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Matchings and Hadwiger's Conjecture

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Abstract

Assuming that a graph G on n vertices is a minimal counterexample to Hadwiger's Conjecture $\chi(G) \leqslant \eta(G)$, we apply the Edmonds–Gallai Structure Theorem to its complement, H, to find that H has a matching of size $\lfloor n/2 \rfloor$. Hence Magyar Tud. Acad. Mat. Kutató Int. Kőzl. 8 (1963) 373: $\chi(G) \leqslant \lceil n/2 \rceil$. Further, H is homeomorphic to a three-connected graph, and is of tree width at least four. The same holds for a minimal counterexample G to Colin de Verdière's Conjecture $\mu(G)+1 \geqslant \chi(G)$. © 2002 Published by Elsevier Science B.V.

1. Introduction

One of the most intriguing conjectures in today's graph theory is the conjecture of Hadwiger [5] linking the chromatic number of a graph G to the maximum size of its clique minor:

Conjecture 1.1 (Hadwiger [5]). Every k-chromatic graph G has a K_k-minor.

Hadwiger's Conjecture can be easily verified for $k \leqslant 3$. For example, for $k=3$, a three-chromatic graph is not bipartite, i.e. it contains an odd cycle, and hence also a K_3-minor. The smallest non-trivial case, $k=4$, was proved by Hadwiger in the same paper [5], and—almost a decade later—by Dirac [3], apparently oblivious of Hadwiger's result.

For $k=5$, the statement of Hadwiger's Conjecture becomes "*Every 5-chromatic graph has a K_5-minor*" and hence, in view of Kuratowski's Theorem [7], its truthfulness implies the Four Color Theorem. In fact, it is *equivalent* to the Four Color Theorem, as proved by Wagner [14] several years before Hadwiger first formulated his conjecture. Thus, by establishing in 1975 the Four Color Theorem, Appel and Haken confirmed Hadwiger's Conjecture for $k=5$. Finally, Robertson et al. [10] proved that for $k=6$ Hadwiger's Conjecture is—again—equivalent to the Four Color Theorem, and hence is true. This is the largest value of k for which Hadwiger's Conjecture has been verified. For a detailed history of Hadwiger's Conjecture, as well as an account of

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Nomenclature

$\alpha(G)$	independence number of G: the maximum size of an independent set in G
$\Delta(G)$	maximum degree of G: the maximum degree of a vertex in G
$\delta(G)$	minimum degree of G: the minimum degree of a vertex in G
$\Gamma(S)$	neighborhood of S: the vertices outside of S, each adjacent to a vertex in S
$\eta(G)$	Hadwiger number of G: the maximum size of a clique minor of G
$\mu(G)$	Colin de Verdière number of G: cf. [2] for definition
$\nu(G)$	matching number of G: the maximum size of a matching in G
π	the ratio of the circumference to the diameter of a circle
$\chi(G)$	chromatic number of G: the minimum number of colors in a proper coloring of G.

recent developments there-about, the reader is refered to a nicely written survey [13] of Toft.

Of course, *Hadwiger's Conjecture is true for almost all graphs*, by a result of Bollobás et al. [1].

2. Main idea

In this section, we describe the main idea of this paper.

Suppose we have been told that some graph G is a counterexample to Hadwiger's Conjecture. We are determined to prove the claimant wrong; in this, our first endeavor is to find a "large" clique minor in G.

Recall that a graph is a *minor* of G if it is either a subgraph of G, or can be obtained from one by a series of edge-contractions. An *edge-contraction* in G can be described as the operation of replacing two adjacent vertices, u and v, by a new vertex, w, and setting the neighborhood $\Gamma(w)$ of w to be the union of the neighborhoods $\Gamma(u)$ and $\Gamma(v)$—minus, of course, the vertices u and v themselves. Respectively, an edge-contraction in G can be described in terms of its complement, H, as the operation of replacing two non-adjacent vertices u and v by a new vertex w, and setting the neighborhood $\Gamma(w)$ of w in H to $\Gamma(u) \cap \Gamma(v)$. Let us call such an operation in H a *co-contraction*. Observe that $\eta(G) \geqslant p$ if and only if we can exhibit in H, perhaps after a series of co-contractions, an independent set of size p.

Suppose now that S is a *cut set* in H, i.e. a set of vertices such that $H - S$ is disconnected. Let $C_1, \dots, C_r$ be the connected components of $H - S$, and set $\alpha_i := \alpha(C_i)$. Clearly, we can immediately exhibit an independent set, A, of size $\alpha_1 + \cdots + \alpha_r$ in H, or even in $H - S$. Using co-contractions, we can typically do even better, still by looking only at the components of $H - S$. For example, if C_1 and C_2 are both non-singleton, then we can always choose a vertex $v_1 \in C_1$ avoiding some maximum independent set in C_1, a similar vertex $v_2 \in C_2$, and co-contract this (independent) pair. By doing so, we create a brand new vertex *whose neighborhood is entirely in* S. This lets us increase the size of the independent set A by one.

To do this in a more systematic fashion, let us set $b_i := |C_i| - \alpha_i$ and consider the complete r-partite[1] graph $K_{b_1,\ldots,b_r}$. The reader will find it easy to argue that $H - S$ can be co-contracted to an independent set of size $p := \alpha_1 + \cdots + \alpha_r + v(K_{b_1,\ldots,b_r})$.

Now, one hopes that p compares favorably to $\chi(G)$, the latter estimated using a large matching, M, in H. In other words, our strategy is to use two matchings "in parallel". The first matching, which we shall denote by N, should help us find a relatively large clique minor of G; the other matching, M, will help us color G in relatively few colors.

Modulo slight variations, this is the main idea of this paper.

3. Edmonds–Gallai Structure Theorem

In this section, we apply our main idea to what is often a very special cut set of H: its Edmonds–Gallai Tutte set. Let us bring out the background.

Given a graph H on n vertices, we write $\mathsf{odd}(H)$ and $\mathsf{ev}(H)$ for the number of odd and even components of H, respectively. For a subset S of the vertex set of H, we set $s := |S|$.

In this notation [the defect version of], Tutte's Theorem asserts the existence of a subset S of the vertex set of H such that $\mathsf{odd}(H - S) = s + n - 2v(H)$. We call such an S a *Tutte set* (of H). Observe that for any maximum matching M in H, the quantity $n - 2v(H)$ is the number of M-exposed vertices.

The Edmonds–Gallai Structure Theorem is a strengthening of Tutte's Theorem. It asserts that a Tutte set S can be [uniquely] chosen so that the odd components of $G - S$ are factor-critical[2] and in the [at most] bipartite graph obtained from G by contracting each odd component of $G - S$, deleting the even components, and deleting the edges spanned by S, S satisfies Hall's condition with surplus one. We call this [unique] Tutte set S as *Edmonds–Gallai*. For the details, we refer the reader to the book of Lovász and Plummer [8, pp. 93–95].

Theorem 3.1. *Suppose G is the only counterexample to Hadwiger's Conjecture among all of its induced subgraphs. Let H denote its complement, and let S be the Edmonds–Gallai set of H. Then* $s + \mathsf{odd}(H - S) + \mathsf{ev}(H - S) = 1$. *In other words, the only component of $H - S$ is H itself.*

[1] By calling this graph r-partite, we are abusing vocabulary here, since some of the b_i's may well be zeros. In fact, $b_i = 0 \Leftrightarrow |C_i| = 1$.

[2] Graph X is *factor-critical* if $X - v$ has a perfect matching for every vertex $v \in X$.

Proof. Let H and S be as in the theorem, and let $C_1, \ldots, C_r$ be the components of $H - S$, $r \geqslant 1$. Suppose, for the sake of contradiction, that C_1 is a *proper* subgraph of H.

Let us set $c_i := \lfloor |C_i|/2 \rfloor$. Observe that, by Tutte's Theorem, c_i is the number of M-edges within C_i for any maximum matching M of H. To mimic our main idea, we consider the complete r-partite graph $K_{c_1,\ldots,c_r}$ and one of its maximum matchings, N. Without loss of generality, we suppose that the N-exposed vertices, say $c \geqslant 0$ altogether, are all in the first partite class.

Let M be a maximum matching in H having no edge with exactly one endpoint is in C_1. Notice that if $|C_1|$ is even, then every maximum matching of H has this property, while if $|C_1|$ is odd, the existence of such a matching is guaranteed by the Edmonds–Gallai Structure Theorem.

Next, let $C \subseteq C_1$ be the subgraph induced by the endpoints of some c M-edges within C_1 plus the M-exposed vertex of C_1, if any. Since the M-edges outside of C induce, in a natural way, a coloring of $G - \bar{C}$, we conclude that G can be properly colored in this many colors: $\chi(\bar{C})$ plus the number of M-edges and M-exposed vertices outside of C.

On the other hand, the minimality of G and our assumption imply that the complement of C satisfies Hadwiger's Conjecture. In other words, C can be co-contracted to an independent set of cardinality at least $\chi(\bar{C})$. We enlarge this set by adding to it one vertex from each odd component of $H - S$ different from C_1—as many vertices altogether as there are M-edges incident with S plus M-exposed vertices outside of C. Furthermore, the matching N induces, in a natural way, a pairing of those vertices in $H - (C \cup S)$ which are not yet in our independent set. Co-contracting each of these $c_1 + \cdots + c_r - c$ pairs (as many as there are M-edges within $C_1 - C, C_2, \ldots, C_r$) let us enlarge our independent set to the size of the coloring of G from the previous paragraph. This is a contradiction. $\square$

As an immediate corollary of Theorem 3.1 and the Edmonds–Gallai Structure Theorem, we obtain the following result:

Theorem 3.2. *Suppose that G is the only counterexample to Hadwiger' s Conjecture among its induced subgraphs, and let H denote its complement. Then H is connected and, depending on the parity of $|G|$, either H has a perfect matching or H is factor-critical.*

Corollary 3.3 (Gallai [4]). *If G on n vertices is the only counterexample to Hadwiger's Conjecture among its induced subgraphs then $\chi(G) \leqslant \lceil n/2 \rceil$.*

Proof. $\chi(G) \leqslant n - \nu(\bar{G})$. $\square$

Exercise 3.4. Suppose that G is the only counterexample to Hadwiger's Conjecture among its induced subgraphs and, in addition, $\alpha(G) = 2$ (i.e. the complement H of G is triangle free). Prove, using only Tutte's Theorem, that H is factor-critical.

4. Slicing it another way

It is my impression that, with respect to Hadwiger's Conjecture, the universe of graphs is traditionally partitioned into the classes of k-chromatic graphs, a class per each value of k. Respectively, aside from proving or disproving the conjecture in general, one establishes a benchmark by settling the conjecture for "the next value of k".

My idea was to look at a different partition, i.e. at the graphs G with $\chi(G)=|G|-p$ for each non-negative integer p. For example, the conjecture is trivially true for $p=0$, since then G is a clique.

In a sense, slicing it this way was a self-deception. Indeed, it is not difficult to show that the conjecture is true for those G for which $\chi(G)=|G|-p$ as soon as it is true for the graphs on at most $2p(p+1)$ vertices. We do not treat this statement in detail, since we are to obtain a better bound presently, in Theorem 4.1. On the other hand, even though these bounds do help for a while (Corollary 4.2), I already list the case $p=6$ as an open question, cf. Problem 5 in Section 8.

Theorem 4.1. *Suppose Hadwiger's Conjecture is false, and let p be the smallest integer such that there is a counterexample G to Hadwiger's Conjecture with $|G|-\chi(G)=p$. Then there is such a counterexample on at most $2p+1$ vertices.*

Proof. Follows immediately from Corollary 3.3 and the trivial inequality $|G|-\chi(G)\geqslant|G'|-\chi(G')$ for any induced subgraph G' of G. □

Corollary 4.2. *Suppose Hadwiger's Conjecture is false, and let p be the smallest integer such that there is a counterexample G to Hadwiger's Conjecture with $|G|-\chi(G)=p$. Then $p\geqslant 6$.*

Proof. By Theorem 4.1, the smallest counterexample is on at most $2p+1$ vertices; by Corollary 3.3, its chromatic number is at most $p+1$. Apply the result of Robertson et al. [10]. □

5. Word on Colin de Verdière's Conjecture

Suppose y is a graph invariant satisfying the following properties:

(Y1) y is minor monotone: $G_1\leqslant G_2\Rightarrow y(G_1)\leqslant y(G_2)$,
(Y2) $y(K_1)\geqslant 1$,
(Y3) $y(G)\geqslant y(G_1)+y(G_2)$, where $\bar{G}=\bar{G}_1\dot{\cup}\bar{G}_2$.

It is easy to see that $\eta(G)$ has these properties. In fact, so does the invariant $c\cdot\eta(G)$ for any $c\geqslant 1$. Moreover, if y has (Y1)–(Y3) then $y\geqslant\eta$. In particular, for such an invariant y, it is "safe" to conjecture $y\geqslant\chi$.

A non-trivial example of such an invariant is the next integer after the number μ introduced by Colin de Verdière[3] [2]; hence the conjecture $\mu + 1 \geqslant \chi$. However, the intriguing thing about this particular conjecture is, as Colin de Verdière proves, that its truthfulness would [still] imply the Four Color Theorem.

It seems that everything that is known about Colin de Verdière's Conjecture is a corollary of the corresponding knowledge on Hadwiger's Conjecture. To follow the trend, we remark that all of the results in this paper translate verbatim from the context of $\eta(G)$ to that of $\mu(G) + 1$—or, for that matter, of any other invariant $y(G)$ with (Y1)–(Y3). In fact, the reader will be able to check that (Y1)–(Y3) are the only properties of η used in this paper.

We conclude that the results of this paper remain valid if the words "Hadwiger's Conjecture" are replaced, throughout, by the words "Colin de Verdière's Conjecture". In particular since the graphs G with planar complements have $\mu(G) \geqslant |G| - 5$ [6], the Colin de Verdière version of Corollary 4.2 implies that *Colin de Verdière's Conjecture is true for graphs with planar complements*. The phrase in italics, were we in sales, could have been chosen as the trademark for this section.

Along these lines, but on a more serious note, we will deduce (Corollary 7.2) that Hadwiger's Conjecture is true for the graphs with series–parallel complements. Perhaps it would be interesting to prove such a statement for the graphs with planar complements as well.

6. Sparsity and connectivity

In the next two sections, we assume that the graph G *is the only counterexample to Hadwiger's Conjecture among its induced subgraphs*, and H is its complement. In this section, we consider "the second iteration" of our main idea. This time, we apply it to an arbitrary cut set in H; the graph H itself being subject to Theorem 3.2.

Theorem 6.1. *Let S be a cut set in H, and let $H_1, \dots, H_r$ denote the connected components of $H - S$. For $i \in \{1, \dots, r\}$, let a_i be a fixed natural number satisfying $a_i \leqslant \alpha(H_i)$, and set $h_i := |H_i|$, $b_i := h_i - a_i$. If, for some $k \leqslant r$, we have $a_1 + \cdots + a_k \geqslant s$ then the complete r-partite graph $K_{b_1,\dots,b_k,h_{k+1},\dots,h_r}$ (this becomes $K_{b_1,\dots,b_r}$ when $k = r$) does not have a matching covering all the vertices of the first k partite classes.*

Proof. In the notations of the theorem, within each H_i, let us fix an independent set A_i of cardinality a_i, and set $B_i := H_i - A_i$. Let us assume, for the sake of contradiction, that the conclusion of the theorem does not hold for some k.

Suppose first that $k = r$. Then, by our assumption, the complete r-partite graph $K_{b_1,\dots,b_r}$ contains a perfect matching. Naturally, this perfect matching induces a pairing of the

[3] Colin de Verdière [2] proved that $\mu(G) + 1$ has (Y1); (Y2) holds trivially; and (Y3) is not difficult to see, especially by using the alternative definition of μ in terms of vector labellings [6].

vertices of $B_1\dot{\cup}\dots\dot{\cup}B_r$, $n-s-(a_1+\dots+a_r)$ vertices altogether. (In particular, if $a_1+\dots+a_r=s$ then n is even.) Co-contracting the vertices of each pair gives us, in combination with the vertices of $A_1\dot{\cup}\dots\dot{\cup}A_r$, an independent set of cardinality $[n-s+(a_1+\dots+a_r)]/2 \geqslant \lceil n/2\rceil$. By Corollary 3.3, the latter quantity is at least as large as $\chi(G)$. This is a contradiction.

Thus, we may assume that $k<r$. Let us write $a{:=}a_1+\dots+a_k$ and $b{:=}b_1+\dots+b_k$.

By Theorem 3.2, either H has a perfect matching, or it is factor-critical. Let us assume the latter; the former case can be treated analogously.

Let v be a vertex of H_r and let M be a perfect matching in $H-\{v\}$. To begin, we claim that no more than b vertices of S are matched by M to vertices of $H'{:=}H_{k+1}\dot{\cup}\dots\dot{\cup}H_r$. Indeed, this is trivially true if $b\geqslant a$, since $a\geqslant s$. Thus assume that $p{:=}a-b>0$. Since $A{:=}A_1\dot{\cup}\dots\dot{\cup}A_k$ is an independent set, the vertices of A are matched by M to a vertices outside of A, at least p of which must be not in $B{:=}B_1\dot{\cup}\dots\dot{\cup}B_k$, and hence are in S. But then at most $s-p=s-a+b\leqslant b$ vertices of S can be matched into H', as claimed.

Let C be the set of vertices in H' matched by M into S; set $c{:=}|C|$. Our next claim is that the complete r-partite graph $K_{b_1,\dots,b_k,h_{k+1},\dots,h_r}$ has a matching, N, covering the b vertices of the first k classes (as guaranteed by our assumption) and, moreover, matching some $d\geqslant c-1$ of these into H'. We leave the proof of this simple claim to the reader.

Next, among all the induced subgraphs of H' on d vertices and inclusionwise comparable to C, let D be one containing the maximum number of M-edges. Set $X{:=}H'-D$. It is crucial to observe that, by construction, $H-X$ possesses, depending on the parity of $|H-X|$, either a perfect, or a near-perfect matching. We conclude, as before, that $\chi(G)\leqslant\chi(\bar{X})+\lceil|H-X|/2\rceil$.

On the other hand, the matching N induces, in a natural way, a pairing of the vertices of $B\cup D$. As before, co-contracting the vertices of each pair gives us, in combination with the vertices of A and an appropriate co-contraction of X, an independent set of cardinality at least $a+(b+d)/2+\chi(\bar{X})\geqslant\lceil(a+b+d+s)/2\rceil+\chi(\bar{X})$. The latter quantity has just been shown to be at least $\chi(G)$. This is a contradiction. □

Corollary 6.2. *Let S be a cut set in H. Then $H-S$ has at most s components.*

Proof. Let $H_1,\dots,H_r$ be the components of $H-S$ labelled in the non-decreasing order of cardinalities: $h_1\leqslant\dots\leqslant h_r$. Suppose, for the sake of contradiction, that $r>s$. Let us set, in the notations of Theorem 6.1, $a_1=\dots=a_s=1$, and $k=s$. Then $a_1+\dots+a_k=s\geqslant s$. Further, it is trivial to see that the complete r-partite graph $K_{h_1-1,\dots,h_s-1,h_{s+1},\dots,h_r}$ has a matching covering the vertices of the first s partite classes (which—as the reader will recall—we allow to be empty). This shows that the graph G violates the conclusion of Theorem 6.1, which is a contradiction. □

Corollary 6.3. *H is two-connected.*

Since Corollary 6.3 is quite an immediate consequence of Corollary 6.2 or, for that matter, of Theorem 6.1 itself, one would expect to hear more about the connectivity of H. Disappointingly, I cannot prove that H is three-connected. In fact, I cannot even prove that $\delta(H) > 2$. What is easy to prove is that a vertex of degree two in H must have non-adjacent neighbors. In light of this, Bert Gerards suggested to me that H might be a subdivision of a three-connected graph. In fact, we prove slightly more in Theorem 6.7 below. But first, we need the following observation whose easy proof we leave to the reader.

Proposition 6.4. *For every two vertices $u \neq v$ of G, we have $\Gamma(u) \not\subset \Gamma(v)$.*

Translated into the terms of the complementary graph H, Proposition 6.4 states that *for every two vertices u and v adjacent in H, there is a vertex $w \neq v$ adjacent to u but not to v.*

Corollary 6.5. *If Q is a clique in H then the collection of sets $\{\Gamma(v) - Q : v \in Q\}$ is an anti-chain with respect to inclusion.*

Corollary 6.6. *H has no* simplicial *vertex (i.e., a vertex whose neighborhood is a clique in H).*

Proof. Assume, on the contrary, that v is simplicial; apply Corollary 6.5 to the clique $Q := \{v\} \cup \Gamma(v)$. □

Theorem 6.7. *Let S be a cut set in H of cardinality two (so that, by Corollary 6.2, $H - S$ has two components). Let H_1 and H_2 be the components of $H - S$ of respective cardinalities $h_1 \leqslant h_2$. Then $H_1 \cup S$ induces a path of length two with the endpoints in S.*

Proof. By Corollary 6.6, it is enough to show that $h_1 = 1$. Thus assume, for the sake of contradiction, that $h_1 \geqslant 2$. Notice that H_1 is a clique, since otherwise H would violate the conclusion of Theorem 6.1 with $k = 1$ and $a_1 = 2$. Applying Corollary 6.5 to the clique $Q := H_1$ shows that the restrictions to S of the neighborhoods of the vertices of H_1 form an anti-chain. Sine a two-element ground set has no anti-chain of size more than two, and only one anti-chain of size two, we conclude that $h_1 = 2$ and $H_1 \cup S$ spans, depending on whether or not S spans an edge, either a four-cycle or a path of length three. Write $S = \{s_1, s_2\}$, $H_1 = \{x_1 x_2\}$, and assume, without loss of generality, that $s_1 x_1 x_2 s_2$ is a path in H.

Case 1: *$s_1 s_2$ is an edge in H.* By the minimality of $G, G - x_1 - x_2$ satisfies Hadwiger's Conjecture. In other words, $H - H_1$ can be co-contracted to an independent set of size $\chi(G - x_1 - x_2)$. If s_1 does not participate in this independent set then, in the corresponding co-contraction of H, this set can be enlarged by x_1. If both s_1 and s_2 participate in this set then, since $s_1 s_2$ is an edge, say s_1 has been co-contracted with another vertex of $H - H_1$. But then again, x_1 can be added to this independent set. Either way, this is a contradiction, since $\chi(G) \leqslant \chi(G - x_1 - x_2) + 1$.

Case 2: s_1s_2 *is not an edge in* H. Again, H_2 can be co-contracted to an independent set of size $\chi(\bar{H}_2)$. In the corresponding co-contraction of H, this set can be enlarged by x_1, and the co-contraction of s_1,s_2, and x_2. This is a contradiction, since $\chi(G) \leqslant \chi(\bar{H}_2)+2$. □

Exercise 6.8. If $S=\{s_1,s_2\}$ is as in Theorem 6.7, prove that $|\Gamma(s_1)\cap\Gamma(s_2)| \geqslant 2$ in H.

Remark 6.9. In his Ph.D. thesis [11] and in paper [12], Toft characterized vertex-critical and k-critical graphs whose complements have a cut set of cardinality at most two. In particular, Theorem 3 of [12] states that the complement of a k-critical graph is two-connected. Though this does not seem to imply our Corollary 6.3 directly, it should be noted that Corollary 6.3 can be proved quite easily using the techniques of [11,12]. Similarly, Corollary 8 of [12] reaches parts of the conclusion of our Theorem 6.7—formulated for a general k-critical graph. (For example, it states that if the complement H of a k-critical graph has a cut set S on two vertices, then $H-S$ has exactly two components.) Again, I do not know whether these results of Toft imply our Theorem 6.7.

To conclude this section, we observe that Theorem 6.1 (respectively, Corollary 6.2) has a useful modification.

Theorem 6.10. *Suppose that, in the notations of Theorem* 6.1, *S is such that for each* $i\in\{1,\dots,r\}$ *there is a vertex in* S *adjacent to nothing in* H_i. *Then, the conclusion of Theorem* 6.1 *holds even if the condition* $a_1+\cdots+a_k \geqslant s$ *for* $k<r$ *is replaced by* $a_1+\cdots+a_k \geqslant s-1$.

Proof. Essentially, we repeat the part of the proof of Theorem 6.1 following the words "Thus, we may assume that $k<r$", stressing only the deviations. Naturally, we assume that $a_1+\cdots+a_k=s-1$, since there is nothing to prove otherwise. Notice that this implies that $b+c$ is odd. Further, b and d have to have the same parity, so that also $c+d$ is odd. If $d=c-1$ then we choose D to be on $d+1=c$ vertices; otherwise, we choose D to be on $d-1$ vertices. In any case, D is picked to contain C and so that $A\cup B\cup S\cup D$ has a perfect matching. The final deviation from the proof of Theorem 6.1 is that now the unique vertex, v, of $B\cup D$ left single under the pairing induced by N, is co-contracted with the vertex of S adjacent to nothing in $H_i \ni v$. This "saves" the counting. □

Corollary 6.11. *If* S *is as in Theorem* 6.10, *then* $H-S$ *has at most* $s-1$ *components.*

7. Tree width

The notion of *tree width*, introduced by Robertson and Seymour in [9], has several equivalent definitions. We recall one.

Definition 7.1. *A graph $\hat{X}$ on n vertices is* maximal tree-width-k *if its vertices can be labelled $v_1,\ldots,v_n$ so that $v_1,\ldots,v_k$ span a clique, and for every i, $k+1 \leqslant i \leqslant n$, the vertex v_i is simplicial of degree k in the induced subgraph of $\hat{X}$ spanned by $v_1,\ldots,v_i$.* The tree width $\mathrm{tw}(X)$ *of a graph X is the least natural number k such that X is a subgraph of a maximal tree-width-k graph.*

We see immediately that the tree-width-one graphs are forests. The tree-width-two graphs are called *series–parallel*, and are characterized by being K_4-minor-free. We leave it to the reader to argue the following consequence of Theorem 6.7.

Corollary 7.2. *Hadwiger's Conjecture holds for the graphs with series–parallel complements.*

Throughout the rest of this section, we let $\hat{H}$ denote a maximal tree-width $k := \mathrm{tw}(H)$ supergraph of H on the same vertex set as H. Notice that $H \neq \hat{H}$ by Corollary 6.6.

Of course, $\hat{H}$ need not be unique for a given H. However, what is important to us is that any cut set S of $\hat{H}$ is a cut set in H. Due to its structure, $\hat{H}$ has several "canonical" cut sets. Namely, these are the k-cliques of $\hat{H}$ shared by two or more $(k+1)$-cliques; other cut sets of interest are $(k+1)$-cliques themselves—namely, those that *are* cut sets. The useful thing about the latter cut sets is that they satisfy the condition of Theorem 6.10. Hence, Corollaries 6.2 and 6.11 give us the following result:

Theorem 7.3. *Let S be a clique in $\hat{H}$. Then H–S has at most k components.*

Suppose now that S is a cutting $(k+1)$-clique in $\hat{H}$. Let $H_1,\ldots, H_r$ be the components of H–S labelled in the non-decreasing order of cardinalities, which we denote by $h_1,\ldots, h_r$. We let the reader argue that S can be chosen so that $h_1+\cdots+h_{r-1} \geqslant h_r - 1$. But then, two applications of Theorem 6.10 show that $\alpha(H_1 \cup \cdots \cup H_{r-1}) < k$ and $\alpha(H_r) < k$, respectively. In particular, $\alpha(H\text{–}S) \leqslant 2(k-1)$ and hence, $|H\text{–}S| \leqslant 2(k-1)(k+1)$. (To derive the last inequality, we used the trivial facts $\chi(H) \leqslant \mathrm{tw}(H)+1$ and $|H| \leqslant \alpha(H)\chi(H)$.) Thus, we have just proved the following result.

Theorem 7.4. $|G| \leqslant 2(k-1)(k+1)+(k+1)=2k^2+k-1$.

Observe that Corollary 7.2 is implied by Theorem 7.4. Indeed, substituting $k=2$ in the claim of the theorem gives us $|G| \leqslant 9$. Then, by Corollary 3.3, $\chi(G) \leqslant 5$ and hence, in light of the result of Wagner [14] and the Four-Color Theorem, G satisfies Hadwiger's Conjecture. Of course, we can prove Corollary 7.2 "directly", without these profound facts.

Refining the argument of Theorem 7.4 for the particular case $k=3$, we obtain the following:

Theorem 7.5. $k \geqslant 4$.

Proof. Suppose, on the contrary, that $k=3$. Let S be a cutting K_4 in $\hat{H}$, and let H_1, H_2, and H_3 be the components of $H-S$ listed in the non-decreasing order of cardinalities: $h_1 \leqslant h_2 \leqslant h_3$. To treat two possibilities at once, we *allow* h_1 *to be zero.* Again, we let the reader argue that we can always choose S so that $h_1+h_2 \geqslant h_3-1$.

Case 1: $h_3 \leqslant 2$. Then the number of vertices n in H is at most 10. In light of Corollary 3.3, this is a contradiction.

Case 2: $h_3=3$. If $h_1<3$ then $n<13$, and we obtain a contradiction as in Case 1. So that $h_1=h_2=h_3=3$. Then $n=13$ and, by Corollary 3.3, $\chi(G) \leqslant 7$. Suppose first that all three H_i's are isomorphic to K_3. Then, as in the proof of Theorem 6.1, H–S can be co-contracted to a $\overline{K_6}$. If $S=\overline{K_4}$ then we can co-contract S to a point not adjacent to anything in H–S, and add it to the $\overline{K_6}$. Else S spans an edge, whence $\chi(G) \leqslant 6$. Either way, we arrive at a contradiction.

Thus, assume that $\alpha(H_3) \geqslant 2$. Choose a vertex from each H_1 and H_2, two independent vertices in H_3, co-contract the third vertex of H_3 with a vertex in S adjacent to nothing in H_3, and co-contract, as usual, two pairs of the unchosen vertices of H_1 and H_2. This yields an independent set of size seven, which is a contradiction.

Case 3: $h_3 \geqslant 4$. Observe that $h_1+h_2 \geqslant 4$ (else $n \leqslant 11$). It follows from Corollary 6.6 that $\alpha(H_1)+\alpha(H_2) \geqslant 2$ and $\alpha(H_3) \geqslant 2$. In the notations of Theorem 6.1, set $a_1+a_2=a_3=2$. Then $b_1+b_2 \geqslant b_3-1$. We conclude that the graph K_{b_1,b_2,b_3} has a perfect, or a near-perfect matching; this, as usual, let us pair-up for co-contraction all $n-8$, but maybe one vertices of B. The un-paired vertex, if any, is then co-contracted with an appropriate vertex in S. This gives us an independent set of cardinality $a_1+a_2+a_3+\lceil (n-8)/2 \rceil = \lceil n/2 \rceil \geqslant \chi(G)$. This contradiction completes the proof. □

8. Open questions

Disappointingly, it was not too difficult for me to discover my limitations:

(1) *Is Hadwiger's Conjecture true for graphs G with* $\alpha(G)=2$? Perhaps it is appropriate to remark here that the weakening $1.5\,\eta(G) \geqslant \chi(G)$ of Hadwiger's Conjecture can be easily shown for these graphs. Notice that, in light of Exercise 3.4, we know $\chi(G)$ *exactly* for a minimal counterexample G.

Hadwiger's Conjecture is "trivially" true for graphs with bipartite complement—for example, because they are perfect. In light of this, we have:

(2) *Is Hadwiger's Conjecture true for graphs G with tripartite complements*?

(3) *Is Hadwiger's Conjecture true for graphs G with triangle-free tripartite complements*? This is, of course, a common weakening of Problems 1 and 2. Again, $1.125\,\eta(G) \geqslant \chi(G)$ can be easily demonstrated in this case.

(4) *Does a minimal counterexample G to Hadwiger's Conjecture satisfy* $\Delta(G) \leqslant |G|-4$? This is the "unsettled" case in Theorem 6.7.

Finally, and perhaps least interesting of all:

(5) *Can the bound in Corollary* 4.2 *be improved to* $p \geqslant 7$? Notice that to answer this question, one "only" has to consider the 7-chromatic graphs on 13 vertices with two-connected, factor-critical complements of tree width at least four.

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