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A Sufficient Condition for a Product Form Distribution of a Queueing Network with Controlled Arrivals

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In this paper we discuss the admission control of customers to separable queueing networks. A network may consist of several service centers, that are visited by customers of different types. These customers arrive according to independent Poisson processes for each type. Controlled arrivals of new customers are modelled by means of state-dependent arrival rates, or more precisely the rates depend on the numbers of customers of the different types present at the moment of arrival. If the arrival rates are restricted to take as values only zero or a non-zero constant (i.e. customers are either rejected or admitted with probability one) and the rates are non-increasing in all arguments (i.e. if a new customer is admitted when the network is in a certain state, then it would also have been admitted in case a customer were removed from the network), then the network is shown to have a product-form stationary distribution. This result is proven both by using a theorem of Kelly and by solving the equilibrium equations.

Key Words & Phrases: Product-form, separable queueing networks, job local balance, admission control.
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1. INTRODUCTION.

The study of *product-form* queueing networks started with Jackson [5] and Gordon and Newell [2]. They showed the existence of the product-form solution for open and closed queueing networks with a single job class, exponential arrival and service times and *First Come First Served* (FCFS) queueing disciplines at each service center. The term product-form refers to the solution of the equilibrium equations having a solution that consists of a product of terms where each term represents a state of the queues, and this result implies that the individual stations behave as separate queues. Baskett *et al.* [1] extended these results to the class of open, closed and mixed queueing networks with multiple job classes, non-exponential service time distributions and different service disciplines, viz. FCFS, *Processor Sharing* (PS), *Last Come First Served Pre-emptive Resume* (LCFSPR) and *Infinite Server* (IS).

In [1] two types of state-dependent arrival processes are considered. In the first case the total arrival rate to the network may depend on the total number of customers in the network. The second type allows the arrivals of a particular type of customer to depend on the number of customers of *the same* type present in the network. In practice one often encounters more general types of arrival processes, e.g. when the arrival rate of one type of customer depends on weighted sums of the numbers of customers of the different types present in the network. In this paper we show that the queueing networks as considered in [1] can be extended with these types of arrival processes while maintaining the product-form stationary distribution.

At a general level a queueing network can be modelled as a continuous-time Markov

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process on a discrete state space. For this process one can derive a set of so-called *global equilibrium equations* as a limit of the Kolmogorov equations. The normalized solution of these equations is the stationary distribution of the network, which is used in computing regular performance measures of the network, such as throughputs, mean queue lengths, mean waiting times etc. The global equilibrium equations have an intuitive explanation, i.e. for each state the probability flow of *all* possible transitions out of that state must be equal to the total probability flow into that state, again considering *all* possible transitions which would result in that particular state. In a network with many different service centers and different customer types the global equilibrium equations can become very complex and may involve a very large number of variables. It turns out, however, that the equations for networks as described in [1] can be solved more easily by separation of the variables (hence the commonly used term *separable queueing networks*). This separation of variables leads to the concept of *local balance*. The corresponding local balance equations also have a very intuitive explanation, i.e. for every state the probability flow out of that state due to transitions concerning a certain type of customers must be equal to the total probability flow into that state restricted to transitions of customers of the same type. Under some restrictions concerning service disciplines and service time distributions the solution of the local balance equations also solves the global balance equations, since these global balance equations are linear combinations of the local balance equations. This idea is exploited further in [4] where the concept of *job local balance* is used. This type of balance equations can be interpreted as: the probability flow out of a state due to all possible transitions of one particular job must be equal to the probability flow into the same state due to all transitions of that same job. One of the consequences of job local balance is insensitivity, i.e. if job local balance holds for a network with exponentially distributed service times, then the solution of the global balance equations is in product-form and holds also for the same network with general service time distributions. Finally there is the concept of *detailed balance*, which states that the probability flow between every two states must be equal to its reverse flow. This concept is extensively used in [6] and guarantees e.g. time-reversibility of the queueing processes in the network.

With the exception of [3,4] the references mentioned assume infinite waiting capacities in each queue of the network, i.e. no blocking can occur. In [3,4] two types of blocking are allowed, viz. the blocking probability for a customer as he is jumping from one queue to another must either be independent of the customer's type or may depend only on the number of customers of the same type present at the queue the customer is jumping to. In this framework it is not allowed that the blocking probability for a jumping customer depends both on its type and on the total number of customers present at the queue he is jumping to.

Consider now a processor sharing queueing system that is visited by two different types of customers according to two independent Poisson processes. Service times are independent and may have different general distributions for both types. In order to keep the sojourn times of customers below a certain level not all new customers are admitted, but depending on the state of the queue (i.e. the numbers of customers of both types present in the queue) a new customer is either admitted or rejected with probability one. If a new customer of a certain type is admitted if and only if the number of customers of that type already present does not exceed a specified level, the queueing process is known to have a product-form stationary distribution (see [1]). The product-form remains valid if a new customer of either type is admitted if and only if the total number of customers present (i.e. the sum of both types) does not exceed a specified level.

Consider now the following admission control law. Let a_1, a_2 and M be three positive integers. There is a pool of M *permission units* (p.u.'s). A newly arrived customer of type i requests a_i p.u.'s from the pool. If this number is available, then these p.u.'s are removed from the pool and the customer is admitted, otherwise he is rejected and lost. After a customer of type i has completed his service at the queue, a_i p.u.'s are put back into the pool. In this paper we show that this queue also has a product-form stationary distribution.

In fact we will prove the existence of a product-form stationary distribution for a general class of admission control laws in open networks, provided the admission control law satisfies the following two conditions:

- A new customer is either rejected of admitted with probability one (i.e. it is not allowed to admit a new customer with a probability strictly between 0 and 1).
- If the network is in such a state that a new customer of a certain type will be admitted, then a customer of that type will also be admitted if there are less customers in the network.

This result is proved in two ways. One approach uses a theorem of Kelly [6], while the other establishes the result by explicitly solving the job local balance equations.

This paper is organised as follows. In section 2 a formalism is introduced to describe the characteristics of an open queueing network. Apart from the state description of the corresponding Markov process this formalism is identical to that in [4]. The concept of job local balance is introduced and sufficient conditions are given for the existence of a product-form stationary distribution. In section 3 admission control laws are introduced by means of state-dependent arrival rates. Sufficient conditions are given for the existence of the product-form equilibrium probabilities. Finally in section 4 some applications of the results are given.

2. Separable Open Queueing Networks.

Consider a queueing network with N service centers or nodes, numbered $1,...,N$ and K different types of customers, numbered $1,..,K$. Customers arrive according to independent Poisson processes with arrival intensities λ_n^k for type k at node n, $k=1,...,K, n=1,...,N$. (Note that in this section we assume the arrival rates to be state-independent.) Customers in the network travel from one node to another according to a routing matrix (R_{nm}^k) for type k, where R_{nm}^k is the probability that a customer who has just completed his service at node n will proceed to node m. Since the network is open, the row sums of the routing matrix need not sum up to one, the difference making up for the probability of leaving the network after departure from a node. The visiting ratios θ_n^k for type k to node n are defined as the solution of

$$\theta_n^k = \lambda_n^k + \sum_{m=1}^{N} \theta_m^k R_{mn}^k , \qquad n=1,...,N\,; k=1,...,K. \tag{2.1}$$

The service requirement of a customer of type k at node n is exponentially distributed with parameter μ_n^k. We denote the workloads by $\rho_n^k = \theta_n^k / \mu_n^k$.

The service disciplines at the different nodes are described by three functions $f_n : \mathbb{N} \to \mathbb{R}_+$, $\phi_n : \mathbb{N}\times\mathbb{N} \to [0,1]$, $\delta_n : \mathbb{N}\times\mathbb{N} \to [0,1]$, that have the following interpretation.

$f_n(k)$ — the speed of the server at the n-th node when k customers are present,

$\phi_n(k,i)$ — the fraction of the service capacity that is awarded to the customer in the i-th position at node n if k customers are present,

$\delta_n(k,i)$ — the probability that a customer arriving at node n is placed in the i-th position if k customers are present at that node.

In addition to these functions we also have to specify what happens to the other customers

on arrival or departure of a particular customer. In this respect we assume that the shift protocol is used, i.e. if a new customer is placed at position i, then the customers at the positions $i,...,k$ shift to positions $i+1,...,k+1$, and if a customer departs from position i, then these customers shift to positions $i-1,...,k-1$. Moreover we assume that the buffers at each node are infinite. With these three functions and the protocol we are able to model all common service disciplines, e.g. FCFS, LCFSPR, PS and IS.
For convenience we assume that the limit of $(f_n(m))^{-1}$ exists if m goes to infinity. Moreover, for the network to be ergodic, we assume

$$\lim_{n\to\infty} \frac{1}{f_n(m)} \geqslant \sum_{k=1}^{K} \rho_n^k \qquad n=1, \ . \ . \ . \ , N.$$

Since the service times are exponentially distributed, the state of the network can be described by specifying the types of the customers at each position in each queue. We will denote a state as

$$X=(X_1, X_2, \ . \ . \ . \ , X_n)$$

with

$$X_n=(x_{n1}, x_{n2}, \ . \ . \ . \ , x_{nk_n^X}) \qquad n=1, \ . \ . \ . \ , N,$$

where k_n^X denotes the number of customers at node n if the state is X, and x_{ns} the type of the customer at position s in queue n. In addition we will also use the following notations for states:

$X-(n,s)$	The state that results if we remove the customer at position s in queue n if the state is X.
$X-(n_1,s_1)+(n_2,s_2)$	State X with the customer at position s_1 at queue n_1 removed and put into position s_2 at queue n_2.
$X+(x_{ns}=k)$	State X with an extra customer of type k put into position s at queue n.

To prevent ambiguity, compound operations as the above mentioned are to be evaluated from left to right.

2.1. The Global Equilibrium Equations.
Suppose the network has a stationary distribution denoted by $p(.)$. This distribution has to satisfy the global equilibrium equations, that state that in equilibrium the probability flow out of a state must equal the probability flow into that state. The probability flow (or mean number of transitions per time unit) out of state X is given by

$$p(X)\{\sum_{n=1}^{N}\sum_{s=1}^{k_n^X} f_n(k_n^X)\phi_n(k_n^X,s)\mu_n^{x_{ns}} + \sum_{n=1}^{N}\sum_{k=1}^{K}\lambda_n^k\} \tag{2.2}$$

The probability flow into state X due to a transition of a customer from one queue to *another* queue is

$$\sum_{n=1}^{N}\sum_{s=1}^{k_n^X}\sum_{\substack{m=1\\ m\neq n}}^{N}\sum_{t=1}^{k_m^X+1} p(X-(n,s)+(m,t))f_m(k_m^X+1)\phi_m(k_m^X+1,t)\mu_m^{x_{ns}}R_{mn}^{x_{ns}}\delta_n(k_n^X-1,s). \tag{2.3}$$

The probability flow due to a transition of a customer from one queue to the *same* queue is given by

$$\sum_{n=1}^{N}\sum_{s=1}^{k_n^X}\sum_{t=1}^{k_n^X} p(X-(n,s)+(n,t))f_n(k_n^X)\phi_n(k_n^X,t)\mu_n^{x_{ns}} R_{nn}^{x_{ns}}\delta_n(k_n^X-1,s). \tag{2.4}$$

The probability flow into state X due to the arrival of a customer is

$$\sum_{n=1}^{N}\sum_{s=1}^{k_n^X} p(X-(n,s))\lambda_n^{x_{ns}}\delta_n(k_n^X-1,s). \tag{2.5}$$

Finally, the probability flow into state X due to the departure of a customer from the network is given by

$$\sum_{n=1}^{N}\sum_{s=1}^{k_n^X+1}\sum_{k=1}^{K} p(X+(x_{ns}=k))f_n(k_n^X+1)\mu_n^k\phi_n(k_n^X+1,s)R_{n0}^k , \tag{2.6}$$

where

$$R_{n0}^k = 1-\sum_{m=1}^{N} R_{nm}^k . \tag{2.7}$$

The global equilibrium equations are equivalent to the statement that (2.2) must be equal to the sum of (2.3)-(2.6). The solution of these equations is unique up to a multiplicative constant, that can be determined by requiring that $p(.)$ must be a proper probability distribution.
Solving for the equilibrium equations is a formidable task, since the number of unknown variables is large (or infinite) and the structure of the solution cannot be readily deduced from the equations. However in many examples of queueing networks one can deduce a simpler set of equilibrium equations, which are far more easily solvable and provide interesting properties of the stationary distribution. One type of these equations are the so-called Job Local Balance equations.

2.2. The Job Local Balance Equations.
The Job Local Balance (JLB) equations state that the probability flow out of a state due to the transition of *one particular* job must be equal to the probability flow into that state due to transitions of *the same* job. Let X be some state and consider a job at position s at queue n that is of type x_{ns}. The probability flow out of X due to service completion of this job is

$$p(X)f_n(k_n^X)\,\phi_n(k_n^X,s)\,\mu_n^{x_{ns}} . \tag{2.8}$$

Similarly to the global equilibrium equations we have to distinguish between the probability flow into state X due to the transition of the same job from one of the other queues to queue n

$$\sum_{\substack{m=1\\ m\neq n}}^{N}\sum_{t=1}^{k_m^X+1} p(X-(n,s)+(m,t))f_m(k_m^X+1)\mu_m^{x_{ns}}\phi_m(k_m^X+1,t)R_{mn}^{x_{ns}}\delta_n(k_n^X-1,s), \tag{2.9}$$

and the probability flow due to a transition of the same job within the same queue, i.e.

$$\sum_{t=1}^{k_n^X} p(X-(n,s)+(n,t))f_n(k_n^X)\mu_n^{x_{ns}}\phi_n(k_n^X,t)R_{nn}^{x_{ns}}\delta_n(k_n^X-1,s). \tag{2.10}$$

Finally, the probability flow into state X due to the arrival of the job is

$$p(X-(n,s))\lambda_n^{x_{ns}}\delta_n(k_n^X-1,s). \tag{2.11}$$

The JLB thus states equality between (2.8) and the sum of (2.9)-(2.11). Similarly to the global balance equations, the JLB equations state a balance in probability flows, but at a

more detailed level. Moreover, one can easily verify that the global balance equations can be written as linear combinations of the JLB equations, so if a solution to the JLB equations exists, it must be the stationary distribution.
The general network as was introduced in the beginning of this section, however, needs an extra condition for solvability of the JLB equations. This condition must be posed on the service characteristics of the queues and uses the concept of symmetric service disciplines.

DEFINITION 2.1. A queue with a service discipline defined through $f(.)$, $\phi(.,.)$ and $\delta(.,.)$ is called *symmetric* if $\phi(k,s)=\delta(k-1,s)$, $k \in \mathbb{N}; s=1,...,k$.

The term symmetric stems from the reversed queueing process, which can be constructed by interchanging $\delta(.,.)$ and $\phi(.,.)$. With this definition a queue can only be symmetric if a new customer immediately receives service, and thus a FCFS queue is not symmetric. The other wellknown types of service disciplines, however, fit well into the definition of symmetric queues, since

$$\delta(k-1,s)=\phi(k,s)=\begin{cases}1, & s=1\\ 0, & s\neq 1\end{cases} \qquad \text{for } LCFSPR$$

$$\delta(k-1,s)=\phi(k,s)=\frac{1}{k} \qquad \text{for } PS,IS$$

The following result is well-known (e.g. cf. [1,4,6]).

THEOREM 2.2. *If the queues are symmetric, then the stationary distribution of the network is given by*

$$p(X)=C\prod_{n=1}^{N}\prod_{s=1}^{k_n^X}\frac{\rho_n^{x_{ns}}}{f_n(s)} \tag{2.12}$$

where C is a normalizing constant, and the stationary distribution thus has the product-form.

PROOF. We prove that (2.12) is the stationary distribution by showing that it satisfies the JLB equations. Note that, since the network is assumed to be ergodic, the normalized solution of the global balance equations is unique. Let $p(.)$ be given by (2.12). Observe that

$$\begin{aligned} &p(X-(n,s)+(m,t))\,/\,p(X)=f_n(k_n^X)\,\rho_m^{x_{ns}}\,/\,f_m(k_m^X+1)\,\rho_n^{x_{ns}}\\ &p(X-(n,s)+(n,t))\,/\,p(X)\;=1\\ &p(X-(n,s))\,/\,p(X)\qquad\;\;=f_n(k_n^X)\,/\,\rho_n^{x_{ns}}. \end{aligned} \tag{2.13}$$

We now prove that $p(.)$ satisfies the JLB equations. Substituting (2.13) in equations (2.8)-(2.11) yields that JLB is equivalent to

$$f_n(k_n^X)\,\phi_n(k_n^X,s)\,\rho_n^{x_{ns}}\,\mu_n^{x_{ns}}$$

$$=\sum_{\substack{m=1\\ m\neq n}}^{N}\sum_{t=1}^{k_m^X+1} f_n(k_n^X)\,\theta_m^{x_{ns}}\,\phi_m(k_m^X+1,t)\,R_{mn}^{x_{ns}}\,\delta_n(k_n^X-1,s)$$

$$+\sum_{t=1}^{k_n^X} f_n(k_n^X)\,\theta_n^{x_{ns}}\,\phi_n(k_n^X,t)\,R_{nn}^{x_{ns}}\,\delta_n(k_n^X-1,s)$$

$$+f_n(k_n^X)\,\lambda_n^{x_{ns}}\,\delta_n(k_n^X-1,s).$$

Using the definition of symmetric queues and

$$\sum_{t=1}^{x} \phi(x,t) = 1$$

this equality is implied by

$$\theta_n^{x_{ns}} = \sum_{m=1}^{N} \theta_m^{x_{ns}} R_{mn}^{x_{ns}} + \lambda_n^{x_{ns}}$$

and this equality holds due to (2.1). □

REMARK 2.3.
If an open network has the JLB property, then the stationary distribution of the network depends only on the mean service times at the nodes and not on the type of the service time distribution [4, Proposition 6.2]. This is called the *insensitivity property* for queueing networks.

REMARK 2.4.
In the terminology of Kelly [6, Chapter 3] the queues in a network that satisfies JLB are called *quasi-reversible*. Basically it means that a quasi-reversible queue considered in isolation which is fed with a Poisson arrival stream, has a Poisson output process.

REMARK 2.5.
In Baskett *et al.* [1] the First-Come-First-Served service discipline is also allowed with the restriction that at FCFS nodes *all* customers must have the *same* exponential servicetime distribution. Obviously the job local balance equations no longer hold for a network with FCFS queues (cf. remark 2.3), but one can prove that there is *station local balance* for each FCFS queue, i.e. the probability flow out of a state due to the service completion of customers at that queue must equal the probability flow into that state due to the arrival of customers.

3. OPEN QUEUEING NETWORKS WITH CONTROLLED ADMISSIONS.

In this section we extend the open networks as considered in [1] with control on the admission of newly arriving customers. The control law is such that, based on the state of the network at the moment of arrival, it is decided whether the new customer is admitted or not. We will not consider randomisation of the control, i.e. customers are accepted or rejected with probability one. Since the admission decision of a new customer is based on the state of the network, controlled admissions can be modelled by means of state-dependent arrival rates.
Let $\mathfrak{S}$, $p(.)$ and $\mathfrak{R}$ denote the state space, the stationary distribution and the set of positive recurrent states of the uncontrolled open network respectively. Define the functions $N_k : \mathfrak{S} \to \mathbb{N}$ by

$$N_k(X) = \sum_{n=1}^{N} \sum_{s=1}^{k_n^x} I(x_{ns} = k) \qquad , X \in \mathfrak{S}, \ k = 1,...,K, \tag{3.1}$$

and the vector-valued function $\mathbf{N} : \mathfrak{S} \to \mathbb{N}^K$ as

$$\mathbf{N}(X) = (N_1(X), \ . \ . \ . \ , N_K(X)) \qquad , X \in \mathfrak{S}, \tag{3.2}$$

where $I(A)$ represents the indicator function of the set A. It is clear from (3.1) that $N_k(X)$ represents the number of customers in the network of type k when the state of the network is X. The vector $\mathbf{N}(X)$ thus represents the numbers of customers specified for all types, and will be referred to as the *population vector* of state X. Finally the functions D

and $D_- : \mathbb{N}^K \rightarrow P(\mathfrak{R})$ are defined as

$$D(\mathbf{m}) = \{ X \in \mathfrak{R} \mid N_k(X) = m_k , k = 1,...,K \} \qquad , \mathbf{m} \in \mathbb{N}^K \tag{3.3}$$

$$D_-(\mathbf{m}) = \{ X \in \mathfrak{R} \mid N_k(X) \leqslant m_k , k = 1,...,K \} \qquad , \mathbf{m} \in \mathbb{N}^K . \tag{3.4}$$

Thus $D(\mathbf{m})$ and $D_-(\mathbf{m})$ represent the sets of all positive recurrent states with population vector equal to $\mathbf{m}$ or componentwise smaller than or equal to $\mathbf{m}$ respectively.
We now introduce the control law for admission of new customers by means of a subset of the state space. A consequence of such a control law is that the network will only have states within this subset.

DEFINITION 3.1. A subset S of $\mathfrak{S}$, that satisfies

$$X \in S \Rightarrow D_-(\mathbf{N}(X)) \subset S \tag{3.5}$$

is called a *set of accessible states.*

The use of a set of accessible states is in determining the set of positive recurrent states in the controlled network. The set determines whether a new customer will be admitted or rejected as follows: Suppose X is the state of the network at the moment a new customer of type k arrives requesting permission to enter the network. If all the states that can occur by admitting this customer (i.e. all elements of the set $D_-(\mathbf{N}(X)+\mathbf{e}_k)$, are elements of S, then the customer is admitted, otherwise he is rejected and lost ($\mathbf{e}_k$ denotes the k-th unit vector).
This control law can be modelled by means of state-dependent arrival intensities.

DEFINITION 3.2. If S is a set of accessible states, then the state-dependent arrival rates defined as

$$\lambda_n^k(X) = \lambda_n^k I(D_-(\mathbf{N}(X)+\mathbf{e}_k) \subset S) \tag{3.6}$$

are called *compatible* with S. The open network with arrival rates as defined in (3.6) is called *controlled* by S.

If S is a set of accessible states, then the open network that is controlled by S has a product-form stationary distribution.

THEOREM 3.3. *Let S be a set of accessible states. The open network that is controlled by S has the stationary distribution*

$$p_S(X) = C(S) I(X \in S) \prod_{n=1}^{N} \prod_{s=1}^{k_n^X} \frac{\rho_n^{x_{ns}}}{f_n(s)} \tag{3.7}$$

where $C(S)$ is a normalizing constant, i.e.

$$C(S) = \left[\sum_{X \in S} \prod_{n=1}^{N} \prod_{s=1}^{k_n^X} \frac{\rho_n^{x_{ns}}}{f_n(s)} \right]^{-1} . \tag{3.8}$$

PROOF. By Kelly [6, Section 3.5.] the network has a product-form stationary distribution if there exists a function $\Psi : \mathbb{N}_K \rightarrow \mathbb{R}_+$ such that

$$\lambda_n^k(X) = \lambda_n^k \frac{\Psi(\mathbf{N}(X)+\mathbf{e}_k)}{\Psi(\mathbf{N}(X))} . \tag{3.9}$$

Indeed, if we take $\Psi(\mathbf{N}) = I(D_-(\mathbf{N}) \subset S)$ then (3.9) is satisfied.
An alternative approach is by proving that $p_S(X)$ given by equation (3.7) satisfies the JLB equations of the controlled network, i.e.

$$p_S(X) f_n(k_n^X) \phi_n(k_n^X, s) \mu_n^{x_{ns}}$$

$$= \sum_{\substack{m=1 \\ m \neq n}}^{N} \sum_{t=1}^{k_m^X+1} p_S(X - (n,s) + (m,t)) f_m(k_m^X+1) \mu_m^{x_{ns}} \phi_m(k_m^X+1, t) R_{mn}^{x_{ns}} \delta_n(k_n^X-1, s) \quad (3.10)$$

$$+ \sum_{t=1}^{k_n^X} p_S(X - (n,s) + (n,t)) f_n(k_n^X) \mu_n^{x_{ns}} \phi_n(k_n^X, t) R_{nn}^{x_{ns}} \delta_n(k_n^X-1, s)$$

$$+ p_S(X - (n,s)) \lambda_n^{x_{ns}} (X - (n,s)) \delta_n(k_n^X - 1, s).$$

(i) Let $X \in S$. According to the definition (3.3) we have for all $j = 1,...,N$, $s = 1,...,k_n^X$, $m = 1,...,N$ and $t = 1,...,k_m^X+1$

$$X - (n,s) + (m,t) \in D(\mathbf{N}(X))$$

so, by $D(\mathbf{N}(X)) \subset D_-(\mathbf{N}(X))$ and definition 3.1,

$$X - (n,s) + (m,t) \in S$$

Analogously we have for all $n = 1,...,N$ and $s = 1,...,k_n^X$

$$X - (n,s) \in S$$

Moreover, since $X \in S$, we have

$$\lambda_n^{x_{ns}} (X - (n,s))$$
$$= \lambda_n^{x_{ns}} I(D_-(\mathbf{N}(X - (n,s)) + \mathbf{e}_{x_{ns}}) \subset S)$$
$$= \lambda_n^{x_{ns}} I(D_-(\mathbf{N}(X)) \subset S)$$
$$= \lambda_n^{x_{ns}}$$

The expression for $p_S(X)$ of (3.7) substituted into (3.10) thus gives the JLB equation of the uncontrolled open network, and since $p_S(X) = C^* p(X)$ for $X \in S$, $p_S(X)$ satisfies the JLB equations of the controlled open network.
(ii) Let $X \in \Re$, $X \notin S$. Since $D(\mathbf{N}(X)) = D(\mathbf{N}(X - (n,s) + (m,t)))$ we have $p_S(X - (n,s) + (m,t)) = p_S(X - (n,s) + (n,t)) = 0$. Moreover, because $X \notin S$,

$$\lambda_n^{x_{ns}} (X - (n,s))$$
$$= \lambda_n^{x_{ns}} I(D_-(\mathbf{N}(X - (n,s)) + \mathbf{e}_{x_{ns}}) \subset S)$$
$$= \lambda_n^{x_{ns}} I(D_-(\mathbf{N}(X)) \subset S)$$
$$= 0$$

so $p_S(X)$ satisfies the JLB equations of the controlled network. □

REMARK 3.4.
Although we have proven theorem 3.3. only for open networks which satisfy job local

balance, the results can analogously be extended to network with FCFS queues, again with the restriction that at the FCFS queues all customers must have the same exponential servicetime distribution.

REMARK 3.5.
The results of theorem 3.3. can also be extended to mixed queueing networks, i.e. networks which may be closed with respect to some types of customers and open with respect to other types.

4. APPLICATIONS.

In this section we present some examples of open networks with sets of accessible states and their associated controls.

EXAMPLE 4.1. Let $\mathbf{m} \in \mathbb{N}^k$ and let S be defined as

$$S = D_-(\mathbf{m}).$$

S is obviously a set of accessible states. The network that is controlled by S, is a network where the number of customers of type k is bounded by $\mathbf{m}_k$, i.e. a new customer of type k is admitted if and only if the number of customers of type k present at the moment of arrival is smaller than $\mathbf{m}_k$.

EXAMPLE 4.2. Let A be an $M \times K$ matrix, for some $M \in \mathbb{N}$, with non-negative coefficients, and let $\mathbf{m} \in (\mathbb{R}_+)^M$. The set S is defined as

$$S = \{ X \in \mathfrak{R} \mid A\,\mathrm{N}(X) \leqslant \mathbf{m} \}, \tag{4.1}$$

where the inequality is to be taken componentwise.

LEMMA 4.3. *The set S of example 4.2. is a set of accessible states.*

PROOF. Let $X \in S$ and $X' \in D_-(\mathrm{N}(X))$. From the definition of (3.4) we have

$$\mathrm{N}(X') \leqslant \mathrm{N}(X)$$

componentwise. A having only non-negative coefficients yields

$$A\mathrm{N}(X') \leqslant A\mathrm{N}(X) \leqslant \mathbf{m},$$

so $X' \in S$ and thus S is a set of accessible states.

Let us now suppose that all coefficients of A and $\mathbf{m}$ are integers. The control law induced by the set S amounts to:

(i) There are M pools of permission units (p.u.'s).
(ii) A customer of type k arriving at the network has to request A_{ik} p.u.'s from pool i.
(iii) If all the requested p.u.'s are present in the pools, they are removed from these pools and the customer is admitted to the network. In all other cases the customer is rejected and lost.
(iv) A customer of type k who leaves the network, puts back A_{ik} p.u.'s into pool i.

With the sets of accessible states as in example 4.2. we are for instance able to model networks of queues with admission control laws that depend on either local or global state information. Consider the network as depicted in figure 4.1. In the network four different types of customers arrive according to independent Poisson processes. The network consists of only one queue that works according to the PS discipline.

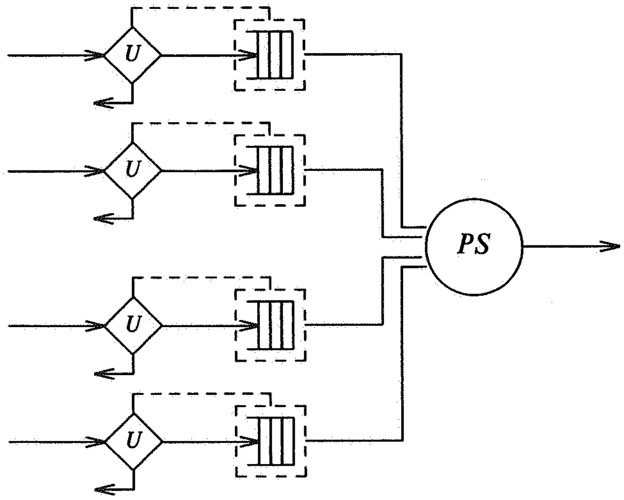

FIGURE 4.1. A Processor Sharing queue with admission control laws based on local state information.

In the network where only local state information is used for the admission control law (*U* in figure 4.1.), the decision whether to admit a new customer is based only on the number of customers of this type that are already present in the queue (depicted by the dashed line and box in figure 4.1.). The admission control law can in this case be established by means of four pools of permission units, one for each type. Customers have to request p.u.'s only from the pool associated with their type.

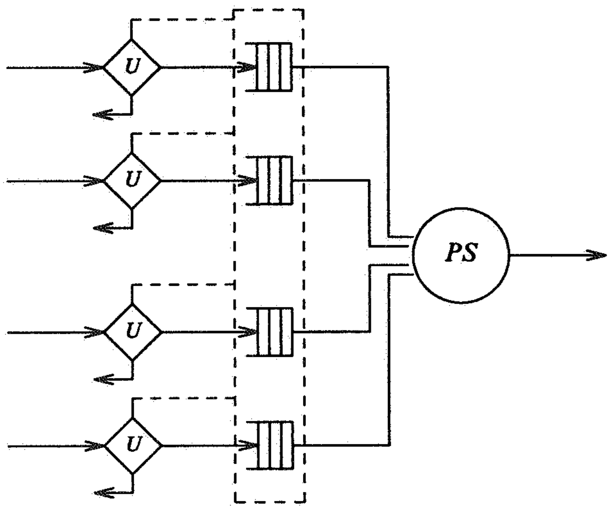

FIGURE 4.2. A Processor Sharing queue with admission control laws based on global state information.

In Figure 4.2. the network is depicted where global state information is used for the admission controller. A controller of this type can e.g. be established by four pools of p.u.'s, where each customer has to request p.u.'s from two or more pools. In this case the controller that has to decide whether a new customer of a certain type is to be admitted, has to know not only the number of customers of this type already present, but also the number of customers of at least one other type. This requires communication between the controllers in the network, but may be advantageous in situations where the distribution of the workload induced by the different customer streams is asymmetric or non-stationary.

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