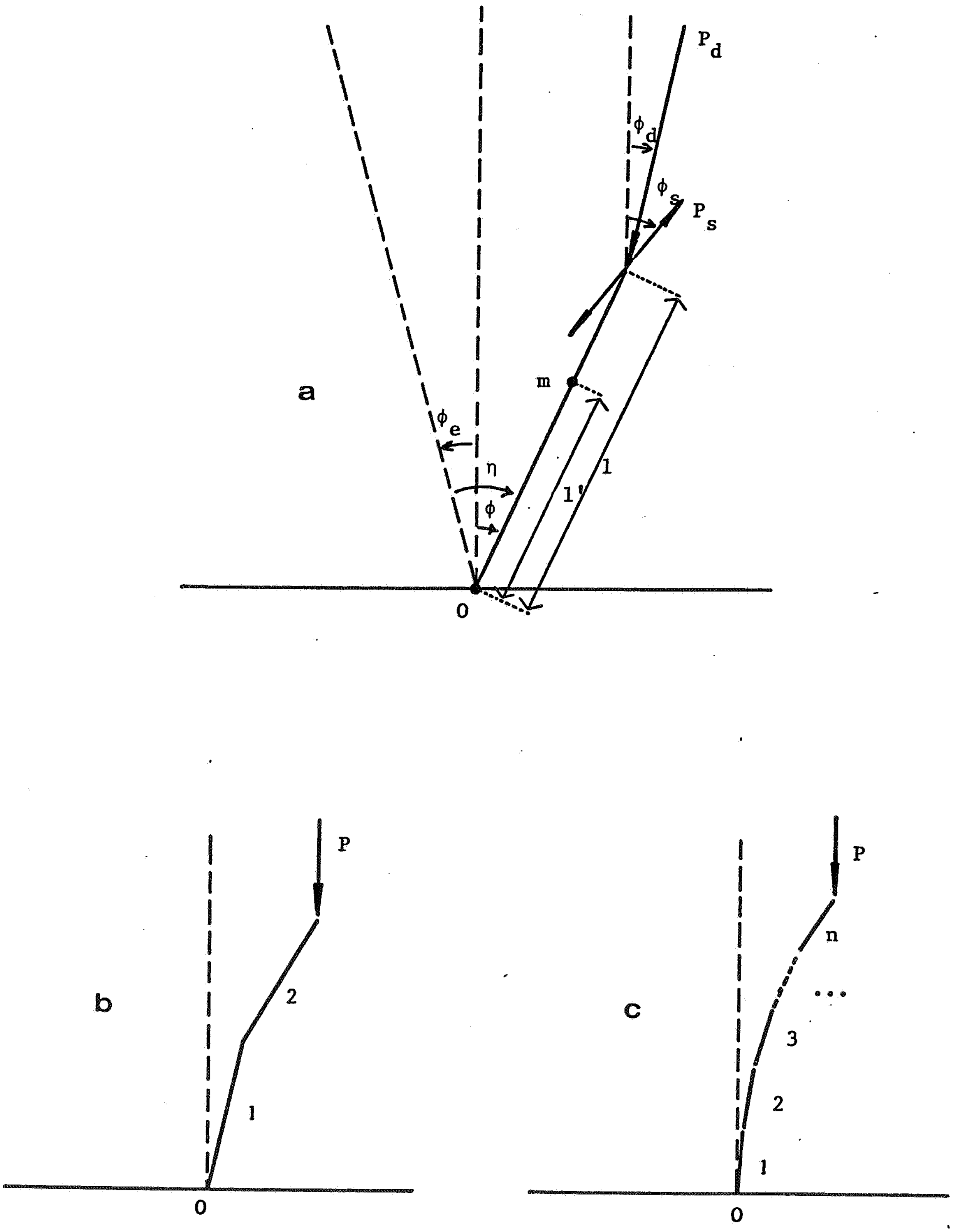


Fig 3. (a) The loaded stiff rod,
(b) The vertically loaded double pendulum,
(c) The vertically loaded n-fold pendulum.

that measures the deviation from the equilibrium position (3.1), leads to:

$$\ddot{\eta} + \epsilon k\dot{\eta} + \eta = \sqrt{\epsilon}\; |k_1\eta - k_0|\; \xi, \tag{3.9a}$$

where

$$k_0 = \varphi_d P_d^* k_1 + \varphi_s, \tag{3.9b}$$

and the dot denotes differentiation with respect to t^*.

3.1 The exit problem

The backward equation corresponding to (3.9) reads:

$$\frac{\partial u}{\partial t^*} = \dot{\eta}\frac{\partial u}{\partial \eta} - (\epsilon k\dot{\eta} + \eta)\frac{\partial u}{\partial \dot{\eta}} + \frac{\epsilon}{2}\,(k_1\eta - k_0)^2\frac{\partial^2 u}{\partial \dot{\eta}^2}. \tag{3.10}$$

This equation is studied asymptotically for small ϵ on a time scale of order ϵ^{-1}. With the expansions

$$t^* = \tilde{t}\epsilon^{-1}, \tag{3.11a}$$
$$u = u^0 + \epsilon u^1, \tag{3.11b}$$

and the transformation $(\eta, \dot{\eta}) \rightarrow (r, \theta)$ defined by

$$\eta = \sqrt{2r}\cos\theta, \qquad \dot{\eta} = \sqrt{2r}\sin\theta, \tag{3.12}$$

we obtain to leading order in ϵ:

$$\frac{\partial u^0}{\partial \theta} = 0, \tag{3.13}$$

implying $u^0 = u^0(r, \tilde{t})$. Note that r^2 is the dimensionless energy of the undisturbed ($\epsilon = 0$) system, $r \in [0, 1]$ and $r^2 = 1$ corresponds to the critical energy. If the initial density is a delta function, i.e. the initial pair $(\eta, \dot{\eta})$ is deterministic, the leading order system is deterministic and Hamiltonian. The trajectories in phase space are energy contours. To the next order in ϵ an equation is obtained in terms of u^0 and u^1. Terms with u^1 vanish by integrating this equation with respect to θ from 0 to 2π and the additional assumption that u^1 is periodic in θ with period 2π. The resulting equation for u^0 reads:

$$\frac{\partial u^0}{\partial \tilde{t}} = \left(\frac{a_0}{r} + a_2 r\right)\frac{\partial u^0}{\partial r} + \left(a_0 + a_1 r^2\right)\frac{\partial^2 u^0}{\partial r^2}, \tag{3.14a}$$

with

$$a_0 = \frac{1}{8}k_0^2, \qquad a_1 = \frac{1}{16}k_1^2, \qquad a_2 = \frac{3}{16}k_1^2 - \frac{1}{2}k. \tag{3.14b}$$

Thus, the description to this order in ϵ includes the damping and stochastic effects. If, as a consequence of the latter effect, the critical energy $r^2 = 1$ is reached in finite time with probability one, starting from $r \in [0, 1]$, the rod is *stochastically unstable*

[15]. In that case, the measure of stability (reliability) of the rod can be indicated by the mean exit time (index of reliability) from the unit interval at 1. Below we discuss this exit problem from the unit interval.

Case 1: $k_0 \neq 0$. In this case $a_0 > 0$. Note that k_1 and so a_1 are always positive. The boundary $r = 0$ of the unit interval is classified as an *entrance* boundary, see for example Roughgarden [14], meaning that $r = 0$ cannot be reached in finite time from the interior of the interval, and the interior can be reached in finite time from $r = 0$. The boundary $r = 1$ is *regular*, it can be reached in finite time from the interior of the interval, and vice versa. Thus, exit from the unit interval can take place only at $r = 1$. Let $u_s(r)$ be the probability of exit at $r = 1$, starting from a point $r \in [0, 1]$. Its leading order part $u_s^0(r)$ satisfies the stationary backward equation (3.14a) with boundary conditions: $u_s^0(0)$ is finite, $u_s^0(1) = 1$. We find $u_s(r) \sim u_s^0(r) \equiv 1$, so that exit at $r = 1$ will occur with probability one. The expected exit time $T(r) \sim \tilde{T}(r)\epsilon^{-1}$, starting from a point r is found by solving the Dynkin equation, see Gardiner [6]:

$$-1 = \left(\frac{a_0}{r} + a_2 r\right) \frac{\partial \tilde{T}}{\partial r} + \left(a_0 + a_1 r^2\right) \frac{\partial^2 \tilde{T}}{\partial r^2}, \tag{3.15a}$$

with the conditions:

$$\tilde{T}(0) \text{ is finite,} \tag{3.15b}$$

$$\tilde{T}(1) = 0. \tag{3.15c}$$

We find:

$$T(r) \sim \frac{1}{\epsilon(a_1 - a_2)} \int_r^1 \left[\left(\frac{a_1}{a_0} s^2 + 1\right)^{-\frac{a_2}{2a_1} + \frac{1}{2}} - 1\right] \frac{1}{s}\, ds, \quad (a_2 \neq a_1). \tag{3.16}$$

If $a_2 = a_1$, this is substituted into equation (3.15a). Solving the corresponding boundary value problem we arrive at:

$$T(r) \sim \frac{1}{2\epsilon a_1} \int_r^1 \frac{1}{s} \log\left(\frac{a_1}{a_0} s^2 + 1\right) ds. \tag{3.17}$$

Case 2: $k_0 = 0$. For $k_0 = 0$ the angles φ_d, φ_s are related by:

$$0 = \varphi_d P_d^* k_1 + \varphi_s. \tag{3.18}$$

An important case is the exactly vertically loaded pendulum ($\varphi_d = \varphi_s = 0$). Since $a_0 = 0$, the behaviour is qualitatively different from that in case 1. At $r = 0$ we now have a *natural* boundary, meaning that $r = 0$ cannot be reached from the interior of the unit interval, and vice versa, in finite time. The point $r = 0$ is an isolated point. Let $u_s(r)$ now be defined as the probability of exit at $r = 1$, starting from a point r of the half open interval $(0, 1]$. This probability is obtained by solving the stationary backward equation (3.14a) with the boundary conditions $u_s^0(1) = 1,\ u_s^0(\delta) = 0$, in which $0 < \delta \ll 1$. In the limit $\delta \to 0$ we obtain:

$$u_s(r) \sim \begin{cases} 1, & a_2 > a_1, \quad \text{(slight damping)}, \\ \left(\dfrac{1}{r}\right)^{\dfrac{a_2 - a_1}{a_1}}, & a_2 < a_1, \quad \text{(heavy damping)}. \end{cases} \tag{3.19}$$

In the first case of (3.19), exit at $r = 1$ occurs with probability one, as in case 1 the rod is stochastically unstable. In the second case of (3.19), the probability of exit at $r = 1$ can be made arbitrarily small by starting close enough (but not at) $r = 0$, the rod is *stochastically stable*. We will continue for the stochastically unstable case. The expected exit time is obtained by solving equation (3.15a) with $a_0 = 0$ under the conditions $\tilde{T}(1) = 0,\ \tilde{T}(\delta) = 0,\ 0 < \delta \ll 1$. In the limit $\delta \to 0$:

$$T(r) \sim \frac{1}{\epsilon(a_2 - a_1)} \log \frac{1}{r}. \tag{3.20}$$

Because the forward and backward equations have a simple form, a variety of interesting expressions can be derived, see the sections 3.3-3.5. Some results are given here. In the remainder of this paper, the dimensionless time t^* is denoted by t, for convenience. Let $\tau(r,t)dt$ be the probability that the time of exit through $r = 1$, starting in $r \in (0,1]$ on $t = 0$, is in $(t, t+dt)$. We have:

$$\frac{1}{\epsilon a_1}\tau(r, \frac{t}{\epsilon a_1}) \sim \frac{r^{-c}}{2\sqrt{\pi}}\left(\log \frac{1}{r}\right) t^{-\dfrac{3}{2}} e^{-c^2 t - \dfrac{(\log r)^2}{4t}}, \tag{3.21a}$$

where

$$c = \frac{a_2 - a_1}{2a_1}, \qquad (0 < c \leq 1, \quad c = 1\text{: no damping}) \tag{3.21b}$$

see Figure 4a. The function $\tau(r,t)$ is a probability density with respect to t. The corresponding n-th cumulant $\kappa_n(r)$ yields (note that $T = \kappa_1$):

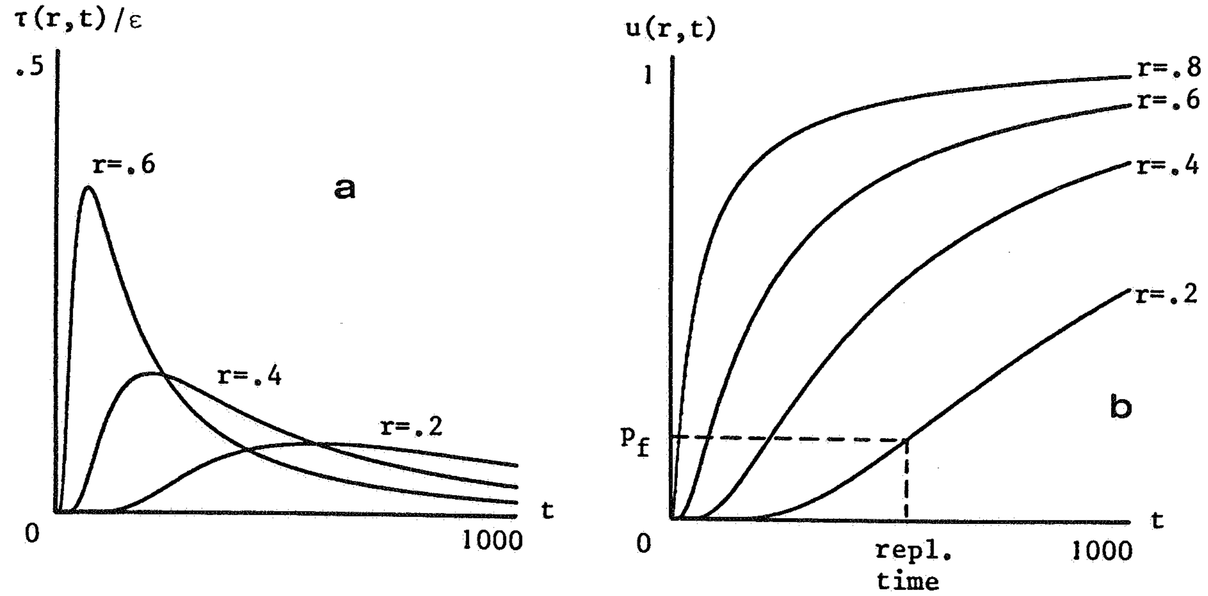
$$\kappa_n(r) \sim \left(4\epsilon a_1 c^2\right)^{-n} \frac{(2n-2)!}{(n-1)!} 2c \log \frac{1}{r}, \qquad n \geq 1. \tag{3.22}$$

Denoting by $u(r,t)$ the probability that at or before time t exit has taken place through $r = 1$, starting in $r \in (0,1]$ on $t = 0$, we have

$$u(r, \frac{t}{\epsilon a_1}) \sim \frac{1}{2}\text{erfc}\left[\frac{\log \frac{1}{r}}{2\sqrt{t}} - c\sqrt{t}\right] + \frac{1}{2} r^{-2c}\text{erfc}\left[\frac{\log \frac{1}{r}}{2\sqrt{t}} + c\sqrt{t}\right], \tag{3.23}$$

where erfc is the complementary error function [1], see Figure 4b. Let $v(r,t)dr$ describe the probability of being in $(r, r+dr)$ at time t, given the probability $v_0(r')dr'$ of being in $(r', r'+dr')$ at $t = 0$, $r, r' \in (0,1]$. We find:

$$v(r, \frac{t}{\epsilon a_1}) \sim \int_0^1 v_0(r')\frac{1}{r}\left(\frac{r}{r'}\right)^c \frac{e^{-c^2 t}}{2\sqrt{\pi t}}\left[e^{-\dfrac{(\log \frac{r}{r'})^2}{4t}} - e^{-\dfrac{(\log r'r)^2}{4t}}\right] dr', \tag{3.24}$$



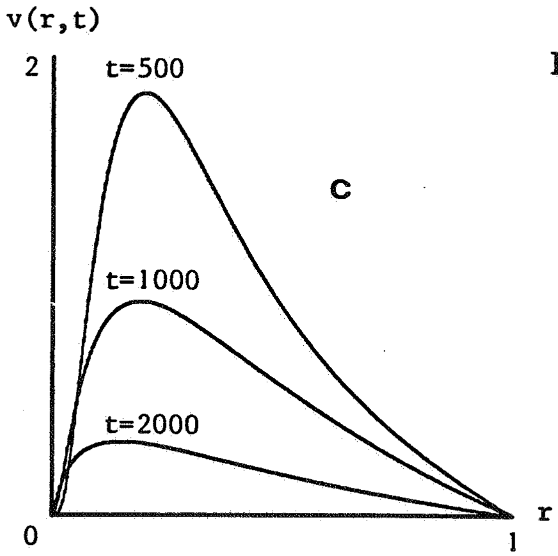


Fig 4. Results for case 2 without damping.

(a) The probability density function of the exit time, depending on the starting point r.

(b) The probability $u(r,t)$. For $r = .2$, the replacement time corresponding to the probability of failure p_f has been indicated.

(c) The probability density $v(r,t)$ with initial density $v_0(r) = \delta(r - .2)$.

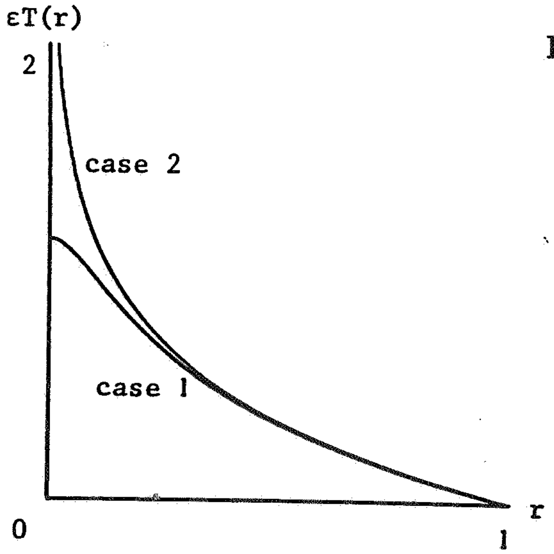


Fig. 5 The expected exit time for the nearly vertically loaded undamped stiff rod, according to case 1 and case 2.

see Figure 4c. Figure 4 refers to the undamped stiff rod with $\epsilon = .01$ and $a_1 = 1/16$. A difficulty in treating problems on a time scale of order ϵ^{-1} is the formulation of initial conditions. The results above have been derived using the original initial conditions, as if nothing happened on the lower time scale.

3.2 Some remarks

For applications, these results have the following relevance. Expressions (3.16), (3.17), (3.20) give the mean time to exit, which can be adopted as a measure for the expected failure time of a system. Formula (3.22) with $n = 2$ gives the variance in exit time. The exit time density (3.21), depicted in Figure 4a, may be compared with functions that are often used to fit the failure density vs. time or the number of cycles, as the exponential, Gamma, Weibull and lognormal densities, see Figure 1 in [13]. A replacement policy can be based on (3.23). On accepting a probability p_f of failure exit, the replacement time is determined by $u(r,t) = p_f$, see Figure 4b. The relation (3.24) can be used for the construction of confidence intervals.

The case 2 that we distinguished above, applies on the condition (3.18). In practice this condition is never satisfied exactly. In view of the asymptotics that we have used, the left side of (3.18) may be replaced by $O(\epsilon^{\frac{1}{2}})$ in order for case 2 to remain applicable. If in practice the left side of (3.18) is of order $O(\epsilon^0)$, then case 2 does not make sense, and all situations are described by case 1. As an example, consider the nearly vertically loaded pendulum. If relation (3.18) is satisfied within $O(\epsilon^{\frac{1}{2}})$, the mean exit time is given by (3.20), otherwise it is given by (3.16) with small a_0 measuring the deviation from exactly vertical. For the undamped case ($a_2 = 3a_1$) these exit times yield respectively:

$$T(r) \sim \frac{1}{2\epsilon a_1} \log \frac{1}{r}, \tag{3.25a}$$

$$T(r) \sim \frac{1}{2\epsilon a_1} \log \sqrt{\frac{a_1 + a_0}{a_1 r^2 + a_0}}, \tag{3.25b}$$

which have been plotted in Figure 5 for $a_0 = .01$, $a_1 = 1$. Moreover, we see that (3.25a) is a good approximation to (3.25b) away from $r = 0$.

Katz and Schuss [10] consider the vertically loaded double and n-fold pendula, see Figure 3bc, as extensions of the vertically loaded stiff rod. They model the elastic continuous column as an n-fold pendulum in the limit for large n, requireing in this limit the kinetic and potential energies of the pendulum to converge to those of the column. Remarkably, the stability of those more complicated models is described by the same boundary value problem (3.14), with $a_0 = 0$, as for the vertically loaded stiff rod (our case 2), though with a different meaning of a_1 and a_2.

In practice, oscillators may arise with a different type of damping (as cubic damping) or noise (red noise, dichotomic noise, etc.), and with a general forcing that can be described by a potential function. The stochastic stability of such oscillators has been treated by Dygas, Matkowsky and Schuss [5].

3.3 The probability $u(r, t)$

In this section we derive the formula (3.23) for $u(r,t)$, the probability that has been defined in section 3.1. This formula concerns the distinguished case 2 (with $a_2 > a_1$), for which the backward equation (3.14a) reads:

$$\frac{\partial u^0}{\partial \tilde{t}} = a_2 r \frac{\partial u^0}{\partial r} + a_1 r^2 \frac{\partial^2 u^0}{\partial r^2}. \tag{3.26a}$$

This equation has to be solved with the conditions:

$$u^0(r,0) = 0, \tag{3.26b}$$
$$u^0(\delta,\tilde{t}) = 0, \qquad \delta \to 0, \tag{3.26c}$$
$$u^0(1,\tilde{t}) = 1. \tag{3.26d}$$

Taking the Laplace transform with respect to $\tilde{t}$:

$$\bar{u}^0(r,s) = \int_0^\infty e^{-s\tilde{t}}\, u^0(r,\tilde{t})\, d\tilde{t}, \tag{3.27}$$

the following boundary value problem is obtained:

$$a_1 r^2 \frac{\partial^2 \bar{u}^0}{\partial r^2} + a_2 r \frac{\partial \bar{u}^0}{\partial r} = s\bar{u}^0 - u^0(r,0) = s\bar{u}^0, \tag{3.28a}$$

$$\bar{u}^0(\delta,s) = 0, \qquad \delta \to 0, \tag{3.28b}$$
$$\bar{u}^0(1,s) = \frac{1}{s}. \tag{3.28c}$$

The equation (3.28a) is an equidimensional or Euler equation [2], that by the change of variable

$$r = e^\rho, \tag{3.29}$$

transforms into:

$$a_1 \frac{\partial^2 \bar{u}^0}{\partial \rho^2} + (a_2 - a_1)\frac{\partial \bar{u}^0}{\partial \rho} - s\bar{u}^0 = 0, \tag{3.30}$$

where the coefficients of the derivatives have become constants. The characteristic equation

$$a_1 k^2 + (a_2 - a_1)k - s = 0 \tag{3.31}$$

corresponding to this equation is solved by:

$$k_1(s) = \frac{-(a_2 - a_1) + \sqrt{(a_2 - a_1)^2 + 4a_1 s}}{2a_1}, \tag{3.32a}$$

$$k_2(s) = \frac{-(a_2 - a_1) - \sqrt{(a_2 - a_1)^2 + 4a_1 s}}{2a_1}. \tag{3.32b}$$

The solution of (3.30) is:

$$\bar{u}^0 = C_1(s)\ e^{k_1(s)\rho} + C_2(s)\ e^{k_2(s)\rho}, \tag{3.33}$$

and that of (3.28*a*):

$$\bar{u}^0 = C_1(s)\ r^{k_1(s)} + C_2(s)\ r^{k_2(s)}. \tag{3.34}$$

The unknown functions C_1 and C_2 follow from the boundary conditions (3.28*bc*):

$$\lim_{\delta\to 0} C_2(s) = \lim_{\delta\to 0} \frac{-1}{s(\delta^{k_2-k_1}-1)} = 0, \tag{3.35a}$$

in which we used the positivity of a_1, and, in the same limit:

$$C_1(s) = \frac{1}{s}. \tag{3.35b}$$

The solution of (3.28) is:

$$\bar{u}^0(r,s) = \frac{r^{k_1(s)}}{s}. \tag{3.36}$$

Using some elementary properties of Laplace transforms and the inverse transform formula 5.129 [12, p.264], we obtain as solution of problem (3.26):

$$u^0(r,\frac{\tilde{t}}{a_1}) = \frac{1}{2}\mathrm{erfc}\left[\frac{\log\frac{1}{r}}{2\sqrt{\tilde{t}}} - c\sqrt{\tilde{t}}\right] + \frac{1}{2}r^{-2c}\mathrm{erfc}\left[\frac{\log\frac{1}{r}}{2\sqrt{\tilde{t}}} + c\sqrt{\tilde{t}}\right], \tag{3.37}$$

and from $u(r,t) \sim u^0(r,\tilde{t})$ formula (3.23) follows. It is easily verified that the results (3.23) and (3.19) are in agreement with

$$\lim_{t\to\infty} u(r,t) = u_s(r). \tag{3.38}$$

3.4 The exit time

Next we will derive the formulas (3.21) and (3.22), successively. The function

$$1 - \int_0^t \tau(r,t')\ dt', \tag{3.39}$$

with τ defined in section 3.1, is known to satisfy the backward equation [6]. By differentiation with respect to t, it then follows that τ satisfies the backward equation as well, so that for its leading order term τ^0 we have:

$$\frac{\partial\tau^0}{\partial\tilde{t}} = a_1 r^2\frac{\partial^2\tau^0}{\partial r^2} + a_2 r\frac{\partial\tau^0}{\partial r}. \tag{3.40}$$

Taking the Laplace transform of τ^0 with respect to $\tilde{t}$:

$$\bar{\tau}^0(r,s) = \int_0^\infty e^{-s\tilde{t}}\ \tau^0(r,\tilde{t})\ d\tilde{t}, \tag{3.41}$$

equation (3.40) becomes:

$$a_1 r^2 \frac{\partial^2 \bar{\tau}^0}{\partial r^2} + a_2 r \frac{\partial \bar{\tau}^0}{\partial r} = s\bar{\tau}^0 - \tau^0(r,0) = s\bar{\tau}^0, \tag{3.42}$$

in which the initial condition

$$\tau^0(r,0) = 0 \tag{3.43}$$

has been used, valid for $\delta < r < 1$, where again $0 < \delta \ll 1$. Note that $\tau(r,t)$ is a probability density with respect to t. For $r = \delta$ and $r = 1$, this density equals the delta function $\delta(t-0)$, or:

$$\mu_0^0(r) = 1, \tag{3.44a}$$
$$\mu_m^0(r) = 0, \quad m > 0, \tag{3.44b}$$

where $\mu_m^0(r)$ is the m-th moment about the origin $\tilde{t} = 0$ of $\epsilon^{-1}\tau^0(r,\tilde{t})$. Expression (3.44$a$) results from the normalisation of the density $\tau(r,t)$ and the expansion:

$$\tau(r,t) \sim \tau^0(r,\tilde{t}). \tag{3.45}$$

By the relation [9]:

$$\epsilon^{-1}\bar{\tau}^0(r,s) = G^0(r,is) = \sum_{m=0}^{\infty} \frac{(-s)^m}{m!}\,\mu_m^0(r), \tag{3.46}$$

in which G^0 is the characteristic or moment generating function of $\epsilon^{-1}\tau^0$, the conditions (3.44) translate into boundary conditions for (3.42):

$$\bar{\tau}^0(1,s) = \bar{\tau}^0(\delta,s) = \epsilon, \quad (\delta \to 0). \tag{3.47}$$

The boundary value problem (3.42, 3.47) is solved in a way similar to that in the previous section. We find:

$$\lim_{\delta\to 0} C_2(s) = \lim_{\delta\to 0} \epsilon\,\frac{\delta^{-k_1} - 1}{\delta^{k_2-k_1} - 1} = 0, \tag{3.48a}$$

because $a_1 > 0$, and, in this limit:

$$C_1(s) = \epsilon, \tag{3.48b}$$

so that:

$$\bar{\tau}^0(r,s) = \epsilon\, r^{k_1(s)}. \tag{3.49}$$

Using the inverse Laplace transform formula 5.85 [12, p.258], the following result is obtained:

$$\frac{1}{\epsilon a_1}\tau^0(r, \frac{\tilde{t}}{a_1}) = \frac{r^{-c}}{2\sqrt{\pi}}\Big(\log\frac{1}{r}\Big)\tilde{t}^{-\frac{3}{2}}\, e^{-c^2\tilde{t} - \frac{(\log r)^2}{4\tilde{t}}}, \tag{3.50}$$

and with (3.45), expression (3.21a) follows. Next we will derive expressions for the moments and cumulants of the density $\tau(r,t)$. The moments $\mu_n(r)$ about $t=0$ of $\tau(r,t)$ are found starting from the definition:

$$\mu_n(r) = \int_0^\infty t^n \tau(r,t)\, dt. \tag{3.51}$$

Using formula (3.21a) for $\tau(r,t)$, formulas 3.471:9 [7, p.340] and 8.468 [7, p.967], it is found after some calculation that:

$$\mu_n(r) \sim (4\epsilon a_1 c^2)^{-n} \sum_{j=1}^{n} \frac{(2n-j-1)!}{(j-1)!(n-j)!} (2c \log r^{-1})^j, \quad n \geq 1. \tag{3.52}$$

With (3.49) the moment generating function G of $\tau(r,t)$ yields:

$$G(r,is) = \bar{\tau}(r,s) \sim \frac{1}{\epsilon}\, \bar{\tau}^0(r, \frac{s}{\epsilon}) = r^{k_1(\frac{s}{\epsilon})}. \tag{3.53}$$

Taylor expanding the logarithm of $G(r,is)$ around $s=0$ we find:

$$\log G(r,is) \sim \sum_{n=1}^{\infty} \frac{(-s)^n}{n!}\, \kappa_n(r), \tag{3.54}$$

with the cumulants $\kappa_n(r)$ given as in (3.22), or, in a different notation [7, p.xliii]:

$$\kappa_n(r) \sim \frac{(2n-3)!!}{(2a_1\epsilon)^n c^{2n-1}} \log \frac{1}{r}. \tag{3.55}$$

3.5 The forward equation

Next follows a derivation of formula (3.24) for $v(r,t)$. The forward Fokker-Planck equation associated with (3.26a) reads:

$$\frac{\partial v^0}{\partial \tilde{t}} = -a_2 \frac{\partial}{\partial r}(r v^0) + a_1 \frac{\partial^2}{\partial r^2}(r^2 v^0), \tag{3.56}$$

which has to be solved with the conditions:

$$\begin{aligned} v^0(r,0) &= v_0(r), && (3.57a)\\ v^0(\delta,\tilde{t}) &= 0, \quad \delta \to 0, && (3.57b)\\ v^0(1,\tilde{t}) &= 0. && (3.57c)\end{aligned}$$

The last two conditions result from erecting absorbing boundaries [6] at $r=\delta$ and $r=1$, and v_0 in (3.57a) is the initial density. Equation (3.56) is rewritten as:

$$\frac{\partial v^0}{\partial \tilde{t}} = a_1 r^2 \frac{\partial^2 v^0}{\partial r^2} + (4a_1 - a_2) r \frac{\partial v^0}{\partial r} + (2a_1 - a_2) v^0. \tag{3.58}$$

Taking the Laplace transform of v with respect to $\tilde{t}$:

$$\bar{v}^0(r,s) = \int_0^\infty e^{-s\tilde{t}}\, v^0(r,\tilde{t})\, d\tilde{t}, \tag{3.59}$$

(3.58), (3.57) changes into the boundary value problem:

$$a_1 r^2 \frac{\partial^2 \bar{v}^0}{\partial r^2} + (4a_1 - a_2) r \frac{\partial \bar{v}^0}{\partial r} + (2a_1 - a_2 - s)\bar{v}^0 = -v_0(r), \tag{3.60a}$$

$$\bar{v}^0(\delta, s) = 0, \quad \delta \to 0, \tag{3.60b}$$
$$\bar{v}^0(1, s) = 0. \tag{3.60c}$$

The change of variable (3.29) turns equation (3.60a) into the equation:

$$a_1 \frac{\partial^2 \bar{v}^0}{\partial \rho^2} + (3a_1 - a_2)\frac{\partial \bar{v}^0}{\partial \rho} + (2a_1 - a_2 - s)\bar{v}^0 = -v_0(e^\rho). \tag{3.61}$$

By solving its characteristic equation, the homogeneous equation associated with (3.61) is found to have the independent solutions

$$e^{(-1-k_1(s))\rho}, \; e^{(-1-k_2(s))\rho}, \tag{3.62}$$

with k_1 and k_2 defined in (3.32), and Wronskian:

$$W(\rho) = (k_1 - k_2)\, e^{(-k_1-k_2-2)\rho}. \tag{3.63}$$

The inhomogeneous equation (3.61) is solved by the method of variation of parameters [2]. In the original variable r its solution reads:

$$\begin{aligned} \bar{v}^0(r,s) = {} & \frac{r^{-1-k_1}}{a_1(k_1-k_2)}\left[C_1(s) + \int_\delta^r v_0(r')\,(r')^{k_1}\, dr'\right] \\ & + \frac{r^{-1-k_2}}{a_1(k_1-k_2)}\left[C_2(s) + \int_r^1 v_0(r')\,(r')^{k_2}\, dr'\right], \end{aligned} \tag{3.64}$$

in which it is assumed that the initial density v_0 is sufficiently regular so that the integrands remain finite in the integration domain (note that $k_1 > 0,\ k_2 < 0$). The functions C_1 and C_2 are determined by the boundary conditions (3.60bc). For C_2 we find:

$$C_2(s) = \frac{\delta^{-k_1} \int_\delta^1 v_0(r')\,(r')^{k_1}\, dr' - \delta^{-k_2} \int_\delta^1 v_0(r')\,(r')^{k_2}\, dr'}{\delta^{-k_2} - \delta^{-k_1}}. \tag{3.65}$$

The second term in the nominator is bounded by $v_0(\delta)$. If v_0 is bounded, this bound is finite. Then, consequently, in the limit for $\delta \to 0$:

$$C_2(s) = -\int_0^1 v_0(r')\,(r')^{k_1}\, dr', \tag{3.66a}$$
$$C_1(s) = 0. \tag{3.66b}$$

The solution of (3.60) can be written in the following form:

$$\bar{v}^0(r,s) = \frac{1}{a_1(k_1-k_2)} \int_0^1 \frac{v_0(r')}{r} \left(\frac{r'}{r}\right)^{\frac{k_1+k_2}{2}} \left[e^{-\left|\log\frac{r}{r'}\right| \frac{k_1-k_2}{2}} - e^{-\left(\log\frac{1}{r'r}\right)\frac{k_1-k_2}{2}} \right] dr'. \tag{3.67}$$

The solution of problem (3.56, 3.57) is obtained using the inverse Laplace transform 5.87 [12, p.258]:

$$v^0(r,\frac{\tilde{t}}{a_1}) = \int_0^1 v_0(r')\frac{1}{r}\left(\frac{r}{r'}\right)^c \frac{e^{-c^2\tilde{t}}}{2\sqrt{\pi\tilde{t}}} \left[e^{-\frac{(\log\frac{r}{r'})^2}{4\tilde{t}}} - e^{-\frac{(\log r'r)^2}{4\tilde{t}}} \right] dr', \tag{3.68}$$

and by

$$v(r,t) \sim v^0(r,\tilde{t}), \tag{3.69}$$

we find (3.24). The results (3.37, 3.50, 3.68) are related by:

$$\frac{\partial}{\partial\tilde{t}} u^0(r,\frac{\tilde{t}}{a_1}) = \frac{1}{\epsilon a_1}\tau^0(r,\frac{\tilde{t}}{a_1}), \tag{3.70a}$$

$$1 - \int_0^1 v^0(r,\frac{\tilde{t}}{a_1})\,dr = u^0(r^*,\frac{\tilde{t}}{a_1}), \quad \text{for } v_0(r) = \delta(r-r^*). \tag{3.70b}$$

4. Stochastic damped oscillation (extended model)

In section 3, all angles φ were assumed to be small. In this section, we allow for loading from an arbitrary direction, i.e. we allow for arbitrary angles φ_d, φ_s, as long as the other angles remain small. Now the equilibrium position φ_e is given by:

$$\varphi_e = \frac{-P_d\sin\varphi_d}{\mu l - P_d\cos\varphi_d}, \quad \text{(by assumption the denominator is positive)}, \tag{4.1}$$

the Lagrange equation of motion analogous to (3.2) reads:

$$m\frac{l'^2}{l^2}\frac{\partial^2 z}{\partial t^2} + \mu z - \left(\frac{z}{l}\cos\varphi_d - \sin\varphi_d\right)P_d = 0, \tag{4.2}$$

and the model with damping and a stochastic load component yields:

$$m\frac{l'^2}{l^2}\frac{\partial^2 z}{\partial t^2} + \alpha\frac{\partial z}{\partial t} + \mu z - \left(\frac{z}{l}\cos\varphi_d - \sin\varphi_d\right)P_d = \left|\frac{z}{l}\cos\varphi_s - \sin\varphi_s\right| \xi(\gamma t)P_s. \tag{4.3}$$

With (3.4a),

$$\tilde{\mu} = \mu - \frac{P_d}{l}\cos\varphi_d, \tag{4.4}$$

and (3.4*cd*), we obtain equation (4.3) in dimensionless form:

$$\frac{\partial^2 z^*}{\partial t^{*2}} + \frac{\alpha}{\sqrt{\tilde{m}\tilde{\mu}}}\frac{\partial z^*}{\partial t^*} + z^* + P_d^* \sin\varphi_d = \frac{P_s^*}{\sqrt{\gamma^*}}\left|\frac{R}{l\sqrt{\tilde{\mu}}}z^* \cos\varphi_s - \sin\varphi_s\right| \xi(t^*). \tag{4.5}$$

The magnitudes of the dimensionless combinations are assumed as:

$$\frac{P_s^*}{\sqrt{\gamma^*}} = \sqrt{\epsilon}, \qquad \frac{R\cos\varphi_s}{l\sqrt{\tilde{\mu}}} = k_1, \qquad \frac{\alpha}{\sqrt{\tilde{m}\tilde{\mu}}} = \epsilon k, \tag{4.6}$$

where $0 < \epsilon \ll 1$ and k, k_1 are order ϵ^0 constants. Equation (4.5) becomes:

$$\frac{\partial^2 z^*}{\partial t^{*2}} + \epsilon k \frac{\partial z^*}{\partial t^*} + z^* + P_d^* \sin\varphi_d = \sqrt{\epsilon}\,|k_1 z^* - \sin\varphi_s|\ \xi(t^*), \tag{4.7}$$

and, with

$$\eta = z^* + P_d^* \sin\varphi_d \tag{4.8}$$

we obtain:

$$\ddot{\eta} + \epsilon k \dot{\eta} + \eta = \sqrt{\epsilon}\ |k_1 \eta - k_0|\ \xi, \tag{4.9}$$

where

$$k_0 = k_1 P_d^* \sin\varphi_d + \sin\varphi_s. \tag{4.10}$$

Equation (4.9) is of the same form as (3.9*a*) in section 3. However, if φ_d or φ_s are large, the expressions for k_0 and k_1 are different from those in section 3. From the expression for k_1 in (4.6) we see that k_1 can become zero now, while in section 3 it was always positive. We will continue to study this case $k_1 = 0$, which corresponds to an exactly horizontal stochastic loading. Since it follows that $k_0^2 = 1$, the backward equation similar to (3.14) reads:

$$\frac{\partial u^0}{\partial \tilde{t}} = \left(\frac{a_0}{r} + a_2 r\right)\frac{\partial u^0}{\partial r} + a_0 \frac{\partial^2 u^0}{\partial r^2}, \tag{4.11a}$$

with

$$a_0 = \frac{1}{8}, \qquad a_2 = -\frac{1}{2}k. \tag{4.11b}$$

As in case 1 of section 3, $r = 0$ is an entrance boundary, $r = 1$ is a regular boundary, and it can be verified that the probability of exit at $r = 1$, starting at any point of the closed unit interval, equals one. For the mean exit time we find:

$$T(r) \sim \frac{1}{\epsilon a_2}\int_r^1 \left[1 - e^{-\frac{a_2}{2a_0}s^2}\right]\frac{1}{s}\,ds, \quad (a_2 \neq 0). \tag{4.12}$$

4.1 The undamped rod with horizontal stochastic loading

In the special case that there is no damping, i.e. $k = 0$ and thus $a_2 = 0$, equation (4.11a) becomes:

$$\frac{\partial u^0}{\partial \tilde{t}} = a_0\Big(\frac{1}{r}\frac{\partial u^0}{\partial r} + \frac{\partial^2 u^0}{\partial r^2}\Big). \tag{4.13a}$$

Moreover, (4.13a) can be used as an approximation to (3.14) for small r. For the leading order term u^0 in the expansion of $u(r,t)$, defined as in section 3.1, this equation is supplemented with the conditions

$$u^0(r,0) = 0, \tag{4.13b}$$
$$u^0(0,\tilde{t}) \text{ is finite}, \tag{4.13c}$$
$$u^0(1,\tilde{t}) = 1. \tag{4.13d}$$

The initial-boundary value problem (4.13) is the same as for axissymmetric heat conduction in a cylinder, which has the solution [16, p.175]:

$$u^0(r,\tilde{t}) = 1 - 2\sum_i \frac{J_0(\xi_i r)\ e^{-a_0\xi_i^2\tilde{t}}}{\xi_i J_1(\xi_i)}, \tag{4.14}$$

where the summation extends over the positive roots of

$$J_0(\xi_i) = 0, \tag{4.15}$$

J_i being the Bessel function of the first kind of order i. The exit time density yields:

$$\epsilon^{-1}\tau^0(r,\tilde{t}) = \frac{\partial}{\partial \tilde{t}}u^0(r,\tilde{t}) = 2a_0\sum_i \frac{\xi_i J_0(\xi_i r)\ e^{-a_0\xi_i^2\tilde{t}}}{J_1(\xi_i)}. \tag{4.16}$$

Next we compute the characteristic function and the first few cumulants corresponding to this density. The differential equation for τ^0 yields:

$$\frac{\partial \tau^0}{\partial \tilde{t}} = a_0\Big(\frac{1}{r}\frac{\partial \tau^0}{\partial r} + \frac{\partial^2 \tau^0}{\partial r^2}\Big). \tag{4.17}$$

Taking the Laplace transform of τ^0 with respect to $\tilde{t}$ we obtain:

$$\frac{\partial^2 \bar{\tau}^0}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\tau}^0}{\partial r} - \frac{s}{a_0}\bar{\tau}^0 = 0, \tag{4.18}$$

where we used the initial condition

$$\tau^0(r,0) = 0. \tag{4.19}$$

The general solution of (4.18) is:

$$\bar{\tau}^0(r,s) = c_1 I_0\Big(\sqrt{\frac{s}{a_0}}r\Big) + c_2 K_0\Big(\sqrt{\frac{s}{a_0}}r\Big), \tag{4.20}$$

where I_0, K_0 are modified Bessel functions of first and second kind, respectively. Using the boundary conditions

$$\bar{\tau}^0(1,s) = \epsilon, \tag{4.21a}$$
$$\bar{\tau}^0(0,s) \text{ is finite,} \tag{4.21b}$$

we find:

$$\bar{\tau}^0(r,s) = \epsilon \, \frac{I_0\left(\sqrt{\frac{s}{a_0}}\,r\right)}{I_0\left(\sqrt{\frac{s}{a_0}}\right)}. \tag{4.22}$$

The characteristic function of $\tau(r,t)$ is:

$$G(r,is) \sim \frac{1}{\epsilon}\bar{\tau}^0(r,\frac{s}{\epsilon}) = \frac{I_0\left(\sqrt{\frac{s}{\epsilon a_0}}\,r\right)}{I_0\left(\sqrt{\frac{s}{\epsilon a_0}}\right)}. \tag{4.23}$$

The cumulants $\kappa_n(r)$ of $\tau(r,t)$ are generated by

$$\log G(r,is) \sim \sum_{n=1}^{\infty} \frac{(-s)^n}{n!}\,\kappa_n(r). \tag{4.24}$$

The first few cumulants are:

$$\kappa_1(r) \sim \frac{1}{4\epsilon a_0}\,(1-r^2), \tag{4.25a}$$
$$\kappa_2(r) \sim \frac{1}{32(\epsilon a_0)^2}\,(1-r^4), \tag{4.25b}$$
$$\kappa_3(r) \sim \frac{1}{96(\epsilon a_0)^3}\,(1-r^6), \tag{4.25c}$$

which are the mean $T(r)$, the variance and the skewness of the exit time density, respectively. Since the author is not aware of a formula for the n-th term in the Taylor series expansion around $s=0$ of $\log I_0(s)$, a general formula for κ_n cannot be given. Instead, exploiting the fact that the Taylor expansion of $I_0(s)$ is given by [7, p.961]

$$I_0(s) = \sum_{k=0}^{\infty} \frac{\left(\frac{s}{2}\right)^{2k}}{(k!)^2}, \tag{4.26}$$

it can be shown that

$$\kappa_n(r) \sim \frac{\alpha_n}{(\epsilon a_0)^n}\,(1-r^{2n}), \tag{4.27}$$

where the α_n can be found from the recurrent relations:

$$\sum_{i=1}^{n} \alpha_i\,(-4)^i\,(i!)\,i\binom{n}{i}^2 = -n, \quad (n \geq 1). \tag{4.28}$$

The first few cumulants can be obtained in a different way as follows. The first few moments $\mu_n^0(r)$ about $\tilde{t} = 0$ of $\epsilon^{-1}\tau^0(r, \tilde{t})$ are found by solving recursively the boundary value problems:

$$-n\mu_{n-1}^0 = a_0\Big(\frac{1}{r}\frac{\partial \mu_n^0}{\partial r} + \frac{\partial^2 \mu_n^0}{\partial r^2}\Big), \quad n \geq 1, \tag{4.29a}$$

$$\mu_n^0(0) \text{ is finite,} \tag{4.29b}$$
$$\mu_n^0(1) = 0, \tag{4.29c}$$

where $\mu_0^0(r) = 1$. We find subsequently:

$$\mu_1^0(r) = \frac{1}{4a_0}\,(1 - r^2), \tag{4.30a}$$
$$\mu_2^0(r) = \frac{1}{32a_0^2}\,(3 - 4r^2 + r^4), \tag{4.30b}$$
$$\mu_3^0(r) = \frac{1}{384a_0^3}\,(19 - 27r^2 + 9r^4 - r^6). \tag{4.30c}$$

With

$$\mu_n(r) = \epsilon^{-n}\mu_n^0(r), \tag{4.31}$$

and the well-known formulas that express the cumulants κ_n in terms of the moments μ_n, see [9, p.7], the results (4.25) are recovered.

Between the density (4.16) and its expectation value the following relation must hold:

$$\int_0^\infty \tilde{t}\,\epsilon^{-1}\tau^0(r,\tilde{t})\,d\tilde{t} = \frac{1}{4a_0}\,(1 - r^2). \tag{4.32}$$

This equality is true indeed as can be shown by the Fourier-Bessel expansion of zero order [3] of the right side of (4.32).

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