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Department of Applied Mathematics Report AM-R8810 September

Bibliotheek
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Amsterdam

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A Hille-Yosida type Theorem for a Class of Weakly ⋆ Continuous Semigroups

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A Hille-Yosida type theorem for a class of weakly ⋆ continuous semigroups on a dual Banach space is proved

1980 Mathematics Subject Classification: 47D05
Key Words & Phrases: weakly ⋆ continuous semigroup, infinitesimal weak ⋆ generator, integral w^*-semigroup, integral generator, Favard class, Hille-Yosida theorem, intertwining formula, generator with non-dense domain.

0. Introduction

In this paper we consider a class of weakly ⋆ continuous semigroups of bounded linear operators on the dual of a Banach space X which are not necessarily the adjoint of a C_0-semigroup on X. Such semigroups arise in a natural way as perturbations (in an appropriate sense) of adjoint C_0-semigroups: see Clement, Diekmann, Gyllenberg, Heijmans and Thieme (part I-IV). There the perturbed semigroup is constructed by exploiting a variation - of - constants formula and duality arguments.

We shall introduce the notion of integral weak ⋆ generator and use this to characterize the aforementioned class of weakly ⋆ continuous semigroups in a one-to-one manner.

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Finally, we refer to JEFFERIES (1986) for some related results.

1. FORMAL CALCULATIONS WITH w^*-SEMIGROUPS

A family $T^\times = \{T^\times(t):t\geqslant 0\}$ of bounded linear operators on a dual Banach space X^* such that

$$\text{(i)}\ T^\times(0) = I \tag{1.1}$$

$$\text{(ii)}\ T^\times(t+s) = T^\times(t)T^\times(s),\ \ t,s\geqslant 0$$

$$\text{(iii)}\ t\mapsto \langle x, T^\times(t)x^*\rangle \text{ is continuous for any given } x\in X \text{ and } x^*\in X^*$$

is called a weakly $*$ continuous semigroup or, in abbreviated form, a w^*-semigroup. The operator $A^\times$ defined by

$$A^\times x^* = w^*-\lim_{h\downarrow 0}\frac{1}{h}(T^\times(h)x^*-x^*) \tag{1.2}$$

with $D(A^\times) = \{x^*:w^*-\lim_{h\downarrow 0}\frac{1}{h}(T^\times(h)x^*-x^*) \text{ exists}\}$ is called the infinitesimal weak $*$ generator or, in abbreviated form, the w^*-generator.

The standard example of a w^*-semigroup is a dual semigroup, i.e.

$$T^\times(t) = T(t)^*$$

where $\{T(t)\}$ is a C_0-semigroup on X. In that case $A^\times = A^*$, where A is the infinitesimal generator of $T(t)$ and one can easily verify all the elegant and powerful relations between semigroup and generator which are familiar from C_0-semigroup theory, provided one replaces strong differentiation and integration by the corresponding weak* analogs (see BUTZER & BERENS, §1.4, 1967). In particular a dual semigroup is uniquely determined by its w^*-generator. It is tempting to conjecture that this situation extends to w^*-semigroups in general.

However, an easy counterexample can be constructed as follows. Consider the C_0-semigroup $T(t)$ of translations on $X = C_0(\mathbb{R})$, the space of continuous functions defined on $\mathbb{R}$ which vanish at infinity. So $(T(t)x)(a) = x(t+a)$ and the dual semigroup T^* on X^* is defined by

$$\langle x, T^*(t)x^*\rangle = \langle T(t)x, x^*\rangle = \int_{\mathbb{R}} x(t+a)x^*(da).$$

It is well known that $X^\odot := \overline{D(A^*)}$ is the maximal subspace of X^* on which $T^*(t)$ is strongly continuous in t. In this particular case $X^\odot$ is the subspace of measures which are Lebesgue absolutely continuous (so $X^\odot \simeq L_1(\mathbb{R})$) and one has the direct sum decomposition

$$X^* = X^\odot \oplus X^\perp$$

where $X^\perp$ denotes the subspace of measures which are singular with respect to the Lebesgue measure. We emphasize that both $X^\odot$ and $X^\perp$ are closed in X^* and invariant under $T^*(t)$. So for any $\alpha\in\mathbb{R}$ we can define a w^*-semigroup $T_\alpha^\times$ on X^* by

$$T_\alpha^\times(t)x^* = \begin{cases} T^*(t)x^* & \text{if } x^*\in X^\odot \\ T^*(\alpha t)x^* & \text{if } x^*\in X^\perp \end{cases} \tag{1.3}$$

Obviously the maximal subspace of strong continuity does not depend on α and on this space $X^\odot$ the action does not depend on α either. So all these semigroups do have the same w^*-generator!

How can one distinguish the 'bad' semigroups $T_\alpha^\times(t)$ with $\alpha\neq 1$ from the 'good' semigroup $T^*(t)$ in a direct way, without invoking duality? The requirement that the semigroup operators are the solution operators corresponding to the Cauchy problem

$$\frac{d^*}{dt}u(t) = A^\times u(t) \tag{1.4}$$

$$u(0) = x^*$$

is as such of not much help since in order to solve (1.4) one has to assume that $x^* \in D(A^\times)$ (and even that does not guarantee that a solution exists since $D(A^\times)$ is not necessarily invariant under $T^\times(t)$). However, if we integrate (1.4) formally we obtain

$$u(t)-x^* = A^\times \int_0^t u(\tau)d\tau \tag{1.5}$$

and it seems reasonable to require that this should hold for $u(t) = T^\times(t)x^*$ and *all* $x^* \in X^*$. But with $T_\alpha^\times(t)$ defined by (1.3) we find

$$T_\alpha^\times(t)x^* - x^* = \begin{cases} A^\times \int_0^t T_\alpha^\times(\tau)x^* d\tau & \text{for } x^* \in X^\odot \\ \alpha A^\times \int_0^t T_\alpha^\times(\tau)x^* d\tau & \text{for } x^* \in X^\perp \end{cases}$$

showing that the requirement is fulfilled iff $\alpha = 1$.

In order to rewrite the requirement in terms of semigroup operators only, we continue our *formal* calculations. If $x^* \in D(A^\times)$ we write

$$A^\times \int_0^t T^\times(\tau)x^* d\tau = \int_0^t T^\times(\tau)A^\times x^* d\tau \tag{1.6}$$

even though a justification cannot be given. If we now consider the identity

$$T^\times(t)x^* = x^* + A^\times \int_0^t T^\times(\tau)x^* d\tau$$

and take x^* of the special form

$$x^* = \int_0^h T^\times(\sigma)y^* d\sigma \in D(A^\times)$$

we obtain

$$\begin{aligned} T^\times(t)\int_0^h T^\times(\tau)y^* d\tau &= \int_0^h T^\times(\tau)y^* d\tau + \int_0^t T^\times(\tau)A^\times \int_0^h T^\times(\sigma)y^* d\sigma d\tau \\ &= \int_0^h T^\times(\tau)y^* d\tau + \int_0^t T^\times(\tau)\{T^\times(h)y^* - y^*\}d\tau \\ &= \int_0^h T^\times(t+\sigma)y^* d\sigma. \end{aligned}$$

This formal calculation motivates the introduction of property

(S1) $\quad T^\times(t) \int_0^h T^\times(\tau)x^* d\tau = \int_0^h T^\times(t+\tau)x^* d\tau, \; x^* \in X^*, \; t,h \geq 0.$

We will call w^*-semigroups with property (S1) 'integral w^*-semigroups'. A straightforward calculation shows that $T_\alpha^\times$ defined by (1.3) is an integral w^*-semigroup iff $\alpha = 1$.

Remark. Define

$$S^\times(t)x^* = \int_0^t T^\times(\tau)x^* d\tau$$

then $\{S^\times(t)\}$ is an 'integrated semigroup' in the sense of ARENDT (to appear), KELLERMANN and HIEBER (to appear) and NEUBRANDER (to appear) iff $\{T^\times(t)\}$ is an integral w^*-semigroup.

Up to now we are neither able to prove that (1.6) holds for all integral w^*-semigroups nor to find a counter example within this class. So we are led to introduce the following concept of a generator:

DEFINITION 1.1. $x^* \in D(A_0^\times)$ and $y^* = A_0^\times x^*$ iff

$$T^\times(t)x^* - x^* = \int_0^t T^\times(\tau)y^* d\tau, \quad \forall t \geqslant 0. \tag{1.7}$$

Note that, for $x^* \in D(A_0^\times)$, y^* is uniquely determined by (1.7). We will call $A_0^\times$ the *integral generator* of $T^\times$. Observe that (1.7) is equivalent to

$$\frac{d^*}{dt} T^\times(t)x^* = T^\times(t)y^*, \quad t \geqslant 0 \tag{1.8}$$

and that automatically $D(A_0^\times)$ is invariant under $T^\times(t)$ and $A_0^\times T^\times(t)x^* = T^\times(t)A_0^\times x^*$. Obviously $A^\times$ is an extension of $A_0^\times$.

One objective of this paper is to single out a large class of integral w^*-semigroups for which the two generators $A^\times$ and $A_0^\times$ are actually the same. The theory of dual semigroups suggests a way to achieve this end. For those we have (BUTZER, BERENS, 1967, Corollary 2.1.5)

$$D(A^*) = \mathrm{Fav}(T^*) = \{x^* \in X^* : t \mapsto T^*(t)x^* \text{ is Lipschitz on } [0,1]\}$$

The fact that $A^\times$ extends $A_0^\times$ and the uniform boundedness principle imply that in general

$$D(A_0^\times) \subset D(A^\times) \subset \mathrm{Fav}(T^\times).$$

Therefore our strategy will be to forget about the w^*-generator for a while and to characterize those integral generators for which the domain coincides with the Favard class. The w^*-generator then coincides with the integral generator automatically.

2. THE CHARACTERIZATION THEOREM

THEOREM 2.1. *Let $A^\times$ be a linear operator on X^*. The following sets (G) and (S) of properties are equivalent:*

(G1) $(\lambda - A^\times)^{-1}$ *is an everywhere defined bounded operator such that for some $M > 0$, $\omega \in \mathbb{R}$,* $\|(\lambda - A^\times)^{-n}\| \leqslant \dfrac{M}{(\lambda - \omega)^n}$ *for $n \in \mathbb{N}$, $\lambda > \omega$.*

(G2) *If (i) $x_n^* \in D(A^\times)$, (ii) $\|x_n^* - x^*\| \to 0$ for $n \to \infty$, and (iii) $\|A^\times x_n^*\| \leqslant C$ for some $C > 0$, then $x^* \in D(A^\times)$ and $A^\times x_n^* \to A^\times x^*$* weakly⋆ *for $n \to \infty$.*

(S) *$A^\times$ is the w^*-generator of an integral w^*-semigroup $T^\times$ which in addition to*

(S1) $T^\times(t) \int_0^h T^\times(\tau)x^* d\tau = \int_0^h T^\times(t+\tau)x^* d\tau$, $x^* \in X^*$, $t,h \geqslant 0$

satisfies

(S2) *If (i) x_n^* is a bounded sequence in X^* and (ii) $S^\times(t)x_n^* = \int_0^t T^\times(\tau)x_n^* d\tau$ converges strongly as $n \to \infty$, uniformly in $t \geqslant 0$ after scaling with a factor $e^{-\lambda t}$ with* Re λ *sufficiently large, then there exists $x^* \in X^*$ such that $x_n^* \to x^*$ weakly ⋆ as $n \to \infty$ and $\|S^\times(t)x_n^* - S^\times(t)x^*\| \to 0$ as $n \to \infty$.*

In the following we shall abbreviate the sentence 'Let $A^\times$ be the w^*-generator of an integral w^*-semigroup such that (G) or, equivalently, (S) in Theorem 2.1 are satisfied' to 'Assume G/S'.

THEOREM 2.2.

Assume G/S. Then

a) $A^{\times}$ *is the integral generator of* $T^{\times}$. *Hence* $D(A^{\times})$ *is invariant under* $T^{\times}(t)$ *and*

$$\frac{d^*}{dt}T^{\times}(t)x^* = A^{\times}T^{\times}(t)x^* = T^{\times}(t)A^{\times}x^* \text{ for } x^*\in D(A^{\times}) \text{ and } t>0.$$

b) $\|T^{\times}(t)\|\leqslant Me^{\omega t}$ *and* $(\lambda-A^{\times})^{-1} = \int_0^{\infty} e^{-\lambda\tau}T^{\times}(\tau)d\tau$ *for* $\lambda>\omega$.

c) $X^{\odot} := \overline{D(A^{\times})}$ *is the maximal subspace of strong continuity of* $T^{\times}$

d) $D(A^{\times}) = \mathrm{Fav}(T^{\times}) = \{x^*:\|T^{\times}(t)x^*-x^*\|\leqslant Ct \text{ for } 0\leqslant t\leqslant 1\} = \{x^*:t\mapsto T^{\times}(t)x^* \text{ is locally Lipschitz on } [0,\infty)\}$

e) *For* $x^*\in X^*$, $\int_0^t T^{\times}(\tau)x^*d\tau\in D(A^{\times})$ *and*

$$A^{\times}\left(\int_0^t T^{\times}(\tau)x^*d\tau\right) = T^{\times}(t)x^*-x^*.$$

In particular $D(A^{\times})$ *is* w^**- dense in* X^*

f) $T^{\times}(t)x^* = w^* - \lim_{n\to\infty}(I-\frac{t}{n}A^{\times})^{-n}x^*$

PROOFS. Let $A^{\odot}$ denote the part of $A^{\times}$ in $X^{\odot} = \overline{D(A^{\times})}$. Assume (G1). The Hille-Yosida theorem shows that $A^{\odot}$ generates a C_0-semigroup $T^{\odot}(t)$ on $X^{\odot}$. We claim that

$$D(A^{\times})\subset \mathrm{Fav}(T^{\odot}) = \{x^{\odot}\in X^{\odot}:\limsup_{t\downarrow 0}\frac{1}{t}\|T^{\odot}(t)x^{\odot}-x^{\odot}\|<\infty\}$$

$$= \{x^{\odot}\in X^{\odot}:t\mapsto T^{\odot}(t)x^{\odot} \text{ is locally Lipschitz on } [0,\infty)\}.$$

Take any $t\geqslant s\geqslant 0$ and $x^{\odot}\in D(A^{\times})$ then

$$T^{\odot}(t)x^{\odot}-T^{\odot}(s)x^{\odot} = \lim_{\lambda\to\infty}(T^{\odot}(t)-T^{\odot}(s))\lambda(\lambda-A^{\odot})^{-1}x^{\odot}$$

$$= \lim_{\lambda\to\infty}\int_s^t T^{\odot}(\tau)A^{\odot}\lambda(\lambda-A^{\odot})^{-1}x^{\odot}d\tau.$$

Since $x^{\odot}\in D(A^{\times})$ we have $A^{\odot}\lambda(\lambda-A^{\odot})^{-1}x^{\odot} = \lambda(\lambda-A^{\times})^{-1}A^{\times}x^{\odot}$ and this remains bounded for $\lambda\to\infty$. Hence $\|T^{\odot}(t)x^{\odot}-T^{\odot}(s)x^{\odot}\|\leqslant C|t-s|$ and the claim is proved.

Any $x^{\odot}\in X^{\odot}$ can be strongly approximated by elements $t^{-1}\int_0^t T^{\odot}(s)x^{\odot}ds\in D(A^{\odot})$. If $x^{\odot}\in\mathrm{Fav}(T^{\odot})$ then

$$A^{\odot}t^{-1}\int_0^t T^{\odot}(s)x^{\odot}ds = t^{-1}(T^{\odot}(t)x^{\odot}-x^{\odot})$$

remains bounded as $t\downarrow 0$. Assume (G2). It follows that any $x^{\odot}\in\mathrm{Fav}(T^{\odot})$ necessarily belongs to $D(A^{\times})$. Hence $D(A^{\times}) = \mathrm{Fav}(T^{\odot})$.

Obviously $\mathrm{Fav}(T^{\odot})$ is invariant under $T^{\odot}$ and so the following definition makes sense:

$$T^{\times}(t)x^* = (\lambda-A^{\times})T^{\odot}(t)(\lambda-A^{\times})^{-1}x^* \tag{2.1}$$

for $\lambda\in\rho(A^{\times})$. The resolvent identity shows that this definition does not depend on the choice of λ. Clearly $\{T^{\times}(t)\}$ is a semigroup. Because of (G1), $\lambda T^{\odot}(t)(\lambda-A^{\times})^{-1}x^*$ remains bounded for $\lambda\to\infty$. Since $T^{\times}(t)x^*$ is independent of λ, $A^{\times}T^{\odot}(t)(\lambda-A^{\times})^{-1}x^*$ has to remain bounded as well. (G1) implies that $T^{\odot}(t)(\lambda-A^{\times})^{-1}x^*$ tends to zero strongly for $\lambda\to\infty$. It then follows from (G2) that $A^{\times}T^{\odot}(t)(\lambda-A^{\times})^{-1}x^*$ tends to zero in the weak $\star$ topology. We conclude that

$$T^{\times}(t)x^* = w^*-\lim_{\lambda\to\infty}\lambda T^{\odot}(t)(\lambda-A^{\times})^{-1}x^*. \tag{2.2}$$

Using (G1) once more we obtain the estimate

$$\|T^{\times}(t)x^*\|\leqslant\|T^{\odot}(t)\|M\|x^*\| \tag{2.3}$$

which shows that $\|T^{\times}(t)\|$ is exponentially bounded. Since $t\mapsto T^{\odot}(t)(\lambda-A^{\times})^{-1}x^*$ is norm continuous we deduce from (G2) that $t\mapsto T^{\times}(t)x^*$ is weak $\star$ continuous. We now know that $\{T^{\times}(t)\}$ is a w^*-semigroup. In order to verify (S1) we need a lemma.

Lemma 2.3. *Let $A^{\times}$ satisfy (G2). Let $x^*:[t_1,t_2]\to X^*$ be continuous with values in $D(A^{\times})$ and such that $\|A^{\times}x^*(t)\|\leqslant C$ for some $C>0$ and $t_1\leqslant t\leqslant t_2$. Then $t\mapsto A^{\times}x^*(t)$ is w^*-continuous on $[t_1,t_2]$,*

$$\int_{t_1}^{t_2}x^*(\tau)d\tau\in D(A^{\times}) \text{ and } A^{\times}\int_{t_1}^{t_2}x^*(\tau)d\tau = \int_{t_1}^{t_2}A^{\times}x^*(\tau)d\tau.$$

Proof. The w^*-continuity of $A^{\times}x^*(t)$ is an immediate consequence of (G2). As $x^*(t)$ is strongly continuous the integral $\int_{t_1}^{t_2}x^*(\tau)d\tau$ is strongly approximated by Riemann sums $\Sigma x^*(t_j)(t_{j+1}-t_j)\in D(A^{\times})$. Similarly $\Sigma A^{\times}x^*(t_j)(t_{j+1}-t_j)$ approximates $\int_{t_1}^{t_2}A^{\times}x^*(\tau)d\tau$ in the weak $\star$ sense since $A^{\times}x^*(t)$ is weakly $\star$ continuous. The assertion now follows from (G2). □

Armed with this lemma we can write

$$\begin{aligned}T^{\times}(t)\int_0^h T^{\times}(\tau)x^*d\tau &= T^{\times}(t)(\lambda-A^{\times})\int_0^h T^{\odot}(\tau)(\lambda-A^{\times})^{-1}x^*d\tau\\ &= (\lambda-A^{\times})T^{\odot}(t)\int_0^h T^{\odot}(\tau)(\lambda-A^{\times})^{-1}x^*d\tau\\ &= (\lambda-A^{\times})\int_0^h T^{\odot}(t+\tau)(\lambda-A^{\times})^{-1}x^*d\tau\\ &= \int_0^h(\lambda-A^{\times})T^{\odot}(t+\tau)(\lambda-A^{\times})^{-1}x^*d\tau = \int_0^h T^{\times}(t+\tau)x^*d\tau\end{aligned}$$

which is exactly (S1). It remains to verify (S2).

The definition (2.1) implies that

$$\int_0^t e^{-\lambda\tau}T^{\times}(\tau)d\tau = (\lambda-A^{\odot})\int_0^t e^{-\lambda\tau}T^{\odot}(\tau)d\tau(\lambda-A^{\times})^{-1}. \tag{2.4}$$

Hence, for Re λ sufficiently large,

$$(\lambda-A^{\times})^{-1} = \int_0^\infty e^{-\lambda\tau}T^{\times}(\tau)d\tau = \lambda\int_0^\infty e^{-\lambda\tau}S^{\times}(\tau)d\tau. \tag{2.5}$$

Consider any bounded sequence x_n^* in X^* such that $e^{-\lambda t}S^{\times}(t)x_n^*$ converges strongly for $n\to\infty$, uniformly in $t\geqslant0$. Put $y_n^* = (\lambda-A^{\times})^{-1}x_n^*$. Then y_n^* converges strongly to a limit, say y^*. Moreover, $A^{\times}y_n^*$ is bounded since x_n^* is bounded. So (G2) implies that $y^*\in D(A^{\times})$ and $A^{\times}y_n^*\to A^{\times}y^*$ weakly $\star$. Hence $x_n^* = (\lambda-A^{\times})y_n^* = \lambda y_n^*-A^{\times}y_n^*\to\lambda y^*-A^{\times}y^*$ weakly $\star$. Put $x^* = \lambda y^*-A^{\times}y^*$ then $y^* = (\lambda-A^{\times})^{-1}x^*$. From (2.1) we deduce

$$S^{\times}(t) = (\lambda - A^{\odot})S^{\odot}(t)(\lambda - A^{\times})^{-1} = (\lambda S^{\odot}(t) - T^{\odot}(t) + I)(\lambda - A^{\times})^{-1}$$

and consequently

$$S^{\times}(t)x_n^* \to (\lambda S^{\odot}(t) - T^{\odot}(t) + I)y^* = (\lambda S^{\odot}(t) - T^{\odot}(t) + I)(\lambda - A^{\times})^{-1}x^* = S^{\times}(t)x^*.$$

Hence (S2) holds. This concludes the (G) $\Rightarrow$ (S) part of the proof of Theorem 2.1.

Let $T^{\times}$ be a w^*-semigroup with *integral* generator $A_0^{\times}$. Applying the uniform boundedness theorem twice we deduce that $\|T^{\times}(t)\|$ is bounded on [0,1]. The semigroup property then implies that $\|T^{\times}(t)\|$ is exponentially bounded. Assume (S1). We claim that $S^{\times}(t)x^* \in D(A_0^{\times})$ and $A_0^{\times}S^{\times}(t)x^* = T^{\times}(t)x^* - x^*$. In order to prove this claim we first note that $S^{\times}(t+h) = S^{\times}(t)T^{\times}(h) + S^{\times}(h)$. Hence (S1) can be rewritten as

$$T^{\times}(t)S^{\times}(h) = S^{\times}(t+h) - S^{\times}(t) = S^{\times}(t)T^{\times}(h) + S^{\times}(h) - S^{\times}(t).$$

Therefore $T^{\times}(t)S^{\times}(h) - S^{\times}(h) = S^{\times}(t)(T^{\times}(h) - I)$ which, by the very definition of an integral generator, proves the claim.

Define $X^{\odot} = \overline{D(A_0^{\times})}$. If $x^* \in D(A_0^{\times})$ then $T^{\times}(t)x^* - x^* = S^{\times}(t)A_0^{\times}x^*$ and consequently $t \mapsto T^{\times}(t)x^*$ is norm continuous. As $T^{\times}(t)$ is exponentially bounded, this property extends to the closure $\overline{D(A_0^{\times})}$. Assume, conversely, that $\|T^{\times}(t)x^* - x^*\| \to 0$ as $t \downarrow 0$. Then $t^{-1}\|S^{\times}(t)x^* - x^*\| \to 0$ as $t \downarrow 0$ as well. Since $S^{\times}(t)x^* \in D(A_0^{\times})$ we conclude that $x^* \in \overline{D(A_0^{\times})}$. So $X^{\odot}$ is the maximal subspace of strong continuity for $T^{\times}$. If we restrict $T^{\times}$ to the invariant subspace $X^{\odot}$ we obtain a C_0-semigroup which we call $T^{\odot}$. The definition of integral generator is such that it immediately follows that $A^{\odot}$ is the part of $A_0^{\times}$ in $X^{\odot}$. We now want to use the Hille-Yosida estimates for $A^{\odot}$ to prove (G1). We show that $\lambda \in \rho(A_0^{\times})$ if Re $\lambda > \omega$. Define, for Re $\lambda > \omega$ and $x^* \in X^*$,

$$R_\lambda^{\times}x^* = \int_0^{\infty} e^{-\lambda s}T^{\times}(s)x^*\,ds.$$

We note that by an approximation argument,

$$T^{\times}(t)\int_0^s T^{\times}(r)f^{\times}(r)dr = \int_0^s T^{\times}(t+r)f^{\times}(r)dr, \quad s,t \geq 0,$$

for every strongly continuous X^*-valued function $f^{\times}$. In particular,

$$T^{\times}(t)\int_0^{\infty} e^{-\lambda s}T^{\times}(s)x^*\,ds = \int_0^{\infty} e^{-\lambda s}T^{\times}(t+s)x^*\,ds = \int_0^{\infty} e^{-\lambda(s-t)}T^{\times}(s)x^*\,ds,$$

which is weak $\star$ differentiable with weak $\star$ derivative $\lambda T^{\times}(t)R_\lambda^{\times}x^* - T^{\times}(t)x^*$. Therefore $R_\lambda^{\times}x^* \in D(A_0^{\times})$ and $A_0^{\times}R_\lambda^{\times}x^* = \lambda R_\lambda^{\times}x^* - x^*$ which yields that $(\lambda - A_0^{\times})R_\lambda^{\times} = I$. On the other hand, if $T^{\times}(t)$ is a weakly $\star$ continuous semigroup satisfying (S_1) then $e^{-\lambda t}T^{\times}(t)$ is a weakly $\star$ continuous semigroup satisfying (S_1) and its integral weak $\star$ generator is $A_0^{\times} - \lambda$ with domain $D(A_0^{\times})$. Thus

$$e^{-\lambda t}T^{\times}(t)x^* - x^* = \int_0^t e^{-\lambda s}T^{\times}(s)(A_0^{\times} - \lambda)x^*\,ds$$

for $x^* \in D(A_0^{\times})$. If Re $\lambda > \omega$ we can take $t \to \infty$ and get that $x^* = R_\lambda^{\times}(\lambda - A_0^{\times})x^*$. This shows that for Re $\lambda > \omega$, $\lambda \in \rho(A_0^{\times})$ and

$$R(\lambda, A_0^{\times})x^* = R_\lambda^{\times}x^* = \int_0^{\infty} e^{-\lambda s}T^{\times}(s)x^*\,ds.$$

Now note that for $\mu \in \rho(A_0^{\times})$ we have

$$(\lambda - A_0^{\times})^{-1} = (\mu - A^{\odot})(\lambda - A^{\odot})^{-1}(\mu - A_0^{\times})^{-1}.$$

We want to control the term $A^{\odot}(\lambda-A^{\odot})^{-1}(\mu-A_0^{\times})^{-1}$. Since

$$A^{\odot}(\lambda-A^{\odot})^{-1}x^{\odot} = \lambda(\lambda-A^{\odot})^{-1}x^{\odot}-x^{\odot} = \lambda\int_0^{\infty} e^{-\lambda\tau}T^{\odot}(\tau)x^{\odot}d\tau - x^{\odot} =$$

$$= \lim_{h\downarrow 0}\int_0^{\infty}\frac{1}{h}(e^{-\lambda(t-h)}-e^{-\lambda t})T^{\odot}(t)x^{\odot}dt - x^{\odot}$$

$$= \lim_{h\downarrow 0}\int_0^{\infty}e^{-\lambda t}\frac{1}{h}(T^{\odot}(t+h)-T^{\odot}(t))x^{\odot}dt$$

$$= \lim_{h\downarrow 0}\int_0^{\infty}e^{-\lambda t}T^{\odot}(t)\frac{1}{h}(T^{\odot}(h)-I)x^{\odot}dt$$

we obtain

$$\|A^{\odot}(\lambda-A^{\odot})^{-1}x^{\odot}\| \leq \frac{C}{\lambda-\omega}\|x^{\odot}\|$$

provided $T^{\odot}(t)x^{\odot}$ is Lipschitz. The definition of integral generator implies at once that $T^{\times}(t)x^{\odot}$ is Lipschitz for $x^{\odot}\in D(A_0^{\times})$. Hence (G1) is a corollary of the Hille-Yosida estimates for $A^{\odot}$.

Assume (S2) . Consider $x_n^*\in D(A_0^{\times})$ such that $x_n^*\to x^*$ strongly while $\|A_0^{\times}x_n^*\|$ is bounded. The identity

$$T^{\times}(t)x_n^*-x_n^* = S^{\times}(t)A_0^{\times}x_n^*$$

and (S2) imply that $A_0^{\times}x_n^*$ converges weakly $*$ to a limit, say y^*, and that

$$T^{\times}(t)x^*-x^* = S^{\times}(t)y^*.$$

By the definition of integral generator this implies that $x^*\in D(A_0^{\times})$ and $y^* = A_0^{\times}x^*$. Hence (G2) holds.

Finally we claim that $D(A_0^{\times}) = \mathrm{Fav}(T^{\odot})$. We know already that $D(A_0^{\times})\subseteq\mathrm{Fav}(T^{\odot})$. The fact that $x^{\odot}\in\mathrm{Fav}(T^{\odot})$ implies $x^{\odot}\in D(A_0^{\times})$ follows from (G2) exactly as before. Let $A^{\times}$ be the w^*-generator of $T^{\times}$ then $D(A_0^{\times})\subseteq D(A^{\times})\subseteq\mathrm{Fav}(T^{\times}) = \mathrm{Fav}(T^{\odot})$. We conclude that $A_0^{\times} = A^{\times}$.

We have now proved Theorem 2.1 but during the proof we have also shown that Theorem 2.2 a,b,c,d,e are true. It remains to prove Theorem 2.2f. From the theory of C_0-semigroups we know that

$$(I-\frac{t}{n}A^{\odot})^{-n}(\lambda-A^{\times})^{-1}x^*\to T^{\odot}(t)(\lambda-A^{\times})^{-1}x^*$$

strongly as $n\to\infty$. By (G1)

$$(\lambda-A^{\times})(I-\frac{t}{n}A^{\odot})^{-n}(\lambda-A^{\times})^{-1}x^* = (I-\frac{t}{n}A^{\times})^{-n}x^*$$

remains bounded as $n\to\infty$. The assertion now follows from (G2) and the intertwining formula (2.1).
□

Remarks. (i) If T is a C_0-semigroup on X with generator A, then T^* satisfies $(S_1)-(S_2)$ and A^* satisfies $(G_1)-(G_2)$.
(ii) If $A^{\times}$ satisfies $(G_1)-(G_2)$ and $B^{\times}:X^{\odot}\to X^*$ is a bounded linear operator then $A^{\times}+B^{\times}$ satisfies $(G_1)-(G_2)$ as well.

3. Duality

Throughout this section we assume that (G_1) is satisfied. Let $A^{\odot}$ be the part of $A^{\times}$ in $X^{\odot}$. Then $A^{\odot}$ is a densely defined operator on $X^{\odot}$ (even more, $A^{\odot}$ is the generator of a C_0-semigroup $T^{\odot}$) and so we can define its adjoint $A^{\odot *}$. Let $X^{\odot\odot} = \overline{D(A^{\odot *})}$ and define $A^{\odot\odot}$ to be the part of $A^{\odot *}$ in $X^{\odot\odot}$.

Then $A^{\odot\odot}$ satisfies the Hille-Yosida conditions and therefore is the generator of a C_0-semigroup $T^{\odot\odot}$ on $X^{\odot\odot}$.

In this section we show that $X^{\odot\odot}$ can be continuously embedded in X^{**} if (G_1) is satisfied and that $T^{\times}$ is the restricted dual of $T^{\odot\odot}$ if G/S is satisfied. To begin let us assume (G_1) and define a pairing between $X^{\odot\odot}$ and X^* in the following way. Choose $\mu\in\rho(A^{\times})$. For $x^*\in X^*$ and $x^{\odot\odot}\in D(A^{\odot\odot})$ we define

$$[x^{\odot\odot},x^*] = \langle(\mu-A^{\odot\odot})x^{\odot\odot},(\mu-A^{\times})^{-1}x^*\rangle \tag{3.1}$$

(note that $(\mu-A^{\times})^{-1}x^*\in D(A^{\times})\subseteq X^{\odot})$. Our first result implies, among other things, that this expression is independent of μ.

Lemma 3.1. *For every $x^*\in X^*$ and $x^{\odot\odot}\in D(A^{\odot\odot})$,*

$$[x^{\odot\odot},x^*] = \lim_{\lambda\to\infty}\langle x^{\odot\odot},\lambda(\lambda-A^{\times})^{-1}x^*\rangle.$$

Proof.

$$\begin{aligned}[x^{\odot\odot},x^*] &= \langle(\mu-A^{\odot\odot})x^{\odot\odot},(\mu-A^{\times})^{-1}x^*\rangle = \\ &\lim_{\lambda\to\infty}\langle(\mu-A^{\odot\odot})x^{\odot\odot},\lambda(\lambda-A^{\times})^{-1}(\mu-A^{\times})^{-1}x^*\rangle = \\ &\lim_{\lambda\to\infty}\langle(\mu-A^{\odot\odot})x^{\odot\odot},(\mu-A^{\odot})^{-1}\lambda(\lambda-A^{\times})^{-1}x^*\rangle = \\ &\lim_{\lambda\to\infty}\langle x^{\odot\odot},\lambda(\lambda-A^{\times})^{-1}x^*\rangle. \qquad \square\end{aligned}$$

Using this characterization the following estimate is easily derived

$$|[x^{\odot\odot},x^*]|\leq M\|x^{\odot\odot}\|\cdot\|x^*\| \tag{3.2}$$

for $x^*\in X^*$ and $x^{\odot\odot}\in D(A^{\odot\odot})$. Since $D(A^{\odot\odot})$ lies dense in $X^{\odot\odot}$ we can extend the continuous linear functional $x^{\odot\odot}\to[x^{\odot\odot},x^*]$ to the whole space $X^{\odot\odot}$. Using the same notation for this extension we find that for every $x^{\odot\odot}\in X^{\odot\odot}$ and $x^*\in X^*$,

$$[x^{\odot\odot},x^*] = \lim_{\lambda\to\infty}\langle x^{\odot\odot},\lambda(\lambda-A^{\times})^{-1}x^*\rangle \tag{3.3}$$

and (3.2) holds. Furthermore

$$[x^{\odot\odot},x^{\odot}] = \langle x^{\odot\odot},x^{\odot}\rangle \tag{3.4}$$

if $x^{\odot}\in X^{\odot}$ and $x^{\odot\odot}\in X^{\odot\odot}$. Let k be the embedding of $X^{\odot\odot}$ into X^{**} given by

$$kx^{\odot\odot}(x^*) = [x^{\odot\odot},x^*], \tag{3.5}$$

then, by (3.2), $\|kx^{\odot\odot}\|\leq M\|x^{\odot\odot}\|$. Furthermore, $\|kx^{\odot\odot}\|\geq\sup_{\|x^{\odot}\|\leq 1}|[x^{\odot\odot},x^{\odot}]| = \|x^{\odot\odot}\|$. Hence

$$1\leq\|k\|\leq M. \tag{3.6}$$

Theorem 3.2. *Assume (G_1). Then*

a) $\langle A^{\odot *}x^{\odot\odot}, x^{\odot}\rangle = [x^{\odot\odot},A^{\times}x^{\odot}]$, $x^{\odot\odot}\in D(A^{\odot *})$, $x^{\odot}\in D(A^{\times})$.

b) $[(\lambda-A^{\odot *})^{-1}x^{\odot *},x^*] = \langle x^{\odot *},(\lambda-A^{\times})^{-1}x^*\rangle$, $x^{\odot *}\in X^{\odot *}$, $x^*\in X^*$.

Proof. We only prove a).
Let $x^{\odot\odot}\in D(A^{\odot *})$ and $x^{\odot}\in D(A^{\times})$. Then

$$\begin{aligned}\langle A^{\odot *}x^{\odot\odot},x^{\odot}\rangle &= \lim_{\lambda\to\infty}\langle A^{\odot *}x^{\odot\odot},\lambda(\lambda-A^{\odot})^{-1}x^{\odot}\rangle \\ &= \lim_{\lambda\to\infty}\langle x^{\odot\odot},\lambda(\lambda-A^{\times})^{-1}A^{\times}x^{\odot}\rangle\end{aligned}$$

$$= [x^{\odot\odot}, A^{\times} x^{\odot}]. \quad \square$$

Our next result gives a rather useful characterization of $A^{\times}$.

THEOREM 3.3. *Assume* (G_1). *Let* $\hat{X}$ *be a closed subspace of* $X^{\odot\odot}$ *which is invariant under* $T^{\odot\odot}$ *and separates points in* X^*. *Let* $x^*, y^* \in X^*$ *be such that*

$$[A^{\odot\odot}\hat{x}, x^*] = [\hat{x}, y^*]$$

for all $\hat{x} \in \hat{X} \cap D(A^{\odot\odot})$. *Then* $x^* \in D(A^{\times})$ *and* $A^{\times} x^* = y^*$.

PROOF. Let $\hat{T}$ be the restriction of $T^{\odot\odot}$ to $\hat{X}$ and let $\hat{A}$ be the generator of $\hat{T}$. Then $D(\hat{A}) = \hat{X} \cap D(A^{\odot\odot})$. Assume that $x^*, y^* \in X^*$ are such that $[\hat{A}\hat{x}, x^*] = [\hat{x}, y^*]$ for all $\hat{x} \in D(\hat{A})$. From Theorem 3.2.b we get that

$$\begin{aligned} <\hat{x}, (\lambda - A^{\times})^{-1} y^*> &= [(\lambda - \hat{A})^{-1}\hat{x}, y^*] \\ &= [\hat{A}(\lambda - \hat{A})^{-1}\hat{x}, x^*] \\ &= [\lambda(\lambda - \hat{A})^{-1}\hat{x} - \hat{x}, y^*] \\ &= [\hat{x}, \lambda(\lambda - A^{\times})^{-1} x^* - x^*] \end{aligned}$$

for all $\hat{x} \in \hat{X}$. Since $\hat{X}$ separates points in X^* this yields

$$(\lambda - A^{\times})^{-1} y^* = \lambda(\lambda - A^{\times})^{-1} x^* - x^*,$$

hence $x^* \in D(A^{\times})$ and $y^* = \lambda x^* - (\lambda - A^{\times}) x^* = A^{\times} x^*$. $\square$

From this point on we assume that G/S is satisfied. Let $T^{\times}$ be the w^* continuous semigroup generated by $A^{\times}$.

THEOREM 3.4. *If* G/S *is satisfied then*

$$[T^{\odot\odot}(t) x^{\odot\odot}, x^*] = [x^{\odot\odot}, T^{\times}(t) x^*], \tag{3.7}$$

for all $x^{\odot\odot} \in X^{\odot\odot}$ *and* $x^* \in X^*$.

PROOF.

$$\begin{aligned} [T^{\odot\odot}(t) x^{\odot\odot}, x^*] &= \lim_{\lambda\to\infty} <T^{\odot\odot}(t) x^{\odot\odot}, \lambda(\lambda - A^{\times})^{-1} x^*> \\ &= \lim_{\lambda\to\infty} <x^{\odot\odot}, T^{\odot}(t)\lambda(\lambda - A^{\times})^{-1} x^*> \\ &= \lim_{\lambda\to\infty} <x^{\odot\odot}, \lambda(\lambda - A^{\times})^{-1} T^{\times}(t) x^*> = [x^{\odot\odot}, T^{\times}(t) x^*]. \end{aligned}$$

Here we have used the intertwining formula (2.1). $\square$

In Sections 1 and 2 we have seen two different characterizations of $A^{\times}$, namely as the w^* generator of $T^{\times}$ and as the integral generator of $T^{\times}$. The next theorem gives a third characterization, namely as the derivative of $T^{\times}(t)$ with respect to the $\sigma(X^*, X^{\odot\odot})$- topology at $t = 0$.

THEOREM 3.5. *Assume* G/S *and let* $x^*, y^* \in X^*$. *Then* $x^* \in D(A^{\times})$ *and* $A^{\times} x^* = y^*$ *if and only if*

$$[x^{\odot\odot}, \frac{1}{h}(T^{\times}(h) x^* - x^*)] \to [x^{\odot\odot}, y^*] \text{ as } h \downarrow 0, \tag{3.8}$$

for every $x^{\odot\odot} \in X^{\odot\odot}$.

Proof. 'if'. Suppose (3.8) is satisfied. If $x^{\odot\odot}\in D(A^{\odot\odot})$ then

$$[x^{\odot\odot},\frac{1}{h}(T^{\times}(h)x^{*}-x^{*})] = [\frac{1}{h}(T^{\odot\odot}(h)x^{\odot\odot}-x^{\odot\odot}),x^{*}] \to [A^{\odot\odot}x^{\odot\odot},x^{*}],\ h\downarrow 0.$$

Hence $[A^{\odot\odot}x^{\odot\odot},x^{*}] = [x^{\odot\odot},y^{*}]$ for $x^{\odot\odot}\in D(A^{\odot\odot})$. Thus by Theorem 3.3 with $\hat{X} = X^{\odot\odot}$, we get that $x^{*}\in D(A^{\times})$ and $A^{\times}x^{*} = y^{*}$.
'only if'. Assume that $x^{*}\in D(A^{\times})$ and $A^{\times}x^{*}=y^{*}$, and let $x^{\odot\odot}\in D(A^{\odot\odot})$. Then

$$[x^{\odot\odot},\frac{1}{h}(T^{\times}(h)x^{*}-x^{*})] = [\frac{1}{h}(T^{\odot\odot}(h)x^{\odot\odot}-x^{\odot\odot}),x^{*}] \to$$

$$[A^{\odot\odot}x^{\odot\odot},x^{*}] = [x^{\odot\odot},A^{\times}x^{*}]$$

as $h\downarrow 0$. Since $D(A^{\odot\odot})$ is dense in $X^{\odot\odot}$ and $\{h^{-1}(T^{\times}(h)x^{*}-x^{*}): 0<h<1\}$ is bounded (recall that $D(A^{\times}) = \mathrm{Fav}(T^{\times})$) this result holds for every $x^{\odot\odot}\in X^{\odot\odot}$ which proves the 'only if' part. □

Theorem 3.6. *Assume G/S. Then*

$$[x^{\odot\odot},\int_0^t T^{\times}(s)x^{*}ds] = \int_0^t [x^{\odot\odot},T^{\times}(s)x^{*}]ds, \tag{3.9}$$

for every $x^{\odot\odot}\in X^{\odot\odot}$ and $x^{*}\in X^{*}$.

Proof. Let $x^{*}\in X^{*}, x^{\odot\odot}\in X^{\odot\odot}$, and $\lambda\in\rho(A^{\times})$. Define $y^{\odot} = (\lambda-A^{\times})^{-1}x^{*}$. Then $y^{\odot}\in D(A^{\times})$. The characterization of $A^{\times}$ as the integral generator of $T^{\times}$ yields that

$$T^{\odot}(t)y^{\odot}-y^{\odot} = \int_0^t T^{\times}(s)A^{\times}y^{\odot}ds =$$

$$\int_0^t T^{\times}(s)(\lambda y^{\odot}-x^{*})ds = \lambda\int_0^t T^{\odot}(s)y^{\odot}ds - \int_0^t T^{\times}(s)x^{*}ds.$$

This yields that

$$[x^{\odot\odot},\int_0^t T^{\times}(s)x^{*}ds] =$$

$$[x^{\odot\odot},\lambda\int_0^t T^{\odot}(s)y^{\odot}ds]-[x^{\odot\odot},T^{\odot}(t)y^{\odot}-y^{\odot}] =$$

$$\int_0^t [x^{\odot\odot},\lambda T^{\odot}(s)y^{\odot}]ds - [A^{\odot\odot}\int_0^t T^{\odot\odot}(s)x^{\odot\odot}ds,y^{\odot}] =$$

$$\int_0^t [x^{\odot\odot},\lambda T^{\odot}(s)y^{\odot}]ds - [\int_0^t T^{\odot\odot}(s)x^{\odot\odot}ds,A^{\times}y^{\odot}] =$$

$$\int_0^t [x^{\odot\odot},\lambda T^{\odot}(s)y^{\odot}]ds - \int_0^t [T^{\odot\odot}(s)x^{\odot\odot},A^{\times}y^{\odot}]ds =$$

$$\int_0^t [T^{\odot\odot}(s)x^{\odot\odot},(\lambda-A^{\times})y^{\odot}]ds = \int_0^t [x^{\odot\odot},T^{\times}(s)x^{*}]ds. \quad \square$$

An immediate consequence of this result is the following characterization of the pairing $[\cdot,\cdot]$:

$$[x^{\odot\odot},x^{*}] = \lim_{t\downarrow 0} \langle x^{\odot\odot},t^{-1}\int_0^t T^{\times}(s)x^{*}ds\rangle, \tag{3.10}$$

for every $x^{\odot\odot} \in X^{\odot\odot}$ and $x^* \in X^*$.

In the practically important case that $A^\times$ is the adjoint of a generator of a C_0-semigroup on X (or a bounded perturbation of it: see CLEMENT et al (Part IV), this space X lies continuously embedded in $X^{\odot\odot}$. Below we present two assumptions, one on $A^\times$ and one on $T^\times$, both of which guarantee that X lies embedded in $X^{\odot\odot}$.

Let $j:X \rightarrow X^{\odot *}$ be the embedding $jx(x^\odot) = <x,x^\odot>$, for $x \in X$, $x^\odot \in X^\odot$. If we give X the new but equivalent norm

$$\|x\|' = \sup\{|<x,x^\odot>| : x^\odot \in X^\odot, \|x^\odot\| \leqslant 1\}$$

then j is an isometry from X onto $j(X)$ (see HILLE AND PHILLIPS, 1957, Chapter XIV). We introduce the following assumptions.

(G0) $\quad \forall_{x \in X}$: $<x, T^\times(t)x^* - x^*> \rightarrow 0$, $t \downarrow 0$, uniformly in $\|x^*\| \leqslant 1$.

(S0) $\quad \forall_{x \in X}$: $<x, T^\times(t)x^* - x^*> \rightarrow 0$, $t \downarrow 0$, uniformly in $\|x^*\| \leqslant 1$.

Note that both (G0) and (S0) are trivially satisfied if $T^\times$ is the adjoint of a C_0-semigroup on X.

LEMMA 3.7. *Assume G/S. For every $x \in X$ and $x^* \in X^*$,*

$$\lim_{\lambda \rightarrow \infty} <x, \lambda(\lambda - A^\times)^{-1}x^* - x^*> = 0.$$

PROOF. Take $x^* \in X^*$. Then $x^* = (\lambda - A^\times)x^*_\lambda$ where $x^*_\lambda = (\lambda - A^\times)^{-1}x^*$. Then

$$\mu(\mu - A^\times)^{-1}x^*_\lambda = x^*_\lambda + (\mu - A^\times)A^\times x^*_\lambda \rightarrow x^*_\lambda, \ \mu \rightarrow \infty,$$

in norm. Furthermore, $A^\times \mu(\mu - A^\times)^{-1}x^*_\lambda = \mu(\mu - A^\times)^{-1}A^\times x^*_\lambda$ is bounded for $\mu \rightarrow \infty$. Thus, by (G2), $x^*_\lambda \in D(A^\times)$ and

$$A^\times \mu(\mu - A^\times)^{-1}x^*_\lambda \rightarrow A^\times x^*_\lambda, \ \mu \rightarrow \infty,$$

with respect to the weak $*$ topology. We already saw that

$$\lambda\mu(\mu - A^\times)^{-1}x^*_\lambda \rightarrow \lambda x^*_\lambda, \ \mu \rightarrow \infty,$$

in norm. By subtraction we get,

$$(\lambda - A^\times)\mu(\mu - A^\times)^{-1}x^*_\lambda \rightarrow (\lambda - A^\times)x^*_\lambda, \ \mu \rightarrow \infty$$

in the weak $*$ sense. Thus

$$\mu(\mu - A^\times)^{-1}x^* \rightarrow x^*, \ \mu \rightarrow \infty$$

in the weak $*$ sense. □

THEOREM 3.8. *Assume G/S. Then (G0) and (S0) are equivalent. Moreover, if one (hence both) of these assumptions is satisfied then $j(X) \subseteq X^{\odot\odot}$ and $[jx, x^*] = <x,x^*>$, for $x \in X$ and $x^* \in X^*$.*

PROOF. Assume (G0). We first show that $j(X) \subseteq X^{\odot\odot}$. For $x \in X$,

$$\|\lambda(\lambda - A^{\odot *})^{-1}jx - jx\| = \sup_{\|x^\odot\| \leqslant 1} |<\lambda(\lambda - A^{\odot *})^{-1}jx - jx, x^\odot>|$$

$$= \sup_{\|x^\odot\| \leqslant 1} |<x, \lambda(\lambda - A^\odot)^{-1}x^\odot - x^\odot>| \rightarrow 0, \ \lambda \rightarrow \infty$$

by (G0), hence $jx \in X^{\odot\odot}$. Furthermore

$$[jx, x^*] = \lim_{\lambda \rightarrow \infty} <jx, \lambda(\lambda - A^\times)^{-1}x^*>$$

$$= \lim_{\lambda\to\infty} <x,\lambda(\lambda-A^{\times})^{-1}x^*> = <x,x^*>$$

by Lemma 3.7.
We show that (S0) is satisfied.

$$\begin{aligned}|<x,T^{\times}(t)x^*-x^*>| &= |[jx,T^{\times}(t)x^*-x^*]| \\ &= |[T^{\odot\odot}(t)jx-jx,x^*]| \\ &\leq \|T^{\odot\odot}(t)jx-jx\|\cdot\|x^*\|\to 0, \quad t\downarrow 0,\end{aligned}$$

uniformly for $\|x^*\|\leq 1$. Thus (S0) is satisfied.
Assume (S0). We first show that $j(X)\subseteq X^{\odot\odot}$ and that $[jx,x^*] = <x,x^*>$.

$$\begin{aligned}\|T^{\odot *}(t)jx-jx\| &= \sup_{\|x^{\odot}\|\leq 1} |<T^{\odot *}(t)jx-jx,x^{\odot}>| \\ &= \sup_{\|x^{\odot}\|\leq 1} |<x,T^{\odot}(t)x^{\odot}-x^{\odot}>|\to 0, \; t\downarrow 0,\end{aligned}$$

by (S0), hence $jx\in X^{\odot\odot}$. Furthermore, by (3.10),

$$\begin{aligned}[jx,x^*] &= \lim_{t\downarrow 0}<x,\frac{1}{t}\int_0^t T^{\times}(s)x^*ds> \\ &= \lim_{t\downarrow 0}\frac{1}{t}\int_0^t <x,T^{\times}(s)x^*>ds = <x,x^*>.\end{aligned}$$

Finally we prove (G0).

$$\begin{aligned}|<x,\lambda(\lambda-A^{\times})^{-1}x^*-x^*>| &= |[\lambda(\lambda-A^{\odot\odot})^{-1}jx-jx,x^*]| \\ &\leq \|\lambda(\lambda-A^{\odot\odot})^{-1}jx-jx\|\cdot\|x^*\|\to 0, \; \lambda\to\infty\end{aligned}$$

uniformly for $\|x^*\|\leq 1$. □

4. An alternative characterization of $X^{\odot\odot}$

In the previous section we have seen that $X^{\odot\odot}$ lies continuously embedded in X^{**}, the embedding operator being denoted by k. In this section we give a direct definition of $k(X^{\odot\odot})$ in terms of the adjoint of $(\lambda-A^{\times})^{-1}$. Throughout this section we assume that (G1) is satisfied.
We define

$$X^{*\odot} = \{x^{**}\in X^{**}: \|\lambda(\lambda-A^{\times})^{-1*}x^{**}-x^{**}\|\to 0, \lambda\to\infty\}. \tag{4.1}$$

From (G1) one easily derives that $X^{*\odot}$ is a closed subspace of X^{**} which is invariant under $(\lambda-A^{\times})^{-1*}$. For future use we prove the following lemma.

Lemma 4.1. *Let $x^{**}\in X^{*\odot}$ satisfy $<x^{**},x^*> = 0$ for every $x^*\in D(A^{\times})$, then $x^{**} = 0$.*

Proof. From the assumption it follows that $<x^{**},(\lambda-A^{\times})^{-1}x^*> = <(\lambda-A^{\times})^{-1*}x^{**},x^*> = 0$ for every $x^*\in X^*$. Taking the supremum over all $x^*\in X^*$ we get that $\|\lambda(\lambda-A^{\times})^{-1}x^{**}\| = 0$. Now letting $\lambda\to\infty$ and using that $x^{**}\in X^{*\odot}$ we find that $x^{**} = 0$. □

Let $p: X^{**}\to X^{\odot *}$ be the projection operator given by

$$px^{**}(x^{\odot}) = <x^{**},x^{\odot}>. \tag{4.2}$$

For a Banach space Y we denote by I_Y the identity operator on Y. We are ready to state the main theorem of this section.

Theorem 4.2.
a) $k(X^{\odot\odot}) \subseteq X^{*\odot}$ and $<kx^{\odot\odot},x^*> = [x^{\odot\odot},x^*]$
b) $p(X^{*\odot}) \subseteq X^{\odot\odot}$ and $[px^{**},x^*] = <x^{**},x^*>$.
c) $k \circ p = I_{X^{*\odot}}$.
d) $p \circ k = I_{X^{\odot\odot}}$

Proof. a) Let $x^{\odot\odot} \in X^{\odot\odot}$. Then

$$\begin{aligned}\|\lambda(\lambda - A^{\times})^{-1*}kx^{\odot\odot} - kx^{\odot\odot}\| &= \sup_{\|x^*\|\leqslant 1} |<\lambda(\lambda - A^{\times})^{-1*}kx^{\odot\odot} - kx^{\odot\odot},x^*>| \\ &= \sup_{\|x^*\|\leqslant 1} |<kx^{\odot\odot},\lambda(\lambda - A^{\times})^{-1}x^* - x^*>| \\ &= \sup_{\|x^*\|\leqslant 1} |[x^{\odot\odot},\lambda(\lambda - A^{\times})^{-1}x^* - x^*]| \\ &= \sup_{\|x^*\|\leqslant 1} |[\lambda(\lambda - A^{\odot\odot})^{-1}x^{\odot\odot} - x^{\odot\odot},x^*]| \\ &\leqslant \|\lambda(\lambda - A^{\odot\odot})^{-1}x^{\odot\odot} - x^{\odot\odot}\| \to 0,\ \lambda \to \infty,\end{aligned}$$

which proves the first assertion. The second assertion follows from definition (3.5).
b) Let $x^{*\odot} \in X^{\odot *}$. Then

$$\begin{aligned}\|\lambda(\lambda - A^{\odot *})^{-1}px^{*\odot} - px^{*\odot}\| &= \sup_{\|x^{\odot}\|\leqslant 1} |<\lambda(\lambda - A^{\odot *})px^{*\odot} - px^{*\odot},x^{\odot}>| = \\ &= \sup_{\|x^{\odot}\|\leqslant 1} |<x^{*\odot},\lambda(\lambda - A^{\odot})^{-1}x^{\odot} - x^{\odot}>| \\ &= \sup_{\|x^{\odot}\|\leqslant 1} |<\lambda(\lambda - A^{\times})^{-1*}x^{*\odot} - x^{*\odot},x^{\odot}>| \\ &\leqslant \|\lambda(\lambda - A^{\times})^{-1*}x^{*\odot} - x^{*\odot}\| \to 0,\ \lambda \to \infty,\end{aligned}$$

which proves the first part of b). The second part is proved by

$$\begin{aligned}[px^{*\odot},x^*] &= \lim_{\lambda\to\infty} <px^{*\odot},\lambda(\lambda - A^{\times})^{-1}x^*> \\ &= \lim_{\lambda\to\infty} <x^{*\odot},\lambda(\lambda - A^{\times})^{-1}x^*> \\ &= \lim_{\lambda\to\infty} <\lambda(\lambda - A^{\times})^{-1*}x^{*\odot},x^*> \\ &= <x^{*\odot},x^*>.\end{aligned}$$

c) For every $x^{*\odot} \in X^{*\odot}$ and $x^* \in X^*$,

$$<k \circ px^{*\odot},x^*> = [px^{*\odot},x^*] = <x^{*\odot},x^*>.$$

Here we have used a) and b).
d) For every $x^{\odot\odot} \in X^{\odot\odot}$ and $x^* \in X^*$,

$$[p \circ kx^{\odot\odot},x^*] = <kx^{\odot\odot},x^*> = [x^{\odot\odot},x^*],$$

and d) is proved. □

This theorem says among other things that $k : X^{\odot\odot} \to X^{*\odot}$ is an isomorphism, and that $k^{-1} = p$. Now suppose that G/S is satisfied, and define $T^{\times *}(t) = T^{\times}(t)^*$, $t \geqslant 0$. One might suspect that

$$X^{*\odot} = \{x^{**} \in X^{**} : \|T^{\times *}(t)x^{**} - x^{**}\| \to 0,\ t \downarrow 0\}.$$

And indeed, the inclusion $\subset$ is proved as follows: by Theorem 4.2b:

$$\|T^{\times *}(t)x^{*\odot}-x^{*\odot}\| = \sup_{\|x^*\|\leq 1}|<T^{\times *}(t)x^{*\odot}-x^{*\odot},x^*>| =$$

$$\sup_{\|x^*\|\leq 1}|<x^{*\odot},T^{\times}(t)x^*-x^*>| = \sup_{\|x^*\|\leq 1}|[px^{*\odot},T^{\times}(t)x^*-x^*]| =$$

$$\sup_{\|x^*\|\leq 1}|[T^{\odot\odot}(t)px^{*\odot}-px^{*\odot},x^*]|\leq\|T^{\odot\odot}(t)px^{*\odot}-px^{*\odot}\|\to 0,\ t\downarrow 0.$$

But the reverse inclusion in general does not hold as the example below shows.

EXAMPLE. Let S^1 be the one-dimensional circle group with $+$ being the addition modulo 2π. For a function $y:S^1\to\mathbb{R}$ we define its translate y_t as: $y_t(\theta) = y(t+\theta)$, $0\leq\theta\leq 2\pi$. Let Y be some vector space of bounded functions on S^1 such that
i) Y contains the constant functions
ii) $y\in Y$ implies $y_t\in Y, t\in\mathbb{R}$.
For example $Y = L^\infty(S^1)$ or $Y = C(S^1)$. (In what follows we mean by $C(S^1)$ the embedding of the space of continuous functions into $L^\infty(S^1)$.) A linear functional y^* on Y is called an *invariant mean* if
1. $y^*(y_t) = y^*(y)$, $y\in Y$, $t\in\mathbb{R}$,
2. $y^*(\mathbb{1}) = 1$
3. $|y^*(y)|\leq\sup_{\theta\in S^1}|y(\theta)|$.
Here $\mathbb{1}$ stands for the element of Y which is identically one. On $C(S^1)$ the only invariant mean is given by the Haar integral. This also defines an invariant mean on $L^\infty(S^1)$, but on this latter space there are many others: see RUDIN (1972). Now let $X = L^1(S^1)$ and let T be the C_0-group of translations on X, i.e.

$$T(t)x = x_t,\ t\in\mathbb{R}.$$

Then $X^* = L^\infty(S^1)$, $X^\odot = C(S^1)$ and $X^{**} = L^\infty(S^1)^*$. By the result of RUDIN (1972) mentioned before there exist at least two different invariant means $x_1^{**},x_2^{**}\in X^{**}$ on X^*. The restrictions of x_1^{**} and x_1^{**} to $X^\odot$ coincide and both correspond with the Haar integral. Let $v^{**} = x_1^{**}-x_2^{**}$. Then $v^{**}\in X^{**}$ and for every $x^*\in X^*$:

$$<T^{**}(t)v^{**}-v^{**},x^*> = <v^{**},T^*(t)x^*-x^*>$$
$$= <v^{**},x^*_{-t}-x^*> = 0$$

by property 1 of an invariant mean. Thus $T^{**}(t)v^{**} = v^{**}$. Suppose $v^{**}\in X^{*\odot}$. Since $<v^{**},x^\odot> = 0$ for every $x^\odot\in X^\odot$, Lemma 4.1 now implies that $v^{**} = 0$, a contradiction. Thus $v^{**}\notin X^{*\odot}$.

We conclude this section with an alternative characterization of $A^{\odot\odot}$. Let the operator $A^{\times\odot}$ on $X^{*\odot}$ be defined as follows: if $x^{*\odot},y^{*\odot}\in X^{*\odot}$ and $<x^{*\odot},A^\times x^*> = <y^{*\odot},x^*>$ for every $x^*\in D(A^\times)$ then $x^{*\odot}\in D(A^{\times\odot})$ and $A^{\times\odot}x^{*\odot} = y^{*\odot}$. Lemma 4.1 guarantees that this is a good definition.

THEOREM. 4.3. *$D(A^{\times\odot}) = k(D(A^{\odot\odot}))$ and $A^{\times\odot}\circ k = k\circ A^{\odot\odot}$ on $D(A^{\odot\odot})$.*

PROOF. '$\supset$': let $x^{\odot\odot}\in D(A^{\odot\odot})$ and $x^*\in D(A^\times)$. From Theorem 3.2a we get that

$$<kx^{\odot\odot},A^\times x^*> = [x^{\odot\odot},A^\times x^*] =$$

$$[A^{\odot\odot}x^{\odot\odot},x^*] = <kA^{\odot\odot}x^{\odot\odot},x^*>,$$

whence it follows that $kx^{\odot\odot}\in D(A^{\times\odot})$ and $A^{\times\odot}kx^{\odot\odot} = kA^{\odot\odot}x^{\odot\odot}$.
'$\subset$' is proved analogously. □

5. Generators with non-dense domain

The generator $A^{\times}$ on X^* satisfying (G1)-(G2) is nothing but a special member of a class of generators with non-dense domain. Let $(X, \|\cdot\|)$ be an arbitrary Banach space and let $A:D(A)\rightarrow X$ be a linear operator satisfying (G1). By setting $\tilde{A} = A - \omega I$ and renormalizing X by the equivalent norm

$$\|x\|' = \sup_{h>0}\sup_{n\geq 0}\|(I - h\tilde{A})^{-n}x\|, \; x\in X,$$

we may replace this assumption by

(H1) A is m −dissipative on $(X, \|\cdot\|)$.

Following Amann (1988), Da Prato & Grisvard (1984), Nagel (1983) and Walther (1986) we define

$$|||x||| = \|(I - A)^{-1}x\|, \quad x\in X$$

as a new norm on X. By (H1)

$$|||x||| \leq \|x\|, \quad x\in X.$$

In general X is not complete with respect to $|||\cdot|||$ (it is if and only if A is bounded), and we define $\hat{X}$ as the completion of X. Obviously, X is densely and continuously embedded in $\hat{X}$.

Let $X_0 = \overline{D(A)}$ and let A_0 be the part of A in X_0. Then A_0 is densely defined and m-dissipative in X_0. Let T_0 be the C_0-contraction semigroup on X_0 generated by A_0. If $D(A)$ is invariant under T_0 we can define

$$T(t) = (I - A)T_0(t)(I - A)^{-1}, \; t\geq 0. \tag{5.1}$$

Then T is a semigroup of bounded linear operators which is not necessarily strongly continuous. Clearly,

$$|||T(t)x||| = \|T_0(t)(I - A)^{-1}x\| \leq \|(I - A)^{-1}x\| = |||x|||, \; x\in X,$$

and

$$|||T(t)x - T(s)x||| = \|T_0(t)(I - A)^{-1}x - T_0(s)(I - A)^{-1}x\| \rightarrow 0 \quad \text{as } |t - s| \rightarrow 0,$$

which yields that T is a C_0-contraction semigroup on X with respect to $|||\cdot|||$. Let $\hat{T}$ be the extension of T to $\hat{X}$. Then $\hat{T}$ is a C_0-contraction semigroup on the Banach space $\hat{X}$. We denote its infinitesimal generator by $\hat{A}$. If $D(A)$ is *not* invariant under T_0, then definition (5.1) makes no sense. However, as the theorem below shows, we still have an extension $\hat{T}(t):\hat{X}\rightarrow\hat{X}$ of T_0.

Theorem 5.1. *Assume (H1). Then*

i) X_0 *is dense in* $(\hat{X}, |||\cdot|||)$
ii) T_0 *has a unique continuous extension* $\hat{T}$ *on* $(\hat{X}, |||\cdot|||)$
iii) $\hat{T}$ *is a* C_0*-contraction semigroup on* $\hat{X}$
iv) $D(\hat{A}) = X_0$
v) A *is the part of* $\hat{A}$ *in* X
vi) $\hat{T}(t) = (I - \hat{A})T_0(t)(I - \hat{A})^{-1}, \; t\geq 0$
vii) $\lim_{h\downarrow 0} |||\hat{T}(t)\hat{x} - T_0(t)(I - h\hat{A})^{-1}\hat{x}||| = 0, \; t\geq 0, \; \hat{x}\in\hat{X}$
viii) $\hat{x}\in D(\hat{A})$ and $\hat{A}\hat{x} = \hat{y}$ iff $\hat{T}(h)\hat{x} - \hat{x} = \int_0^h \hat{T}(s)\hat{y}ds, \; h>0$
ix) X *is invariant under* $\hat{T}$ *iff* $D(A)$ *is invariant under* T_0.

From (viii) it follows that for every $\hat{x}\in\hat{X}$ and $t\geq 0$,

$$\hat{S}(t)\hat{x} := \int_0^t \hat{T}(s)\hat{x}ds \in D(\hat{A}) = X_0$$

and

$$\hat{A}\hat{S}(t)\hat{x} = \hat{T}(t)\hat{x}-\hat{x}.$$

Let $S(t)$ be the restriction of $\hat{S}(t)$ to X. Then $S(t)$ is the integrated semigroup associated with A. We assume

(H2) $\{x \in X : \|x\| \leq 1\}$ is closed in $(\hat{X}, |||\cdot|||)$.

REMARK. One can easily show that (H2) is equivalent with
(H2′) $x_n \in D(A)$, $n \geq 1$, $x_n \to x$, $n \to \infty$, and $\|Ax_n\|$ bounded implies that $x \in D(A)$ and

$$\|(I-A)x\| \leq \liminf_{n\to\infty} \|(I-A)x_n\|.$$

THEOREM 5.2. *Assume (H1)-(H2). Then*

i) $D(A) = \mathrm{Fav}(T_0)$
So in particular, $D(A)$ is invariant under T_0 and X is invariant under $\hat{T}$. Let T be the restriction of $\hat{T}$ to X.
ii) $\|T(t)x\| \leq \|x\|$, $t \geq 0$, $x \in X$
iii) $T(t)S(h)x = S(h)T(t)x$
iv) $x \in D(A)$ and $y = Ax$ iff $T(h)x - x = S(h)y$, $h > 0$
v) *If $\{x_n\}$ is a bounded sequence in X such that $\{e^{-t}S(t)x_n\}$ converges uniformly as $n \to \infty$, then there exists an $x \in X$ such that $|||x_n - x||| \to 0$ and $\|S(h)x_n - S(h)x\| \to 0$, $h > 0$.*

Weakly $\star$ continuous semigroups satisfying (S1)-(S2) fit into this framework surprisingly well. Let $A^{\times}$ be a linear operator on the dual Banach space X^* satisfying (G1)-(G2) (with $M = 1$, and $\omega = 0$). Then (H1) holds. Let $\hat{X}^*$ be the completion of X^* with respect to the norm $|||\cdot|||$.

LEMMA 5.3. *Let $y_n^* \in X^*$, $\|y_n^*\| \leq M$ and $|||y_n^* - \hat{y}||| \to 0$ as $n \to \infty$ for some $\hat{y} \in \hat{X}^*$. Then $\hat{y} \in X^*$ and $y_n^* \to \hat{y}$ weakly $\star$ as $n \to \infty$.*

PROOF. Define $x_n^* \in D(A^{\times})$ by $x_n^* = (I-A^{\times})^{-1}y_n^*$. By (G1), $\|x_n^*\| \leq \|y_n^*\| \leq M$, and $\|A^{\times}x_n^*\| = \|-y_n^* + x_n^*\| \leq 2M$. Since $\{y_n^*\}$ is a Cauchy sequence with respect to $|||\cdot|||$, $\{x_n^*\}$ is a Cauchy sequence with respect to $\|\cdot\|$, hence there exists a $x^* \in X^*$ such that $\|x_n^* - x^*\| \to 0$ as $n \to \infty$. Now (G2) implies that $x^* \in D(A^{\times})$ and $A^{\times}x_n^* \to A^{\times}x^*$ weakly $\star$ as $n \to \infty$. Thus $y_n^* \to (I-A^{\times})x^*$ weakly $\star$ $n \to \infty$. From $\|x_n^* - x^*\| \to 0$ we also deduce that $|||y_n^* - (I-A^{\times})x^*||| \to 0$ as $n \to \infty$, hence $\hat{y} = (I-A^{\times})x^*$. □

This lemma shows in particular that (H2) is satisfied. Thus from Theorem 5.1 and 5.2 it follows that $A^{\times}$ generates a semigroup $T^{\times}$ on X^* which is continuous with respect to $|||\cdot|||$, hence weakly $\star$ continuous by Lemma 5.3. Furthermore (S1) follows from Theorem 5.2(iii) and (S2) from Theorem 5.2-(v).

ACKNOWLEDGEMENT

We would like to thank A.C.M. van Rooy for the reference: RUDIN (1972).

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