Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

J.M. Schumacher

State representations of linear systems with output constraints

Department of Operations Research and System Theory Report OS-R8807 March

The Centre for Mathematics and Computer Science is a research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a nonprofit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

State Representations of Linear Systems with Output Constraints

J. M. Schumacher
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

We derive representations in state space form for linear systems that are described by input/state/output equations, and that are subjected to a number of constant linear constraints on the outputs. In the case of a linear system without further structure, the state representation of the constrained system turns out to be essentially non-unique in general. For linear Hamiltonian systems, however, there is a 'natural' choice of the representation which preserves the Hamiltonian structure, and which leads to a unique solution (under a certain nondegeneracy condition). As a by-product, we obtain an algebraic proof of the rule 'a system with n degrees of freedom under k constraints becomes a system with $n-k$ degrees of freedom.' Similar results are obtained for linear systems with a gradient structure.

1980 Mathematics Subject Classification (1985): 93B20, 70F20, 70J15, 94C05.

Key Words and Phrases: linear systems, gradient systems, Hamiltonian systems, state representation.

Note: This report will be submitted for publication elsewhere.

1. Introduction and preliminaries

The purpose of this paper is to contribute to the understanding of the relation between *realization theory* and *physical modeling*. Here, we understand 'realization theory' in a broad sense, as the theory of equivalent system representations. 'Physical modeling' is understood as the construction of dynamical models for physical systems using constitutive equations and element connections.

Methods for physical modeling, in the above sense, are basic ingredients in every engineering curriculum. Although this is not always made very explicit, the central issue addressed by these techniques is the *transformation* of a given system of differential and algebraic relations to another — more suitable — form. For instance, one may obtain a system description for an electrical network immediately by writing down the equations that follow from the laws of Kirchhoff, Ohm, and Faraday. The system of equations that one gets will in general involve algebraic as well as differential relations. The modeling problem is to obtain an equivalent description in standardized (for instance, input/state/output) form. The setting up of Lagrangian equations for constrained mechanical systems follows essentially the same route.

The problem of transforming a system of algebraic and differential equations to a standardized form has also been considered in system theory, for instance by Rosenbrock [14] and Luenberger [9], and more recently by J. C. Willems [20] and the author [18]. However, it appears that these methods, which were developed for the class of general linear systems, are not able to bring out the specific properties that one should expect to see in the context of physical modeling. A simple example may help to clarify this point. Consider two masses attached to springs, each subject to an external force, and suppose that the motion is constrained by a rigid connection between the two masses. Equations may be written down as follows:

$$m_1\ddot{y}_1(t) + k_1 y_1(t) = u_1(t) \tag{1.1}$$

Report OS-R8807
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

$$m_2\ddot{y}_2(t) + k_2 y_2(t) = u_2(t) \tag{1.2}$$

$$y_1(t) = y_2(t) \tag{1.3}$$

In the framework of Willems [20], the 'behavior' defined by these equations is simply the set of all trajectories $(y_1(\cdot), y_2(\cdot), u_1(\cdot), u_2(\cdot))^{\mathrm{T}}$ that satisfy (1.1-1.3). Among these trajectories, there are obviously also harmonic solutions, for instance

$$y_1(t) = y_2(t) = \sin \omega t \tag{1.4}$$

$$u_1(t) = (k_1 - m_1\omega^2)\sin \omega t \tag{1.5}$$

$$u_2(t) = (k_2 - m_2\omega^2)\sin \omega t. \tag{1.6}$$

These solutions show no value of ω that has a special significance. The situation changes, however, if we associate with each trajectory the function

$$W(t) = u_1(t)\dot{y}_1(t) + u_2(t)\dot{y}_2(t) \tag{1.7}$$

which expresses the work done one the system. For the harmonic solutions above, one gets

$$W_h(t) = [(k_1 + k_2) - (m_1 + m_2)\omega^2]\,\omega \sin \omega t \cos \omega t. \tag{1.8}$$

From this, we see that there is one particular value of ω that leads to a nontrivial harmonic solution in which no work is done on the system:

$$\omega_n = \left[\frac{k_1 + k_2}{m_1 + m_2}\right]^{1/2}. \tag{1.9}$$

Of course, ω_n is, according to generally accepted rules, the *natural frequency* of the connected system.

Our example is sufficiently generic to warrant the conclusion that the natural frequencies of a connected system will in general not be recognized if one follows the viewpoint of [20]. This is no surprise since the framework of this reference is developed for the class of general linear systems, and, at this level, there is no notion of *energy* which serves to distinguish the 'natural' frequencies from other frequencies. The same phenomenon occurs in any other setting for transformations of general systems of linear equations, such as the one provided by [14].

The present paper aims to explain the seeming discrepancy between modeling with the techniques of [20] and [18] on the one hand, and 'physical modeling' on the other hand. We shall show that a link between the two can be formulated in *geometric* terms, i. e., using particular structures of vector spaces, in the spirit of modern treatments of classical mechanics (cf. [1]). Since our primary purpose is to establish the existence of this link, full generality will not be pursued; in particular, the treatment here will be restricted to linear systems, and, even more in particular, to systems of algebraic and differential equations that can be written as standard input/state/output systems together with a number of static constraints on the outputs. Furthermore, we will feel free to use additional assumptions (although not unreasonable ones) when this is convenient. Although we shall concentrate on the connection between the *general linear* case and the more specific cases of *Hamiltonian* linear systems and *gradient* linear systems, one can reasonably expect that the links established in the linear situation can be extended to the nonlinear case (cf. [16], [17].)

We will now quickly review a number of definitions and results from linear algebra and from linear system theory that will be needed below. Consider first a linear system in input/state/output representation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.10}$$

$$y(t) = Cx(t) + Du(t) \tag{1.11}$$

where $x(t)$, $u(t)$, and $y(t)$ take values in finite-dimensional linear spaces X, U, and Y, and where A, B, C, and D are linear mappings between the appropriate spaces. To alleviate the notational burden, the time argument will often be suppressed below. With the system (1.10-1.11), we associate the *transfer matrix*

$$G(s) = C(sI - A)^{-1}B + D. \tag{1.12}$$

The transfer matrix $G(s)$ can be considered as a matrix over the field $\mathbb{R}(s)$ of rational functions with real coefficients. The system (1.10-1.11) is said to be *invertible* if $G(s)$ is invertible as a matrix over $\mathbb{R}(s)$; similar definitions are used for left and right invertibility. Because $G(s)$ as defined by (1.12) is regular at infinity, we can also consider $G(s)$ as a matrix over the ring $\mathbb{R}_\infty(s)$ of proper rational functions with real coefficients. The ring $\mathbb{R}_\infty(s)$ is a principal ideal domain with a unique maximal ideal generated by the function s^{-1}; one can therefore define a *Smith normal form* of $G(s)$ (see, for instance, [10], p. 42) in which all nonzero elements are of the form s^{-k_j}, $k_j \geqslant 0$ (see [6]). The indices k_j are called the *orders of the zeros at infinity* of $G(s)$.

The orders of the zeros at infinity can also be expressed more directly in terms of the mappings A, B, C, and D. For this, we need the 'V^*-algorithm' (given in [21], p. 91, for the case $D = 0$, and in [2] for the more general case in which D may be nonzero). Given A, B, C and D as mappings between the state space X, the output space Y, and the input space U, define

$$V^0 = X \tag{1.13}$$

$$V^{k+1} = \{x \in V^k \mid \exists u \in U \text{ s.t. } Ax + Bu \in V^k \text{ and } Cx + Du = 0\}. \tag{1.14}$$

Clearly, one defines in this way a decreasing sequence of subspaces of X. Because $\dim X$ is finite, there must be some value of k for which V^{k+1} equals V^k, and then V^{k+j} will be equal to V^k for all $j \geqslant 0$. This limit subspace will be denoted by $V^*(A, B, C, D)$ or simply by V^* if the reference is clear. Now, it has been shown in [11] (see also [12]) that the number of zeros at infinity of order $\geqslant k$ of the system (1.10-1.11) is, for $k \geqslant 1$, equal to

$$p_k = \dim(V^{k-1} \cap B[\ker D]) - \dim(V^* \cap B[\ker D]). \tag{1.15}$$

When $D = 0$, there are no zeros at infinity of order zero so that p_1 is equal to the rank of the transfer matrix:

$$\operatorname{rank} G(s) = \dim \operatorname{im} B - \dim(\operatorname{im} B \cap V^*). \tag{1.16}$$

This formula was first proved, by a different method, in [5]. For the '$D \neq 0$' case, one has

$$\dim \ker G(s) = \dim\{u \in U \mid Bu \in V^* \text{ and } Du = 0\}. \tag{1.17}$$

The subspace V^* is the largest of the subspaces V of X having the property that there exists a feedback mapping $F: X \to U$ such that

$$(A + BF)V \subset V \tag{1.18}$$

and

$$V \subset \ker(C + DF) \tag{1.19}$$

(see [21], [2]).

The system (1.10-1.11) is said to have *uniformly k-th order zeros at infinity* if it is invertible and all its zeros at infinity are of order k. If $k = 0$, this simply means that D must be invertible. For larger values of k, it is easily verified that (1.10-1.11) has uniformly k-th order zeros at infinity if and only if $D = 0$, $CA^jB = 0$ for $j = 0, \cdots, k-2$, and $CA^{k-1}B$ is invertible. Also, in this case, it is seen from the defining algorithm (1.13-1.14) that

$$V^* = V^k = \ker C \cap \ker CA \cap \cdots \cap \ker CA^{k-1}. \tag{1.20}$$

Now, let us recall some definitions from linear algebra that will be needed below (see, for instance, [8], Ch. XIII). Let X be a vector space over a field K of characteristic $\neq 2$. A *symmetric form* on X is a bilinear mapping $f: X \times X \to K$ that satisfies $f(x_1, x_2) = f(x_2, x_1)$ for all x_1 and x_2 in X. The form is said to be *nondegenerate* if $f(x_1, x_2) = 0$ for all x_2 in X only if $x_1 = 0$. We shall use square brackets below as a notation for symmetric forms, so we shall write $[x_1, x_2]$ rather than $f(x_1, x_2)$. Note that positivity is not required in the definition of a symmetric form. A linear mapping $A: X \to X$ is said to be *symmetric* with respect to $[\cdot\,,\cdot\,]$ if

$$[Ax_1, x_2] = [Ax_2, x_1] \tag{1.21}$$

for all x_1 and x_2 in X.

A bilinear mapping $g: X \times X \to K$ is said to be an *alternating form* if $g(x_1, x_2) = -g(x_2, x_1)$ for all x_1 and x_2 in X. The form is said to be *nondegenerate* under the same condition as in the symmetric case. We shall use round brackets to denote alternating forms, writing (x_1, x_2) rather than $g(x_1, x_2)$. A vector space equipped with a nondegenerate alternating form is called a *symplectic space*; such spaces are always even-dimensional ([8], p. 371). For a symplectic space, one can always find a *symplectic basis*, i. e., a basis in which the alternating form (x_1, x_2) can be written as $x_1^T J x_2$ with

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}. \tag{1.22}$$

Given a subspace V of a symplectic space X, its *symplectic orthoplement* is

$$V^{(\perp)} = \{x \in X \mid (v, x) = 0 \text{ for all } v \in V\}. \tag{1.23}$$

Simple rules such as $\dim V^{(\perp)} = \operatorname{codim} V$ and $(V \cap W)^{(\perp)} = V^{(\perp)} + W^{(\perp)}$ will be freely used.

Now, let U and Y be vector spaces over a field K. A bilinear mapping $h: U \times Y \to K$ is called a *duality* between U and Y if it is nondegenerate in the sense that $h(u, y) = 0$ for all $u \in U$ implies $y = 0$, and $h(u, y) = 0$ for all $y \in Y$ implies $u = 0$. Dualities will be denoted below by sharp brackets: we shall write $<u, y>$ rather than $h(x, y)$. Spaces that are connected by a duality must have equal dimension. If Y_1 is a subspace of Y, its *orthogonal space* is the subspace of U that is defined by

$$Y_1^{\perp} = \{u \in U \mid <u, y> = 0 \text{ for all } y \in Y_1\}. \tag{1.24}$$

Again, simple rules concerning dimensions and concerning sums and intersections will be used without comment.

If X is a space equipped with a symmetric form, then every subspace of X can also be equipped with a symmetric form in a natural way; one simply takes the restriction of the form to the given subspace. The same is not true for symplectic spaces, as is already evident from the fact that symplectic spaces must have even dimension. However, we do have the following result.

LEMMA 1.1 *Let X be a symplectic space and let V be a subspace of X. Let W be a complement of $V \cap V^{(\perp)}$ in V. Under these conditions, W is a symplectic space with respect to the restriction of the alternating form on X to W.*

PROOF There is a natural isomorphism between W and the factor space $V/(V \cap V^{(\perp)})$, and this isomorphism makes the restricted alternating form on W correspond to the induced alternating form on the factor space. It is easily seen that $V/(V \cap V^{(\perp)})$ is symplectic with respect to the induced form.

In a similar fashion, one can prove that if U and Y are dual spaces and Y_1 is a subspace of Y, then there is a natural induced duality between Y_1 and any complement of $Y_1^{\perp}$ in U.

2. THE GENERAL LINEAR CASE

Consider a linear system in input/state/output form:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.1}$$

$$y(t) = Cx(t) + Du(t) \tag{2.2}$$

Now assume that we constrain the outputs to lie in a certain subspace of the output space Y:

$$y(t) \in Y_1, \quad Y_1 \subset Y. \tag{2.3}$$

The three equations (2.1-2.3) still describe a linear system in the sense of [19] (the set of input/output functions $(u(\cdot), y(\cdot))$ for which there exists a state function $x(\cdot)$ such that (2.1), (2.2), and (2.3) are satisfied forms a linear subspace of the vector space of all input/output functions), but it is, of course, not a description in state space form. So one may ask how to obtain minimal state representations for the system described by (2.1-2.3). In order to solve this, we note that the question in this form is a particular instance of a problem for which a solution algorithm was given in [18]. We review the procedure from [18] for the special case at hand.

Let H be a linear mapping acting on Y such that $\ker H = Y_1$. Let $V^*(Y_1)$ denote the subspace $V^*(A, B, HC, HD)$ of X. In other words, $V^*(Y_1)$ is the largest subspace V of X for which there exists a feedback mapping $F: X \to U$ such that V is $(A+BF)$-invariant, and $(C+DF)V \subset Y_1$. Decompose X as $X = X_1 \oplus X_2$ where $X_1 = V^*(Y_1)$. Let the mapping $F: X \to U$ be such that $V^*(Y_1)$ is $(A+BF)$-invariant and $(C+DF)V^*(Y_1) \subset Y_1$; we can always arrange that $\ker F$ contains X_2 (this simplifies the notation somewhat, but is otherwise inessential). The equations (2.1-2.2) may now be rewritten in the form

$$\dot{x}_1 = (A_{11} + B_1 F_1)x_1 + A_{12}x_2 + B_1(u - F_1 x_1) \tag{2.4}$$

$$\dot{x}_2 = A_{22}x_2 + B_2(u - F_1 x_1) \tag{2.5}$$

$$y = (C_1 + DF_1)x_1 + C_2 x_2 + D(u - F_1 x_1). \tag{2.6}$$

By construction, we have $H(C_1 + DF_1) = 0$, so that the restriction (2.3) can be written as

$$HC_2 x_2 + HD(u - F_1 x_1) = 0. \tag{2.7}$$

We temporarily introduce new inputs by defining

$$u - F_1 x_1 = [G_1 \quad G_2] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \tag{2.8}$$

where $G = [G_1 \quad G_2]$ is an invertible mapping satisfying

$$\operatorname{im} G_1 = \{u \in U \mid Bu \in V^*(Y_1),\ Du \in Y_1\}. \tag{2.9}$$

This gives us the equations $B_2 G_1 = 0$ and $HDG_1 = 0$. We can now rewrite (2.5) as

$$\dot{x}_2 = A_{22}x_2 + B_2 G_2 v_2 \tag{2.10}$$

whereas the constraints can be formulated as

$$HC_2 x_2 + HDG_2 v_2 = 0. \tag{2.11}$$

It is shown in [18] that the only solution of the set of differential and algebraic equations given by (2.10-2.11) is the zero solution $x_2(t) = 0$, $v_2(t) = 0$. So, the input/output trajectories of the system (2.1-2.2) that satisfy the constraint (2.3) are described by

$$\dot{x}_1 = (A_{11} + B_1 F_1)x_1 + B_1 G_1 v_1 \tag{2.12}$$

$$y = (C_1 + DF_1)x_1 + DG_1 v_1 \tag{2.13}$$

$$u = G_1 v_1 + F_1 x_1. \tag{2.14}$$

Now, we want to eliminate the auxiliary input v_1 in order to arrive at a description in standard state space form. This can be done from (2.14) since G_1 is injective. Write

$$[G_1 \quad G_2]^{-1} = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} \tag{2.15}$$

so that $v_1 = K_1 u - K_1 F_1 x_1$. The equations (2.12-2.13) can be written as:

$$\dot{x}_1 = (A_{11} + B_1(I - G_1 K_1)F_1)x_1 + B_1 G_1 K_1 u \tag{2.16}$$

$$y = (C_1 + D(I - G_1K_1)F_1)x_1 + DG_1K_1u \tag{2.17}$$

$$K_2u = K_2F_1x_1. \tag{2.18}$$

Note that the original inputs $u(t)$ have been split up into certain combinations $K_1u(t)$ which still function as inputs, and other combinations $K_2u(t)$ whose values are determined by the constraints. The latter variables are, therefore, described as outputs.

By construction, we have $K_1G_1 = I$ so that $I - G_1K_1$ is a projection with kernel

$$\ker(I - G_1K_1) = \operatorname{im} G_1 = \{u \in U \mid Bu \in V^*(Y_1),\ Du \in Y_1\}. \tag{2.19}$$

The image of the projection is $\operatorname{im} G_2$, which may be any complement to $\operatorname{im} G_1$ in U. Note that if $F: X \to U$ is any mapping which acts on X_1 like $(I - G_1K_1)F_1$, then F has the properties

$$(A + BF)V^*(Y_1) \subset V^*(Y_1) \tag{2.20}$$

and

$$(C + DF)V^*(Y_1) \subset Y_1, \tag{2.21}$$

and satisfies $\operatorname{im} F \subset \operatorname{im} G_2$. Our conclusions can be summarized as follows.

THEOREM 2.1 *Let $\mathscr{B}$ be the external behavior of the system (2.1-2.3), i. e., $\mathscr{B}$ is the set of input/output trajectories $(u(\cdot), y(\cdot))$ for which there exists a state trajectory $x(\cdot)$ such that (2.1-2.3) are satisfied. Let $V^*(Y_1)$ be the limit of the decreasing sequence of subspaces defined by*

$$V^0 = X \tag{2.22}$$

$$V^{k+1} = \{x \in V^k \mid \exists u \in U \text{ s.t. } Ax + Bu \in V^k,\ Cx + Du \in Y_1\}. \tag{2.23}$$

Write $U_1 = \{u \in U \mid Bu \in V^*(Y_1),\ Du \in Y_1\}$, and let U_2 be any complement of U_1 in U. Then it is always possible to choose F such that

$$(A + BF)V^*(Y_1) \subset V^*(Y_1) \tag{2.24}$$

$$(C + DF)V^*(Y_1) \subset Y_1 \tag{2.25}$$

and

$$\operatorname{im} F \subset U_2. \tag{2.26}$$

Take such a mapping F. Write F_{21} for the restriction of F to $V^*(Y_1)$ considered as a mapping into U_2, $(A + BF)_{11}$ for the restriction of $A + BF$ to $V^*(Y_1)$, B_{11} for the restriction of B to U_1 taken as a mapping into $V^*(Y_1)$, and C_1 for the restriction of C to $V^*(Y_1)$. With these definitions, the set $\mathscr{B}$ is described by the following equations in i/s/o form:

$$\dot{x}_1 = (A + BF)_{11}x_1 + B_{11}u_1 \tag{2.27}$$

$$y = (C_1 + D_2F_{21})x_1 + D_1u_1 \tag{2.28}$$

$$u_2 = F_{21}x_1. \tag{2.29}$$

Moreover, the action of F on $V^*(Y_1)$ is determined uniquely by the requirements (2.24), (2.25) and (2.26).

PROOF It remains to prove the uniqueness claim. Let F and F' both satisfy the conditions (2.24), (2.25), and (2.26). Take $x \in V^*(Y_1)$, and write $Fx = u$, $F'x = u'$. It follows from (2.24) and (2.25) that $B(u - u') \in V^*(Y_1)$, and $D(u - u') \in Y_1$, so that $u - u' \in U_1$ by the definition of U_1. However, from (2.26) we see that also $u - u' \in U_2$. Because $U_1 \cap U_2 = \{0\}$, it follows that $u = u'$, and the claim is proved.

COROLLARY 2.2 *In the situation of the theorem, the reduction of the number of inputs that results from imposing the constraint (2.3) is equal to*

$$m_{red} = \operatorname{codim} Y_1 - \operatorname{codim}(Y_1 + \operatorname{im} G(s)). \tag{2.30}$$

REMARK 2.3 The number of independent constraints is given by $\operatorname{codim} Y_1$. So, the corollary states in particular that the number of inputs can never be reduced by an amount larger than the number of independent constraints.

PROOF The theorem that we just proved shows that the number of inputs in the constrained system is equal to

$$m_c = \dim\{u \in U \mid Bu \in V^*(Y_1),\ Du \in Y_1\}. \tag{2.31}$$

Let H denote any mapping such that $\ker H = Y_1$; then (1.17) shows that the above quantity is equal to the dimension of the kernel of the rational mapping $HG(s)$. Denoting the original number of inputs by $m = \dim U$, we can write

$$\begin{aligned} m_{red} &= m - m_c = \dim U - \dim\ker HG(s) = \dim\operatorname{im} HG(s) = \\ &= \dim\operatorname{im} G(s) - \dim[\operatorname{im} G(s) \cap \ker H] = \\ &= \dim[\operatorname{im} G(s) + Y_1] - \dim Y_1, \end{aligned} \tag{2.32}$$

which is, of course, equivalent to (2.30).

It is seen that the constraints will not reduce the number of inputs if and only if the transfer matrix $G(s)$ maps into the constraint subspace Y_1. In general, one can say that only constraints on outputs that depend on the *controllable* part of the system will reduce the number of inputs.

An important thing to note in the theorem is that, once a choice has been made for a complement of $U_1 = \{u \in U \mid Bu \in V^*(Y_1),\ Du \in Y_1\}$ in U, the minimal state representation of (2.1-2.3) is essentially unique. So, the ambiguity of choosing a state representation for a system under output constraints is parametrized by the freedom one has in selecting a complement to a given subspace of the input space.

3. HAMILTONIAN SYSTEMS

We use the following definition of a linear Hamiltonian system in state space form, which is easily seen to be compatible with the definition given in [15] (p. 111, p. 150).

DEFINITION 3.1 Consider a linear system in input/state/output form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3.1}$$

$$y(t) = Cx(t). \tag{3.2}$$

Assume that the state space X is equipped with a symplectic form denoted by $(\cdot\,,\cdot)$, and that the input and output spaces U and Y are dual with respect to a duality denoted by $<\cdot\,,\cdot>$. The system (3.1-3.2) is said to be *Hamiltonian* if the following conditions are satisfied:

(i) (Ax_1, x_2) defines a symmetric form on X;

(ii) $(x, Bu) = <Cx, u>$ for all $x \in X$ and $u \in U$.

We shall also assume that B is injective.

The assumption on the injectivity of the input mapping B helps to avoid some uninteresting singularities. Note that this assumption, under the condition (ii), also implies that the output mapping C is surjective: $u \in \ker B$ is equivalent to $0 = (x, Bu) = <Cx, u>$ for all x, i. e., $u \in \ker B$ if and only if $u \in (\operatorname{im} C)^{\perp}$.

For a Hamiltonian system (3.1-3.2), the quadratic form $H(x) = ½(Ax, x)$ is called the *Hamiltonian* of the system, or the *energy*. The system is said to be *time-reversible* if $CA^{2k}B = 0$ for all nonnegative integers k (see [15] (p. 200) for a motivation of the terminology).

The main result of this section is the following theorem, which gives the equations of motion in state space form for the constrained Hamiltonian system. Similar state space equations were given in [17] (for a much more general situation than considered here) and in [7], but in these references it was not proven that these equations do indeed give an equivalent representation of the original system with constraints.

THEOREM 3.2 *Let (3.1-3.2) be a Hamiltonian system with uniformly second order zeros at infinity (i. e., $CB=0$ and CAB is invertible). Let Y_1 be a subspace of Y, and consider the system (3.1-3.2) under the constraint*

$$y(t)\in Y_1 \text{ for all } t. \tag{3.3}$$

Define U_1 by

$$U_1=\{u\in U \mid Bu\in V^*(Y_1)\}. \tag{3.4}$$

Under these conditions, the subspace $Y_1^\perp$ is a complement of U_1 in U if and only if

$$CABY_1^\perp \cap Y_1=\{0\}. \tag{3.5}$$

Assume that (3.5) holds; then there exists a feedback mapping $F:X\rightarrow U$ satisfying $\operatorname{im} F\subset Y_1^\perp$ such that $V^*(Y_1)$ is $(A+BF)$-invariant and is mapped into Y_1 by $C+DF$. Take such an F. Write F_{21} for the resctriction of F to $V^*(Y_1)$ taken as a mapping into $Y_1^\perp$, $(A+BF)_{11}$ for the restriction of $A+BF$ to $V^*(Y_1)$, B_{11} for the restriction of B to U_1 taken as a mapping into $V^*(Y_1)$, and C_{11} for the restriction of C to $V^*(Y_1)$ taken as a mapping into Y_1. Let Y_2 be a complement of Y_1 in Y. A state space description of the system (3.1-3.3), with state space $X_1=V^*(Y_1)$, is then given by

$$\dot{x}_1=(A+BF)_{11}x_1+B_{11}u_1 \tag{3.6}$$

$$y_1=C_{11}x \tag{3.7}$$

$$u_2=F_{21}x_1 \tag{3.8}$$

$$y_2=0. \tag{3.9}$$

Moreover, the space X_1 is a symplectic space with respect to the form it inherits from X, and the system given by (3.6-3.7) is Hamiltonian with respect to this induced form and the induced duality between U_1 and Y_1. Finally, the energy function of (3.6-3.7) is the energy function of (3.1-3.2) restricted to X_1.

REMARK 3.3 It should be noted that (3.5) will hold for every subspace Y_1 of Y if the matrix CAB is positive definite. This is easy to see: suppose $u\in Y_1^\perp$ is such that $CABu\in Y_1$, then we will have

$$<CABu, u>=0. \tag{3.10}$$

If CAB is positive definite, this implies that $u=0$. The positive definiteness condition can be interpreted in physical terms. If $u(t)$ has the dimension of a force and $y(t)$ that of a displacement, then $(CAB)^{-1}$ will have the dimension of a mass. Under the conditions of the theorem (uniformly second order zeros at infinity), the matrix $(CAB)^{-1}$ is the leading term in the power series development around infinity of the inverse transfer function ('mechanical impedance'). Therefore, the requirement that CAB is positive definite can be interpreted as the condition that the effective mass matrix at infinite frequency should be positive definite.

PROOF (of the theorem) Following the 'V^* algorithm', one gets (because $CB=0$):

$$V^1(Y_1)=\{x\in X \mid Cx\in Y_1\}, \tag{3.11}$$

$$V^2(Y_1)=\{x\in X \mid Cx\in Y_1 \text{ and } CAx\in Y_1\}, \tag{3.12}$$

$$V^3(Y_1)=\{x\in V^2(Y_1) \mid \exists u\in U \text{ s.t. } CA^2x+CABu\in Y_1\}. \tag{3.13}$$

Because CAB is invertible, we have $V^3(Y_1)=V^2(Y_1)$, so that $V^*(Y_1)=V^2(Y_1)$ is given by (3.12). Therefore, we have (again using $CB=0$)

$$U_1=\{u\in U \mid CABu\in Y_1\}=(CAB)^{-1}Y_1 \tag{3.14}$$

which shows that the dimensions of U_1 and Y_1 are equal. So, $Y_1^\perp$ is a complement to U_1 if and only if

$Y_1^\perp \cap U_1 = \{0\}$; by (3.14), this is equivalent to (3.5). If the complementarity condition holds, then it follows from Thm. 2.1 that the constrained system is described by (3.6-3.9).

We now have to show that $X_1 = V^*(Y_1)$ inherits a symplectic structure from X. We use lemma 1.1 with $\{x \mid Cx \in Y_1\}$ in the role of the subspace V and $V^*(Y_1)$ in the role of the subspace W. First, we note that the symplectic orthoplement of $\{x \mid Cx \in Y_1\}$ is equal to $BY_1^\perp$, because

$$\{x \mid Cx \in Y_1\} = \{x \mid \langle Cx, u\rangle = 0 \text{ for all } u \in Y_1^\perp\} = \\ = \{x \mid (x, Bu) = 0 \text{ for all } u \in Y_1^\perp\} = (BY_1^\perp)^{(\perp)}. \tag{3.15}$$

It follows from $CB = 0$ that $BY_1^\perp$ is contained in $\{x \mid Cx \in Y_1\}$, so that, obviously,

$$\{x \mid x \in Y_1\} \cap \{x \mid Cx \in Y_1\}^{(\perp)} = BY_1^\perp. \tag{3.16}$$

Therefore, lemma 1.1 will give us the result we want if we can show that $V^*(Y_1)$ is a complement of $BY_1^\perp$ in $\{x \mid Cx \in Y_1\}$. To show that the two subspaces intersect only in 0, assume that $u \in Y_1^\perp$ is such that $Bu \in V^*(Y_1)$. It follows from (3.12) that $CABu \in CABY_1^\perp \cap Y_1 = \{0\}$. Because CAB is invertible, this proves indeed that $u = 0$. To complete this step in the proof, we have to show that $\{x \mid Cx \in Y_1\} = V^*(Y_1) + BY_1^\perp$. Take x such that $Cx \in Y_1$; we have to find $u \in Y_1^\perp$ such that $x - Bu \in V^*(Y_1)$. Because $CABY_1^\perp$ is a complement of Y_1 in the output space Y, there exists a $u \in Y_1^\perp$ such that

$$CAx - CABu \in Y_1. \tag{3.17}$$

Because we also have $C(x - Bu) = Cx \in Y_1$, we see that $x - Bu \in V^2(Y_1) = V^*(Y_1)$, as desired.

Now, since F maps into $Y_1^\perp$ and $BY_1^\perp$ is contained in the symplectic orthoplement of $V^*(Y_1)$, we have

$$((A + BF)x_1, x_2) = (Ax_1, x_2) \tag{3.18}$$

for x_1 and x_2 from $V^*(Y_1)$. It follows that condition (i) in the definition of a Hamiltonian system is satisfied, and that the Hamiltonian of the constrained system is the restriction to $V^*(Y_1)$ of the Hamiltonian of the original system. Because U_1 is a complement to $U_2 = Y_1^\perp$, the duality between U and Y can be restricted to a duality between U_1 and Y_1. It follows immediately that condition (ii) in the definition of a Hamiltonian system is also satisfied by the constrained system.

Remark 3.4 Suppose that the dimension of the original state space is $2n$, and that we impose k constraints; i.e., the codimension of Y_1 is k. Under the conditions of the theorem, we know that the state space of the constrained system, $V^*(Y_1)$, is a complement to $BY_1^\perp$ in $\{x \mid Cx \in Y_1\}$. Because B is injective and C is surjective, it follows from this that the dimension of $V^*(Y_1)$ is $2n - 2k$. This is the well-known property that 'a system with n degrees of freedom under k constraints becomes a system with $n - k$ degrees of freedom'. We have given here an *algebraic* proof of this fact; in textbooks, one usually finds proofs that are based on some limit argument (cf. the classical reference [13], but also the more recent treatment in [3]).

We have seen that imposing output constraints on a Hamiltonian system leads, in general, to a new Hamiltonian system. Some properties of the original system will go over to the constrained system; such properties will be called 'hereditary'. In the following proposition, we list a number of hereditary properties.

Proposition 3.5 *Under the conditions of Thm. 3.2, the constrained system (3.6-3.7) will have uniformly second order zeros at infinity. If the original system (3.1-3.2) has a positive definite infinite-frequency effective mass matrix, then the same will be true for the constrained system. If the original system is time-reversible, then so is the constrained system.*

Proof We use the notation of the theorem. Because $CB = 0$, one has $C(A + BF)B = CAB$. This shows immediately that $C_{11}(A + BF)_{11}B_{11}$ is injective and hence invertible. The fact that $CB = 0$ also implies that $C_{11}B_{11} = 0$, and so we have shown that the constrained system has uniformly second order zeros at infinity.

Now, assume that the infinite-frequency effective mass matrix of the original system, which is given by $(CAB)^{-1}$, is positive definite. Take u from U_1. Using (3.18), one can write

$$<C_{11}(A+BF)_{11}B_{11}u,u> = ((A+BF)Bu,Bu) = (ABu,u) = <CABu,u> \geqslant 0, \quad (3.19)$$

with equality if and only if $u=0$. We see that the matrix $(C_{11}(A+BF)_{11}B_{11})^{-1}$ is also positive definite, which proves our second claim.

It remains to show that the property of time-reversibility is hereditary. We first show that the following property holds for all $k \geqslant 0$:

$$((A+BF)^k x_1, x_2) = (A^k x_1, x_2) \qquad (x_1, x_2 \in V^*(Y_1)). \quad (3.20)$$

This property is trivially true for $k=0$, and its validity for $k=1$ is asserted by (3.18). The general case is proved by induction: suppose that (3.20) holds for certain k, then, for x_1 and x_2 from $V^*(Y_1)$:

$$\begin{aligned}((A+BF)^{k+1}x_1,x_2) &= (A(A+BF)^k x_1, x_2) = (Ax_2,(A+BF)^k x_1) = \\ &= -((A+BF)^k x_1, Ax_2) = -(A^k x_1, Ax_2) = (Ax_2, A^k x_1) = \\ &= (A^{k+1}x_1, x_2). \end{aligned} \quad (3.21)$$

In this derivation, we used the validity of the formula for $k=1$, condition (i) in the definition of a Hamiltonian system, the symplectic property, the induction assumption, the symplectic property again, and condition (i) again. Now, suppose that the original system is time-reversible, i. e., the mappings appearing in (3.1-3.2) satisfy $CA^{2k}B=0$ for all $k \geqslant 0$. By condition (ii) of the definition and the property that we just proved, it then follows that

$$<C(A+BF)^{2k}Bu_1,u_2> = 0 \quad (3.22)$$

for all u_1 and u_2 from U_1, which, by the fact that U_1 and Y_1 are dual spaces, is enough to show that $C_{11}((A+BF)_{11})^{2k}B_{11}=0$ for all $k \geqslant 0$.

This proposition allows us to conclude, for instance, that any system that is obtained by putting linear constraints on the outputs in the Hamiltonian system $M\ddot{y}+Ky=u$ (M and K symmetric, M positive definite) will be a time-reversible Hamiltonian system with a positive definite effective mass matrix at infinite frequency.

4. Gradient systems

The following definition of a linear gradient system in input/state/output form will be used here; it is easily seen to be equivalent to the one given in [15] (p. 224).

Definition 4.1 Consider a linear system in input/state/output form

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (4.1)$$

$$y(t) = Cx(t). \quad (4.2)$$

Assume that the state space X is equipped with a nondegenerate quadratic form denoted by $[\cdot,\cdot]$, and that the input and output space are dual with respect to a duality denoted by $<\cdot,\cdot>$. The system (4.1-4.2) is said to be a *gradient system* (with respect to the quadratic form $[\cdot,\cdot]$ and the duality $<\cdot,\cdot>$) if the following requirements hold:

(i) A is symmetric with respect to $[\cdot,\cdot]$;

(ii) $[Bu,x] = <u,Cx>$ for all $u \in U$ and $x \in X$.

We also assume that B is injective.

As in the case of Hamiltonian systems, injectivity of B implies surjectivity of C. We shall call the quadratic form ½$[Ax,x]$ the *generalized potential* of the system; note that Ax may be seen as the gradient of ½$[Ax,x]$ with respect to the symmetric form $[\cdot,\cdot]$.

We now consider the constraint (2.3) in this context. As before, we will be looking for conditions under which we can define a constrained system which inherits, in a natural way, the special structure of the original system. We define $U_1 = \{u \mid Bu \in V^*(Y_1)\}$ as in the Hamiltonian case, and look for complements of U_1 in U. Again, a natural candidate is $Y_1^\perp$. It turns out that, as soon as this candidate qualifies, the description that one derives from it has all the desired properties.

THEOREM 4.2 *Consider the gradient system (4.1-4.2) under the constraint (2.3), and suppose that $Y_1^\perp$ is a complement to $U_1 = \{u \mid Bu \in V^*(Y_1)\}$ in U. Let $F: X \to U$ be a feedback mapping such that $\operatorname{im} F \subset Y_1^\perp$ and such that $V^*(Y_1)$ is $(A+BF)$-invariant. Write F_{21} for the restriction of F to $V^*(Y_1)$ taken as a mapping into $Y_1^\perp$, $(A+BF)_{11}$ for the restriction of $A+BF$ to $V^*(Y_1)$, B_{11} for the restriction of B to U_1 taken as a mapping into $V^*(Y_1)$, and C_{11} for the restriction of C to $V^*(Y_1)$ taken as a mapping into Y_1. Let Y_2 be a complement of Y_1 in Y. A state space description of the system (4.1-4.2) under the constraint (2.3), with state space $X_1 = V^*(Y_1)$, is then given by*

$$\dot{x}_1 = (A+BF)_{11}x_1 + B_{11}u_1 \tag{4.3}$$

$$y_1 = C_{11}x \tag{4.4}$$

$$u_2 = F_{21}x_1 \tag{4.5}$$

$$y_2 = 0. \tag{4.6}$$

Moreover, the system (4.3-4.4) is a gradient system with respect to the restriction of the form $[\cdot,\cdot]$ to $V^*(Y_1)$ and of the duality $<\cdot,\cdot>$ to the pair of spaces (U_1, Y_1). The generalized potential function of this system coincides with the generalized potential of the original system restricted to $V^*(Y_1)$.

PROOF The fact that (4.3-4.6) is a representation of the system (4.1-4.2) under the constraint (2.3) follows from the general theory. Because F maps into $Y_1^\perp$ and $V^*(Y_1)$ is mapped by C into Y_1, we have

$$[(A+BF)x_1, x_2] = [Ax_1, x_2] + <Fx_1, Cx_2> = [Ax_1, x_2] \tag{4.7}$$

for all x_1 and x_2 from $V^*(Y_1)$. This shows that condition (i) of the definition of a gradient system is satisfied, and also that the generalized potentials of the unconstrained system and the constrained system are equal on $V^*(Y_1)$. The fact that the duality between U and Y can be restricted to U_1 and Y_1 follows from the assumption that U_1 is complementary to $Y_1^\perp$, and condition (ii) for the constrained system is then immediate from the corresponding property of the original system.

If the symmetric form on X is definite (i.e., $[x, x] = 0$ implies $x = 0$ — this means that the form is either positive or negative definite), then we can show that the complementarity condition is satisfied for all possible restrictions.

PROPOSITION 4.3 *Suppose that the system (4.1-4.2) is a gradient system with respect to a nondegenerate symmetric form $[\cdot,\cdot]$ in the state space X and a duality $<\cdot,\cdot>$ between U and Y. Assume that the symmetric form $[\cdot,\cdot]$ is definite. Under these conditions, the subspace $U_1 = \{u \in U \mid Bu \in V^*(Y_1)\}$ is complementary to $Y_1^\perp$ for any subspace Y_1 of Y.*

PROOF We first show that the definiteness of the symmetric form on X implies that CB is invertible. Because CB is square, it is sufficient to prove the injectivity. So, suppose that $CBu = 0$; then

$$[Bu, Bu] = <CBu, u> = 0 \tag{4.8}$$

which implies that $u = 0$. From the invertibility of CB, it follows that

$$V^*(Y_1) = \{x \in X \mid Cx \in Y_1\}. \tag{4.9}$$

As a consequence, we have

$$U_1 = \{u \in U \mid CBu \in Y_1\} = (CB)^{-1}Y_1. \tag{4.10}$$

This shows that $\dim U_1 + \dim Y_1^\perp = \dim U$, so that the complementarity condition will hold if $U_1 \cap Y_1^\perp = \{0\}$.

For gradient systems in general, one has

$$\{x \mid Cx \in Y_1\} = (BY_1)^{[\perp]} \tag{4.11}$$

(proof as in the Hamiltonian case, see (3.15)). When the form is definite, this implies that $V^*(Y_1)$ has zero intersection with $BY_1^{\perp}$. Now, take $u \in U_1 \cap Y_1^{\perp}$; then $Bu \in V^*(Y_1) \cap Y_1^{\perp} = \{0\}$, so that $u = 0$.

In gradient systems that arise as descriptions of RLC networks, the form on the state space X is usually *not* definite, unless one has either no capacitors or no inductors in the network. In order to obtain suitable sufficient conditions for output constraints to be well-behaved in the context of general RLC networks, a further analysis of the gradient systems defined by such networks should be undertaken (following up on the work in [4]); however, we won't do this here.

The invertibility of the mapping CB, which, as we have seen, holds automatically when the form on X is definite, is in itself already enough to obtain the expression (4.9) for $V^*(Y_1)$. This allows us to draw the conclusion that a gradient system with uniformly first order zeros at infinity and with state space dimension n ('n degrees of freedom') becomes a system with $n-k$ degrees of freedom if k linear constraints are imposed on the outputs. Note that the property of having uniformly first-order zeros at infinity is 'hereditary': if the original system (4.1-4.2) has this property, then the same holds for the constrained system (4.3-4.4). For, C maps $V^*(Y_1)$ into Y_1, and this implies that if $C_{11}B_{11}u = 0$ then $CBu = 0$, so that $C_{11}B_{11}$ will be invertible if CB is. (We also use here the fact that the invertibility of CB implies that $C_{11}B_{11}$ is square.)

5. Examples

We shall work out two simple examples in order to illustrate the abstract theory and to show that the theory leads to the answers that one should expect.

Mechanical example

Our first example is the same as in Section 1. State space equations for a mass on an ideal spring can be written as follows:

$$\frac{d}{dt}\begin{bmatrix} q \\ p \end{bmatrix}(t) = \begin{bmatrix} 0 & m^{-1} \\ -k & 0 \end{bmatrix}\begin{bmatrix} q \\ p \end{bmatrix}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t) \tag{5.1}$$

$$y(t) = [1 \quad 0]\begin{bmatrix} q \\ p \end{bmatrix}(t). \tag{5.2}$$

The force (input) is $u(t)$, the displacement (output) is $y(t)$. It is easily verified that the above equations constitute a Hamiltonian system with respect to the symplectic form

$$\left(\begin{bmatrix} p \\ q \end{bmatrix}, \begin{bmatrix} \tilde{p} \\ \tilde{q} \end{bmatrix}\right) = q\tilde{p} - \tilde{q}p \tag{5.3}$$

on $X = \mathbb{R}^2$ and the duality

$$<u, y> = uy \tag{5.4}$$

between $U = \mathbb{R}$ and $Y = \mathbb{R}$. The system has a second order zero at infinity. Let us now take two of such systems and connect them by requiring that the outputs (displacements) must be the same, i.e., the two masses are firmly attached to eachother. The system before connection is described by the matrices

$$A = \begin{bmatrix} 0 & m_1^{-1} & 0 & 0 \\ -k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2^{-1} \\ 0 & 0 & -k_2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{5.5}$$

and it is a Hamiltonian system having uniformly second order zeros at infinity with respect to the symplectic form

$$\left(\begin{bmatrix} q_1 \\ p_1 \\ q_2 \\ p_2 \end{bmatrix}, \begin{bmatrix} \tilde{q}_1 \\ \tilde{p}_1 \\ \tilde{q}_2 \\ \tilde{p}_2 \end{bmatrix}\right) = q_1\tilde{p}_1 - \tilde{q}_1 p_1 + q_2\tilde{p}_2 - \tilde{q}_2 p_2 \tag{5.6}$$

on $X = \mathbb{R}^4$ and the duality

$$< \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} > = y_1 u_1 + y_2 u_2 \tag{5.7}$$

between $Y = \mathbb{R}^2$ and $U = \mathbb{R}^2$. The connection constraint is expressed by:

$$y(t) \in Y_1 = \text{span}\,\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \}. \tag{5.8}$$

One readily computes that a basis for $V^*(Y_1)$ is given by the two vectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ m_1 \\ 0 \\ m_2 \end{bmatrix}. \tag{5.9}$$

We also have

$$Y_1^\top = \text{span}\,\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \}, \tag{5.10}$$

and so we are looking for a mapping $F: X \to U$ which ranges in this subspace and which is such that $V^*(Y_1)$ becomes $A+BF$-invariant. Upon computing, one finds that this requires

$$F \begin{bmatrix} 1 & 0 \\ 0 & m_1 \\ 1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} [\frac{m_2 k_1 - m_1 k_2}{m_1 + m_2} \quad 0]. \tag{5.11}$$

Here, we need that $m_1 + m_2$ is not equal to zero; note that this is precisely the condition for $Y_1^\perp$ to be complementary to U_1. We assume that this condition is fulfilled. It then follows from (5.11) that the action of $A+BF$ on $V^*(Y_1)$ is given by

$$(A+BF) \begin{bmatrix} 1 & 0 \\ 0 & m_1 \\ 1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & m_1 \\ 1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -\dfrac{k_1 + k_2}{m_1 + m_2} & 0 \end{bmatrix}. \tag{5.12}$$

In order to obtain a matrix representation for the connected system, we have to select basis vectors in $V^*(Y_1)$, U_1, and Y_1. It is not difficult to show (cf. [15], p. 200) that 'canonical' bases may be chosen in the following way for every Hamiltonian system (3.1-3.2) satisfying $CB = 0$. First select an arbitrary basis for the output space Y. Next, determine the dual basis $u_1, \cdots, u_m$ for the input space U. Finally, it is possible to find a symplectic basis $\{q_1, \cdots, q_n, p_1, \cdots, p_n\}$ for the state space X in such a way that $Bu_i = p_{n-m+i}$ for $i = 1, \cdots, m$.

We can carry out this program in the case at hand. A basis vector for the output space Y_1 is $[1 \quad 1]^\top$. The dual basis vector in U_1 is $(m_1 + m_2)^{-1}[m_1 \quad m_2]^\top$. A corresponding symplectic basis for X is given by

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, (m_1+m_2)^{-1} \begin{bmatrix} 0 \\ m_1 \\ 0 \\ m_2 \end{bmatrix} \right\}. \tag{5.13}$$

The matrices describing the connected system are then

$$(A+BF)_{11} = \begin{bmatrix} 0 & (m_1+m_2)^{-1} \\ -(k_1+k_2) & 0 \end{bmatrix}, \; B_{11} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_{11} = [1 \quad 0]. \tag{5.14}$$

An electrical network

For our second example, let us consider the parallel connection of two capacitors. Before connection, the two capacitors can be described by the equations

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t) = 0 \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}(t) \tag{5.15}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}(t) = \begin{bmatrix} C_1^{-1} & 0 \\ 0 & C_2^{-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t) \tag{5.16}$$

where the inputs are currents and the outputs are voltages, and C_1 and C_2 are the capacitances of the two capacitors. This system is a gradient system with respect to the quadratic form

$$\left[\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right] = C_1^{-1}x_1^2 + C_2^{-1}x_2^2 \tag{5.17}$$

on X and the duality

$$\left< \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right> = y_1u_1 + y_2u_2 \tag{5.18}$$

between Y and U. Now, establishing a parallel connection between the two capacitors means that the voltages across the two capacitors must be equal, which leads to the output constraint

$$y_1(t) = y_2(t). \tag{5.19}$$

So, the constraint subspace Y_1 is given by

$$Y_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}. \tag{5.20}$$

One easily computes that

$$U_1 = \text{span}\left\{ \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \right\} \tag{5.21}$$

so that U_1 is complementary to $Y_1^{\perp} = \text{span}\,\{[1 \quad -1]^{\mathrm{T}}\}$ if and only if $C_1 \neq -C_2$. Assuming this, we have to find a feedback mapping F ranging in $Y_1^{\perp}$ such that $V^*(Y_1) = \text{span}\,\{[C_1 \quad C_2]^{\mathrm{T}}\}$ becomes $(A+BF)$-invariant. This is satisfied by taking $F = 0$, which leads to $(A+BF)_{11} = 0$. We can take $[1 \quad 1]^{\mathrm{T}}$ as a basis vector for Y_1; the dual basis vector in U_1 is $(C_1+C_2)^{-1}[C_1 \quad C_2]^{\mathrm{T}}$. Taking (nominally) the same vector as a basis vector for X_1, we obtain $B_{11} = 1$. Finally, C_{11} is then given by $(C_1+C_2)^{-1}$, because

$$\begin{bmatrix} C_1^{-1} & 0 \\ 0 & C_2^{-1} \end{bmatrix} \frac{1}{C_1+C_2} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{C_1+C_2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{5.22}$$

The equations for the connected system now take the form

$$\dot{x}(t) = u(t) \tag{5.23}$$

$$y(t) = (C_1 + C_2)^{-1} x(t). \tag{5.24}$$

Here, the old inputs (currents) are expressed in terms of the new input by

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{C_1 + C_2} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} u \tag{5.25}$$

so that the new input can be written in terms of the old inputs as

$$u = u_1 + u_2. \tag{5.26}$$

6. Conclusions

It has been shown in this paper that the transformation of a linear with output constraints to standard i/s/o form leads to an essentially non-unique result, and that the indeterminacy can be described by the freedom one has in selecting a complement to a given subspace in the input space. For the more specific categories of linear Hamiltonian systems and linear gradient systems, the notion of energy, expressed through special vector space structures, serves to remove the indeterminacy, and leads to results that are familiar from physical modeling. We have thus shown how the general theory of transformations of linear systems can be connected to the standard methods of physical modeling, at least for the case of linear systems under output constraints.

The analysis has also revealed a 'nondegeneracy' condition, which appears in geometric terms as the requirement that two given subspaces should be complementary. The viewpoint used in this paper seems less suitable for a treatment of the particular cases in which the complementarity condition does not hold. An alternative framework for analysis can be set up by prescribing new external variables in conjunction with the output constraints. This point of view is currently under investigation.

References

1. R. A. Abraham, J. E. Marsden (1978). *Foundations of mechanics* (2nd ed.), Benjamin/Cummings, Reading, Mass.
2. B. D. O. Anderson (1975). Output-nulling invariant and controllability subspaces. *Proc. IFAC Sixth World Congress, Boston/Cambridge, 1975*, paper 43.6.
3. V. I. Arnold (1978). *Mathematical Methods of Classical Mechanics*, Springer, New York.
4. R. K. Brayton, J. K. Moser (1964). A theory of nonlinear networks. *Quart. Appl. Math. 22*, 1-33 (Part I), 81-104 (Part II).
5. J. P. Corfmat, A. S. Morse (1976). Control of linear systems through specified input channels. *SIAM J. Contr. Optimiz. 14*, 163-175.
6. M. L. J. Hautus (1976). The formal Laplace transform for smooth linear systems. G. Marchesini, S. K. Mitter (eds.). *Mathematical Systems Theory* (Proc. Intern. Symp., Udine, Italy, 1975), Lect. Notes Econ. Math. Syst. 131, Springer, New York, 29-47.
7. J. Hoekstra (1988). *Hamiltonse systemen met beperkingen op de uitgang* (Hamiltonian systems with output restrictions), M. Sc. thesis, Dept. of Appl. Math., Twente Univ. (In Dutch.)
8. S. Lang (1965). *Algebra*, Addison-Wesley, Reading, Mass.
9. D. G. Luenberger (1977). Dynamic systems in descriptor form. *IEEE Trans. Automat. Contr. AC-22*, 312-321.
10. C. C. MacDuffee (1956). *The Theory of Matrices*, Chelsea, New York. (Reprint of original, 1933.)
11. M. Malabre (1982). Structure à l'infini des triplets invariants. Application à la poursuite parfaite de modèle. A. Bensoussan, J. L. Lions (eds.). *Analysis and Optimization of Systems*, Lecture Notes in Control and Information Sciences 44, Springer, New York, 43-53.
12. H. Nijmeijer, J. M. Schumacher (1985). On the inherent integration structure of nonlinear systems. *IMA J. Math. Contr. Inf. 2*, 87-107.
13. J. W. S. Rayleigh (1945). *The Theory of Sound (Vol. 1)*, Dover, New York. (Reprint of original,

1894.)

14. H. H. ROSENBROCK (1970). *State Space and Multivariable Theory*, Wiley, New York.
15. A. J. VAN DER SCHAFT (1984). *System Theoretic Descriptions of Physical Systems*, CWI Tract 3, CWI, Amsterdam.
16. A. J. VAN DER SCHAFT (1986). *On realization of nonlinear systems described by higher-order differential equations*, Memo. nr. 569, Dept. of Appl. Math., Twente Univ.
17. A. J. VAN DER SCHAFT (1987). Equations of motion for Hamiltonian systems with constraints. *J. Phys. A: Math. Gen. 20*, 3271-3277.
18. J. M. SCHUMACHER (1988). Transformations of linear systems under external equivalence. *Lin. Alg. Appl. 101*.
19. J. C. WILLEMS (1983). Input-output and state-space representations of finite-dimensional linear time-invariant systems. *Linear Algebra Appl. 50*, 581-608.
20. J. C. WILLEMS (1986). From time series to linear system. Part I: Finite dimensional linear time invariant systems. *Automatica 22*, 561-580.
21. W. M. WONHAM (1979). *Linear Multivariable Control: a Geometric Approach* (2nd ed.), Springer, New York.