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Homotopy and crossings of systems of curves on a surface

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Let $C_1, \ldots, C_k$ and $C_1', \ldots, C_k'$ be closed curves on a compact orientable surface S. We characterize (in terms of counting crossings) when there exists a permutation π of $\{1, \ldots, k\}$ so that, for each $i = 1, \ldots, k$, $C'_{\pi(i)}$ is freely homotopic to C_i or C_i^{-1}. The characterization is equivalent to the nonsingularity of a certain infinite symmetric matrix.

We prove the following theorem:

THEOREM. Let $C_1,\ldots,C_k$ and $C'_1,\ldots,C'_{k'}$ be *primitive closed curves on a compact orientable surface* S. *Then the following are equivalent:*

(i) $k = k'$ *and there exists a permutation* π *of* $\{1,\ldots,k\}$ *so that for each*

$i=1,\ldots,k$: $C'_{\pi(i)} \sim C_i$ *or* $C'_{\pi(i)} \sim C_i^{-1}$;

(ii) *for each closed curve* D *on* S:

(1) $$\sum_{i=1}^{k} \mathrm{mincr}(C_i,D) = \sum_{i=1}^{k'} \mathrm{mincr}(C'_i,D).$$

Here we use the following terminology and notation. A *closed curve* C *on* S is a continuous function $C:S_1 \longrightarrow S$, where S_1 is the unit circle $\{z \in \mathbb{C} \mid |z|=1\}$. Two closed curves C and C' are called *(freely) homotopic*, in notation: $C \sim C'$, if there exists a continuous function $\Phi:[0,1] \times S_1 \longrightarrow S$ so that $\Phi(0,z)=C(z)$ and $\Phi(1,z)=C'(z)$ for all $z \in S_1$. A closed curve C is *primitive* if there do not exist a closed curve D and an integer $n \geq 2$ so that $C \sim D^n$. For a closed curve D and integer n, D^n is the closed curve defined by $D^n(z):=D(z^n)$ for $z \in S_1$.

Finally, for closed curves C and D:

(2) $\mathrm{cr}(C,D) := \left|\{(y,z) \in S_1 \times S_1 \mid C(y) = D(z)\}\right|$ if $C \neq D$,

$\quad\quad\; \mathrm{cr}(C,C) := \left|\{(y,z) \in S_1 \times S_1 \mid C(y) = C(z), y \neq z\}\right|$,

$\quad\quad\; \mathrm{mincr}(C,D) := \min\{\mathrm{cr}(\tilde{C},\tilde{D}) \mid \tilde{C} \sim C, \tilde{D} \sim D\}.$

PROOF. The implication (i) $\Rightarrow$ (ii) being trivial (as $\mathrm{mincr}(C^{-1},D)=\mathrm{mincr}(C,D)$), we show (ii) $\Rightarrow$ (i). By symmetry we may assume

(3) $$\sum_{i=1}^{k'} \sum_{j=1}^{k'} \mathrm{mincr}(C'_i,C'_j) \leq \sum_{i=1}^{k} \sum_{j=1}^{k} \mathrm{mincr}(C_i,C_j).$$

By a result of Baer [1] there exist $\tilde{C}_1 \sim C'_1, \ldots, \tilde{C}_{k'} \sim C'_{k'}$, so that:

(4) $\mathrm{cr}(\tilde{C}_i,\tilde{C}_j) = \mathrm{mincr}(C'_i,C'_j)$ for $i,j=1,\ldots,k'$.

We may assume that in fact $\tilde{C}_i = C'_i$ for $i=1,\ldots,k'$, that $C'_i \neq C'_j$ if $i \neq j$, and that each point of S is passed at most twice by the C'_i. (So no two crossings of the C'_i coincide.)

Let $G = (V,E)$ be the graph made up by the curves C'_i. So G is a graph embedded on S. Each point of S passed twice by the C'_i is a vertex, of degree 4, of G.

Moreover, we take as vertices some of the points of S passed exactly once by the C_i', in such a way that G will be a graph without loops or parallel edges. So each vertex of G has degree 2 or 4.

Now by (1), for each closed curve $D: S_1 \longrightarrow S \setminus V$:

$$(5) \qquad \sum_{i=1}^{k} \text{mincr}(C_i, D) = \sum_{i=1}^{k'} \text{mincr}(C_i', D) \leq \sum_{i=1}^{k'} \text{cr}(C_i', D) = \text{cr}(G, D),$$

where $\text{cr}(G, D) := |\{z \in S_1 \mid D(z) \in G\}|$. Hence, by the 'homotopic circulation theorem' in [2], there exist cycles $C_{11}, \ldots, C_{1t_1}, \ldots, C_{k1}, \ldots, C_{kt_k}$ in G and rationals $\lambda_{11}, \ldots, \lambda_{1t_1}, \ldots, \lambda_{k1}, \ldots, \lambda_{kt_k} > 0$ so that:

$$(6) \qquad \text{(i) } C_{ij} \sim C_i \qquad\qquad\qquad (i=1,\ldots,k; \ j=1,\ldots,t_i);$$

$$\text{(ii) } \sum_{j=1}^{t_i} \lambda_{ij} = 1 \qquad\qquad\qquad (i=1,\ldots,k);$$

$$\text{(iii) } \sum_{i=1}^{k} \sum_{j=1}^{t_i} \lambda_{ij} \chi^{C_{ij}}(e) \leq 1 \qquad\qquad (e \in E).$$

Here a *cycle* in G is a sequence

$$(7) \qquad C = (v_0, e_1, v_1, e_2, v_2, \ldots, v_{d-1}, e_d, v_d),$$

where $v_0, \ldots, v_d$ are vertices of G, $v_0 = v_d$ and e_i is the edge connecting v_{i-1} and v_i $(i=1,\ldots,d)$. $(v_1, \ldots, v_d, e_1, \ldots, e_d$ need not be distinct.) With each cycle in G we can associate in the obvious way a closed curve on S - unique up to homotopy. For any cycle (7) and any edge e of G:

$$(8) \qquad \chi^C(e) := \text{number of } i \in \{1, \ldots, d\} \text{ with } e_i = e.$$

We call a cycle (7) *not-returning* if $e_i \neq e_{i-1}$ for $i=1,\ldots,d$, and $e_1 \neq e_d$. Clearly, we may assume the C_{ij} to be not-returning.

Consider now any vertex of G of degree 4, and denote the edges incident with v by e_1, e_2, e_3, e_4 in cyclic order:

$$(9)$$

We call edges e_1 and e_3 *opposite in* v, and similarly, we call e_2 and e_4 *opposite in* v.

The remainder of this proof consists of showing:

(10)   (i) for each edge e, equality holds in (6)(iii);

(ii) for each cycle C_{ij} and each vertex v of degree 4, each time when C_{ij} passes v it goes from one edge to the edge opposite in v.

Having shown this, it follows that each C_{ij} belongs to $\{C_1',(C_1')^{-1},\ldots,C_k',(C_k')^{-1}\}$, and hence we have (i) in our theorem.

In order to prove (10), we first show two lemmas. For each vertex v of degree 4, we fix one choice e_1,e_2,e_3,e_4 as in (9). For any cycle C in G, any vertex of degree 4 in G and any $i,j \in \{1,2,3,4\}$, let

(11) $\alpha_{ij}^{v}(C) :=$ number of times C passes v by going from e_i to e_j or from e_j to e_i.

LEMMA A. *For any pair of not-returning cycles C,D in G:*

(12) $\text{mincr}(C,D) \leq \displaystyle\sum_{v \in W} (\beta_{13}\gamma_{24} + \beta_{24}\gamma_{13} + \tfrac{1}{2}((\beta_{12}+\beta_{34})(\gamma_{13}+\gamma_{14}+\gamma_{23}+\gamma_{24}) +$

$+ (\beta_{13}+\beta_{24})(\gamma_{12}+\gamma_{14}+\gamma_{23}+\gamma_{34}) + (\beta_{14}+\beta_{23})(\gamma_{12}+\gamma_{13}+\gamma_{24}+\gamma_{34})))$,

where $\beta_{ij} := \alpha_{ij}^{v}(C)$ *and* $\gamma_{ij} := \alpha_{ij}^{v}(D)$, *and* $W:=\{v \in V \mid v$ *has degree* $4\}$.

[Note that the term in (12) with factor $\tfrac{1}{2}$ contains all products $\beta_{gh}\gamma_{ij}$ with $g \neq h$, $i \neq j$ and $|\{g,h\} \cap \{i,j\}| = 1$.]

PROOF OF LEMMA A. We can represent C and D as:

(13) $C = (v_0,f_1,v_1,f_2,v_2,\ldots,v_{s-1},f_s,v_s)$,
$D = (w_0,g_1,w_1,g_2,w_2,\ldots,w_{t-1},g_t,w_t)$,
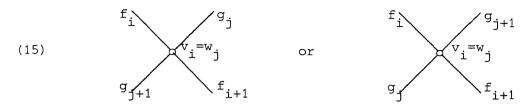
where $v_0,\ldots,v_s$ and $w_0,\ldots,w_t$ are vertices of G, with $v_s=v_0$ and $w_t=w_0$, f_i is an edge of G connecting v_{i-1} and v_i (i=1,...,s) and g_i is an edge of G connecting w_{i-1} and w_i (i=1,...,t), so that $f_i \neq f_{i-1}$ for i=1,...,s and $g_i \neq g_{i-1}$ for i=1,...,t (taking indices of v and f mod s, and indices of w and g mod t).

Let λ be the number of pairs $(i,j) \in \{1,\ldots,s\} \times \{1,\ldots,t\}$ so that:

(14) $v_i = w_j \in W$, f_i and f_{i+1} are opposite in v_i, g_j and g_{j+1} are opposite in w_j, while $\{f_i, f_{i+1}\} \neq \{g_j, g_{j+1}\}$.

So (14) corresponds to:

(15)

or

Let μ be the number of pairs $(i,j) \in \{1,\ldots,s\} \times \{1,\ldots,t\}$ so that:

(16) $v_i = w_j \in W$, $f_{i+1} = g_{j+1}$ and $f_i \neq g_j$.

So μ is also equal to the number of pairs $(i,j) \in \{1,\ldots,s\} \times \{1,\ldots,t\}$ so that:

(17) $v_i = w_j \in W$, $f_i = g_j$ and $f_{i+1} \neq g_{j+1}$.

Similarly, let ν be the number of pairs $(i,j) \in \{1,\ldots,s\} \times \{1,\ldots,t\}$ so that:

(18) $v_i = w_j \in W$, $f_{i+1} = g_j$ and $f_i \neq g_{j+1}$.
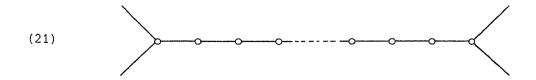
Again, ν is also equal to the number of pairs $(i,j) \in \{1,\ldots,s\} \times \{1,\ldots,t\}$ so that:

(19) $v_i = w_j \in W$, $f_i = g_{j+1}$ and $f_{i+1} \neq g_j$.

Note that the right-hand side of (12) is equal to $\lambda + \mu + \nu$. To see that $mincr(C,D) \leq \lambda + \mu + \nu$, note that μ is equal to the number of pairs $(i,j) \in \{1,\ldots,s\} \times \{1,\ldots,t\}$ so that there exists a number $b \geq 1$ with:

(20) $f_i \neq g_j, v_i = w_j, f_{i+1} = g_{j+1}, v_{i+1} = w_{j+1}, f_{i+2} = g_{j+2}, \ldots, v_{i+b} = w_{j+b}, f_{i+b+1} \neq g_{j+b+1}$,

which corresponds to pictures of type

(21)

Similarly, ν is equal to the number of pairs $(i,j) \in \{1,\ldots,s\} \times \{1,\ldots,t\}$ so that there exists a number $b \geq 1$ with:

(22) $\quad f_i \neq g_{j+1}, v_i = w_j, f_{i+1} = g_j, v_{i+1} = w_{j-1}, f_{i+2} = g_{j-1}, \ldots, v_{i+b} = w_{j-b}, f_{i+b+1} \neq g_{j-b}.$

Again, this corresponds to a picture of type (21).

Since each of the intersections of type (21) can be replaced by parts that have one crossing or none at all, we obtain $\mathrm{mincr}(C,D) \leq \lambda + \mu + \nu.$ □

Next we study the pattern of the C_{ij} at one fixed vertex v of degree 4. Again, let the neighbourhood of v be as in (9), and denote for $g, h \in \{1,2,3,4\}$:

(23) $\quad \alpha_{gh}^v := \alpha_{gh} := \sum\limits_{i=1}^{k} \sum\limits_{j=1}^{t_i} \lambda_{ij} \alpha_{gh}^v (C_{ij}).$

Then:

LEMMA B. *For each fixed vertex v of degree 4:*

(24) $\quad 2\alpha_{13}\alpha_{24} + \alpha_{12}\alpha_{13} + \alpha_{12}\alpha_{14} + \alpha_{12}\alpha_{23} + \alpha_{12}\alpha_{24} + \alpha_{13}\alpha_{14} + \alpha_{13}\alpha_{23} + \alpha_{13}\alpha_{34} +$
$\quad + \alpha_{14}\alpha_{24} + \alpha_{14}\alpha_{34} + \alpha_{23}\alpha_{24} + \alpha_{23}\alpha_{34} + \alpha_{24}\alpha_{34} \leq 2,$

with equality only if $\alpha_{13} = \alpha_{24} = 1$ *and* $\alpha_{12} = \alpha_{23} = \alpha_{34} = \alpha_{14} = 0.$

PROOF OF LEMMA B. The left-hand side of (24) is not larger than the first expression in the following series of inequalities (as the latter is obtained by adding $2\alpha_{12}\alpha_{34} + 2\alpha_{14}\alpha_{23}$):

(25) $\quad 2\alpha_{13}\alpha_{24} + 2\alpha_{12}\alpha_{34} + 2\alpha_{14}\alpha_{23} + \alpha_{12}\alpha_{13} + \alpha_{12}\alpha_{14} + \alpha_{12}\alpha_{23} + \alpha_{12}\alpha_{24} + \alpha_{13}\alpha_{14} + \alpha_{13}\alpha_{23} +$
$\quad + \alpha_{13}\alpha_{34} + \alpha_{14}\alpha_{24} + \alpha_{14}\alpha_{34} + \alpha_{23}\alpha_{24} + \alpha_{23}\alpha_{34} + \alpha_{24}\alpha_{34} =$

$\quad = \frac{1}{2}[\alpha_{12}(\alpha_{13} + \alpha_{14} + \alpha_{23} + \alpha_{24} + 2\alpha_{34}) + \alpha_{13}(\alpha_{12} + \alpha_{14} + \alpha_{23} + \alpha_{34} + 2\alpha_{24}) +$
$\quad\quad + \alpha_{14}(\alpha_{12} + \alpha_{13} + \alpha_{24} + \alpha_{34} + 2\alpha_{23}) + \alpha_{23}(\alpha_{12} + \alpha_{24} + \alpha_{13} + \alpha_{34} + 2\alpha_{14}) +$
$\quad\quad + \alpha_{24}(\alpha_{12} + \alpha_{23} + \alpha_{14} + \alpha_{34} + 2\alpha_{13}) + \alpha_{34}(\alpha_{13} + \alpha_{23} + \alpha_{14} + \alpha_{24} + 2\alpha_{12})] =$

$\quad = \frac{1}{2}[\alpha_{12}(\delta_3 + \delta_4) + \alpha_{13}(\delta_2 + \delta_4) + \alpha_{14}(\delta_2 + \delta_3) + \alpha_{23}(\delta_1 + \delta_4) + \alpha_{24}(\delta_1 + \delta_3) + \alpha_{34}(\delta_1 + \delta_2)],$

where, for $g \in \{1,2,3,4\}$:

(26) $\quad \delta_g := \sum\limits_{i=1}^{k} \sum\limits_{j=1}^{t_i} \lambda_{ij} \chi^{C_{ij}}(e_g).$

So by (6)(iii), $\delta_g \leq 1$ for each $g \in \{1,2,3,4\}$. Moreover:

(27) $\quad \delta_1 = \alpha_{12} + \alpha_{13} + \alpha_{14}, \quad \delta_2 = \alpha_{12} + \alpha_{23} + \alpha_{24},$
$\quad \delta_3 = \alpha_{13} + \alpha_{23} + \alpha_{34}, \quad \delta_4 = \alpha_{14} + \alpha_{24} + \alpha_{34}.$

Hence the last expression in (25) is not larger than the first expression in:

(28) $\qquad \alpha_{12} + \alpha_{13} + \alpha_{14} + \alpha_{23} + \alpha_{24} + \alpha_{34} = \frac{1}{2}(\delta_1 + \delta_2 + \delta_3 + \delta_4) \leq 2.$

This proves inequality (24). In order to have equality we should have:

(29) $\qquad \alpha_{12}\alpha_{34} = 0, \quad \alpha_{14}\alpha_{23} = 0, \quad \delta_1 = \delta_2 = \delta_3 = \delta_4 = 1.$

Now (27) and (29) imply:

(30) $\qquad \alpha_{12} = \frac{1}{2}(\delta_1 + \delta_2 - \delta_3 - \delta_4) + \alpha_{34} = \alpha_{34},$

$\qquad \alpha_{14} = \frac{1}{2}(\delta_1 + \delta_4 - \delta_2 - \delta_3) + \alpha_{23} = \alpha_{23}.$

Hence $\alpha_{12} = \alpha_{34} = \alpha_{14} = \alpha_{23} = 0$ and $\alpha_{13} = \alpha_{24} = 1.$ $\qquad \square$

From Lemmas A and B we derive:

(31) $\displaystyle \sum_{i=1}^{k} \sum_{i'=1}^{k} \text{mincr}(C_i, C_{i'}) = \sum_{i=1}^{k} \sum_{j=1}^{t_i} \sum_{i'=1}^{k} \sum_{j'=1}^{t_{i'}} \lambda_{ij}\lambda_{i'j'} \cdot \text{mincr}(C_{ij}, C_{i'j'}) \leq$

$\displaystyle \sum_{i=1}^{k} \sum_{j=1}^{t_i} \sum_{i'=1}^{k} \sum_{j'=1}^{t_{i'}} \lambda_{ij}\lambda_{i'j'} \sum_{v \in W} [\alpha_{13}^{v}(C_{ij})\alpha_{24}^{v}(C_{i'j'}) + \alpha_{24}^{v}(C_{ij})\alpha_{13}^{v}(C_{i'j'}) +$

$+ \frac{1}{2}[(\alpha_{12}^{v}(C_{ij}) + \alpha_{34}^{v}(C_{ij}))(\alpha_{13}^{v}(C_{i'j'}) + \alpha_{14}^{v}(C_{i'j'}) + \alpha_{23}^{v}(C_{i'j'}) + \alpha_{24}^{v}(C_{i'j'})) +$

$+ (\alpha_{13}^{v}(C_{ij}) + \alpha_{24}^{v}(C_{ij}))(\alpha_{12}^{v}(C_{i'j'}) + \alpha_{14}^{v}(C_{i'j'}) + \alpha_{23}^{v}(C_{i'j'}) + \alpha_{34}^{v}(C_{i'j'})) +$

$+ (\alpha_{14}^{v}(C_{ij}) + \alpha_{23}^{v}(C_{ij}))(\alpha_{12}^{v}(C_{i'j'}) + \alpha_{13}^{v}(C_{i'j'}) + \alpha_{24}^{v}(C_{i'j'}) + \alpha_{34}^{v}(C_{i'j'}))]] =$

$\displaystyle \sum_{v \in W} [\alpha_{13}^{v}\alpha_{24}^{v} + \alpha_{24}^{v}\alpha_{13}^{v} + \frac{1}{2}[(\alpha_{12}^{v} + \alpha_{34}^{v})(\alpha_{13}^{v} + \alpha_{14}^{v} + \alpha_{23}^{v} + \alpha_{24}^{v}) +$

$+ (\alpha_{13}^{v} + \alpha_{24}^{v})(\alpha_{12}^{v} + \alpha_{14}^{v} + \alpha_{23}^{v} + \alpha_{34}^{v}) + (\alpha_{14}^{v} + \alpha_{23}^{v})(\alpha_{12}^{v} + \alpha_{13}^{v} + \alpha_{24}^{v} + \alpha_{34}^{v})]] =$

$\displaystyle \sum_{v \in W} (2\alpha_{13}^{v}\alpha_{24}^{v} + \alpha_{12}^{v}\alpha_{13}^{v} + \alpha_{12}^{v}\alpha_{14}^{v} + \alpha_{12}^{v}\alpha_{23}^{v} + \alpha_{12}^{v}\alpha_{24}^{v} + \alpha_{13}^{v}\alpha_{14}^{v} + \alpha_{13}^{v}\alpha_{23}^{v} + \alpha_{13}^{v}\alpha_{34}^{v} +$

$+ \alpha_{14}^{v}\alpha_{24}^{v} + \alpha_{14}^{v}\alpha_{34}^{v} + \alpha_{23}^{v}\alpha_{24}^{v} + \alpha_{23}^{v}\alpha_{24}^{v} + \alpha_{23}^{v}\alpha_{34}^{v} + \alpha_{24}^{v}\alpha_{34}^{v}) \leq$

$\displaystyle 2|W| = \sum_{i=1}^{k'} \sum_{i'=1}^{k'} \text{mincr}(C_i', C_{i'}').$

By our assumption (3), we should have equality throughout in (31). Hence by Lemma B, for each vertex v in W:

(32) $\qquad \alpha_{13}^{V} = \alpha_{24}^{V} = 1, \quad \alpha_{12}^{V} = \alpha_{23}^{V} = \alpha_{34}^{V} = \alpha_{14}^{V} = 0.$

So (10) holds, and hence we have (i) in our theorem. $\qquad$ ⊔⊓

A linear algebraic formulation.

Our theorem above can be formulated equivalently as the nonsingularity of a certain infinite symmetric matrix. Let C be the family of equivalence classes of closed curves on S, with respect to the equivalence relation ∼. For $\Gamma, \Delta \in C$ we define

(33) $\qquad \text{mincr}(\Gamma, \Delta) := \text{mincr}(C, D)$

for (arbitrary) $C \in \Gamma$ and $D \in \Delta$. So mincr is considered also as a function from $C \times C$ to $\mathbb{Z}_{+}$ (= set of nonnegative integers). We can represent this function as an infinite symmetric matrix M, with both rows and columns indexed by C.

The rows of matrix M are not linearly independent. First of all, the row corresponding to the trivial class <0> is all-zero (where 0 is a homotopically trivial closed curve, and where <..> denotes the equivalence class containing ..). Moreover, the rows corresponding to <C> and to <C^{-1}> are the same, since $\text{mincr}(C^{-1}, D) = \text{mincr}(C, D)$ for each closed curve D. More generally, it is shown in [2] that for each pair of closed curves C,D on S and each $n \in \mathbb{Z}$:

(34) $\qquad \text{mincr}(C^{n}, D) = |n| \cdot \text{mincr}(C, D).$

So the row corresponding to <C^{n}> is a multiple of the row corresponding to <C>.

Now the theorem above actually says that (34) generate *all* linear dependencies of rows of M. To explain this, we mention the following result of [2]:

(35) $\qquad$ for each homotopically nontrivial closed curve C on S there exists

$\qquad$ a primitive closed curve D on S and an integer $n \geq 1$ so that $C \sim D^{n}$.

$\qquad$ Integer n and closed curve D are unique (up to homotopy).

Let $C_{p} \subseteq \{<C> \mid C \text{ primitive closed curve}\}$ so that for each primitive closed curve C exactly one of <C> and <C^{-1}> belongs to C_{p}, which we denote by [C]. Let M' be the $C_{p} \times C_{p}$ submatrix of M. Then the following is equivalent to our theorem above:

EQUIVALENT FORM OF THE THEOREM. *The matrix* M' *is nonsingular, i.e., the rows of* M' *are linearly independent.*

<u>PROOF OF EQUIVALENCE</u>. I. We first derive the equivalent form from the theorem.

Suppose M' has linearly dependent rows. That is, there are distinct $\Gamma_1, \ldots, \Gamma_t$
$\in C_p$ and $\lambda_1, \ldots, \lambda_t \in \mathbb{R} \setminus \{0\}$ (with $t \geq 1$), so that for each $\Delta \in C_p$:

$$(36) \qquad \sum_{i=1}^{t} \lambda_i \cdot \text{mincr}(\Gamma_i, \Delta) = 0.$$

Since M' is an integer matrix, we may assume that the λ_i are rational, and hence

integer. By repeating each Γ_i $|\lambda_i|$ times, we obtain $\Gamma'_1, \ldots, \Gamma'_{t'}$ and $\Gamma''_1, \ldots, \Gamma''_{t''}$

in C_p (with $t'+t'' \geq 1$), so that for each $\Delta \in C_p$:

$$(37) \qquad \sum_{i=1}^{t'} \text{mincr}(\Gamma'_i, \Delta) = \sum_{i=1}^{t''} \text{mincr}(\Gamma''_i, \Delta),$$

and so that $\{\Gamma'_1, \ldots, \Gamma'_{t'}\} \cap \{\Gamma''_1, \ldots, \Gamma''_{t''}\} = \emptyset$. Now (34) and (35) imply that (37)

in fact holds for every $\Delta \in C$. But then our theorem gives that $t'=t''$, and there

exists a permutation π of $\{1, \ldots, t'\}$ so that $\Gamma'_i = \Gamma''_{\pi(i)}$ for each $i=1, \ldots, t'$.

This is a contradiction.

II. To see the reverse implication, note that condition (ii) of the Theorem

implies that for each $\Delta \in C_p$:

$$(38) \qquad \sum_{i=1}^{k} \text{mincr}([c_i], \Delta) = \sum_{i=1}^{k'} \text{mincr}([c'_i], \Delta).$$

Since by Theorem' the rows of M' are linearly independent, we must have $k=k'$

and $[c_i] = [c'_{\pi(i)}]$ for each $i=1, \ldots, k$, for some permutation π of $\{1, \ldots, k\}$. $\Box$

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