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Meixner-Pollaczek Polynomials and the Heisenberg Algebra

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An alternative proof is given for the connection between a system of continuous Hahn polynomials and identities for symmetric elements in the Heisenberg algebra which was first observed by Bender, Mead and Pinsky. The continuous Hahn polynomials turn out to be Meixner-Pollaczek polynomials. Use is made of the connection between Laguerre polynomials and Meixner-Pollaczek polynomials, the Rodrigues formula for Laguerre polynomials, an operational formula involving Meixner-Pollaczek polynomials and the Schrödinger model for the irreducible unitary representations of the three-dimensional Heisenberg group.

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1. Introduction

In two recent papers [6], [7] Bender, Mead and Pinsky discussed the connection between certain continuous Hahn polynomials and symmetrizations of elements in the Heisenberg algebra. They showed that, if

$$[q,p]=i$$

and $T_{m,n}$ is the sum of all possible terms containing m factors of p and n factors of q then

$$T_{n,n} = \text{const. } S_n(T_{1,1}), \tag{1.1}$$

for some polynomial S_n of degree n, which turns out to be the orthogonal polynomial of degree n on $\mathbb{R}$ with respect to the weight function $x \mapsto 1/\mathrm{ch}(\pi x/2)$. However, the actual proof of this result is not very clear from these two papers.

In the present note we give an alternative proof of (1.1). First, in section 2, we observe a transformation connecting certain continuous Hahn polynomials, in particular the above polynomials S_n to certain Meixner-Pollaczek polynomials. Next, in section 3 we use a Mellin transform relating Laguerre polynomials and Meixner-Pollaczek polynomials and the Rodrigues formula for Laguerre polynomials in order to derive an operational formula involving Meixner-Pollaczek polynomials. Finally, in section 4 we use this operational formula in order to derive formula (1.1). Here we make use of the Schrödinger model for the irreducible unitary representations of the Heisenberg group.

I thank G. Gasper for a reference to (2.8).

2. On continuous Hahn polynomials expressible as Meixner-Pollaczek polynomials

Continuous Hahn polynomials are defined by

$$p_n(x;a,b,c,d) := i^n \frac{(a+c)_n(a+d)_n}{n!}\, {}_3F_2\left[\begin{matrix}-n, n+a+b+c+d-1, a+ix \\ a+c, a+d\end{matrix}; 1\right]. \tag{2.1}$$

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If $c=\bar{a}$, $d=\bar{b}$ and $\operatorname{Re} a, \operatorname{Re} b>0$ then they are orthogonal on $(-\infty,\infty)$ with respect to the weight function

$$w(x) := \Gamma(a+ix)\Gamma(b+ix)\Gamma(c-ix)\Gamma(d-ix). \tag{2.2}$$

See Atakishiyev and Suslov [3] and Askey [1], but read $a+ix$ instead of $a-ix$ in [1, formula (3)].

Meixner-Pollaczek polynomials are defined by

$$P_n^{(a)}(x;\phi) := e^{in\phi}\,{}_2F_1(-n, a+ix; 2a; 1-e^{-2i\phi}). \tag{2.3}$$

If $a>0$ and $0<\phi<\pi$ they are orthogonal on $(-\infty,\infty)$ with respect to the weight function

$$w(x) = e^{(2\phi-\pi)x}\,|\Gamma(a+ix)|^2. \tag{2.4}$$

See Meixner [10], Pollaczek [11] and, for standardized notation, Askey and Wilson [2, Appendix].

For $a=c=b-½=d-½>0$ the weight function (2.2) becomes

$$w(x) = 2^{-4a+2}\pi\,|\Gamma(2a+2ix)|^2. \tag{2.5}$$

On comparing with (2.4) we conclude that

$$p_n(x;a,a+½,a,a+½) = \text{const.}\; P_n^{(2a)}(2x;½\pi).$$

The constant can be computed by comparing coefficients of x^n. We obtain:

$$p_n(x;a,a+½,a,a+½) = \frac{(2a)_n\,(2a+½)_n}{n!}\,P_n^{(2a)}(2x;½\pi). \tag{2.6}$$

In terms of hypergeometric functions this formula reads

$${}_3F_2\left[\begin{matrix}-n, n+4a, a+ix \\ 2a, 2a+½\end{matrix}\,\middle|\,1\right] = {}_2F_1(-n, 2a+2ix; 4a; 2). \tag{2.7}$$

This identity can also be obtained from Bailey's [4, p.502] formula

$${}_4F_3\left[\begin{matrix}a, b, n+2c, -n \\ a+b+½, c, c+½\end{matrix};1\right] = {}_3F_2\left[\begin{matrix}2a, 2b, -n \\ a+b+½, 2c\end{matrix};1\right] \tag{2.8}$$

by letting $b\to\infty$.

For $a:=¼$ the weight function (2.5) becomes

$$w(x) = \frac{2\pi^2}{\operatorname{ch}(2\pi x)}.$$

In particular, we find for the polynomials S_n introduced in §1, which Bender, Mead and Pinsky [7] identified with special continuous Hahn polynomials, that they can be written as Meixner-Pollaczek polynomials:

$$S_n(x) = \text{const.}\, P_n^{(1/2)}(½x, ½\pi). \tag{2.9}$$

3. An operational formula involving Meixner-Pollaczek polynomials

Recall that we can obtain the *Mellin transform pair*

$$\begin{cases} G(\lambda) = \int\limits_0^\infty F(\tau)\,\tau^{-1-i\lambda}\,d\tau \\ F(\tau) = (2\pi)^{-1}\int\limits_{-\infty}^{\infty} G(\lambda)\,\tau^{i\lambda}\,d\lambda \end{cases} \tag{3.1}$$

from the *Fourier transform pair*

$$\begin{cases} g(\lambda) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i\lambda t}\, dt \\ f(t) = \int_{-\infty}^{\infty} g(\lambda)\, e^{2\pi i\lambda t}\, d\lambda \end{cases} \tag{3.2}$$

by making the substitutions

$$\tau = e^{2\pi t},\ F(\tau) = f(t),\ G(\lambda) = 2\pi\, g(\lambda)$$

in (3.2). In particular, Mellin inversion in (3.1) is valid if the function $t \mapsto F(e^{2\pi t})$ belongs to the class $\mathcal{S}$ of rapidly decreasing C^∞-functions on $\mathbb{R}$. If F_1, F_2 are two such functions and G_1, G_2 their Mellin transforms then we have the *Parseval formula*

$$\int_0^\infty F_1(\tau)\overline{F_2(\tau)}\frac{d\tau}{\tau} = \int_{-\infty}^{\infty} G_1(\lambda)\overline{G_2(\lambda)}\frac{d\lambda}{2\pi}. \tag{3.3}$$

Proposition 3.1. For $a>0$ and $0<\phi<\pi$ Laguerre polynomials $x \mapsto L_n^{2a-1}(x)$ and Meixner-Pollaczek polynomials $\lambda \mapsto P_n^{(a)}(\lambda;\phi)$ are mapped onto each other by the Mellin transform in the following way:

$$\int_0^\infty \frac{n!\,e^{-in\phi}}{(2a)_n} e^{-\frac{1}{2}x(1+i\operatorname{cotg}\phi)}\, x^a\, L_n^{2a-1}(x)\, x^{-1-i\lambda}\, dx$$
$$= e^{(ia-\lambda)(\phi-\frac{1}{2}\pi)}\,(2\sin\phi)^{a-i\lambda}\,\Gamma(a-i\lambda)\,P_n^{(a)}(\lambda;\phi). \tag{3.4}$$

Proof. The left hand side can be rewritten as

$$e^{-in\phi}\sum_{k=0}^{n}\frac{(-n)_k}{(2a)_k\,k!}\int_0^\infty e^{-\frac{1}{2}x(1+i\operatorname{cotg}\phi)}\, x^{k+a-i\lambda-1}\, dx$$
$$= e^{-in\phi}\sum_{k=0}^{n}\frac{(-n)_k}{(2a)_k\,k!}\,\frac{\Gamma(a-i\lambda+k)}{(\frac{1}{2}+\frac{1}{2}i\operatorname{cotg}\phi)^{a-i\lambda+k}}$$
$$= e^{-in\phi}\,\Gamma(a-i\lambda)(1-e^{2i\phi})^{a-i\lambda}\,{}_2F_1(-n,a-i\lambda;2a;1-e^{2i\phi})$$
$$= e^{in\phi}\,\Gamma(a-i\lambda)(1-e^{2i\phi})^{a-i\lambda}\,{}_2F_1(-n,a+i\lambda;2a;1-e^{-2i\phi}),$$

which can be rewritten as the right hand side of (3.4). □

It is possible to give an interpretation of the above proposition in the context of matrix elements of discrete series representations of $SL(2,\mathbb{R})$, cf. Koornwinder [9, §7] and Basu and Wolf [5].

Corollary 3.2. For $a>0$ and $0<\phi<\pi$ Laguerre polynomials can be expressed by the differentiation formula

$$\frac{n!\,e^{-in\phi}}{(2a)_n} e^{-\frac{1}{2}x(1+i\operatorname{cotg}\phi)}\, x^a\, L_n^{2a-1}(x)$$
$$= P_n^{(a)}(-ix\,d/dx\,;\phi)\,(e^{-\frac{1}{2}x(1+i\operatorname{cotg}\phi)}\,x^a). \tag{3.5}$$

Proof. In the left hand side of (3.4) Mellin transform is taken of a function which belongs to the class $\mathcal{S}$ as a function of t, where $x=e^t$. Hence we can apply Mellin inversion (cf. (3.1)) and we can write the left hand side of (3.5) as

$$(2\pi)^{-1} \int_{-\infty}^{\infty} e^{(ia-\lambda)(\phi-\frac{1}{2}\pi)} (2\sin\phi)^{a-i\lambda}\, \Gamma(a-i\lambda)\, P_n^{(a)}(\lambda;\phi)\, x^{i\lambda}\, d\lambda$$
$$= P_n^{(a)}(-ix\, d/dx\, ;\phi) \left[e^{(ia-\lambda)(\phi-\frac{1}{2}\pi)} (2\sin\phi)^{a-i\lambda}\, \Gamma(a-i\lambda)\, x^{i\lambda}\right],$$

which equals the right hand side of (3.5). □

By substitution of the Rodrigues formula

$$n!\, e^{-x} x^{\alpha} L_n^{\alpha}(x) = \left[\frac{d}{dx}\right]^n (e^{-x} x^{n+\alpha})$$

into (3.5) we obtain

$$\left[\frac{d}{dx}\right]^n (e^{-x} x^{n+2a-1})$$
$$= (2a)_n\, e^{in\phi}\, e^{-\frac{1}{2}x(1-i\operatorname{cotg}\phi)}\, x^{a-1}\, P_n^{(a)}(-ix\, d/dx\, , \phi)\, [e^{-\frac{1}{2}x(1+i\operatorname{cotg}\phi)}\, x^{a}]. \tag{3.6}$$

In particular, for $\phi=\frac{1}{2}\pi$ and $a=\frac{1}{2}$ we obtain

$$\left[i\frac{d}{dx}\right]^n (e^{-x} x^n) = n!\, e^{-\frac{1}{2}x}\, P_n^{(\frac{1}{2})}(ix\, d/dx + \tfrac{1}{2}i\, , \tfrac{1}{2}\pi)\, [e^{-\frac{1}{2}x}].$$

Hence, for arbitrary $\nu \in \mathbb{C}$:

$$e^{i\nu x} \left[i\frac{d}{dx}\right]^n (x^n e^{-2i\nu x}) = n!\, P_n^{(\frac{1}{2})}(ix\, d/dx + \tfrac{1}{2}i\, , \tfrac{1}{2}\pi)\, [e^{-i\nu x}]. \tag{3.7}$$

4. Proof of the Bender-Mead-Pinsky result

Consider the *Heisenberg group* H_1 which is $\mathbb{R}^3$ equipped with the multiplication rule

$$(\xi,\eta,\tau)(\xi',\eta',\tau') = (\xi+\xi',\eta+\eta',\tau+\tau'+\tfrac{1}{2}(\xi'\eta-\xi\eta')). \tag{4.1}$$

Let $\lambda \in \mathbb{R} \setminus \{0\}$ and let π_λ denote the unique (up to equivalence) irreducible unitary representation of H_1 such that

$$\pi_\lambda(0,0,\tau) = e^{i\lambda\tau} I, \quad \tau \in \mathbb{R}.$$

Then, with $\mu := |\lambda|^{\frac{1}{2}}$ and $\epsilon := \operatorname{sign}(\lambda)$, π_λ can be realized on $L^2(\mathbb{R})$ by

$$(\pi_\lambda(\xi,\eta,\tau) f)(x) = e^{i\mu\xi x}\, e^{i\mu^2(\epsilon\tau+\frac{1}{2}\xi\eta)}\, f(x+\mu\eta), \quad f \in L^2(\mathbb{R}). \tag{4.2}$$

Let X and Y be the infinitesimal generators of the one-parameter subgroups of elements $(\xi,0,0)$ and $(0,\eta,0)$, respectively. Let σ denote the symmetrization mapping from the symmetric algebra to the universal enveloping algebra of the Lie algebra of H_1, i.e.

$$\sigma(X_1 \cdots X_k) := \frac{1}{k!} \sum_s X_{s(1)} \cdots X_{s(k)}, \tag{4.3}$$

where s runs over all permutations of $\{1,\ldots,k\}$, cf. for instance Helgason [8, Ch.2, Theorem 4.3]. Let f be a C^∞-function locally defined on $\mathbb{R}$. Then

$$(\pi_\lambda(X) f)(x) = i\mu x f(x),$$
$$\pi_\lambda(Y) f)(x) = \mu f'(x)$$

and

$$(\pi_\lambda(\sigma(X^n Y^n))f)(x)$$
$$= \left[\frac{\partial}{\partial\xi}\right]^n \left[\frac{\partial}{\partial\eta}\right]^n (e^{i\mu\xi x}\, e^{i\mu^2(\epsilon\tau+\frac{1}{2}\xi\eta)}\, f(x+\mu\eta))\Big|_{\xi,\eta,\tau=0}$$
$$= \left[i\mu\frac{\partial}{\partial\eta}\right]^n ((x+\tfrac{1}{2}\mu\eta)^n\, f(x+\mu\eta))\Big|_{\eta=0}$$

Hence

$$(\pi_\lambda(\sigma(X^n Y^n))f)(x) = |\lambda|^n \left[i\frac{\partial}{\partial y}\right]^n ((x+\tfrac{1}{2}y)^n\, f(x+y))\Big|_{y=0}. \tag{4.4}$$

For $n=1$ this simplifies to

$$(\pi_\lambda(\sigma(XY))f)(x) = |\lambda|\, (ix\,\partial/\partial x + \tfrac{1}{2}i)\, f(x). \tag{4.5}$$

Let

$$f_\nu(x) := e^{-i\nu x}.$$

Then we obtain from (4.4), (3.7) and (4.5) that

$$(\pi_\lambda(\sigma(X^n Y^n))\, f_\nu)(x)$$
$$= |\lambda|^n \left[i\frac{\partial}{\partial y}\right]^n ((x+\tfrac{1}{2}y)^n\, e^{-i\nu(x+y)})\Big|_{y=0}$$
$$= 2^{-n}\, |\lambda|^n\, e^{i\nu x} \left[i\frac{\partial}{\partial x}\right]^n (x^n\, e^{-2i\nu x})$$
$$= 2^{-n}\, n!\, |\lambda|^n\, P_n^{(\frac{1}{2})}(ix\, d/dx + \tfrac{1}{2}i,\, \tfrac{1}{2}\pi)[e^{-i\nu x}]$$
$$= 2^{-n}\, n!\, |\lambda|^n\, P_n^{(\frac{1}{2})}(|\lambda|^{-1}\, \pi_\lambda(\sigma(XY)),\, \tfrac{1}{2}\pi)[f_\nu(x)].$$

Hence, by integrating both sides against suitable functions of ν we obtain:

$$\pi_\lambda(\sigma(X^n Y^n)) = 2^{-n}\, n!\, |\lambda|^n\, P_n^{(\frac{1}{2})}(|\lambda|^{-1}\, \pi_\lambda(\sigma(XY)),\, \tfrac{1}{2}\pi). \tag{4.6}$$

In view of (2.9) and (4.3) this becomes for $\lambda=1$ the result (1.1) of Bender, Mead and Pinsky [6].

References

1. R. Askey, *Continuous Hahn polynomials*, J. Phys. A **18** (1985) L1017-L1019.
2. R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Mem. Amer. Math. Soc. 319 (1985).
3. N.M. Atakishiyev and S.K. Suslov, *The Hahn and Meixner polynomials of an imaginary argument and some of their applications*, J. Phys. A **18** (1985) 1583-1596.
4. W.N. Bailey, *Transformations of generalized hypergeometric series*, Proc. London Math. Soc. **(2) 29** (1929) 495-502.
5. D. Basu and K.B. Wolf, *The unitary irreducible representations of SL(2,R) in all subgroup reductions*, J. Math. Phys. **23** (1982) 189-205.
6. C.M Bender, L.R. Mead, and S.S. Pinsky, *Resolution of the operator-ordering problem by the method of finite elements*, Phys. Rev. Lett. **56** (1986) 2445-2448.

7. C.M. Bender, L.R. Mead, and S.S. Pinsky, *Continuous Hahn polynomials and the Heisenberg algebra*, J. Math. Phys. **28** (1987) 509-513.

8. S. Helgason, *Groups and geometric analysis*, Academic Press (1984).

9. T.H. Koornwinder, *Group theoretic interpretations of Askey's scheme of hypergeometric orthogonal polynomials*, Report PM-R8703, Centre for Math. and Computer Science, Amsterdam (1987).

10. J. Meixner, *Orthogonale Polynomsysteme mit einer besonderen Gestalt der erzeugenden Funktion*, J. London Math. Soc. **9** (1934) 6-13.

11. F. Pollaczek, *Sur une famille de polynômes orthogonaux qui contient les polynômes d'Hermite et de Laguerre comme cas limites*, C. R. Acad. Sci. Paris **230** (1950) 1563-1565.