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Jacobi functions as limit cases of q-ultraspherical polynomials

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Jacobi Functions as Limit Cases of q-ultraspherical Polynomials

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It is shown that Jacobi functions of equal indices can be obtained as limit cases of q-ultraspherical polynomials as q tends to 1. In particular, if the Jacobi function can be interpreted as a spherical function on a hyperboloid then the degree of the q-ultraspherical polynomial tends in the limit to the geodesic distance to a fixed point on the hyperboloid. Two appendices contain rigorous proofs of the limit formulas as q tends to 1 for the q-binomial series and the q-gamma function.

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1. Introduction

q-Ultraspherical polynomials are given by the finite Fourier series

$$C_n(\cos\theta;\beta\,|\,q) := \sum_{k=0}^{n} \frac{(\beta;q)_k\,(\beta;q)_{n-k}}{(q;q)_k\,(q;q)_{n-k}}\, e^{i(n-2k)\theta}, \tag{1.1}$$

cf. Askey & Ismail [4, (3.1)]. They have a limit as ultraspherical polynomials:

$$\lim_{q\uparrow 1} C_n(\cos\theta;q^\lambda\,|\,q) = C_n^\lambda(\cos\theta), \tag{1.2}$$

cf. [4, (2.11)]. This can for instance be proved from (1.1) and

$$C_n^\lambda(\cos\theta) = \sum_{k=0}^{n} \frac{(\lambda)_k\,(\lambda)_{n-k}}{k!\,(n-k)!}\, e^{i(n-2k)\theta}, \tag{1.3}$$

cf. Szegö [24, (4.9.19)].

For $q=0$ and $\beta^{-1}\in\mathbb{N}$ the q-ultraspherical polynomials (1.1) have an interpretation as spherical functions on homogeneous trees, such that each vertex is adjacent to exactly $\beta^{-1}+1$ edges. Then n is the variable which lives on the homogeneous space: it denotes the graph distance on the tree to some fixed point (cf. Cartier [6] and Askey & Ismail [4, §8]). On the other hand, the ultraspherical polynomials (1.3) have (for $\lambda=\frac12 d-\frac12$, $d\in\mathbb{N}$) an interpretation as spherical functions on the sphere S^d, where θ lives on the sphere and denotes the geodesic distance to some fixed point (cf. for instance Vilenkin [26, Ch.9]). Hence, in the passage from $q=0$ to $q=1$ there has been some interchange of the geometric roles of n and θ.

Jacobi functions

$$\phi_\mu^{(\alpha,\alpha)}(t) := \frac{2^{\frac12-\alpha}\,\Gamma(2\alpha+1)}{\Gamma(\alpha+\frac12)\,\Gamma(\alpha+\frac12)}\,(\mathrm{sh}\,2t)^{-2\alpha}\int_{-t}^{t} e^{i\mu s}\,(\mathrm{ch}\,2t-\mathrm{ch}\,2s)^{\alpha-\frac12}\,ds, \tag{1.4}$$

cf. Koornwinder [14], [15], Faraut [8], have (for $\alpha=\frac12 d-1$, $d\in\mathbb{N}$) an interpretation as spherical functions on the hyperboloids $SO_0(d,1)/SO(d)$, where t is the geometric variable denoting the geodesic distance. The purpose of the present note is to derive a limit transition from the q-ultraspherical

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polynomials to the Jacobi functions such that this preserves the variable with the geometric meaning.

Our result is easier described in terms of renormalized q-ultraspherical polynomials

$$R_n(e^{i\theta};\beta\mid q):=\frac{\beta^{n/2}(q;q)_n}{(\beta^2;q)_n}\,C_n(\cos\theta;\beta\mid q), \tag{1.5}$$

which take the value 1 for $e^{i\theta}=\beta^{1/2}$. Then we will prove:

$$\lim_{q\uparrow 1} R_{4t/(\log q^{-1})}(q^{1/4 i\mu};q^{\alpha+1/2}\mid q)=\phi_\mu^{(\alpha,\alpha)}(t),\quad t\in[0,\infty),\ \mu\in\mathbb{C}. \tag{1.6}$$

For the moment, we will only give a proof for the case $\alpha\geqslant 1/2$. This will be done in section 2, while section 3 contains some speculations about further extensions of this result. In section 2 we need the limit formulas as $q\uparrow 1$ for the q-binomial formula and for the q-gamma function. It seemed useful to me to complement the usual formal proofs of these limits with rigorous proofs. This is the topic of Appendices A and B.

I thank W. van Assche for suggesting me the second part of the proof of Theorem B.2 and W.A. Al-Salam for giving me references on the historical origin of the q-gamma function, cf. Appendix B.

2. Proof of the limit formula

By (1.1) and (1.5) we have

$$R_n(e^{i\theta};\beta\mid q)=\frac{\beta^{n/2}(q;q)_n}{(\beta^2;q)_n}\sum_{k=0}^{n}\frac{(\beta;q)_k\,(\beta;q)_{n-k}}{(q;q)_k\,(q;q)_{n-k}}\,e^{i(n-2k)\theta}. \tag{2.1}$$

Now fix $t\in(0,\infty)$, $\mu\in\mathbb{C}$, $\alpha\geqslant 1/2$ and make in (2.1) the substitutions

$$n=\frac{4t}{\log q^{-1}},\quad e^{i\theta}=q^{-1/4 i\mu},\quad \beta=q^{\alpha+1/2}, \tag{2.2}$$

where we choose q such that $4t/(\log q^{-1})\in\mathbb{Z}_+$. In this way q can approach 1 from below while n approaches ∞ in $\mathbb{Z}_+$. Also replace the summation variable k in (2.1) by

$$s=t-1/2 k\log q^{-1}. \tag{2.3}$$

Then the left hand side of (2.1) becomes the expression after the limit sign in (1.6), while the right hand side of (2.1) becomes

$$\frac{e^{-(2\alpha+1)t}(q;q)_{4t/(\log q^{-1})}}{(q^{2\alpha+1};q)_{4t/(\log q^{-1})}}\sum_{\substack{s=-t\\ \text{step } 1/2\log q^{-1}}}^{t}\frac{(q^{\alpha+1/2};q)_{2(t-s)/(\log q^{-1})}\,(q^{\alpha+1/2};q)_{2(t+s)/(\log q^{-1})}}{(q;q)_{2(t-s)/(\log q^{-1})}\,(q;q)_{2(t+s)/(\log q^{-1})}}\,e^{i\mu s}. \tag{2.4}$$

Now use that

$$\frac{(q^{b+1};q)_{a/(\log q^{-1})}}{(q;q)_{a/(\log q^{-1})}}=\frac{(e^{-a}q;q)_\infty}{(e^{-a}q^{b+1};q)_\infty}\,\frac{(1-q)^{-b}}{\Gamma_q(b+1)}, \tag{2.5}$$

where

$$\Gamma_q(b+1):=\frac{(1-q)^{-b}(q;q)_\infty}{(q^{b+1};q)_\infty}. \tag{2.6}$$

Hence (2.4) can be rewritten as

$$\frac{2\Gamma_q(2\alpha+1)(1-q)}{\Gamma_q(\alpha+1/2)\Gamma_q(\alpha+1/2)\log q^{-1}}\,\frac{(e^{-4t}q^{2\alpha+1};q)_\infty}{(e^{-4t}q;q)_\infty}\,e^{-(2\alpha+1)t}$$

$$\times\, 1/2\log q^{-1}\sum_{\substack{s=-t\\ \text{step } 1/2\log q^{-1}}}^{t}\frac{(e^{-2(t-s)}q;q)_\infty\,(e^{-2(t+s)}q;q)_\infty}{(e^{-2(t-s)}q^{\alpha+1/2};q)_\infty\,(e^{-2(t+s)}q^{\alpha+1/2};q)_\infty}\,e^{i\mu s}. \tag{2.7}$$

We assumed that $\alpha \geqslant \frac{1}{2}$. Hence the summand in (2.7) can be majorized by $e^{|\mathrm{Im}\mu||t|}$. We can rewrite the part

$$\tfrac{1}{2}\log q^{-1} \sum_{s=-t}^{t} \cdots$$

of (2.7) as

$$\int_{-t}^{t} f_q(\sigma)\, d\sigma,$$

where

$$f_q(\sigma) = \frac{(e^{-2(t-s)}q;q)_\infty\,(e^{-2(t+s)}q;q)_\infty}{(e^{-2(t-s)}q^{\alpha+\frac{1}{2}};q)_\infty\,(e^{-2(t+s)}q^{\alpha+\frac{1}{2}};q)_\infty}\, e^{i\mu s}$$

if $s = t - \frac{1}{2}k\log q^{-1}$ ($k \in \mathbb{Z}_+$) and $\sigma \in (s - \frac{1}{4}k\log q^{-1}, s + \frac{1}{4}k\log q^{-1})$. Because of Proposition A.1 there is pointwise convergence:

$$\lim_{q\uparrow 1} f_q(\sigma) = (1-e^{-2(t-\sigma)})^{\alpha-\frac{1}{2}}\,(1-e^{-2(t+\sigma)})^{\alpha-\frac{1}{2}}\, e^{i\mu\sigma}.$$

Hence, by dominated convergence and by (A.4) and (B.2), (2.7) converges to

$$\frac{2\Gamma(2\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\,\Gamma(\alpha+\frac{1}{2})}\, e^{-(2\alpha+1)t}\,(1-e^{-4t})^{-2\alpha} \int_{-t}^{t} (1-e^{-2(t-s)})^{\alpha-\frac{1}{2}}\,(1-e^{-2(t+s)})^{\alpha-\frac{1}{2}}\, e^{i\mu s}\, ds$$

as $q\uparrow 1$. This can be rewritten as the left hand side of (1.4). Thus we have completed the proof of (1.6).

3. Discussion of the results

Remark 1. The q-ultraspherical polynomials are special cases of q-Wilson polynomials and can thus be written as ${}_4\phi_3$ q-hypergeometric functions:

$$R_n(e^{i\theta};\beta\,|\,q) = {}_4\phi_3\left[\begin{matrix} q^{-n}, q^n\beta^2, \beta^{\frac{1}{2}}e^{i\theta}, \beta^{\frac{1}{2}}e^{-i\theta} \\ \beta q^{\frac{1}{2}}, -\beta q^{\frac{1}{2}}, -\beta \end{matrix}; q, q\right], \tag{3.1}$$

cf. [4, (3.10)]. If we make the substitutions (2.2) in (3.1) then the right hand side of (3.1) becomes

$${}_4\phi_3\left[\begin{matrix} e^{4t}, e^{4t}q^{2\alpha+1}, q^{\frac{1}{2}\alpha+\frac{1}{4}-\frac{1}{4}i\mu}, q^{\frac{1}{2}\alpha+\frac{1}{4}+\frac{1}{4}i\mu} \\ q^{\alpha+1}, -q^{\alpha+1}, -q^{\alpha+\frac{1}{2}} \end{matrix}; q, q\right]. \tag{3.2}$$

Formally, as $q\uparrow 1$, (3.2) tends to the ordinary hypergeometric function

$${}_2F_1\left[\begin{matrix} \frac{1}{2}\alpha+\frac{1}{4}-\frac{1}{4}i\mu, \frac{1}{2}\alpha+\frac{1}{4}+\frac{1}{4}i\mu \\ \alpha+1 \end{matrix}; \frac{(1-e^{4t})(1-e^{-4t})}{4}\right], \tag{3.3}$$

which we can write as a Jacobi function

$$\phi_{\frac{1}{2}\mu}^{(\alpha,-\frac{1}{2})}(2t) = \phi_\mu^{(\alpha,\alpha)}(t), \tag{3.4}$$

by [15, (2.7),(5.32)]. Of course, this is the way we obtained (1.6) first, but it is not valid as a proof.

Remark 2. One would expect that Jacobi functions of general order (α,β)

$$\phi_\mu^{(\alpha,\beta)}(t) := {}_2F_1(\tfrac{1}{2}(\alpha+\beta+1+i\mu), \tfrac{1}{2}(\alpha+\beta+1-i\mu); \alpha+1; -\mathrm{sh}^2 t) \tag{3.5}$$

can be similarly obtained as limits of q-Wilson polynomials (cf. Askey and Wilson [5]). Indeed, write the q-Wilson polynomial as

$$P_n(e^{i\theta};a,b,c,d\,|\,q) := {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{n-1}abcd, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q\right]. \tag{3.6}$$

Make the substitutions

$$n = \frac{2t}{\log q^{-1}},\ e^{i\theta} = q^{-\frac{1}{2}i\mu},\ a = q^{\frac{1}{2}(\alpha+\beta+1)},\ b = q^{\frac{1}{2}(\alpha-\beta+1)},\ c = -q^{\gamma},\ d = -q^{\delta}. \tag{3.7}$$

Then the right hand side of (3.6) becomes

$${}_4\phi_3\left[\begin{matrix} e^{2t}, e^{-2t}q^{\alpha+\beta+\gamma+\delta}, q^{\frac{1}{2}(\alpha+\beta+1+i\mu)}, q^{\frac{1}{2}(\alpha+\beta+1-i\mu)} \\ q^{\alpha+1}, -q^{\gamma}, -q^{\delta} \end{matrix}; q, q\right] \tag{3.8}$$

As $q\uparrow 1$ this converges formally to the right hand side of (3.5). It would be interesting to give a rigorous proof that

$$\phi_\mu^{(\alpha,\beta)}(t) = \lim_{q\uparrow 1} P_{2t/(\log q^{-1})}(q^{-\frac{1}{2}i\mu}; q^{\frac{1}{2}(\alpha+\beta+1)}, q^{\frac{1}{2}(\alpha-\beta+1)}, -q^{\gamma}, -q^{\delta}\,|\,q). \tag{3.9}$$

It is remarkable that the Jacobi functions of general order cannot be obtained as limits of q-Jacobi polynomials (cf. Askey and Wilson [5, (4.16),(4.17)]). One would only obtain order $(\alpha,-\alpha)$ or $(\alpha,-\frac{1}{2})$ for the Jacobi functions.

Formally one can obtain many other limit transitions to Jacobi functions of the above type, for instance starting with continuous dual Hahn polynomials, cf. [16, (5.14)].

Remark 3. The limit formula (1.6), which we proved in section 2 only if $\alpha \geqslant \frac{1}{2}$, can be extended to $\alpha > -1$ by use of the q-difference formula [5, (5.14)] specialized to the case of the q-ultraspherical polynomials and a similar difference formula in μ for the Jacobi functions.

Remark 4. The limit formula (1.6) might be used in order to derive new formulas for Jacobi functions from known ones for q-ultraspherical polynomials. In particular, an expression for the product

$$\phi_\lambda^{(\alpha,\alpha)}(t)\,\phi_\mu^{(\alpha,\alpha)}(t)$$

as an integral over ν of $\phi_\nu^{(\alpha,\alpha)}(t)$ might be obtained from the product formula for q-ultraspherical polynomials in Rahman and Verma [22]. Until now such a formula is only known for the cases $\alpha=0$ or $\frac{1}{2}$, cf. Mizony [20].

Remark 5. Macdonald [17], [18], [19] has given multivariable generalizations of the q-ultraspherical polynomials. These are associated with root systems. Similarly, Heckman and Opdam [10], [9] have studied multi-variable analogues of Jacobi polynomials and Jacobi functions associated with root systems. Macdonald [17] already observed that some Heckman-Opdam polynomials occur as limits of Macdonald's polynomials as $q\uparrow 1$ and some very deep results in the Heckman-Opdam papers could thus be proved in a much more elementary way. It would be very interesting to prove that an analogue of (1.6) would also be valid for functions associated with root systems.

Remark 6. It would be of interest to prove the Plancherel formula for the Jacobi function transform (cf. [15]) from the orthogonality relations for the q-ultraspherical polynomials by use of (1.6). For this the q-ultraspherical polynomial transform should converge strongly to the Jacobi function transform. See Ruijsenaars [23], where such a strong limit result has been proved for some other class of orthogonal special functions. A next step would be to do the same in the multivariable situation of the previous remark.

Appendix A: On the convergence of the q-binomial series to the binomial series

Ramanujan observed in his second notebook (Entry I in Chapter 16, cf. [1, p.4]) that

$$\lim_{q\uparrow 1}\frac{(q^\lambda x;q)_\infty}{(x;q)_\infty}=(1-x)^{-\lambda}. \tag{A.1}$$

As pointed out by Askey & Ismail [4, (2.12)], this follows formally from the q-binomial formula

$$\frac{(q^\lambda x;q)_\infty}{(x;q)_\infty}=\sum_{n=0}^{\infty}\frac{(q^\lambda;q)_n}{(q;q)_n}x^n, \tag{A.2}$$

and the binomial formula

$$(1-x)^{-\lambda}=\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}x^n \tag{A.3}$$

by taking termwise limits. In order to make this formal proof into a rigorous one, we have to majorize the series in the right hand side of (A.2) by a convergent series not depending on q.

Lemma A.1 Let $\lambda,\mu,k\in\mathbb{R}$, $0\leqslant\mu-\lambda\leqslant k$, $\mu+\lambda\geqslant 1$,

$$f(t):=\frac{e^{-\mu t}-e^{-(\lambda+k)t}}{1-e^{-(k+1)t}},\quad t>0.$$

Then $f'(t)\leqslant 0$ if $t>0$.

Proof. Put

$$\begin{aligned}g(t)&:=(1-e^{-(k+1)t})^2 f'(t)\\&=-\mu e^{-\mu t}+(\lambda+k)e^{-(\lambda+k)t}+(\mu-k-1)e^{-(\mu+k+1)t}+(1-\lambda)e^{-(\lambda+2k+1)t}.\end{aligned}$$

Then a calculation shows that

$$\begin{aligned}g(0)&=g'(0)=0,\\g''(0)&=-(k+1)(k+\lambda-\mu)(\lambda+\mu-1)\leqslant 0\end{aligned}$$

and

$$\begin{aligned}&d/dt\,(e^{(k+1)t}\,d/dt\,(e^{(-\lambda-2k-1+\mu)t}\,d/dt\,(e^{(\lambda+2k+1)}\,g(t))))\\&=-(k+1)(k+\lambda-\mu)(k+\lambda)(\mu-\lambda+1)e^{(\mu-\lambda+1)t}\leqslant 0.\end{aligned}$$

Hence $g(t)\leqslant 0$. □

Proposition A.2. Let $\lambda,\mu\in\mathbb{R}$. Then

$$\lim_{q\uparrow 1}\frac{(q^\lambda z;q)_\infty}{(q^\mu z;q)_\infty}=(1-z)^{\mu-\lambda}, \tag{A.4}$$

uniformly on $\{z\in\mathbb{C}\mid |z|\leqslant 1\}$ if $\mu\geqslant\lambda$, $\mu+\lambda\geqslant 1$ and uniformly on compacta of $\{z\in\mathbb{C}\mid |z|\leqslant 1,\ z\neq 1\}$ for other choices of λ, μ.

Proof. It is sufficient to prove the Proposition in the case $\mu\geqslant\lambda$, $\mu+\lambda\geqslant 1$, since we can always find $m,l\in\mathbb{Z}_+$ such that $\mu+m\geqslant\lambda+l$, $(\mu+m)+(\lambda+l)\geqslant 1$. If we then write

$$\frac{(q^\lambda z;q)_\infty}{(q^\mu z;q)_\infty}=\frac{(q^\lambda z;q)_l}{(q^\mu z;q)_m}\,\frac{(q^{\lambda+l}z;q)_\infty}{(q^{\mu+m}z;q)_\infty},$$

we observe that the first quotient tends to $(1-z)^{l-m}$ as $q\uparrow 1$, uniformly on compacta of $\{z\in\mathbb{C}\mid |z|\leqslant 1,\ z\neq 1\}$.

So assume $\mu>\lambda$, $\mu+\lambda\geqslant 1$, $0<q<1$, the result being trivial if $\mu=\lambda$. Then, for $|z|\leqslant 1$:

$$\frac{(q^\lambda z;q)_\infty}{(q^\mu z;q)_\infty} = \frac{(q^{\lambda-\mu}q^\mu z;q)_\infty}{(q^\mu z;q)_\infty} = \sum_{n=0}^{\infty} \frac{(q^{\lambda-\mu};q)_n}{(q;q)_n}(q^\mu z)^n$$
$$= \sum_{n=0}^{\infty} \frac{q^\mu-q^\lambda}{1-q}\,\frac{q^\mu-q^{\lambda+1}}{1-q^2}\cdots\frac{q^\mu-q^{\lambda+n-1}}{1-q^n}\,z^n. \tag{A.5}$$

By Lemma A.1 we have, for $k\geqslant\mu-\lambda$,

$$0\leqslant \frac{q^\mu-q^{\lambda+k}}{1-q^{k+1}} \leqslant \lim_{q\uparrow 1}\frac{q^\mu-q^{\lambda+k}}{1-q^{k+1}} = \frac{\lambda-\mu+k}{k+1}.$$

Put $m:=[\mu-\lambda+1]$. Then, for $n\geqslant m$,

$$\left|\frac{(q^{\lambda-\mu};q)_n}{(q;q)_n}(q^\mu z)^n\right| \leqslant M\frac{(\lambda-\mu+m)_{n-m}}{(m+1)_{n-m}},$$

where

$$M:=\sup_{0<q<1}\left|\frac{q^\mu-q^\lambda}{1-q}\,\frac{q^\mu-q^{\lambda+1}}{1-q^2}\cdots\frac{q^\mu-q^{\lambda+m-1}}{1-q^m}\right|.$$

Now the series $\sum_{n=m}^{\infty}\frac{(\lambda-\mu+m)_{n-m}}{(m+1)_{n-m}}$ converges. Hence the series (A.5) converges uniformly to its termwise limit

$$\sum_{n=0}^{\infty}\frac{(\lambda-\mu)_n}{n!}z^n = (1-z)^{\mu-\lambda}. \quad \Box$$

Appendix B: On the convergence of the q-gamma function to the gamma function

In (2.6) we introduced the q-gamma function

$$\Gamma_q(z) := \frac{(1-q)^{1-z}(q;q)_\infty}{(q^z;q)_\infty},\quad z\in\mathbb{C},\ z\neq 0,-1,-2,\cdots,\ 0<q<1. \tag{B.1}$$

A version of this function, namely

$$\Omega(q,a) := \frac{(q;q)_\infty}{(q^{a+1};q)_\infty},$$

cf. Heine [11, p.109], seems to go back to a paper by Heine in Crelle's J. **24** (1847), p.309. Jackson [12] introduced a q-gamma function similar to (B.1) except for a factor $q^{z(z+1)/2}$ and he claimed that it reduces for $q=1$ to Gauss's product expansion (B.5) for the gamma function. In Jackson's paper [13] of 1905 the q-gamma function essentially occurs as in (B.1), with $|q|<1$. Jackson there rewrites $1/\Gamma_q(z)$ as an infinite product resembling Weierstrass's product expansion [7, (1.1.3)] for $1/\Gamma(z)$ and he claims that his expression converges to Weierstrass's expression as $q\uparrow 1$. The same limit

$$\lim_{q\uparrow 1}\Gamma_q(x) = \Gamma(x),\quad x>0, \tag{B.2}$$

was given (without proof) in Ramanujan's second notebook (Entry 1 in Chapter 16), cf. [1, p.4]. In more recent times Askey [3] extensively discussed the q-gamma function, together with a rigorous proof of (A.1). In Andrews [2, Appendix A] one finds W. Gosper's proof of (B.2), which is very short, but not completely satisfactory, as it is only formal. Here we add some material which makes Gosper's proof rigorous. Moreover we extend the proof to complex values of the argument. The proof of this extension was suggested by W. van Assche.

Define for $n=1,2,3,\cdots$

$$f_{z,n}(q) := \begin{cases} \dfrac{(1-q^{n+1})^z}{(1-q^{n+z})(1-q^n)^{z-1}}, & 0\leqslant q<1, \\ \dfrac{n}{n+z}\left[\dfrac{n+1}{n}\right]^z, & q=1. \end{cases} \tag{B.3}$$

Then, by (B.1) and following [2, Appendix A] we can write

$$\Gamma_q(z+1) = \prod_{n=1}^{\infty} f_{z,n}(q), \quad 0<q<1, \tag{B.4}$$

while for $\Gamma(z+1)$ we have Gauss's formula

$$\Gamma(z+1) = \prod_{n=1}^{\infty} f_{z,n}(1), \tag{B.5}$$

cf. for instance Olver [21, Ch.2, §1.3].

Lemma B.1. Let $q\in[0,1]$, $x\in(0,\infty)$, $n\in\mathbb{N}$. Then $f_{x,n}$ is continuous on $[0,1]$, $f_{1,n}(q) = 1$, $f_{x,n}(0)=1$ and $f_{x,n}$ is monotonically decreasing or increasing on $[0,1]$ according to whether $0<x<1$ or $x>1$, respectively.

Proof. Put

$$g_{x,n}(t) := \log(f_{x,n}(e^{-2t})),$$
$$h(y) := y\coth y.$$

Then

$$g'_{x,n}(t) = t^{-1}(x\,h((n+1)t) - h((n+x)t) - (x-1)h(nt))$$

and $h''(y)>0$. Hence h is convex and $g'_{x,n}(t)<0$ or >0 for $t\in(0,\infty)$ according to whether $0<x<1$ or $x>1$, respectively. □

Theorem B.2. For all complex $z\neq -1,-2,\cdots$ we have

$$\lim_{q\uparrow 1}\Gamma_q(z+1) = \Gamma(z+1). \tag{B.6}$$

Proof. First we take $z\in(1,\infty)$. Then, by Lemma B.1, the factors $f_{z,n}(q)$ lie between 1 and $f_{z,n}(1)$. Hence, the limit as $q\uparrow 1$ in (B.4) can be taken factorwise and the result follows from (B.5).

Next we take $z\in\mathbb{C}$ with $\mathrm{Re}\,z>0$. Then it is easily seen from (B.3) that

$$|f_{z,n}(q)| \leqslant f_{\mathrm{Re}\,z,\,n}(q)$$

Hence

$$|\Gamma_q(z+1)| \leqslant \Gamma_q(\mathrm{Re}\,z+1) \leqslant \max\{\Gamma(\mathrm{Re}\,z+1), 1\}.$$

So the family of functions $\{z\mapsto\Gamma_q(z)\}$ $(0<q<1)$ is uniformly bounded on compacta of $\{z \mid \mathrm{Re}\,z>0\}$, while it tends to a limit as $q\uparrow 1$ if $z>0$. Hence, by Vitali's convergence theorem (cf. for instance Titchmarsh [25, §5.21]) the family tends to a limit on the whole right half plane as $q\uparrow 1$. This limit is necessarily the analytic continuation of $z\mapsto\Gamma(z+1)$.

Finally the limit formula can be proved for other complex $z\neq -1,-2,\cdots$ by use of the two recurrence formulas

$$\Gamma_q(z+1) = \frac{1-q^z}{1-q}\Gamma_q(z) \quad \text{and} \quad \Gamma(z+1) = z\,\Gamma(z).$$

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