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N.M. Temme

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Uniform Asymptotic Expansions of a Class of Integrals in Terms of Modified Bessel Functions, with Application to Confluent Hypergeometric Functions

N. M. Temme

Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

The integral

$$F_\lambda(z,\alpha)=\int_0^\infty t^{\lambda-1}e^{-zt-\alpha/t}f(t)\,dt$$

is considered for large values of z; α and λ are uniformity parameters in $[0,\infty)$. The asymptotic expansion is given in terms of the modified Bessel function $K_\lambda(2\sqrt{\alpha z})$. The asymptotic nature of the expansion is discussed and error bounds are constructed for the remainders in the expansions. An example is given for confluent hypergeometric or Whittaker functions. In this example the integrals are transformed to standard forms and the mappings are investigated.

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1. Introduction

We consider integrals of the form

$$F_\lambda(z,\alpha)=\int_0^\infty t^{\lambda-1}e^{-zt-\alpha/t}f(t)\,dt, \tag{1.1}$$

which reduces to a modified Bessel function in the case that f is a constant. We have

$$2(\alpha/z)^{\lambda/2}K_\lambda(2\sqrt{\alpha z})=\int_0^\infty t^{\lambda-1}e^{-zt-\alpha/t}\,dt. \tag{1.2}$$

The integral in (1.1) is considered with $\alpha,\lambda\geq 0$ and large positive values of z. We aim to derive asymptotic expansions for $F_\lambda(z,\alpha)$ that hold uniformly with respect to both α and λ in the interval $[0,\infty)$. To handle the transition of the case $\alpha=0$ to $\alpha>0$, the modified Bessel function (1.2) is needed. Observe that when $\alpha=0$ the essential singularity in the integrand of (1.1) disappears and that (1.1) becomes a more familiar Laplace integral, that can be expanded by using Watson's lemma.

First we consider fixed values of λ. To describe the asymptotic features we introduce the positive number β defined by

$$\beta=\sqrt{\alpha/z}. \tag{1.3}$$

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The saddle points of $\exp(-zt - \alpha/t)$ are located at $t = \pm\beta$. When β is bounded away from zero, we can use the familiar Laplace method, since at the point $t = \beta$ the integrand has the form of a Gaussian function. When, however, $\alpha \to 0$, that is, $\beta \to 0$, the internal saddle point coalesces with the point $t = 0$, where the argument of the exponential function has a pole. In addition, there is an algebraic singularity (if $\lambda \neq 1$), but the influence of the essential singularity due to the pole is more significant. Observe that in the limit $\alpha = 0$, as mentioned earlier, the pole disappears; also, both saddle points coalesce with the pole. These asymptotic features are typical for certain integrals defining Bessel functions. For this reason the modified Bessel function in (1.2) serves as a basic approximant in the uniform asymptotic expansions in this paper. In §4 we show how an integral with the same phenomena can be transformed into the standard form (1.1).

The integral in (1.1) is the simplest case with the asymptotic features described above, especially when the parameters are in the indicated intervals. We apply the results to a confluent hypergeometric function. By allowing different intervals of integration, say a contour in the complex plane, we can also consider negative values of α. Then the ordinary Bessel function $J_\nu(z)$ shows up. This case is more difficult, but the applications are very interesting in the theory of special functions.

Consider as an analogue of (1.1) a loop integral in the form

$$G_\lambda(z,\alpha) = \frac{1}{2\pi i}\int_{-\infty}^{(0^+)} t^{-\lambda-1}e^{zt+\alpha/t}f(t)\,dt. \tag{1.4}$$

This notation means that the contour of integration starts from $-\infty, \arg t = -\pi$, describes a circle counter-clockwise around the origin, and returns to $-\infty, \arg t = +\pi$. The integral (1.4) has the modified Bessel function $I_\lambda(2\sqrt{\alpha z})$ as approximant. When $f = 1$ we have

$$G_\lambda(z,\alpha) = (z/\alpha)^{\lambda/2}I_\lambda(2\sqrt{\alpha z}). \tag{1.10}$$

When α is negative this function is an ordinary J-Bessel function. In [2] and [6] integrals of the type (1.4) are treated and the method is used for obtaining a uniform expansion of Laguerre polynomials. We plan to return to this problem in a future paper.

The starting point (1.1) is of interest since it has a real interval of integration. So the transformation to the standard form (1.1) involves a real mapping. This makes the first steps of the analysis rather simple, since we do not need to trace the transformed contour in the complex plane. For studying the asymptotic nature of the expansion, we use complex variables, however.

The plan of the paper is as follows. In §2 we construct a series expansion based on an integration by parts procedure, and we give estimates for the remainder in the expansion. In §3 we consider an expansion that is based on expanding f at the internal saddle point. In §4 we give an application to confluent hypergeometric functions. In §5 the parameter λ is considered as a second uniformity parameter in $[0,\infty)$, and again we apply the methods on a confluent hypergeometric function. Especially, we pay attention to the mappings needed for a transformation to the standard form.

TERMINOLOGY. We call a parameter *fixed* when it does not depend on the parameters z, α, λ.

2. AN INTEGRATION BY PARTS PROCEDURE

The procedure of this section takes into account both saddle points $\pm\beta$ of the exponential function (where β is given in (1.3)), although $-\beta$ lies outside the interval of integration. For this reason we assume that f is also defined at negative values of its argument, and that f is sufficiently smooth for the operations to be used here. Further conditions on f will be given later.

2.1. Construction of the formal series

The first step is the representation

$$f(t) = a_0 + b_0(t-\beta) + (t-\beta/t)g(t), \tag{2.1}$$

where a_0, b_0 follow from substitution of $t = \pm\beta$. We have

$$a_0 = f(\beta), \qquad b_0 = \frac{1}{2\beta}[f(\beta) - f(-\beta)].$$

Inserting (2.1) into (1.1) we obtain

$$F_\lambda(z,\alpha) = a_0 A_\lambda(z,\beta) + b_0 B_\lambda(z,\beta) + F_\lambda^{(1)}(z,\alpha),$$

where A_λ, B_λ are combinations of the modified Bessel functions introduced in (1.2). It is straightforward to verify that

$$A_\lambda(z,\beta) = 2\beta^\lambda K_\lambda(2\beta z), \quad B_\lambda(z,\beta) = 2\beta^{\lambda+1}[K_{\lambda+1}(2\beta z) - K_\lambda(2\beta z)]. \tag{2.2}$$

An integration by parts gives

$$\begin{aligned} F_\lambda^{(1)}(z,\alpha) &= -\frac{1}{z}\int_0^\infty t^\lambda g(t)\, d\, e^{-z(t+\beta^2/t)} \\ &= \frac{1}{z}\int_0^\infty t^{\lambda-1} e^{-z(t+\beta^2/t)} f_1(t)\, dt \end{aligned}$$

with

$$f_1(t) = t^{1-\lambda}\frac{d}{dt}[t^\lambda g(t)] = \lambda g(t) + t g'(t).$$

We see that $zF_\lambda^{(1)}(z,\alpha)$ is of the same form as $F_\lambda(z,\alpha)$. The above procedure can now be applied to $zF_\lambda^{(1)}(z,\alpha)$ and we obtain for (1.1) the formal expansion

$$F_\lambda(z,\alpha) \sim A_\lambda(z,\beta)\sum_{s=0}^\infty a_s z^{-s} + B_\lambda(z,\beta)\sum_{s=0}^\infty b_s z^{-s}, \quad \text{as} \quad z\to\infty, \tag{2.3}$$

where we define inductively $f_0 = f, g_0 = g$ and for $s = 1, 2, \ldots$

$$\begin{aligned} f_s(t) &= t^{1-\lambda}\frac{d}{dt}[t^\lambda g_{s-1}(t)] = a_s + b_s(t-\beta) + (t-\beta^2/t)g_s(t), \\ a_s &= f_s(\beta), \qquad b_s = \frac{1}{2\beta}[f_s(\beta) - f_s(-\beta)]. \end{aligned} \tag{2.4}$$

REMARK 2.1. As mentioned earlier, for this procedure we need function values of f and derivatives at negative values, although the integral(1.1) is defined only for $t-$values in $[0,\infty)$. When we consider analytic functions f, as we do later, we assume that f is analytic in a domain Ω in the complex plane that contains the real line. When, however, f is supposed to belong to $C^k[0,\infty)$, we assume in the above procedure that f has been smoothly continued on $(-\infty,\infty)$.

2.2. The remainder of the expansion

We introduce a remainder for the expansion in (2.3) by writing

$$F_\lambda(z,\alpha) = A_\lambda(z,\beta)\sum_{s=0}^{n-1} a_s z^{-s} + B_\lambda(z,\beta)\sum_{s=0}^{n-1} b_s z^{-s} + z^{-n}R_n, \tag{2.5}$$

where $n = 0, 1, \ldots$. When $n = 0$ the sums are empty and $R_0 = F_\lambda(z,\alpha)$. The integration by parts procedure yields for R_n the representation

$$R_n = \int_0^\infty t^{\lambda-1} e^{-z(t+\beta^2/t)} f_n(t)\, dt, \tag{2.6}$$

where f_n is defined by (2.4).

When a bound for $|f_n(t)|$ is available, say,

$$|f_n(t)| \le M_n, \qquad t \ge 0, \qquad n = 0, 1, \ldots, \tag{2.7}$$

then a bound for R_n reads

$$|R_n| \le M_n A_\lambda(z,\beta). \tag{2.8}$$

Since f_n depends on β, the quantity M_n may also depend on β. It follows that for bounded values of β, say $\beta \in [0,\beta_0], \beta_0$ fixed and finite, the estimate (2.8) of the remainder R_n shows the asymptotic nature of the expansion (2.5), provided that (2.7) is satisfied.

We must point out that, in general, it is rather difficult to find realistic numbers M_n in order to obtain sharp estimates in (2.8). Also, the estimate in (2.7) is rather global, since it takes into account values of f_n in the complete interval $[0,\infty)$.

A sharper and more realistic bound for R_n may be obtained as follows. Let

$$w_\sigma(t) = e^{\sigma(t+\beta^2/t-2\beta)}, \qquad t > 0, \qquad \sigma \ge 0. \tag{2.9}$$

Observe that $w_\sigma(\beta) = 1$ and that

$$\lim_{t\to 0} w_\sigma(t) = \lim_{t\to+\infty} w_\sigma(t) = +\infty.$$

We assume that we can assign quantities σ_n and M_n, which may depend on β and which satisfy

$$\sigma_n \ge 0, \qquad M_n \ge 1+\epsilon_n, \qquad \epsilon_n \text{ fixed and positive}, \tag{2.10}$$

such that for all $t > 0$ we have

$$|f_n(t)| \le M_n\, |f_n(\beta)|\, w_{\sigma_n}(t). \tag{2.11}$$

Then in stead of (2.8) we obtain

$$|R_n| \le M_n\, |f_n(\beta)|\, \widetilde{A}_\lambda(z,\beta), \qquad z > \sigma_n, \tag{2.12}$$

where

$$\widetilde{A}_\lambda(z,\beta) = A_\lambda(z-\sigma_n,\beta)\, e^{-2\beta\sigma_n}. \tag{2.13}$$

When $f_n(\beta) = 0$ a slight modification is needed. The idea about this approach is that in (2.11) function values outside a neighborhood of $t = \beta$ may be estimated very roughly, and that the integral, which results after inserting the right-hand side of (2.11) into (2.6), can be written in terms of one of the approximants in front of the series in (2.5).

A possible approach to compute M_n and σ_n of (2.11) is to start with trial values of M_n satisfying (2.10). Then we compute

$$\sigma_n = \sup_{t\geq 0} \widetilde{f}_n(t), \qquad \beta \text{ fixed in} [0,\infty),$$

where

$$\widetilde{f}_n(t) = \frac{\ln|f_n(t)/[M_n f_n(\beta)]|}{t+\beta^2/t-2\beta}, \qquad t\neq\beta, \qquad f_n(\beta)\neq 0.$$

Observe that the function defined in (2.13) satisfies

$$\widetilde{A}_\lambda(z,\beta)/A_\lambda(z,\beta) = 1+o(1) \quad \text{as} \quad z\to\infty,$$

uniformly with respect to $\beta\in[0,\infty)$. This follows from (2.2) and well-known asymptotic relations for the Bessel function.

3. Expansion at the internal saddle point

In the expansion (2.3) we have used function values of f at the negative saddle point $-\beta$. These values appear in the coefficients a_s, b_s of the expansion. The form of the expansion is very attractive, since only two special functions arise, and also since the parameters β and z are nicely separated in both series. Although the expansion (2.3) has a canonical form, there remains the drawback that the function f must be defined at $(-\infty,0]$ in order to obtain for β a uniformity domain $[0,\infty)$. For example, it is not possible to obtain such a uniformity domain when $f(t)=1/(t+1)$. In this section we only expand the function f at the internal saddle point and we formulate further conditions on f in order to obtain an optimal domain for β.

3.1. The functions Q_s and $\widetilde{Q}_s(\zeta)$

We expand f in the form

$$f(t) = \sum_{s=0}^{\infty} a_s(\beta)(t-\beta)^s, \quad a_s = \frac{f^{(s)}(\beta)}{s!}. \tag{3.1}$$

Substituting (3.1) in (1.1), we obtain after interchanging the order of summation and integration the formal result

$$F_\lambda(z,\alpha) \sim z^{-\lambda}\sum_{s=0}^{\infty} a_s(\beta)Q_s(\zeta)z^{-s}, \quad \text{as} \quad z\to\infty, \tag{3.2}$$

where

$$Q_s(\zeta) = \zeta^{\lambda+s}\int_0^\infty t^{\lambda-1}(t-1)^s e^{-\zeta(t+1/t)}\,dt, \tag{3.3}$$

$$\zeta = \beta z. \tag{3.4}$$

The functions $Q_s(\zeta)$ can be expressed in terms of the modified Bessel functions defined in (1.2). It is easily verified that

$$Q_s(\zeta) = 2\zeta^{\lambda+s}\sum_{r=0}^{s}(-1)^{s-r}\binom{s}{r}K_{\lambda+r}(2\zeta). \tag{3.5}$$

On the other hand, integrating by parts in (3.3), we obtain the recursion relation

$$Q_{s+2} = (s+\lambda+1-2\zeta)Q_{s+1} + \zeta(2s+\lambda+1)Q_s + s\zeta^2 Q_{s-1}, \quad s=0,1,2,\ldots \tag{3.6}$$

For proving the asymptotic properties of (3.2) it is useful to introduce the functions

$$\widetilde{Q}_s(\zeta) = \zeta^{\lambda+s}\int_0^\infty t^{\lambda-1}|t-1|^s e^{-\zeta(t+1/t)}\,dt. \tag{3.6}$$

By applying Laplace's method it is found that for large positive values of ζ

$$\widetilde{Q}_s(\zeta) \sim \zeta^{\lambda+(s-1)/2} e^{-2\zeta} \Gamma(\frac{s+1}{2}), \quad s = 0, 1, 2, \ldots \tag{3.7}$$

Furthermore, we have when z is fixed

$$\lim_{\beta \to 0} \widetilde{Q}_s(\zeta) = \Gamma(\lambda + s). \tag{3.8}$$

3.2. Error bounds, interpretation of the expansion

We introduce a remainder in the expansion (3.2) by writing

$$f(t) = \sum_{s=0}^{n-1} a_s(\beta)(t-\beta)^s + R_n(t,\beta)(t-\beta)^n, \quad n = 0, 1, 2, \ldots. \tag{3.9}$$

Then we obtain for (3.2)

$$F_\lambda(z,\alpha) = z^{-\lambda}\Big[\sum_{s=0}^{n-1} a_s(\beta) Q_s(\zeta) z^{-s} + E_n(z,\alpha) z^{-n}\Big], \tag{3.10}$$

where

$$E_n(z,\alpha) = z^{\lambda+n} \int_0^\infty t^{\lambda-1}(t-\beta)^n R_n(t,\beta) e^{-z(t+\beta^2/t)}\, dt. \tag{3.11}$$

Let f be analytic in a connected domain Ω of the complex plane; Ω may depend on β and we assume that the radius of convergence R_β of the expansion (3.1) satisfies the condition

$$R_\beta \geq \rho(1+\beta)^\kappa, \quad \beta \geq 0, \quad (\rho, \kappa \text{ fixed}, \rho > 0, \kappa \geq 1/2). \tag{3.12}$$

This condition says that the singularities of f should have a distance of $\mathcal{O}(\beta^\kappa)$, uniformly with respect to $\beta \in [0,\infty)$. When $\kappa < 1/2$ the singularities of f are too close to the saddle point. Furthermore, we assume that f has the following growth condition in Ω: there is a real fixed number p such that

$$\sup_{t \in \Omega} (1+|t|)^{-p} |f(t)| \tag{3.13}$$

is bounded for $\beta \in [0,\infty)$.

The coefficients $a_s(\beta)$ of (3.1) can be written as

$$a_s(\beta) = \frac{1}{2\pi i} \int_{C_r} \frac{f(t)}{(t-\beta)^{s+1}} dt, \tag{3.14}$$

where C_r is a circle with centre β and radius $r(1+\beta)^\kappa$; r may depend on β, but should be uniformly bounded from zero and small enough to keep C_r inside Ω. Using (3.14) we obtain the following form of Cauchy's inequality

$$|a_s(\beta)| \leq r^{-s} M_r(\beta)(1+\beta)^{-s\kappa}, \tag{3.15}$$

where

$$M_r(\beta) = \sup_{t \in C_r} |f(t)|. \tag{3.16}$$

In the next theorem we introduce an asymptotic sequence $\{\phi_s\}$, which is constructed on the basis of the estimates in (3.7) and (3.14). For the concept of asymptotic scale and (generalized) asymptotic expansion we refer to [4 p. 25].

THEOREM 3.1. *Let $\zeta = \beta z$, $\kappa \geq 1/2$ and let*

$$\phi_s = \phi_s(z,\beta) = M_r(\beta)(1+\beta)^{-s\kappa}(1+\zeta)^{\lambda+(s-1)/2}e^{-2\zeta}\, z^{-s}, \quad s = 0,1,2,\ldots. \tag{3.17}$$

Then $\{\phi_s\}$ is an asymptotic scale as $z \to \infty$, uniformly with respect to $\beta \in [0,\infty)$.

PROOF.

$$\frac{\phi_{s+1}}{\phi_s} = (1+\beta)^{-\kappa}\sqrt{\zeta+1}\, z^{-1} \leq \frac{1}{\sqrt{z}}, \quad \text{if } z \geq 1. \tag{3.18}$$

■

Now we write the expansion (3.2) in the notation

$$z^{\lambda} F_{\lambda}(z,\alpha) \sim \sum_{s=0}^{\infty} a_s(\beta) Q_s(\zeta) z^{-s}; \qquad \{\phi_s\} \quad \text{as} \quad z \to \infty, \tag{3.19}$$

and we have

THEOREM 3.2. *The expansion (3.19) is a uniform asymptotic expansion as $z \to \infty$, uniformly with respect to $\beta \in [0,\infty)$.*

PROOF. According to the definition of generalized (uniform) asymptotic expansions, we have to prove

$$z^{-n} E_n(z,\beta) = \mathcal{O}(\phi_n), \quad n = 0,1,2,\ldots, \tag{3.20}$$

as $z \to \infty$, uniformly with respect to $\beta \in [0,\infty)$. The interval of integration in (3.11) is split up as follows

$$[0,\infty) = \Delta_- \cup [t_-,t_+] \cup \Delta_+, \tag{3.21}$$

where

$$\Delta_- = [0,t_-], \quad \Delta_+ = [t_+,\infty), \quad t_{\pm} = \beta \pm r_1(1+\beta)^{\kappa}, \quad 0 < r_1 < r, \quad r_1 \text{ fixed}, \tag{3.22}$$

with r as in (3.14). When t_- happens to be negative, we replace it by 0. For $t \in [t_-,t_+]$ we can write

$$R_n(t,\beta) = \frac{1}{2\pi i}\int_{C_r} \frac{f(\tau)}{(\tau-t)(\tau-\beta)^n}\, d\tau, \tag{3.23}$$

with C_r as in (3.14). If $\tau \in C_r$, we have $|\tau - t| \geq (r-r_1)(1+\beta)^{\kappa}$. Thus we obtain as in (3.15)

$$|R_n(t,\beta)| \leq \frac{M_r(\beta)(1+\beta)^{-n\kappa}}{r^{n-1}(r-r_1)}. \tag{3.24}$$

Hence the integral over $[t_-,t_+]$ in (3.11) gives a contribution which is bounded by

$$\begin{aligned} &\frac{z^{\lambda} M_r(\beta)(1+\beta)^{-n\kappa}}{r^{n-1}(r-r_1)} \int_{t_-}^{t_+} t^{\lambda-1}|t-\beta|^n e^{-z(t+\beta^2/t)}\, dt = \\ &M_r(\beta)(1+\beta)^{-n\kappa} z^{-n} \tilde{Q}_n(\zeta)\mathcal{O}(1), \quad \text{as} \quad z \to \infty, \end{aligned} \tag{3.25}$$

uniformly with respect to $\beta \in [0,\infty)$. Using (3.7), (3.8) and (3.17), we conclude that

$$z^{-n} E_n(z,\lambda) = I_- + I_+ + \mathcal{O}(\phi_n), \quad \text{as} \quad z \to \infty, \tag{3.26}$$

uniformly with respect to $\beta \in [0,\infty)$, where $I_{\pm}$ are the contributions to (3.11) from $\Delta_{\pm}$. For $t \in \Delta_{\pm}$ we write

$$(t-\beta)^n R_n(t,\beta) = f(t) - \sum_{s=0}^{n-1} a_s(\beta)(t-\beta)^s$$

and the proof is finished when we have shown that

$$z^{\lambda} \int_{\Delta_{\pm}} t^{\lambda-1} e^{-z(t+\beta^2/t)}\, g(t)\, dt = \mathcal{O}(\phi_n), \quad \text{as} \quad z \to \infty, \tag{3.27}$$

uniformly with respect to $\beta \in [0,\infty)$, where $g(t)$ is $|f(t)|$ or $|a_s(t-\beta)^s|\,(0 \leq s \leq n-1)$. In fact it possible to prove that

$$I_{\pm} \sim 0; \qquad \{\phi_s\}, \quad \text{as} \quad z \to \infty, \tag{3.28}$$

uniformly with respect to $\beta \in [0,\infty)$. That is, $I_{\pm}$ are asymptotically equal to zero with respect to the scale $\{\phi_s\}$. The proof of (3.28) is similar to that given for another type of integrals in [5, Lemma 3.3], and will not be repeated here. ■

The above theorem only gives an order estimate in terms of ϕ_n for the remainder defined in (3.11) and gives an interpretation of the asymptotic nature of the expansions (3.2) and (3.19). To obtain a numerical upper bound for $E_n(z,\lambda)$ we proceed as in the previous section. Since $f(t)$ satisfies the growth condition (3.13), it is possible to find numbers M_n, σ_n satisfying (2.10), such that

$$|R_n(t,\beta)| \leq M_n |a_n(\beta)| w_{\sigma_n}(t), \qquad 0 < t < \infty. \tag{3.29}$$

Using this in (3.11), we obtain the bound

$$|E_n(z,\lambda)| \leq M_n |a_n(\beta)| e^{-2\beta\sigma_n} \widetilde{Q}_n(\zeta - \beta\sigma_n), \qquad z > \sigma_n. \tag{3.30}$$

When $a_n(\beta)$ happens to vanish as a function of $\beta \in [0,\infty)$, this approach needs a slight modification.

4. Application to confluent hypergeometric functions

We start with the confluent hypergeometric function defined by

$$\Gamma(a)U(a,b,x) = \int_0^\infty u^{a-1}(1+u)^{b-a-1}e^{-xu}\,du. \tag{4.1}$$

We consider a as the large parameter and x as a uniformity parameter in $[0,\infty)$; b is a fixed real parameter. We take $b \leq 1$; the relation

$$U(a,b,x) = x^{1-b}U(a+1-b, 2-b, x)$$

can be used when $b > 1$.

4.1. Transformation to the standard form

First we give a simple intermediate transformation. The function $[u/(u+1)]^a$ assumes its maximal value (on $[0,\infty)$) at $u = \infty$. This function controls the asymptotic behaviour of the integrand and, hence, we transform it to an exponential function by writing $u/(u+1) = \exp(-w)$. Then (4.1) becomes

$$\Gamma(a)U(a,1-\lambda,x) = \int_0^\infty w^{\lambda-1}e^{-aw-x/(e^w-1)}\widetilde{f}(w)\,dw, \tag{4.2}$$

where

$$\widetilde{f}(w) = \Big[\frac{1-e^{-w}}{w}\Big]^{\lambda-1}. \tag{4.3}$$

We transform (4.2) into (1.1) with the help of the transformation

$$w + \frac{\nu}{e^w - 1} = t + \frac{\beta^2}{t} + A, \tag{4.4}$$

where $\nu = x/a$ and β, A are to be determined. We compute them on the following condition on the mapping: the critical points of the $w-$function in (4.4) must correspond with the critical values of the $t-$function. Critical $w-$ and $t-$values are $\pm w_0$, $\pm t_0$, where

$$t_0 = \beta, \quad w_0 = \cosh^{-1}(1+\nu/2) = \ln\big(1 + \frac{\nu + W_0}{2}\big), \quad W_0 = \sqrt{\nu^2 + 4\nu}. \tag{4.5}$$

It follows that

$$A = -\frac{\nu}{2}, \quad \beta = \frac{w_0 + \sinh w_0}{2} = \frac{1}{2}\ln\big(1 + \frac{\nu + W_0}{2}\big) + \frac{1}{4}W_0. \tag{4.6}$$

From the simple differential equation

$$\frac{d\beta}{d\nu} = \frac{1}{4}\sqrt{(\nu+4)/\nu}$$

it follows that β^2 of (4.4) is an analytic function of ν. Conversely, ν is an analytic function of β^2.

With these values of A, β the mapping $w \mapsto t$ is regular at $w = \pm w_0$ and at $w = 0$. In fact it is regular in $(-\infty, \infty)$ and as a conformal mapping in a large domain Ω of the complex plane. We have the correspondances

$$t(\pm\infty) = \pm\infty, \quad t(\pm w_0) = \pm\beta, \quad t(0) = 0. \tag{4.7}$$

More details on the mapping are given in the next subsection.

Using transformation (4.4) in (4.2), we arrive at the standard form

$$F_\lambda(z,\alpha) = \Gamma(a)e^{-x/2}U(a,b,x) = \int_0^\infty t^{\lambda-1}e^{-zt-\alpha/t}f(t)\,dt, \tag{4.8}$$

with $z = a$, $\alpha = z\beta^2$, $\lambda = 1-b$, β defined in (4.6) with $\nu = x/a$ and

$$f(t) = \Big(\frac{1-e^{-w}}{t}\Big)^{\lambda-1}\frac{dw}{dt}, \quad \frac{dw}{dt} = \Big(\frac{e^w-1}{t}\Big)^2\frac{t^2-\beta^2}{(e^w-1)^2-\nu e^w}. \tag{4.9}$$

The function $t(w)$ defined in (4.4) is an odd function of w. This easily follows from rewriting (4.4) in the form

$$\frac{1}{2}\nu + w + \frac{\nu}{e^w-1} = t + \frac{\beta^2}{t}. \tag{4.10}$$

After these preparations the expansion of (3.2) can be constructed. The expansion holds uniformly with respect to $\beta \in [0,\infty)$; that is, uniformly with respect to $x \in [0,\infty)$.

The asymptotic nature of the expansion follows from combining (3.20) and (3.17). For this particular case we can derive an upper bound for $M_r(\beta)$ of (3.16). The $t-$ values on the circle are written as $t = \beta + \tau\sqrt{\beta+1}$, with $|\tau| = r$, r fixed. When β is large, we derive from (4.6) $\nu \sim 4\beta$. So, for large values of β, we obtain from (4.10) $w + \nu/(e^w-1) \sim \tau^2$, or $w \sim \ln(\nu/\tau^2)$. It follows from (4.9) that $f(t) \sim 2\beta^{1/2-\lambda}$, $t \in C_r$. Consequently, we can find a fixed number K, such that

$$M_r(\beta) \le K(\beta+1)^{1/2-\lambda}, \quad \beta \in [0,\infty). \tag{4.11}$$

To conclude this subsection, we give the first coefficient $a_0(\beta)$ of (3.2). A few calculations based on (4.9) and l'Hôpital's rule yield

$$\frac{dw}{dt}|_{t=\pm\beta} = \sqrt{2\tanh(w_0/2)/\beta}.$$

So we obtain

$$a_0(\beta) = \sqrt{2\tanh(w_0/2)/\beta}\,\Big(\frac{1-e^{-w_0}}{\beta}\Big)^{\lambda-1}. \tag{4.12}$$

4.2. Analytical properties of the mapping (4.4)

We now consider the mapping (4.4) in more detail. We restrict w to the strip

$$H = \{w \mid -\infty < \Re w < \infty, -\pi \le \Im w \le \pi\}, \tag{4.13}$$

and we prove the following

THEOREM 4.1. *Let Ω be the image of H under the mapping $w \mapsto t$ defined in (4.4). Let $\nu \in [0,\infty)$ and let A, β be defined by (4.6). Then $t(w,\beta)$ is analytic in H.*

PROOF. We prove that $t(w,\beta)$ and $w(t,\beta)$ are analytic in a fixed neighborhood of (0,0). Accordingly, we concentrate on small (complex) values of the parameters. For remaining values the proof is much easier. From (4.6) it follows that

$$w_0 = \beta[1+o(1)], \quad \nu = \beta^2[1+o(1)], \quad \text{as} \quad \beta \to 0.$$

Recall that $t(w)$ is an odd function of w (see (4.10)). We introduce a function $y = y(w,\beta)$ by writing

$$t = w[\frac{\beta}{w_0} + (w^2 - w_0^2)y]. \tag{4.14}$$

This matches the points $w = 0 \Longleftrightarrow t = 0$ and also the critical points $w = \pm w_0 \Longleftrightarrow t = \pm\beta$; y is an even function of w and should vanish with β. Substituting (4.14) in (4.10), we obtain

$$\frac{\nu t}{w}\frac{\phi(w)-\phi(w_0)}{w^2-w_0^2} + \frac{\beta}{w_0}(1-\frac{\beta}{w_0}) + (w^2-w_0^2)(1-\frac{2\beta}{w_0})y - w^2(w^2-w_0^2)^2y^2 = 0,$$

where $\phi(w) = w/(\exp w - 1) - 1 + w/2$. We expand

$$\frac{\phi(w)-\phi(w_0)}{w^2-w_0^2} = \sum_{s=1}^{\infty} b_s(w^2-w_0^2)^{s-1}, \quad b_1 = \frac{\beta - w_0}{\nu w_0}.$$

Since $\phi(w)$ is analytic if $|w| < 2\pi$, the series converges if β and w are small. Finally, we obtain the equation $F(y,w,\beta) = 0$, where F is given by

$$\frac{\nu\beta}{w_0}\sum_{s=2}^{\infty} b_s(w^2-w_0^2)^{s-2} + \nu y\sum_{s=2}^{\infty} b_s(w^2-w_0^2)^{s-1} + (1-\frac{2\beta}{w_0})y - w^2y^2.$$

The series represent analytic functions of w, w_0. Since w_0 is an analytic function of β, F is analytic in a fixed neighborhood of $(0,0,0)$, $F(0,0,0) = 0$, and $F_y(0,0,0) = -1$. After these preparations we can use an implicit function theorem (see, for instance, [1, p. 36]) and solve for $y(w,\beta)$; it is analytic in a fixed neighborhood of $(0,0)$. By using (4.14) it follows that the same holds for $t(w,\beta)$. ∎

The first terms in the expansion

$$t(w,\beta) = c_1(\beta)w + c_3(\beta)w^3 + \ldots$$

easily follow from (4.10). We have

$$c_1(\beta) = \beta^2/\nu, \quad c_3(\beta) = (c_1^2 + \beta^2/6 - c_1 - \nu c_1/4)/\nu.$$

THEOREM 4.2. *The mapping (4.4) is univalent in H.*

PROOF. When $\Re w \to \pm\infty$ we have $w \sim t$. Hence the mapping is one-to-one at ∞. Next we show that the mapping is univalent on $\mathcal{L}_+ = \{w = u + iv \,|\, u \in (-\infty,\infty),\ v = \pi\}$, which is the upper part of the boundary ∂H of H. The remaing part follows by symmetry of the mapping. We write $t = re^{i\theta}$. The image of $\mathcal{L}_+$ in the $t-$plane is defined by the equations

$$\pi = r\sin\theta(1-\frac{\beta^2}{r^2}), \quad \Psi(u) = \Phi(\theta), \tag{4.15}$$

where

$$\Psi(u) = \nu/2 + u - \nu/(e^u+1), \quad \Phi(\theta) = r\cos\theta(1+\frac{\beta^2}{r^2}).$$

The first equation in (4.15) defines a curve given by

$$r(\theta) = \frac{\pi + \sqrt{\pi^2 + 4\beta^2 \sin^2\theta}}{2\sin\theta}, \quad 0 \le \theta \le \pi.$$

It follows that $r > \pi/\sin\theta$. Furthermore, we have

$$\Psi(-\infty) = \Phi(\pi) = -\infty, \quad \Psi(+\infty) = \Phi(0) = +\infty.$$

The function $\Psi(u)$ is one-to-one on $(-\infty, \infty)$. The same is true for $\Phi(\theta)$ on $[0, \pi]$, but the proof requires a little extra work. We have, using the first equation in (4.15),

$$\frac{dr}{d\theta} = -\frac{r\cos\theta(r^2 - \beta^2)}{\sin\theta(r^2 + \beta^2)}.$$

It follows that

$$\frac{\Phi(\theta)}{d\theta} = -\frac{\sin\theta(r^2 + \beta^2)}{r^3}[r^2 + (\frac{dr}{d\theta})^2], \tag{4.16}$$

which shows that $\Phi(\theta)$ is one-to-one on $[0, \pi]$. We infer that for each value $u \in (-\infty, \infty)$ we can find one and only one value $\theta \in [0, \pi)$, such that $\Psi(u) = \Phi(\theta)$, and, hence, one and only one value $r(\theta)$. So, the mapping $w \mapsto t$ is one-to-one on ∂H. We now use a well-known result of complex function theory, that says that consequently the mapping is also univalent in the interior of H, since it is analytic there. See [3, Vol. II, p. 118]. ∎

For the uniform expansion of (4.8) we take Ω as the image of the strip H under the mapping $w \mapsto t$. From $f(t)$ defined in (4.9) it follows that (3.13) is bounded in Ω if $p = 1 - \lambda$ and that $M_r(\beta)$ of (3.16) is well defined. There remains to show that the radius of convergence R_β of the series in (3.1) satisfies (3.12). It appears that we must take $\kappa = 1/2$. In fact we show that Ω contains a disc around β with radius $\rho\sqrt{\beta + 1}$ (ρ fixed), for all $\beta \ge 0$. We try to intersect the curve defined by the first equation of (4.15) with a circle around $t = \beta$. We write $t = \sigma + i\tau$. This gives the equations

$$(\sigma - \beta)^2 + \tau^2 = r^2, \quad \pi = \tau\big(1 - \frac{\beta^2}{\sigma^2 + \tau^2}\big), \quad \tau > \pi,$$

with the "touching" condition

$$\frac{\sigma - \beta}{\tau} = \frac{2\sigma(\tau - \pi)^2}{\pi\beta^2 + 2\tau(\tau - \pi)^2}.$$

For large values of β the solution of these three equations reads

$$\tau = \pi + a\sqrt{\beta}[1 + o(1)], \quad \sigma = \beta + b\sqrt{\beta}[1 + o(1)], \quad r = c\sqrt{\beta}[1 + o(1)],$$

with $a = b = \sqrt{\pi/2}$, $c = \sqrt{\pi}$.

This shows that Ω is large enough to apply Theorem 3.2. From a further analysis it follows that the value $\kappa = 1/2$ is best possible in this case. Apart from the real critical points $\pm w_0$ given in (4.5), which are regular points for the mapping, we have other ones located at $\pm w_0 \pm 2\pi n i$, $n = 1, 2, \ldots$. For large values of β those are mapped at a distance $\mathcal{O}(\sqrt{\beta})$ from the critical point $t = \beta$.

REMARK 4.1. The behaviour of $f(t)$ of (4.9) in the left half-plane $\Re t < 0$ is quite different from that in $\Re t > 0$, except when $\lambda = 1$. Consequently, the approach of §2 is less attractive. See also Remark 2.1.

REMARK 4.2. When $b = 1/2$, (4.1) is a parabolic cylinder function and the functions $Q_s(\zeta)$ defined in (3.3), (3.5) are elementary functions ($\lambda = 1/2$). Then (3.2) gives an expansion of the parabolic cylinder function $D_\nu(z)$, as $\nu \to -\infty$, which is uniformly valid with respect to $x \in [0, \infty)$.

5. A SECOND UNIFORMITY PARAMETER

In this section we consider (1.1) with λ as a second uniformity parameter in $[0,\infty)$. Thus we take more advantage of the fact that the modified Bessel function is a function of two variables. In this case it is convenient to put the reciprocal gamma function in front of the integral. So, now we write

$$F_\lambda(z,\alpha) = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt-\alpha/t} f(t)\, dt, \tag{5.1}$$

In [5] we considered (5.1) with $\alpha = 0$, again with $z \to \infty$ and λ as a uniformity parameter in $[0,\infty)$. In [6] we applied the present method for a loop integral (without proofs) to the case of Laguerre polynomials.

We write $\lambda = \mu z$. The critical points of the integrand are now defined as the points were the derivative of $t + \beta^2/t - \mu \ln t$ vanishes. This gives the real critical points

$$t_\pm = \frac{\mu \pm T}{2}, \quad T = \sqrt{\mu^2 + 4\beta^2}. \tag{5.2}$$

Observe that also in this case one of the real saddle points is outside the interval of integration, and that the "phase function" that is used to compute the critical points has a logarithmic singularity at $t = 0$. The two critical points coincide with this singularity when β and μ both vanish. At the same moment, however, the logarithmic singularity disappears.

First we construct an expansion by using the integration by parts procedure of §2. The modification of (2.1) is

$$f(t) = c_0 + d_0(t - t_+) + (t - \mu - \beta^2/t)h(t). \tag{5.3}$$

Using this in (5.1) we obtain, after repeating the procedure,

$$F_\lambda(z,\alpha) = C(z,\beta,\mu) \sum_{s=0}^{n-1} c_s z^{-s} + D(z,\beta,\mu) \sum_{s=0}^{n-1} d_s z^{-s} + z^{-n} R_n. \tag{5.4}$$

The functions in front of the series are again combinations of Bessel functions as in (2.2). We have

$$C(z,\beta,\mu) = \frac{2\beta^\lambda}{\Gamma(\lambda)} K_\lambda(2\beta z), \quad D(z,\beta,\mu) = \frac{2\beta^\lambda}{\Gamma(\lambda)} [\mu K_{\lambda+1}(2\beta z) - t_+ K_\lambda(2\beta z)].$$

The coefficients c_s, d_s follow from the recursion relation

$$f_0(t) = f(t), \quad f_s(t) = t\frac{d}{dt} h_{s-1}(t) = c_s + d_s(t - t_+) + (t - \mu - \beta^2/t)h_s(t),$$

$$c_s = f_s(t_+), \quad d_s = \frac{f(t_+) - f(t_-)}{t_+ - t_-}.$$

The remainder R_n in (5.4) can be written in the form

$$R_n = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-zt-\alpha/t} f_n(t)\, dt.$$

A bound can be constructed by using constants σ_n, M_n satisfying (2.10), and using a function

$$w_\sigma(t) = e^{\sigma(t+\beta^2/t-\mu \ln t - t_+ - \beta^2/t_+ + \mu \ln t_+)}$$

such that, as in (2.11), for all $t > 0$

$$|f_n(t)| \le M_n\, |f_n(t_+)|\, w_{\sigma_n}(t).$$

Then we obtain

$$|R_n| \leq M_n\,|f_n(t_+)|\,\widetilde{C}(z,\beta,\mu), \qquad z > \sigma_n,$$

where

$$\widetilde{C}(z,\beta,\mu) = C(z-\sigma_n,\beta,\mu)\,e^{-\sigma_n(t_+ + \beta^2/t_+ - \mu \ln t_+)}$$

When $f_n(t_+) = 0$ a slight modification is needed. An optimal value of σ_n follows from the method described in §2.2.

The analogue of the expansion of §3 is obtained by substituting

$$f(t) = \sum_{s=0}^{n-1} c_s(\beta,\mu)(t-t_+)^s + R_n(t,\beta,\mu)(t-t_+)^n, \quad c_s = \frac{f^{(s)}(t_+)}{s!}. \tag{5.5}$$

So we obtain

$$F_\lambda(z,\alpha) = z^{-\lambda}\Big[\sum_{s=0}^{n-1} c_s(\beta,\mu)P_s z^{-s} + E_n(z,\alpha,\lambda)z^{-n}\Big], \tag{5.6}$$

where

$$E_n(z,\alpha,\lambda) = z^{\lambda+n}\int_0^\infty t^{\lambda-1}(t-t_+)^n R_n(t,\beta,\mu)e^{-z(t+\beta^2/t)}\,dt,$$

$$P_s = \frac{z^{\lambda+s}}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}(t-t_+)^s e^{-z(t+\beta^2/t)}\,dt = \frac{2z^{\lambda+s}\beta^\lambda}{\Gamma(\lambda)}\sum_{r=0}^{s}\binom{s}{r}(-t_+)^{s-r}\beta^r K_{\lambda+r}(2\beta z).$$

A recursion relation for P_s follows from the above integral representation.

The asymptotic properties can be described in terms of the functions

$$\begin{aligned}\widetilde{P}_s &= \frac{z^{\lambda+s}}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}|t-t_+|^s e^{-z(t+\beta^2/t)}\,dt \\ &\sim \frac{\eta^{\lambda+(s-1)/2}}{\Gamma(\lambda)}e^{-\eta(1+\beta^2/t_+^2)}\Big[\frac{\beta^2+t_+^2}{2t_+^2}\Big]^{-(s+1)/2}\Gamma\Big(\frac{s+1}{2}\Big), \quad \text{as} \quad \eta\to\infty,\end{aligned} \tag{5.7}$$

where $\eta = zt_+$. Since z is the large parameter, η is large if at least one of the uniformity parameters β, μ is bounded away from zero.

The coefficients c_s and the remainder R_n can be written as

$$c_s(\beta,\mu) = \frac{1}{2\pi i}\int_{C_r}\frac{f(\tau)}{(\tau-t_+)^s}\,d\tau, \quad R_n(t,\beta,\mu) = \frac{1}{2\pi i}\int_{C_r}\frac{f(\tau)}{(\tau-t)(\tau-t_+)^n}\,d\tau,$$

where C_r is a circle around t_+ with radius $r(1+t_+)^\kappa$, $\kappa \geq 1/2$, $r > 0$. We accept that f depends on both uniformity parameters β, μ, and we assume that the domain of analyticity Ω is large enough to contain such a circle for all $\beta, \mu \geq 0$.

As in §3 we have the following theorems. The quantity $M_r(\beta,\mu)$ is defined as in (3.16); we also assume that (3.13) is bounded for all β, $\mu \in [0,\infty)$.

THEOREM 5.1. *Let $\eta = zt_+$, $\kappa \geq 1/2$ and let*

$$\chi_s = \frac{M_r(\beta,\mu)}{\Gamma(\lambda)}(1+t_+)^{-s\kappa}(1+\eta)^{(s-1)/2}e^{-\eta(1+\beta^2/t_+^2)}\Big[\frac{\beta^2+t_+^2}{2t_+^2}\Big]^{-(s+1)/2}, \quad s = 0,1,2,\ldots. \tag{5.8}$$

Then $\{\chi_s\}$ is an asymptotic scale as $z\to\infty$, uniformly with respect to $\beta,\mu \in [0,\infty)$.

THEOREM 5.2. *The expansion*

$$z^\lambda F_\lambda(z,\alpha) \sim \sum_{s=0}^{\infty} c_s(\beta,\mu)P_s z^{-s}; \quad \{\chi_s\} \quad \text{as} \quad z\to\infty, \tag{5.9}$$

is a uniform asymptotic expansion as $z\to\infty$, uniformly with respect to $\beta,\mu \in [0,\infty)$.

A bound for the remainder E_n of (5.6) can be constructed by combining the methods used for (3.30) and the above estimate for the remainder of (5.4).

5.1. Application to a confluent hypergeometric function

Our starting point is (cf. (4.2))

$$\frac{\Gamma(a)}{\Gamma(\lambda)}U(a,1-\lambda,x)=\frac{1}{\Gamma(\lambda)}\int_0^\infty e^{-z[-\mu\ln(1-e^{-w})+w+\nu/(e^w-1)]}\,\frac{dw}{1-e^{-w}}, \tag{5.10}$$

with $z=a,\mu=\lambda/z,\nu=x/z$. The real critical points of the "phase function" are

$$w_\pm=\ln\Big(1+\frac{\mu+\nu\pm W}{2}\Big),\quad W=\sqrt{(\mu+\nu)^2+4\nu}. \tag{5.11}$$

The transformation to the standard form (5.1) reads

$$-\mu\ln(e^w-1)+(\mu+1)w+\frac{\nu}{e^w-1}=t+\frac{\beta^2}{t}-\mu\ln t+A; \tag{5.12}$$

A,β are determined by substituting $w_\pm$ and $t_\pm$, where $t_\pm$ are the critical points defined in (5.2). We have the correspondances

$$t(\pm\infty)=\pm\infty,\quad t(w_\pm)=t_\pm,\quad t(0)=0.$$

Observe that the introduction of a second parameter (here in the form of μ) does not require a third constant in the equation (5.12). It has the same number of constants as (4.4). In fact, in order to obtain a regular mapping $w\mapsto t$, the constants multiplying the log-functions in the left- and right-hand side of (5.12) must be the same. We assume that the log-functions take their principal branches.

Elimination of A from the two equations (5.12) (with $w=w_\pm$, $t=t_\pm$) gives a relation for the unknown parameter β in terms of μ,ν:

$$(\mu+1)\ln\frac{2+\mu+\nu+W}{2+\mu+\nu-W}-\mu\ln\frac{W+\mu+\nu}{W-\mu-\nu}+W=2T-\mu\ln\frac{T+\mu}{T-\mu}. \tag{5.13}$$

By considering $\mu\in[0,\infty)$ as a fixed parameter, we obtain a more transparent relation for $\beta(\nu)$ in the form of a differential equation:

$$\frac{d\beta(\nu)}{d\nu}=\frac{\beta W}{2\nu T},\quad \beta(0)=0. \tag{5.14}$$

The value of A follows from (5.12) by substituting $w=t_+$, $t=t_+$. We have

$$A=(\mu+1)w_+-\mu\ln\frac{\mu+\nu+W}{\mu+T}-\frac{\mu+\nu-W}{2}-T.$$

Using (5.13), we can eliminate $W/2-T$ and we obtain

$$A=\frac{1}{2}\big[(\mu+1)\ln(\mu+1)+\mu\ln\frac{\beta^2}{\nu}-\mu-\nu\big]. \tag{5.15}$$

The transformation (5.12) is discussed in the next subsection. By using it in (5.10) we obtain the standard form (5.1):

$$F_\lambda(z,\alpha)=\frac{e^{zA}\Gamma(a)}{\Gamma(\lambda)}U(a,1-\lambda,x)=\frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-zt-\alpha/t}f(t)\,dt,$$

where $z=a,\alpha=z\beta^2$; β^2 follows from (5.13) with $\mu=\lambda/z,\nu=x/z$. Furthermore,

$$f(t)=\frac{t}{1-e^{-w}}\frac{dw}{dt}=\frac{e^w(e^w-1)}{t}\,\frac{t^2-\mu t-\beta^2}{(e^w-1)^2-(\mu+\nu)(e^w-1)-\nu}. \tag{5.16}$$

The first coefficient of (5.9) equals $f(t_+)$. A few computations give

$$c_0(\beta,\mu) = e^{w_+/2}\sqrt{T/W}.$$

The function f satisfies $f(t) \sim t$ as $t \to +\infty$, whereas f is exponentially small at $-\infty$. This time we can also derive an expansion based on (5.4).

5.2. Analytical properties of the mapping (5.12)

The mapping $w \mapsto t$ defined in (5.12) is one-to-one on the strip H given in (4.13). First we prove this property for the boundary. The equations for the image of the upper part of ∂H are given by (cf. (4.15))

$$\pi = r\sin\theta\left(1-\frac{\beta^2}{r^2}\right) - \mu\theta, \quad \Psi(u) = \Phi(\theta),$$

where

$$\Psi(u) = -A + (\mu+1)u - \nu/(e^u+1), \quad \Phi(\theta) = r\cos\theta\left(1+\frac{\beta^2}{r^2}\right) - \mu\ln r.$$

It follows that the image is given by

$$r(\theta) = \frac{\mu\theta + \pi + \sqrt{(\mu\theta+\pi)^2 + 4\beta^2\sin^2\theta}}{2\sin\theta}, \quad 0 \le \theta \le \pi.$$

The function $\Psi(u)$ is one-to-one on $(-\infty,\infty)$. When we compute $d\Phi(\theta)/d\theta$, we find the same expression as in (4.16). Hence the mapping is univalent on the boundary for all β, $\mu \in [0,\infty)$.

It remains to show that the mapping is analytic inside H. The interesting question is: Is $t(w)$ analytic at $t = 0$, $t = w_\pm$, uniformly with respect to the parameters ν, μ? Especially interesting are small values of the parameters, since then the critical points coalesce with the pole and log-singularity at $w = 0$. When one of the parameters is bounded away from the origin, the critical points $w_\pm$ are well separated. In that case the problem is simpler. Here we prove that $t(w,\nu,\mu)$ is analytic for complex values of the three arguments in a fixed neighborhood of (0,0,0). The proof follows the idea of §4.2.

First we have

THEOREM 5.1. *$\beta^2 = \beta^2(\nu,\mu)$ defined by (5.13) is an analytic function of ν, μ.*

PROOF. As remarked earlier, we concentrate on small values of the parameters. For $\mu = 0$ the relation between ν and β is given in (4.6), and we have mentioned there that $\beta^2(\nu,0)$ is analytic in the domain of interest. On the other hand, we have the expansion

$$\beta^2(\nu,\mu) \sim c_1(\mu)\nu + c_2(\mu)\nu^2 + c_3(\mu)\nu^3 + \dots, \quad \text{as} \quad \nu \to 0.$$

The coefficients c_s are analytic functions of μ. The first few easily follow from (5.13):

$$c_1(\mu) = e^{(\mu+1)\ln(\mu+1)/\mu - 1} = 1 + \frac{1}{2}\mu - \frac{1}{24}\mu^2 + \mathcal{O}(\mu^3), \quad \text{as} \quad \mu \to 0,$$

$$c_2(\mu) = c_1(\mu)[\mu + 2 - 2c_1(\mu)]/\mu^2 = \frac{1}{12} + \mathcal{O}(\mu), \quad \text{as} \quad \mu \to 0.$$

Next we observe that the quantity T of (5.2) is singular at $\beta^2 = -\mu^2/4$ and that W of (5.11) has singular points at $\nu = \nu_0$, $\nu = \nu_1$, where

$$\nu_0 = -(\mu+2) + 2\sqrt{\mu+1}, \quad \nu_1 = -(\mu+2) - 2\sqrt{\mu+1}. \tag{5.17}$$

It is obvious that the singularities at $-\mu^2/4$, ν_0 must correspond. That is, a necessary condition for β^2 to be regular for small values of $|\mu|$ is $\beta^2(\nu_0,\mu) = -\mu^2/4$. Note that $\nu_0 \sim -\mu^2/4$ as $\mu \to 0$ and that (5.13) is satisfied when we substitute $T = W = 0$.

We "remove" the singularity at $\nu = \nu_0$ from (5.13) and we introduce a function $X = X(q,\mu)$ by writing

$$\frac{T+\mu}{T-\mu}\frac{W-\mu-\nu}{W+\mu+\nu} = \frac{1+\sqrt{q}X}{1-\sqrt{q}X}, \quad q = \nu - \nu_0. \tag{5.18}$$

In other words,

$$\sqrt{q}X = \frac{\mu W - (\nu+\mu)T}{WT - \mu(\mu+\nu)}, \quad T = \mu[W + (\nu+\mu)\sqrt{q}X]/D, \quad D = \nu + \mu + \sqrt{q}XW. \tag{5.19}$$

Now we can rewrite (5.13) in the form $K + L + M = 0$, with

$$K = (W - 2T)D = W(\nu - \mu) + \sqrt{q}X(\nu^2 + 4\nu - \mu^2),$$

$$L = D(\mu+1)\ln\frac{2+\mu+\nu+W}{2+\mu+\nu-W} = D(\mu+1)\ln\frac{1+\sqrt{q}Z}{1-\sqrt{q}Z},$$

$$M = D\mu\ln\frac{T+\mu}{T-\mu}\frac{W-\mu-\nu}{W+\mu+\nu} = D\mu\ln\frac{1+\sqrt{q}X}{1-\sqrt{q}X},$$

where

$$Z = \frac{\sqrt{\nu-\nu_1}}{2+\mu+\nu} = \frac{\sqrt{q+\nu_0-\nu_1}}{q+2+\mu+\nu_0}.$$

We expand $K + L + M$ in powers of q. A first observation is that $F(q,X,\mu) := (K + L + M)/\sqrt{q}$ is a function of q, X, μ, the factor $\sqrt{q}$ being completely removed. We expand F in powers of q. We have

$$F(q,X,\mu) = F_0 + F_1 q + F_2 q^2 + \dots,$$

where $F_s(X,\mu)$ do not explicitly depend on q (or ν). We compute

$$F_0 = (\nu_0 - \mu)\sqrt{\nu_0 - \nu_1} - 2\mu(\nu_0+\mu)X + \frac{2\sqrt{\nu_0-\nu_1}(\nu_0+\mu)(\mu+1)}{2+\mu+\nu_0} + 2\mu(\nu_0+\mu)X.$$

It appears that $F_0(X,\mu) \equiv 0$, and that, hence, we can continue with the equation $G(q,X,\mu) := F/q = F_1 + F_2 q + \dots = 0$. We claim that the equation $G(q,X,\mu) = 0$ can be solved for $X = X(q,\mu)$, and that X is analytic for small values of both arguments. By calculating some limits, it follows from (5.18) or (5.19) that $X(0,0) = -1/2$. This is used to show that $G(0,-1/2,0) = F_1(0,0) = 0$. In order to apply an implicit function theorem (see [1, p. 36]), we need to show that G is analytic in a neighborhood of $(0,-1/2,0)$ and that $G(0,-1/2,0) = 0$, $G_X(0,-1/2,0) \neq 0$. It is straightforward to verify that $G(q,X,\mu)$ is analytic in a neighborhood of $(0,-1/2,0)$. Furthermore, $G_X(0,-1/2,0) = \partial F_1/\partial X = 4$ at $(X,\mu) = (-1/2,0)$. We have shown that we can solve the equation $G = 0$ and that the solution $X(q,\mu)$ is analytic in a fixed neighborhood of $(0,0)$.

It remains to show that β^2 is analytic. We consider T of (5.2) given in the middle of (5.19). We are done when we have shown that μ/D is bounded away from zero when μ is small, since then we can divide the denominator of T by μ. From the above result it follows that we can expand

$$X(q,\mu) = X_0(q) + X_1(q)\mu + \dots,$$

where the coefficients X_s are analytic functions of q. From the first equation of (5.19) we compute $X_0 = -1/\sqrt{\nu-\nu_1} = -1/\sqrt{\nu+4}$. Hence $D = \nu + \mu + (\nu-\nu_0)\sqrt{\nu-\nu_1}X_0 + \mathcal{O}(\mu) = \mathcal{O}(\mu)$ as $\mu \to 0$. It now follows that T^2 is an analytic function of q,μ in a fixed neighborhood of $(0,0)$, and, consequently, that β^2 is analytic. This proves the theorem. ∎

REMARK 5.1. It is possible to base a proof on the differential equation (5.14). The condition $\beta(0) = 0$ is not enough to prove the theorem, since the ratio β^2/ν (at $\nu = 0$) turns out to be undefined. Requiring that this ratio equals $c_1(\mu)$ of (5.17) is sufficient, however.

In Theorem 4.1 we expanded the functions of (4.10) at the critical points $\pm w_0$, and in (4.14) we used a representation of t in which y can be viewed as a part of the complete expansion. In fact (4.14) is a change of variables. In the present case we expand at the critical points $w_\pm$, and the expansions have the form

$$\phi(w) = \sum_{k=0}^{\infty} [a_k + w b_k] V^k, \quad V = V(w) = (w - w_-)(w - w_+). \tag{5.20}$$

When ϕ is sufficiently smooth, the coefficients a_k, b_k are uniquely defined. The first few are given by

$$a_0 = \frac{\phi_- w_+ - \phi_+ w_-}{w_+ - w_-}, \quad b_0 = \frac{\phi_+ - \phi_-}{w_+ - w_-},$$

$$a_1 = \frac{b_0 w_0 - \phi'_+ w_- - \phi'_- w_+}{(w_+ - w_-)^2}, \quad b_1 = \frac{\phi'_+ + \phi'_- - 2b_0}{(w_+ - w_-)^2},$$

where $w_0 = w_+ - w_-$, and $\phi_+ = \phi(w_+)$, etc. For analytic functions the coefficients can be represented as Cauchy-type integrals. We have

$$a_k = \frac{1}{2\pi i} \int_C (w - w_0) V^{-k-1}(w) \phi(w)\, dw, \quad b_k = \frac{1}{2\pi i} \int_C V^{-k-1}(w) \phi(w)\, dw, \tag{5.21}$$

where C is a contour around the two critical points; ϕ must be analytic inside C and continuous on C. This can be verified by substituting a new variable $w = v + w_0/2$. Then we have

$$f(v + w_0/2) = \sum c_k V^k + v \sum b_k V^k, \quad c_k = a_k + \frac{1}{2} w_0 b_k.$$

By separating odd and even parts (with respect to v), and representing c_k, b_k as Cauchy integrals in the $V-$plane, we arrive at (5.21). (Note that a circle around the origin in the $w-$plane is traversed twice in the $V-$plane.) For MacLaurin series the domain of convergence is a disc. For expansions as in (5.20) the domain of convergence is defined by $|V(w)| < |V(w_s)|$, where w_s is a singularity of ϕ; this domain is bounded by a Cassini's oval with foci at $w_\pm$. See also [7, p. 149, Exercise 24].

The parameter t of (5.12) is represented in the form

$$t = w[B + Cw + V(w)y], \tag{5.22}$$

where B, C do not depend on w, and we require that the points $\{w_-, 0, w_+\}$ correspond with $\{t_-, 0, t_+\}$. This gives for B, C the values

$$B = \frac{w_+^2 t_- - w_-^2 t_+}{w_+ w_- (w_+ - w_-)}, \quad C = \frac{t_+ w_- - t_- w_+}{w_- w_+ (w_+ - w_-)}. \tag{5.23}$$

The critical points $w_\pm, t_\pm$ are not analytic for small values of the parameters. However, we have

LEMMA 5.1. *$B, C, w_+ w_-, w_0 = w_+ + w_-$ are analytic functions of μ, ν in a fixed neighborhood of $(0,0)$. Moreover, $B = 1 + o(1)$, $C = o(1)$ near $(0,0)$.*

PROOF. We use the notation of Theorem 5.1. We have $w_0 = \ln(1+\mu)$ and the product $w_+ w_-$ is an even function of W. So the singularity in $W = \sqrt{q}\sqrt{\nu - \nu_1}$ is removed when we expand $w_+ w_-$ in powers of W. Using (5.2), we can write

$$2C = -\frac{\mu}{w_+ w_-} \left[1 - \frac{T \ln(1+\mu)/\mu}{w_+ - w_-}\right]. \tag{5.24}$$

We introduce a parameter η by writing

$$\beta^2 = \nu[E + (\nu - \nu_0)\eta], \quad E = -\frac{\mu^2}{4\nu_0} = -\frac{\nu_1}{4}.$$

Then we have $T = 2\sqrt{q}\sqrt{E + \nu\eta}$; $\eta = \eta(\nu, \mu)$ is analytic in a neighborhood of $(0,0)$. Next we use $w_+ - w_- = \ln[(1 + \sqrt{q}Z)/(1 - \sqrt{q}Z)]$. Since the factor $\sqrt{q}$ can be removed, we infer that the fraction $T/(w_+ - w_-)$ is regular. It is easily verified that the expression between square brackets in (5.24) vanishes when $\nu \to 0$ and that $w_+ w_- = -\nu F$, where $F = F(\nu, \mu)$ is analytic at $(0,0)$, with $F = 1 + \mathcal{O}(\nu + \mu)$, as $\nu, \mu \to 0$. This proves that C is analytic at $(0,0)$; the factor μ in the first fraction of (5.24) takes care of the vanishing of C at $(0,0)$. A more detailed analysis shows that $C \sim -\mu/24$, $\mu \to 0$, $\nu = 0$. The proof for B now follows from the representation $B + w_0 C = (t_+ - t_-)/(w_+ - w_-) = T/(w_+ - w_-)$. At $\nu = 0$ this expression reduces to $\mu/\ln(\mu + 1) = 1 + \mathcal{O}(\mu)$, as $\mu \to 0$. ∎

COROLLARY 5.1. *Let ϕ of (5.20) be analytic in a domain containing the points $w_\pm$. Then the coefficients a_k, b_k are analytic functions of the parameters μ, ν.*

PROOF. This follows from the fact that sum and product of $w_\pm$ occur in $V(w)$ and that the Cauchy type integrals in (5.21) are analytic functions of $w_+ + w_-$ and $w_+ w_-$. ∎

After these preparations we are ready to consider

THEOREM 5.2. *The function $t(w, \nu, \mu)$ defined by (5.12), with β^2 defined in (5.13), is analytic in a fixed neighborhood of $(0, 0, 0)$.*

PROOF. We write (5.12) in the form

$$F(t, w, \mu, \nu) = tH(w) - S(t) = 0, \tag{5.25}$$

where

$$H(w) = -\mu \ln \frac{e^w - 1}{w} + (\mu + 1)w + \frac{\nu}{e^w - 1} - A, \quad S(t) = t^2 + \beta^2 - \mu t \ln \frac{t}{w}. \tag{5.26}$$

Using (5.22) we can consider F as a function of w, with two known parameters μ, ν and one unknown parameter y. We expand F as in (5.20):

$$F = \sum_{k=0}^{\infty} [u_k + w v_k] V^k(w), \tag{5.27}$$

where the coefficients u_k, v_k do not depend on w and t; they do depend on y, however. The first coefficients are

$$\begin{aligned}
u_0 &= -C^2b^2a - C^2a^2 - aB^2 - 2aBCb + f_0B + g_0aC - \beta^2 \\
&\quad + \mu a(Cc_0 + Bd_0 + Cbd_0), \\
v_0 &= -2aBC - C^2b^3 - bB^2 + f_0C + g_0B - 2C^2ab - 2b^2BC + g_0bC \\
&\quad + \mu(Cad_0 + Bc_0 + Cbc_0 + bBd_0 + Cb^2d_0), \\
u_1 &= -B^2 - C^2b^2 - 2aBy - 2Cbay - 2C^2a + f_0y + f_1B - 2BCb + g_0C + g_1aC \\
&\quad + \mu(aBd_1 + Cbad_1 + Cac_1 + Cc_0 + Bd_0 + Cbd_0 + ayd_0), \\
v_1 &= -2bBy - 2Cb^2y - 2BC - 2Cay - 2C^2b + g_0y + f_1C + g_1B + g_1bC \\
&\quad + \mu(bBd_1 + Cb^2d_1 + Bc_1 + Cad_1 + Cbc_1 + Cd_0 + yc_0 + byd_0), \\
u_2 &= -ay^2 + g_1C + f_2B + f_1y - 2By + g_2aC - C^2 - 2Cby \\
&\quad + \mu(Cac_2 + aBd_2 + Bd_1 + Cc_1 + yd_0 + Cbd_1 + Cbad_2 + ayd_1), \\
v_2 &= g_2bC - by^2 + g_1y - 2Cy + g_2B + f_2C \\
&\quad + \mu(Cb^2d_2 + byd_1 + Cbc_2 + Cad_2 + Bc_2 + bBd_2 + yc_1 + Cd_1),
\end{aligned}$$

where a, b are defined by $w^2 = a + bw + V(w)$, i.e., $a = -w_+w_-$, $b = w_+ + w_-$ and the coefficients c_k, d_k, f_k, g_k occur in the expansions

$$\begin{aligned}
wH(w) &= f_0 + g_0w + f_1V + g_1wV + f_2V^2 + g_2wV^2 + \ldots, \\
\ln \frac{t}{w} &= c_0 + d_0w + c_1V + d_1wV + c_2V^2 + d_2wV^2 + \ldots.
\end{aligned}$$

The coefficients u_0, v_0 vanish identically. This can be verified by straightforward manipulations. It also follows from the observation that the representation (5.22) can be viewed as a truncated expansion for t, in which the first coefficients B, C are defined properly. When more coefficients D, E, ... would have been included in y (and defined properly), more and more coefficients u_k, v_k would vanish

identically. When using (5.22), only a few coefficients will vanish. Although u_1, v_1 contain the parameter y (also via c_1, d_1), these coefficients vanish too. Again, this can be verified by straightforward manipulations.

It follows that we can proceed with the equation $G = 0$, where

$$G = G(y, w, \mu, \nu) = F(t, w, \mu, \nu)/V^2(w)$$

. The coefficient u_2 contains a term $-2By$, with B given in (5.23). From Lemma 5.1 it follows that B is bounded away from zero when the parameters μ, ν are small. The remaining contributions to u_2 containing the parameter y tend to zero as $\mu, \nu \to 0$. All coefficients u_k, v_k are analytic functions of μ, ν, and the convergent infinite series (including coefficients v_2 and higher) represents a function of y, w, μ, ν that is analytic in a neighborhood of (0,0,0,0). Consequently, since $\partial G(0,0,0,0)/\partial y = -2$, we can solve for y and this solution is an analytic function of w, μ, ν in a fixed neighborhood of (0,0,0). The same holds for t given in (5.22). ∎

REMARK 5.3. A simpler version ($\mu = 0$) of the above theorem is considered in Theorem 4.1. Another simpler version ($\nu = 0$) is given in [5, Theorem 2.1].

We still have to show that Ω (the image of strip H of (4.13) under the mapping $w \mapsto t$ defined in (5.12)) is large enough to contain a disc around t_+ with radius $\rho(1 + t_+)^\kappa, \kappa \geq 1/2$, ρ fixed. It is not difficult to verify that when $\beta > \mu$ the proof runs as in §4.2. If μ is much larger than β, the situation improves, and we can take $\kappa = 1$.

We conclude with computing a bound for the quantity $M_r(\beta, \mu)$ used in (5.8), and defined as in (3.16). The $t-$ values on the circle C_r are written as $t = t_+ + \tau\sqrt{t_+ + 1}$, with $|\tau| = r$, r fixed. We assume that at least one of the parameters ν, μ is large. We have

$$t + \beta^2/t - \mu \ln t \sim t_+ + \beta^2/t_+ - \mu \ln t_+ + \frac{T(1 + t_+)}{t_+(\mu + T)}\tau^2 + \mathcal{O}(t_+^{-1}).$$

We denote the factor multiplying τ^2 by q. Observe that, roughly speaking, q belongs to the interval $[1/2, 1]$. Using this in (5.12), we obtain

$$q\tau^2 \sim -\mu \ln \frac{e^w - 1}{e^{w_+} - 1} + (\mu + 1)(w - w_+) + \nu\Big[\frac{1}{e^w - 1} - \frac{1}{e^{w_+} - 1}\Big].$$

Denoting the right-hand side by $\phi(w)$, we see that $\phi(w_+) = \phi'(w_+) = 0$. A few computations give

$$\phi''(w_+) = 1 + \frac{2(\nu + W)}{(\nu + \mu + w)^2} = 1 + o(1).$$

To solve the equation $\phi(w) = q\tau^2$ we expand $\phi(w_+ + v) = \frac{1}{2}v^2\phi''(w_+) + \ldots$. We can take the fixed number r as small as we please. Then the solution of the above equation reads $w \sim w_+ + \tau\sqrt{2q}$. Using this in (5.16), we infer that $f(t) \sim 4\sqrt{t_+}/\tau$, under the condition that $t \in C_r$ and that at least one of the parameters ν, μ is large. Consequently, we can find a fixed number K, such that

$$M_r(\beta, \mu) \leq K\sqrt{1 + t_+}/r, \quad \nu, \mu \in [0, \infty).$$

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