Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

B. Koren

Robustness improvement of point Gauss-Seidel relaxation for steady, hypersonic flow computations

Department of Numerical Mathematics Note NM-N8805 October

The Centre for Mathematics and Computer Science is a research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a nonprofit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

Robustness Improvement of Point Gauss-Seidel Relaxation for Steady, Hypersonic Flow Computations

Barry Koren
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

In hypersonic flow computations, Newton iteration in a local relaxation procedure may easily fail. A remedy is proposed for overcoming this problem. The remedy consists of a switch to local, explicit time stepping in case of Newton's failure. Promising results are shown for a hypersonic flow computation around a blunt body, using the steady, 2D Euler equations.

1980 Mathematics Subject Classifications: 65N10, 65N20, 76K05.
Key Words & Phrases: relaxation method, explicit time stepping method, steady Euler equations, hypersonic flows.
Note: This work was supported by the European Space Agency (ESA), via Avions Marcel Dassault - Bréguet Aviation (AMD-BA).

1. Introduction

1.1. *Governing equations*

The flow equations considered are the perfect gas, steady, 2D Navier-Stokes equations with Re very large ($Re = 10^{100}$). So, in fact, the equations considered are the Euler equations

$$\frac{\partial}{\partial x}\begin{pmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ \rho u(e+p/\rho)\end{pmatrix}+\frac{\partial}{\partial y}\begin{pmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ \rho v(e+p/\rho)\end{pmatrix}=0, \tag{1.1}$$

with

$$e=\frac{1}{\gamma-1}\frac{p}{\rho}+\frac{1}{2}(u^2+v^2). \tag{1.2}$$

So far, real gas effects are not taken into account. Even the specific heat ratio γ of the di-atomic gas considered is assumed to be constant and determined by fully excited translational and rotational energies only. (Though it could be easily introduced as a function ranging from zero up to the full equipartition value, the vibrational energy is assumed to be zero.)

1.2. *Existing computational method*

For a detailed account of the existing computational method which is taken as a point of departure, we refer to [2,3]. Here we give an overview of its main characteristics only.

Discretized equations are obtained by subdividing the computational domain Ω into quadrilateral finite volumes $\Omega_{i,j}$, and by requiring that the conservation laws, (1.1) in integral form, hold for each finite volume separately. This discretization requires an evaluation of the convective flux vector at each volume wall. A proper evaluation of this vector is of great importance. For this, we prefer an upwind approach which follows the Godunov principle [1]. The 1D Riemann problem thus arising at each volume wall is solved in an approximate way by using Osher's scheme [5] in its so-called P-variant [2]. The approximation of the left and right state in the 1D Riemann problem determines the accuracy of the convective discretization. Here we consider first-order accuracy only, which is obtained, in the standard way, by taking the left and right state equal to that in the corresponding adjacent volumes.

For the solution of the nonlinear system of first-order accurate discretized equations, collective symmetric point Gauss-Seidel relaxation is used. In this relaxation, one or more (exact) Newton steps are used for the collective update of the four state vector components in each finite volume. Given the good smoothing properties of symmetric point Gauss-Seidel relaxation applied to first-order upwind discretizations, multigrid techniques may be applied for accelerating the solution process.

Note NM-N8805
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

2. Extension to hypersonics

2.1. Discretization method

The existing upwind finite volume discretization is maintained. The only modification to be introduced in the existing discretization method consists of a protection of Osher's boundary condition treatment at solid, impermeable walls [2,4] against an extrapolation leading to an unphysical expansion beyond vacuum; cavitation. For the situation with a solid wall boundary at the left e.g., Osher's scheme yields

$$(u_B, v_B, c_B, z_B)^T = (0, v, c - \frac{\gamma-1}{2}u, z)^T, \tag{2.1}$$

where $(u_B, v_B, c_B, z_B)^T$ is the unknown boundary state, and $(u,v,c,z)^T$ the known inner state. In a qualitatively correct way, (2.1) leads to a compression towards the wall for $u<0$, and vice versa an expansion for $u>0$. However, for $\gamma=7/5$ and $u/c>5$ e.g., a situation that occurs at the beginning of the solution process when considering a di-atomic gas and a uniform, hypersonic initial solution, (2.1) yields: $c_B<0$. As a safeguard against cavitation we introduce a switch from (2.1) to the less sophisticated but safer mirror principle

$$(u_B, v_B, c_B, z_B)^T = (-u, v, c, z)^T, \tag{2.2}$$

if $u/c>2/(\gamma-1)$. Notice that in general this will not change the solution because the switch is not made if $(u,v,c,z)^T$ is sufficiently accurate.

2.2. Solution method

In the present paper we do *not yet* consider the possibility of accelerating hypersonic flow computations by multigrid techniques. Here, we restrict ourselves to the relaxation method only, and more in particular to its robustness.

Single-grid, hypersonic blunt body flow computations with the revised discretization method, though with the non-revised solution method, break down in the very first visit to the stagnation domain.

Starting with a poor initial solution, one may gain in robustness by introducing a continuation process preceding to the nested iteration. In such a process, usually a single upstream boundary condition, for instance M_∞, is increased from some low initial value to its correct high value, while performing relaxation sweeps. Continuation processes like this require a tuning of both the initial value and the increment. For hypersonic flow problems, proper tuning is difficult, given the fact that in these flows the condition number of the derivative matrices used may be quite large. (The larger the condition numbers, the larger are the perturbations in the iterands induced by perturbations in the righthand sides; righthand side perturbations which may already be quite large by themselves in hypersonic flow computations.) The ill-conditionedness occurring in hypersonic flow computations can be illustrated for the derivative matrix $\nabla(\rho u, \rho u^2+p, \rho uv, \rho u(e+p/\rho))$, where $\nabla=(\partial/\partial u, \partial/\partial v, \partial/\partial c, \partial/\partial z)^T$, the differential operator applied in our solution method, and where $c=\sqrt{\gamma p/\rho}$, and $z=\ln(p\rho^{-\gamma})$. Considering for simplicity $v=0$ and $p=1$, it clearly appears from Fig. 2.1 that the condition of $\nabla(\rho u, \rho u^2+p, \rho uv, \rho u(e+p/\rho))$ becomes worse for $u/c\gtrsim 1.5$. Notice further that the condition becomes worse also for increasing c with u close to zero. The latter indicates that stagnation flows become harder to relax with increasing upstream Mach number.

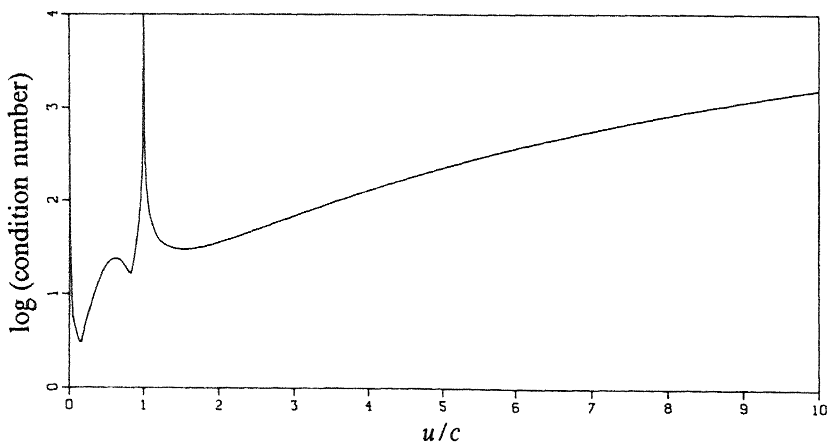


Fig. 2.1. Condition of typical derivative matrix to be inverted.

In all our (final) algorithms we want to avoid any tuning. Therefore, as a remedy for failure of the Newton process in the point relaxation method, we propose a switched-relaxation-evolution technique. In this technique, we simply start applying the existing relaxation method, and take measures only as soon as the local Newton iteration fails. To discuss these measures for robustness improvement, we consider the local, first-order discrete system

$$\begin{aligned}\mathcal{F}_{i,j}(q_{i,j}) \equiv\ & T^{-1}(\phi_{i+½,j})F(T(\phi_{i+½,j})q_{i,j},\ T(\phi_{i+½,j})q_{i+1,j})l_{i+½,j} - \\ & T^{-1}(\phi_{i-½,j})F(T(\phi_{i-½,j})q_{i-1,j},\ T(\phi_{i-½,j})q_{i,j})l_{i-½,j} + \\ & T^{-1}(\phi_{i,j+½})F(T(\phi_{i,j+½})q_{i,j},\ T(\phi_{i,j+½})q_{i,j+1})l_{i,j+½} - \\ & T^{-1}(\phi_{i,j-½})F(T(\phi_{i,j-½})q_{i,j-1},\ T(\phi_{i,j-½})q_{i,j})l_{i,j-½} = 0,\end{aligned} \tag{2.3}$$

where $T(\phi)$ denotes the matrix for rotation to a local coordinate system, $F(q^l, q^r)$ the numerical flux function and l the length of a finite volume wall. (For details we refer to [2,3].)

2.2.1. Failing Newton iteration. As a non-failing Newton iteration to solve $q_{i,j}$ from (2.3) we define: a Newton iteration for which: (i)

$$\frac{|\mathcal{F}^{(k)}_{i,j}(q^{n+1}_{i,j})|}{|\mathcal{F}^{(k)}_{i,j}(q^{n}_{i,j})|} \leqslant 1, \quad k=1,2,3,4, \quad \forall_{i,j}, \tag{2.4}$$

for any n-th Newton iterand ($n=0,1,2,\cdots,N$) and each of the four residual components, and for which (ii) each iterand $q^{n+1}_{i,j}$ is physically correct, with physical correctness defined in the following way. Considering the local solution vector $q_{i,j}=(u_{i,j}, v_{i,j}, c_{i,j}, z_{i,j})^T$ and the corresponding hypersonic, upstream state vector $q_\infty=(u_\infty, v_\infty, c_\infty, z_\infty)^T$, we know that the flow speed may not exceed the value corresponding with adiabatic expansion to vacuum, departing from upstream conditions:

$$u^2_{i,j}+v^2_{i,j} \leqslant u^2_\infty+v^2_\infty+\frac{2}{\gamma-1}c^2_\infty, \quad \forall_{i,j}. \tag{2.5}$$

Further, we know that after this expansion, the speed of sound equals zero, its minimally allowable value:

$$c_{i,j} \geqslant 0, \quad \forall_{i,j}. \tag{2.6}$$

The maximally allowable value of the speed of sound is that corresponding with the stagnation temperature (which is the same for both isentropic and non-isentropic compression). For adiabatic flows we can write:

$$c_{i,j} \leqslant \sqrt{c^2_\infty+\frac{\gamma-1}{2}(u^2_\infty+v^2_\infty)}, \quad \forall_{i,j}. \tag{2.7}$$

For $z_{i,j}$ we can directly write with the entropy condition:

$$z_{i,j} \geqslant z_\infty, \quad \forall_{i,j}. \tag{2.8}$$

For the upper limit of $z_{i,j}$ we have to consider the state q_2 at the downstream side of a normal shock wave which has at its upstream side a state q_1 which has expanded to vacuum, departing from upstream conditions. Given the gasdynamical relations

$$p_2 = \frac{2\gamma M_1^2-(\gamma-1)}{\gamma+1}p_1, \tag{2.9a}$$

$$\rho_2 = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2}\rho_1, \tag{2.9b}$$

$$p_1\rho_1^{-\gamma} = p_\infty\rho_\infty^{-\gamma}, \tag{2.9c}$$

it is clear that

$$z_{i,j} \leqslant \lim_{M_1\to\infty}\ln(p_2\rho_2^{-\gamma}) = \infty, \quad \forall_{i,j}. \tag{2.10}$$

Summarizing, we see that in adiabatic flows both the flow speed and the speed of sound have a physical lower and upper limit. The entropy only has a lower limit.

In the algorithm, (2.4)-(2.8) are checked after each update in the Newton iteration. As soon as one or more of these five requirements are not satisfied, the Newton iteration is said to have failed, and any correction made is rejected.

2.2.2. Evolution technique. As the alternative for a failing Newton iteration, we apply now one or eventually two explicit time stepping schemes to the local, semi-discrete system

$$\frac{\partial q_{i,j}}{\partial t}+\frac{1}{A_{i,j}}\mathcal{F}_{i,j}(q_{i,j}) = 0. \tag{2.11}$$

where $A_{i,j}$ denotes the area of finite volume $\Omega_{i,j}$. As time stepping scheme to be applied first, we take the following version of Wambecq's explicit two-step rational Runge-Kutta scheme [6]

$$q_{i,j}^{n+1} = q_{i,j}^{n}-\frac{\tau_{i,j}}{A_{i,j}}\frac{\mathcal{F}_{i,j}^{2}(q_{i,j}^{n})}{2\mathcal{F}_{i,j}(q_{i,j}^{n})-\mathcal{F}_{i,j}(q_{i,j}^{n}+\frac{1}{2}\tau_{i,j}\mathcal{F}_{i,j}(q_{i,j}^{n}))},\quad n=0,1,2,\cdots,N. \tag{2.12}$$

In here, $\tau_{i,j}$ denotes the local time step, for which, to start with, we safely take the one which is maximally allowed for the forward Euler scheme:

$$\tau_{i,j}=\frac{A_{i,j}h_{i,j}}{\sup\left\{\frac{d\mathcal{F}_{i,j}(q_{i,j})}{dq_{i,j}}\right\}}, \tag{2.13}$$

with $h_{i,j}$ a characteristic local mesh size. With our upwind discretization, (2.13) may be rewritten by good approximation as

$$\tau_{i,j}=\frac{h_{i,j}}{\sqrt{u_{i,j}^2+v_{i,j}^2}+c_{i,j}}. \tag{2.14}$$

For the evaluation of the denominator in (2.12) we use the Samelson inverse of a vector:

$$\mathcal{G}^{-1} \equiv |\mathcal{G}|^{-2}\,\mathcal{G}, \tag{2.15}$$

$\mathcal{G}$ being a vector, whereas for the norm of a vector, we simply use the Cartesian inner product. As initial solution we take the same $q_{i,j}^0$ that just failed for the Newton iteration. The motivation for applying Wambecq's scheme is its good stability as demonstrated in [6] for a stiff and coupled system of four equations, which is precisely what we have here in hypersonics. However, a potential danger of (2.12) is that there is no guarantee for the denominator to be non-zero.

To protect Wambecq's scheme against a possibly too large time step, and against a (nearly) zero denominator, in each time step we require both the predictor and corrector to satisfy (2.5)-(2.8). As soon as a physically unrealistic value occurs, the time stepping is stopped immediately, rejecting any update made. Then, at first we assume that the unphysical result is due to a too large time step. Therefore, as a remedy, we halve $\tau_{i,j}$ and restart the time stepping with Wambecq's scheme, using the same $q_{i,j}^0$. In case of re-occurrence of something unphysical, we assume that the denominator was the problem. Therefore, as a new remedy, we restart with an explicit time stepping scheme which is safe in this sense; the simple forward Euler scheme

$$q_{i,j}^{n+1} = q_{i,j}^{n}-\frac{\tau_{i,j}}{A_{i,j}}\mathcal{F}_{i,j}(q_{i,j}^{n}),\quad n=0,1,2,\cdots,N. \tag{2.16}$$

As $\tau_{i,j}$ we apply the one which was latest used with Wambecq's scheme, and as $q_{i,j}^0$ still the same as before. When a physically unrealistic value (according to (2.5)-(2.8)) occurs again, $\tau_{i,j}$ is halved for the second time and the time stepping with forward Euler is restarted, still using the same $q_{i,j}^0$. In case of something unphysical once more, the time stepping is stopped and the finite volume visited is quit without any update being made. (Notice that for both time stepping schemes, we do not require (2.4) to be satisfied.)

With the present switched-relaxation-evolution approach we expect that in those volumes where Newton fails, the local evolution technique will finally bring the solution into the attraction domain of Newton (for the next sweep), and so make itself superfluous at the end of the solution process.

3. NUMERICAL RESULTS

3.1. *Flow problem*

As test case we consider a hypersonic flow around a blunt forebody with canopy. The forebody is composed out of two ellipse segments (Fig. 3.1), given by

$$\left.\begin{aligned}\left[\frac{x}{0.06}\right]^2+\left[\frac{y}{0.015}\right]^2&=1\\ \left[\frac{x}{0.035}\right]^2+\left[\frac{y}{0.025}\right]^2&=1\end{aligned}\right\}\ x<0, \tag{3.1a}$$

and a parallel part, given by

$$\left.\begin{aligned}y&=-0.015\\ y&=0.025\end{aligned}\right\}\ 0\leqslant x\leqslant 0.016. \tag{3.1b}$$

The upstream flow conditions are: $M_\infty=8.15$, $\alpha=30°$, α denoting the angle of attack. (So the flight situation considered is a reentry situation.)

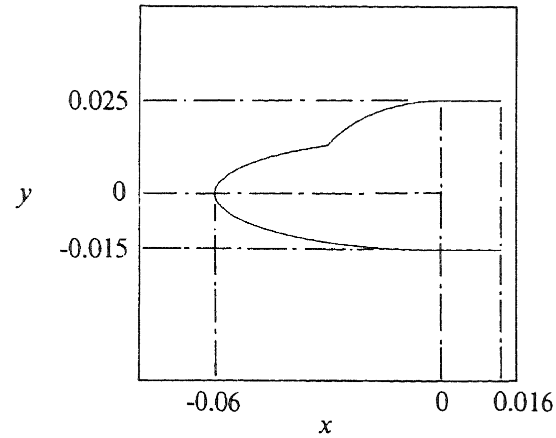
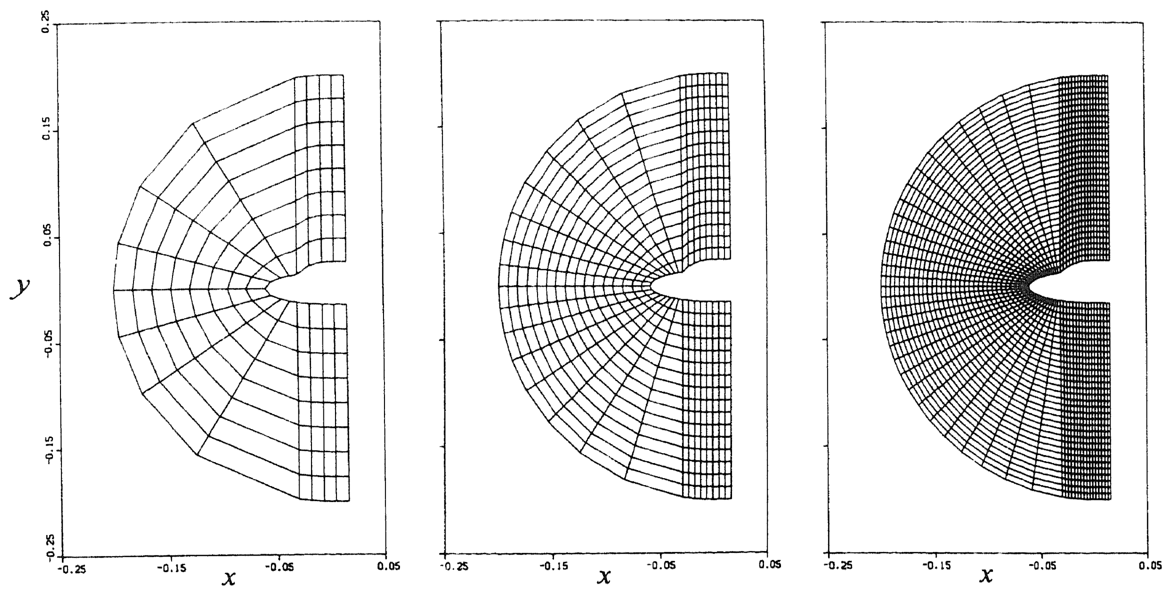


Fig. 3.1. Double ellipse.

As grids we use the C-type grids shown in Fig. 3.2. The grids are exactly equidistant in radial direction per radial column separately, and nearly equidistant in tangential direction, at the body.



a. 16×8-grid.

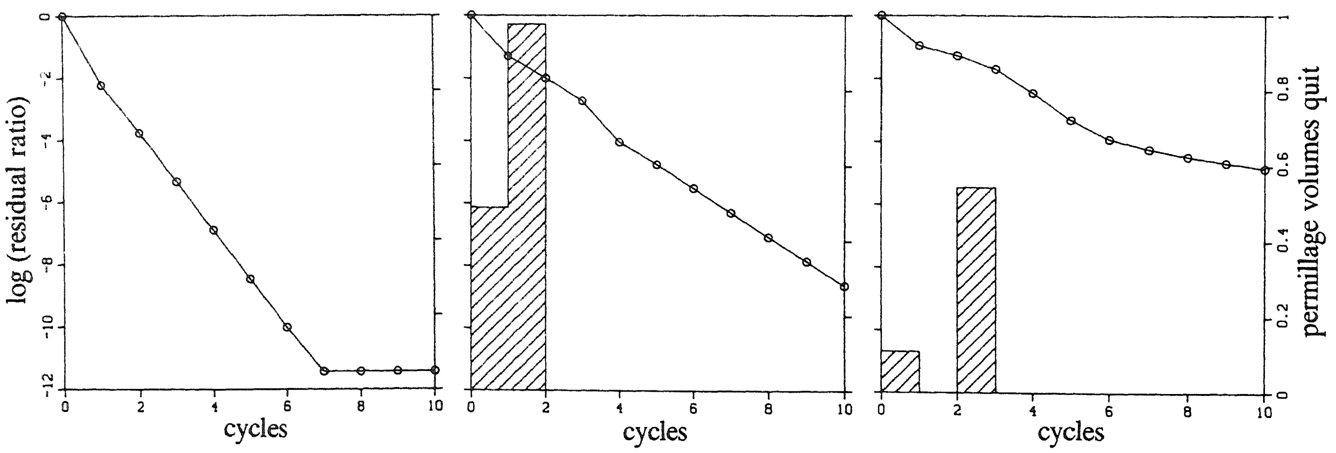
b. 32×16-grid.

c. 64×32-grid.

Fig. 3.2. Grids double ellipse.

In Fig. 3.3, we show for the 16×8-, 32×16-, and 64×32-grid, respectively, the behaviour of the switched-relaxation-evolution technique. The residual ratio along the left vertical axes is the ratio $\Sigma_{i=1}^{4}|F_h(q_h^n)|_i/\Sigma_{i=1}^{4}|F_h(q_h^0)|_i$, where $|F_h(q_h)|$ denotes the summation over all volumes of the absolute value of the discrete Euler defect, i the i-th defect component and q_h^n the solution after the n-th cycle. The solution q_h^0 is the uniformly constant initial solution. The quantity along the right vertical axes is the permillage of quit volumes in the total number of finite volumes visited during one cycle (one cycle being defined as two diagonally opposite, symmetric relaxation sweeps). In all three grid cases, the initial solution continuously fits to the hypersonic upstream boundary conditions. The robustness of the switched-relaxation-evolution technique is clear. For none of the cases considered there is an abortion of the solution process due to overflow or such. We even have convergence for all three grid cases. Further, from Fig. 3.3b and 3.3c, it appears that the evolution technique makes itself superfluous indeed.

Notice that the convergence slow down with decreasing mesh size is expected for a plain relaxation method and is supposed to be repaired by application of a suitable multigrid technique.



a. 16×8-grid. b. 32×16-grid. c. 64×32-grid.

Fig. 3.3. Convergence results switched-relaxation-evolution technique.

4. Conclusions

The essential element for robustness is the continuous checking of both the local relaxation and the local evolution. The essential element for convergence is the triple-combination of Newton iteration, Wambecq's explicit, two-step rational Runge-Kutta scheme and the explicit Euler scheme.

For steady Navier-Stokes flow computations at a finite Reynolds number, the proposed checks on physical correctness can be maintained as long as the flow remains adiabatic. Only the time step needs to be reconsidered for diffusion.

Convergence acceleration is expected to be ready to hand by incorporation of multigrid.

Acknowledgement

The author wants to thank P.W. Hemker and W.J. Bannink for their advice.

References

1. S.K. Godunov, *Math. Sbornik* **47** , 271 (1959).
2. P.W. Hemker, and S.P. Spekreijse, *Appl. Num. Math.* **2** , 475 (1986).
3. B. Koren, *Int. J. Numer. Methods in Fluids* (to appear, 1988).
4. S. Osher, and S. Chakravarthy, *J. Comput. Phys.* **50** , 447 (1983).
5. S. Osher, and F. Solomon, *Math. Comp.* **38** , 339 (1982).
6. A. Wambecq, *Computing* **20** , 333 (1978).