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Realization of Autoregressive Equations in Pencil and Descriptor Form

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A linear system described by autoregressive equations with a given input/output structure cannot be transformed to standard state space form if the implied input/output relation is nonproper. Instead, one has to use a realization in descriptor form. In this paper, we show how to obtain minimal descriptor realizations from autoregressive equations without separating finite and infinite frequencies, and without going through a reduction process. We work under external equivalence, so that we can consider even situations in which there is no transfer matrix. Our approach is based on the so-called *pencil representation* of linear systems, and we show that there is a natural realization of autoregressive equations in pencil form. In this way, we can also clarify the link between the realization theories of J. C. Willems and P. A. Fuhrmann.

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1. Introduction and preliminaries

In this paper, we study methods for obtaining state representations for linear systems given by higher-order equations in the external variables, with special attention to the so-called 'non-proper' situation. Suppose that relations between input variables u and output variables y are specified by equations of the form

$$R_1(\sigma)y + R_2(\sigma)u = 0 \tag{1.1}$$

where $R_1(\sigma)$ and $R_2(\sigma)$ are polynomial matrices, σ denotes differentation or shift (depending on whether one works in continuous time or in discrete time), and y and u are functions of time. Here, as well as in most cases below, the time argument is suppressed to alleviate the notation. Following the terminology of J. C. Willems [16], we shall refer to (1.1) as a set of *autoregressive equations*. Inputs and outputs are jointly referred to as *external variables*, and (1.1) may be rewritten as

$$R(\sigma)w = 0 \tag{1.2}$$

where $R(s) = [R_1(s) \quad R_2(s)]$ is sometimes called an *AR matrix*, and $w = [y^{\mathrm{T}} \quad u^{\mathrm{T}}]^{\mathrm{T}}$ is the *vector of external variables*. Of course, it is also possible to take (1.2) as a starting point, without distinction between 'inputs' and 'outputs' in the external variables. The *behavior* defined by (1.2) is the set of all time

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functions w that satisfy (1.2). A behavior may also be specified by other means, for instance by representations that involve *auxiliary (internal) variables*, such as the state representations to be defined below. Two representations will be said to be *externally equivalent* [1, 15] if their induced behaviors are the same. In this paper, we will be looking for minimal representations under external equivalence. In comparison with the notion of transfer equivalence which has been used more commonly in realization theory, external equivalence is both stronger and more general — more general, because transfer equivalence can be defined only for systems with a given input/output structure that is such that a transfer matrix exists, and stronger, because when both notions are applicable, external equivalence implies transfer equivalence but not the other way around.

The standard realization theory presupposes that the matrix $R_1(s)$ is square and nonsingular, and that $R_1^{-1}(s)R_2(s)$ is proper rational. Under these assumptions, it is well-known that an equivalent representation can be found in the usual state space form

$$\begin{aligned} \sigma x &= Ax + Bu \\ y &= Cx + Du. \end{aligned} \tag{1.3}$$

A powerful and elegant method to obtain such a state space realization was devised by Fuhrmann [4] who stated his result under transfer equivalence, and a similar procedure under external equivalence was given by J. C. Willems [16]. However, the standard assumptions mentioned above are not always satisfied. Examples of situations in which this occurs can for instance be found in circuit models [11], econometric models [9], and system inversion [6]. An often used modification of (1.3), which enables one to cover also these so-called *non-proper* situations, is the *descriptor* form [8]

$$\begin{aligned} \sigma Ex &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1.4}$$

where the matrix E is not necessarily invertible. Algorithms to go from (1.1) to (1.4) which follow the line of [4] have been presented in [19] and [3]. These papers both work under transfer equivalence and so there is still the assumption that the matrix $R_1(s)$ is invertible. The realization procedure is then based on a decomposition of the transfer matrix $R_1^{-1}(s)R_2(s)$ into a strictly proper and a polynomial part. For the strictly proper part, a representation in standard state space form is obtained by the usual means, and the polynomial part is realized in a special descriptor form by using a modification of Fuhrmann's procedure; finally, the two realizations are put together again to create a representation in descriptor form.

One of the important uses of realization theory is the translation of properties of and statements about linear systems from polynomial terms to state space terms and *vice versa*, as is extensively shown in [5]. The realization procedure for nonproper systems by cutting and pasting, as just described, is somewhat indirect, and is therefore less suitable for such translation purposes. In this paper, we shall show how to obtain a realization in descriptor form without separation of finite and infinite frequencies. The realization will be obtained under external equivalence, and will be minimal in the appropriate sense. As an application, we shall establish the relations between basic indices associated with the representation (1.1) and with the representation (1.4). The realization procedure will be motivated along the lines of [16], and our discussion will also clarify the relation between the realization algorithm in [16] and the one in [4].

The development below will be based on what we call the *pencil representation* of a linear system. This is a representation of the form

$$\begin{aligned} \sigma Gz &= Fz \\ w &= Hz \end{aligned} \tag{1.5}$$

where w is a vector of external variables containing both inputs and outputs, and σ again denotes either differentiation or shift. It appears that the state representation that is most naturally connected to the autoregressive representation (1.2) is in the pencil form. A similar form has been used before in [1], and pencil techniques in general are popular tools in numerical system theory (see for instance [14]). It may also be noted that the form (1.5) has been used for systems with partial differential equations in which control is exerted through the boundary conditions ('boundary control systems'; cf. [12]).

Formally, a pencil representation is given by a six-tuple $(Z, X, W; F, G, H)$ in which W is the space of external variables, Z is the space of internal variables, X is the equation space, F and G are linear mappings from Z to X, and H is a linear mapping from Z to W. We shall consider only pencil representations that are finite-dimensional in the sense that both $\dim Z$ and $\dim X$ are finite. Also, $\dim W$ will always be finite. Two pencil representations $(Z, X, W; F, G, H)$ and $(\tilde{Z}, \tilde{X}, W; \tilde{F}, \tilde{G}, \tilde{H})$ will be called *isomorphic* if there exist isomorphisms $S: Z \rightarrow \tilde{Z}$ and $T: X \rightarrow \tilde{X}$ such that $\tilde{G} = TGS^{-1}$, $\tilde{F} = TFS^{-1}$, and $\tilde{H} = HS^{-1}$. The *behavior* given by a pencil representation is the set of all w for which there exists a z such that (1.5) holds. (One has to select suitable function classes here; this will be discussed later.) A pencil representation is said to be *minimal* (under external equivalence) if both $\dim Z$ and $\dim X$ are minimal in the class of equivalent representations. Let us quickly review what can be inferred about minimality of pencil representations from the existing literature.

PROPOSITION 1.1 *A pencil representation $(Z, X, W; F, G, H)$ is minimal under external equivalence if and only if the following conditions hold:*

(i) *G is surjective;*

(ii) *$[G^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$ is injective;*

(iii) *the matrix $[sG^{\mathrm{T}} - F^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$ has full column rank for all $s \in \mathbb{C}$.*

Moreover, a minimal representation is unique up to isomorphism.

PROOF If G is not surjective in a representation of the form (1.5), then 'Step One' of the realization algorithm in [13] may be used to find an equivalent representation with a smaller equation space X. So in every minimal representation the mapping G must be surjective. By a suitable choice of bases in X and Z, a matrix representation of G may then be given as $G = [I \quad 0]$; with respect to these bases, write $F = [A \quad B]$, and $H = [C' \quad D']$. Writing z correspondingly as a vector with components ξ and η, the representation (1.5) takes the form

$$\begin{aligned} \sigma\xi &= A\xi + B\eta \\ w &= C'\xi + D'\eta. \end{aligned} \tag{1.6}$$

This is called the *general state space form* in [13]. It follows from Cor. 4.2 of that paper (see also [15], Thm. 4.5 and [16], Section 5) that such a system is minimal if and only if $V^*(A, B, C', D') = \{0\}$ and D' is injective. The condition on V^* and the injectivity of D' together imply that the associated system pencil

$$\begin{bmatrix} sI - A & B \\ C' & D' \end{bmatrix} \tag{1.7}$$

has full column rank for all s (see [7, p. 544]), so that (iii) holds. Because D' is injective, the matrix

$$\begin{bmatrix} I & 0 \\ C' & D' \end{bmatrix}$$

is injective too; this implies (ii). Conversely, if the conditions (i-iii) hold, then it follows from (ii) and (iii) that the system pencil has full column rank for all s, so that V^* in the equivalent state space form must be zero. The injectivity of D' in the equivalent state space form is immediate from (ii), by reversing the argument used above.

Now, consider two minimal representations $(Z, X, W; F, G, H)$ and $(\tilde{Z}, \tilde{X}, W; \tilde{F}, \tilde{G}, \tilde{H})$ of the same system. As above, both representations can be rewritten in driving-variable form; the resulting state space representations will be denoted by (A, B, C', D') and $(\tilde{A}, \tilde{B}, \tilde{C}', \tilde{D}')$, respectively. Because these are minimal representations of the same behavior, it follows from Thm. 7.1 in [15] that there exist invertible mappings Q and R and a mapping F such that $\tilde{A} = Q(A + BF)Q^{-1}$, $\tilde{B} = QBR$, $\tilde{C}' = (C' + D'F)Q^{-1}$ and $\tilde{D}' = DR$. So we can write the following equations:

$$[I \quad 0] = Q[I \quad 0] \begin{bmatrix} Q^{-1} & 0 \\ FQ^{-1} & R \end{bmatrix} \tag{1.8}$$

$$[\tilde{A} \quad \tilde{B}] = Q[A \quad B]\begin{bmatrix} Q^{-1} & 0 \\ FQ^{-1} & R \end{bmatrix} \tag{1.9}$$

$$[\tilde{C}' \quad \tilde{D}'] = [C' \quad D']\begin{bmatrix} Q^{-1} & 0 \\ FQ^{-1} & R \end{bmatrix}. \tag{1.10}$$

This shows that the two given representations are isomorphic.

The condition (ii) in the above proposition can also be formulated in a different way, which brings out an analogy with condition (iii).

PROPOSITION 1.2 *Suppose that* $M(s) = [sG^{\mathrm{T}} - F^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$ *has full column rank, and that* G *is surjective. Under these conditions, the matrix* $[G^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$ *is injective if and only if* $M(s)$ *has no zeros at infinity.*

The proof of this is immediate from the following two lemmas.

LEMMA 1.3 *Let the matrices* G *and* H *be such that* $[G^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$ *is injective, and let* F *be any matrix of the same size as* G. *Under these conditions,* $M(s) = [sG^{\mathrm{T}} - F^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$ *has no zeros at infinity.*

PROOF Let $[K \quad L]$ be a left inverse of $[G^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$; then $(I - s^{-1}KF)^{-1}[s^{-1}K \quad L]$ is a proper rational left inverse of $M(s)$, and it follows that $M(s)$ can have no zeros at infinity (see for instance [7, Exc. 6.5-14]).

LEMMA 1.4 *If* $M(s) = [sG^{\mathrm{T}} - F^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$ *is of full column rank and has no zeros at infinity, then*

$$\ker \begin{bmatrix} G \\ H \end{bmatrix} \cap F^{-1}[\operatorname{im} G] = \{0\}. \tag{1.11}$$

PROOF The proof is by contradiction. Suppose that (1.11) would not hold; then there would exist a $z_0 \neq 0$ and a z_1 such that $Gz_0 = 0$, $Hz_0 = 0$, and $Fz_0 = Gz_1$. Write $z(s) = z_0 + z_1 s^{-1}$; then

$$M(s)z(s) = \begin{bmatrix} sG - F \\ H \end{bmatrix}(z_0 + z_1 s^{-1}) = \begin{bmatrix} Gz_2 - Fz_1 \\ Hz_1 \end{bmatrix} s^{-1} + \cdots \tag{1.12}$$

The proper, but not strictly proper function $z(s)$ is therefore mapped by $M(s)$ to a strictly proper function, which implies (see for instance [10]) that $M(s)$ must have a zero at infinity.

2. PENCIL REPRESENTATIONS FROM A GIVEN BEHAVIOR: DISCRETE TIME

In this section, we shall discuss the pencil representation for systems that are given directly through their (discrete-time) behavior. Our treatment here is close to the development in [16]; however, we emphasize the pencil representation rather than the driving-variable representation, and we derive some results that do not depend on the assumption that the behavior is closed in the topology of pointwise convergence.

Following the definition in [16], a *linear, time-invariant, discrete-time behavior* is a shift-invariant subspace of the space $W^{\mathbb{Z}_+}$ of all functions from $\mathbb{Z}_+$ to a vector space $W \simeq \mathbb{R}^q$. The following mappings are defined on $W^{\mathbb{Z}_+}$: the *shift*

$$\sigma: (w_0, w_1, \cdots) \mapsto (w_1, w_2, \cdots), \tag{2.1}$$

the *forward shift*

$$\sigma^*: (w_0, w_1, \cdots) \mapsto (0, w_0, w_1, \cdots), \tag{2.2}$$

and the *evaluation mapping at time 0*

$$\chi: (w_0, w_1, \cdots) \mapsto w_0. \tag{2.3}$$

Now, let $\mathcal{B}$ be a given behavior. Following [16], we introduce the subspaces

$$\mathcal{B}^0 = \{w \in \mathcal{B} \mid (\sigma^*)^k w \in \mathcal{B} \ \ \forall k \geq 0\} \tag{2.4}$$

and

$$\mathscr{B}^1 = \{w \in \mathscr{B}^0 \mid \chi w = 0\} \tag{2.5}$$

of $\mathscr{B}$. Intuitively, $\mathscr{B}^0$ contains the trajectories that start from the zero state; so the quotient space $\mathscr{B}/\mathscr{B}^0$ should be (isomorphic to) the state space. The quotient space $\mathscr{B}/\mathscr{B}^1$ describes the freedom that arises at each point in time because of the freedom one has in choosing an 'input' variable (or rather, a value of the 'driving variable'). So, $\mathscr{B}/\mathscr{B}^1$ is the candidate for the space of driving variables. The following facts are trivially verified:

$$\sigma\mathscr{B}^1 \subset \mathscr{B}^0 \tag{2.6}$$

$$\mathscr{B}^1 \subset \ker\chi. \tag{2.7}$$

Because of (2.6), we can properly define a mapping $M_1 : \mathscr{B}/\mathscr{B}^1 \to \mathscr{B}/\mathscr{B}^0$ by

$$M_1 : w \bmod \mathscr{B}^1 \mapsto \sigma w \bmod \mathscr{B}^0. \tag{2.8}$$

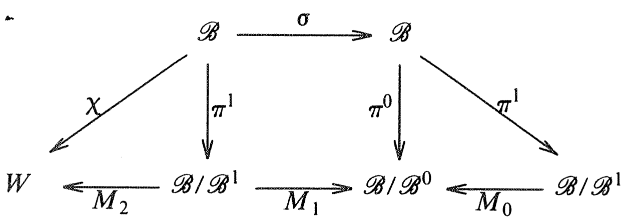
Because of (2.7), there is also a mapping $M_2 : \mathscr{B}/\mathscr{B}^1 \to W$ defined by

$$M_2 : w \bmod \mathscr{B}^1 \mapsto \chi w. \tag{2.9}$$

Furthermore, we introduce the projection mapping $M_0 : \mathscr{B}/\mathscr{B}^1 \to \mathscr{B}/\mathscr{B}^0$, defined simply by

$$M_0 : w \bmod \mathscr{B}^1 \mapsto w \bmod \mathscr{B}^0. \tag{2.10}$$

If elements of $\mathscr{B}/\mathscr{B}^1$ are seen as 'state + driving variable', then M_0 deletes the driving variable. The mappings M_0, M_1 and M_2 could also have been introduced by requiring that Diagram 1 below commutes, where π^0 denotes projection modulo $\mathscr{B}^0$ and π^1 projection modulo $\mathscr{B}^1$.



— Diagram 1 —

The discrete-time behavior described by a *pencil representation* such as (1.5) will be denoted by $\mathscr{B}_p(Z, X, W; F, G, H)$. More explicitly,

$$\mathscr{B}_p(Z, X, W; F, G, H) = \{w : \mathbb{Z}_+ \to W \mid \exists z : \mathbb{Z}_+ \to Z \text{ s.t. } \sigma Gz = Fz \text{ and } Hz = w\}. \tag{2.11}$$

We can now formulate the following proposition.

PROPOSITION 2.1 *For any linear, time-invariant, discrete-time behavior $\mathscr{B}$, one has*

$$\mathscr{B} \subset \mathscr{B}_p(\mathscr{B}/\mathscr{B}^1, \mathscr{B}/\mathscr{B}^0, W; M_1, M_0, M_2). \tag{2.12}$$

PROOF Take $w \in \mathscr{B}$. Define $z : \mathbb{Z}_+ \to \mathscr{B}/\mathscr{B}^1$ by

$$z_k = \pi^1 \sigma^k w. \tag{2.13}$$

One easily verifies from the definitions that $\sigma M_0 z = M_1 z$ and that $M_2 z = w$. This proves that $w \in \mathscr{B}_p(\mathscr{B}/\mathscr{B}^1, \mathscr{B}/\mathscr{B}^0, W; M_1, M_0, M_2)$.

The *closure* of a behavior $\mathscr{B}$ (in the topology of pointwise convergence) will be denoted by $\mathscr{B}^{cl}$. A sequence w belongs to $\mathscr{B}^{cl}$ if and only if for every $k \geqslant 0$ there exists a $\tilde{w} \in \mathscr{B}$ such that $w_j = \tilde{w}_j$ for all

$0 \leqslant j \leqslant k$.

PROPOSITION 2.2 *For any linear, time-invariant, discrete-time behavior $\mathcal{B}$, one has*

$$\mathcal{B}^{\mathrm{cl}} \supset \mathcal{B}_{\mathrm{p}}(\mathcal{B}/\mathcal{B}^1, \mathcal{B}/\mathcal{B}^0, W; M_1, M_0, M_2). \tag{2.14}$$

PROOF Take $w \in \mathcal{B}_{\mathrm{p}}(\mathcal{B}/\mathcal{B}^1, \mathcal{B}/\mathcal{B}^0, W; M_1, M_0, M_2)$, and let $z: \mathbb{Z}_+ \to \mathcal{B}/\mathcal{B}^1$ be such that $\sigma M_0 z = M_1 z$ and $M_2 z = w$. To show that $w \in \mathcal{B}^{\mathrm{cl}}$, we shall prove by induction that for every k there exists a $\tilde{w}^k \in \mathcal{B}$ such that $w_i = \tilde{w}_i^k$ for $0 \leqslant i \leqslant k$. First, let $\hat{w}^k \in \mathcal{B}$ be such that

$$z_k = \pi^1 \hat{w}^k. \tag{2.15}$$

Next, define $\tilde{w}^k$ by

$$\tilde{w}^k = (\hat{w}_0^0, \hat{w}_0^1, \cdots, \hat{w}_0^k, \hat{w}_1^k, \hat{w}_2^k, \cdots). \tag{2.16}$$

For $0 \leqslant i \leqslant k$, one has

$$w_i = M_2 z_i = M_2 \pi^1 \hat{w}^i = \chi \hat{w}^i = \hat{w}_0^i = \tilde{w}_i^k. \tag{2.17}$$

It remains to prove that $\tilde{w}^k \in \mathcal{B}$ for all k. For $k = 0$, this is trivial since $\tilde{w}^0 = \hat{w}^0 \in \mathcal{B}$. Since

$$\begin{aligned} \tilde{w}^{k+1} - \tilde{w}^k &= (0, 0, \cdots, 0, \hat{w}_0^{k+1} - \hat{w}_1^k, \hat{w}_1^{k+1} - \hat{w}_2^k, \cdots) = \\ &= (\sigma^*)^k (\hat{w}^{k+1} - \sigma \hat{w}^k), \end{aligned} \tag{2.18}$$

the proof will follow by induction if we can show that $\hat{w}^{k+1} - \sigma \hat{w}^k \in \mathcal{B}^0$ for all k. But this follows from

$$\pi^0 \hat{w}^{k+1} = M_0 \pi^1 \hat{w}^{k+1} = M_0 z_{k+1} = M_1 z_k = M_1 \pi^1 \hat{w}^k = \pi^0 \sigma \hat{w}^k. \tag{2.19}$$

COROLLARY 2.3 [16] *If $\mathcal{B} = \bar{\mathcal{B}}^{\mathrm{cl}}$, then $\mathcal{B}_{\mathrm{p}}(\mathcal{B}/\mathcal{B}^1, \mathcal{B}/\mathcal{B}^0, W; M_1, M_0, M_2) = \mathcal{B}$.*

The above corollary states that every closed, linear, time-invariant behavior admits a pencil representation. Moreover, as shown in [16, Thm. 9], the spaces $\mathcal{B}/\mathcal{B}^1$ and $\mathcal{B}/\mathcal{B}^0$ that appear in the representation $\mathcal{B}_{\mathrm{p}}(\mathcal{B}/\mathcal{B}^1, \mathcal{B}/\mathcal{B}^0, W; M_1, M_0, M_2)$ are *finite-dimensional*. For completeness, we shall offer a proof of this fact which we think is more straightforward than the two proofs that were already given for essentially the same fact in [16]. Some notation will be needed. Let $[w]_k$ denote the *k-truncation* of an element w of $W^{\mathbb{Z}_+}$: if

$$w = (w_1, w_2, \cdots, w_k, w_{k+1}, \cdots), \tag{2.20}$$

then

$$[w]_k = (w_1, w_2, \cdots, w_k). \tag{2.21}$$

For subspaces $\mathcal{B}$ of $W^{\mathbb{Z}_+}$, write

$$\mathcal{B}_k = \{[w]_k \mid w \in \mathcal{B}\}. \tag{2.22}$$

Define a sequence of subspaces of W by

$$W_k^0(\mathcal{B}) = \{w \in W \mid (0, 0, \cdots, 0, w) \in \mathcal{B}_k\}. \tag{2.23}$$

We shall let $\mathcal{B}$ be a fixed linear time-invariant behavior, and write W_k^0 rather than $W_k^0(\mathcal{B})$. It is immediate from $\sigma\mathcal{B} \subset \mathcal{B}$ that $W_{k+1}^0 \subset W_k^0$ for all k. Because W is finite-dimensional, the sequence of subspaces $W_1^0 \supset W_2^0 \supset \cdots$ must reach a limit after a finite number of steps; the limit subspace will be denoted by W^0. We now prove:

LEMMA 2.4 *Suppose that $\mathcal{B}$ is closed. Let k_0 be such that $W_{k_0}^0 = W^0$, and let $\Phi: \mathcal{B} \to \mathcal{B}_{k_0}$ denote the mapping $w \mapsto [w]_{k_0}$. Under these conditions, one has*

$$\ker \Phi \subset \mathcal{B}^0. \tag{2.24}$$

PROOF Since $\mathscr{B}^0$ is by definition the largest σ^*-invariant subspace of $\mathscr{B}$, it suffices to show that $\ker\Phi$ is σ^*-invariant. Take $w\in\ker\Phi$; we want to show that also $\sigma^*w\in\ker\Phi$, which will follow if we can prove that $\sigma^*w\in\mathscr{B}$. For this, it is sufficient to show that

$$[\sigma^*w]_j \in \mathscr{B}_j \qquad \forall j\geqslant 0, \tag{2.25}$$

by the closedness of $\mathscr{B}$. For $0\leqslant j\leqslant k_0+1$, $[\sigma^*w]_j=0$ and so the condition (2.25) is certainly satisfied. To proceed by induction, suppose that $[\sigma^*w]_i\in\mathscr{B}_i$ for some $i\geqslant k_0+1$. Let $\tilde{w}\in\mathscr{B}$ be such that $[\tilde{w}]_i=[\sigma^*w]_i$. We then have $[w-\sigma\tilde{w}]_{i-1}=0$, and therefore,

$$w_i-\tilde{w}_{i+1}\in W_i^0=W_{i+1}^0. \tag{2.26}$$

From (2.26) and the fact that $[\sigma^*w-\tilde{w}]_i=0$, it follows that

$$[\sigma^*w-\tilde{w}]_{i+1}\in\mathscr{B}_{i+1}. \tag{2.27}$$

Since $[\tilde{w}]_{i+1}$ obviously belongs to $\mathscr{B}_{i+1}$, we may conclude that $[\sigma^*w]_{i+1}\in\mathscr{B}_{i+1}$, which is what we wanted to prove.

REMARK 2.5 From the lemma, one easily derives that W^0 is equal to $\chi\mathscr{B}^0$.

PROPOSITION 2.6 *If a linear, time-invariant behavior $\mathscr{B}$ is closed, then $\mathscr{B}/\mathscr{B}^0$ is finite-dimensional.*

PROOF By the lemma, we have

$$\dim\mathscr{B}/\mathscr{B}^0\leqslant\dim\mathscr{B}/\ker\Phi=\dim\operatorname{im}\Phi\leqslant\dim W^{k_0}=q(k_0+1). \tag{2.28}$$

It is not hard to show directly that the pencil representation obtained above is, in fact, minimal.

LEMMA 2.7 *If $(Z, X, W; F, G, H)$ is a pencil representation of the linear, time-invariant behavior $\mathscr{B}$, then*

$$\dim X\geqslant\dim\mathscr{B}/\mathscr{B}^0 \tag{2.29}$$

and

$$\dim Z\geqslant\dim\mathscr{B}/\mathscr{B}^1. \tag{2.30}$$

PROOF Introduce the behavior of the auxiliary variables

$$\mathscr{Z}=\{z:\mathbb{Z}_+\mapsto Z \mid \sigma Gz=Fz\}. \tag{2.31}$$

By definition of a pencil representation, we have

$$H\mathscr{Z}=\mathscr{B}. \tag{2.32}$$

In analogy with $\mathscr{B}^0$, we also introduce

$$\mathscr{Z}^0=\{z\in\mathscr{Z}\mid(\sigma^*)^kz\in\mathscr{Z}\ \forall k\geqslant 0\}. \tag{2.33}$$

Obviously, one has

$$H\mathscr{Z}^0\subset\mathscr{B}^0. \tag{2.34}$$

It is easily verified that, in fact,

$$\mathscr{Z}^0=\{z\in\mathscr{Z}\mid Gz_0=0\}, \tag{2.35}$$

which shows that $\mathscr{Z}^0$ is the kernel of the mapping which assigns the element Gz_0 of X to a given $z\in\mathscr{Z}$. As a consequence, we get

$$\dim(\mathscr{Z}/\mathscr{Z}^0)\leqslant\dim X. \tag{2.36}$$

Because of (2.34), we can unambiguously define a mapping $\Psi:\mathscr{Z}/\mathscr{Z}^0\rightarrow\mathscr{B}/\mathscr{B}^0$ by

$$\Psi : z \bmod \mathscr{Z}^0 \mapsto Hz \bmod \mathscr{B}^0. \tag{2.37}$$

Moreover, (2.32) shows that this map is surjective. Therefore,

$$\dim \mathscr{B}/\mathscr{B}^0 \leqslant \dim \mathscr{Z}/\mathscr{Z}^0 \leqslant \dim X. \tag{2.38}$$

For the proof of the second inequality, one introduces

$$\mathscr{Z}^1 = \{z \in \mathscr{Z}^0 \mid z_0 = 0\} = \{z \in \mathscr{Z} \mid z_0 = 0\} \tag{2.39}$$

and proceeds analogously, noting that $H\mathscr{Z}^1 \subset \mathscr{B}^1$ and that $\dim(\mathscr{Z}/\mathscr{Z}^1) \leqslant \dim Z$.

We summarize the main results in the following theorem.

THEOREM 2.8 *Let $\mathscr{B}$ be a closed, linear, time-invariant, discrete-time behavior. Then a finite-dimensional minimal pencil representation of $\mathscr{B}$ is given by $(\mathscr{B}/\mathscr{B}^1, \mathscr{B}/\mathscr{B}^0, W; M_1, M_0, M_2)$, where $\mathscr{B}^0$ and $\mathscr{B}^1$ are defined by (2.4) and (2.5), respectively, and the mappings M_0, M_1 and M_2 are defined by requiring that Diagram 1 commutes.*

A behavior $\mathscr{B}$ will rarely be given 'as such', and consequently the construction of a pencil representation as given above is mainly of theoretical value. Two important ways of prescribing a behavior are the following:

▷ by *data*: $\mathscr{B}$ is determined as the smallest closed, linear, shift-invariant subspace of $W^{\mathbb{Z}_+}$ that contains a given (finite) set of trajectories. This leads to realization procedures involving generalizations of the Hankel matrix: see [17] and, for the case of approximate modeling, [18].

▷ by *equations*: $\mathscr{B}$ is determined as the set of all trajectories that satisfy a certain set of differential or difference equations. For the purpose of describing a closed, linear, time-invariant behavior, such equations may always be rewritten in the form $R(\sigma)w = 0$, where $R(s)$ is a polynomial matrix [15, Prop. 3.3].

We shall be concerned with the second option in this paper. In the next section, we shall consider systems given by a set of equations $R(\sigma)w = 0$, and we shall construct a pencil representation by expressing the spaces $\mathscr{B}/\mathscr{B}^0$ etc. in terms of the polynomial matrix $R(s)$.

3. PENCIL REPRESENTATIONS FROM AUTOREGRESSIVE EQUATIONS: DISCRETE TIME

Let a behavior be given by

$$R(\sigma)w = 0 \tag{3.1}$$

where $R(s)$ is a polynomial matrix of size $k \times q$, and σ denotes the shift. We shall continue to work in discrete time in order to employ the results of the previous section to give a representation in pencil form for the behavior described by (1.1). Similar results can be obtained for systems in continuous time, but these require a different proof technique and will be handled in the next section.

It will be convenient to use an alternative notation for time series, more adapted to the description in terms of a polynomial matrix. Via the correspondence

$$(w_0, w_1, \cdots) \leftrightarrow w_0\lambda^{-1} + w_1\lambda^{-2} + \cdots, \tag{3.2}$$

we can identify $W^{\mathbb{Z}_+}$ with the set of formal power series (with vanishing constant term) in the parameter λ^{-1}. This set, to be denoted by ΩW, is a subset of the set ΛW of formal Laurent series around infinity in λ, of which a typical element is

$$w_{-i-1}\lambda^i + w_{-i}\lambda^{i-1} + \cdots + w_{-1} + w_0\lambda^{-1} + w_1\lambda^{-2} + \cdots.$$

The natural projection of ΛW onto ΩW, effected by 'deleting the polynomial part', will be denoted by π_-. Elements of ΩW will be written as $w(\lambda)$ or sometimes also simply as w.

The action of the shift σ on $W^{\mathbb{Z}_+}$ corresponds on ΩW to multiplication by λ followed by projection:

$$\sigma w \leftrightarrow \pi_-(\lambda w(\lambda)). \tag{3.3}$$

Consequently, the behavior $\mathscr{B}$ given by (1.1) is represented in ΩW by the set X^R that is defined by

$$X^R = \{w \in \Omega W \mid \pi_-(R(\lambda)w(\lambda)) = 0\}. \tag{3.4}$$

The right shift σ^* is represented in ΩW by multiplication by λ^{-1}. Therefore, $\mathscr{B}^0$ corresponds to the subspace N^R defined by

$$N^R = \{w \in \Omega W \mid \pi_-(\lambda^{-k}R(\lambda)w(\lambda)) = 0 \ \forall k \geq 0\} = \{w \in \Omega W \mid R(\lambda)w(\lambda) = 0\}. \tag{3.5}$$

Finally, $\mathscr{B}^1$ is equal to $\sigma^*\mathscr{B}^0$, which corresponds to $\lambda^{-1}N^R$.

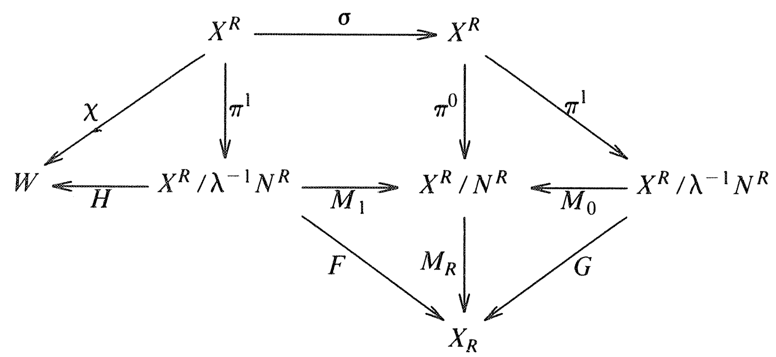
The quotient space $\mathscr{B}/\mathscr{B}^0$, which plays a role in the pencil representation of the previous section as the space in which the dynamic equation 'takes place', is represented as X^R/N^R. We can consider multiplication by $R(\lambda)$ as a mapping from X^R to $\mathbb{R}^k[\lambda]$, the set of polynomials with coefficients in $\mathbb{R}^k$. The space N^R is then precisely the kernel of this mapping, which suggests replacing the quotient space X^R/N^R by the isomorphic space

$$X_R = \{p(\lambda) \in \mathbb{R}^k[\lambda] \mid \exists w(\lambda) \in \Omega W \text{ s.t. } R(\lambda)w(\lambda) = p(\lambda)\}. \tag{3.6}$$

The isomorphism is given, of course, by the mapping M_R defined as follows:

$$M_R: w(\lambda) \bmod N^R \mapsto R(\lambda)w(\lambda) \qquad (w(\lambda) \in X^R). \tag{3.7}$$

With some of the notation used in Diagram 1 unchanged, we now introduce the mappings F, G and H by requiring that the diagram below commutes. We then obtain the following theorem.



— Diagram 2 —

THEOREM 3.1 *The behavior given by (1.1) is equal to* $\mathscr{B}_p(X^R/\lambda^{-1}N^R, X_R, W; F, G, H)$; *and this pencil representation is minimal.*

PROOF Apart from changes of notation, all we did was replacing the representation derived in the previous section by an isomorphic one. The result is therefore immediate from the previous section.

Bases for the vector spaces X_R and $X^R/\lambda^{-1}N^R$ may be found by taking $R(s)$ to row reduced form, and then concrete matrix representations for the mappings F, G, and H can be obtained. This is worked out in section 8.

4. PENCIL REPRESENTATIONS FROM AUTOREGRESSIVE EQUATIONS: CONTINUOUS TIME

In the discrete-time context, many system properties are conveniently expressed in terms of the behavior itself, and we have used this fact extensively in the previous sections to prove properties of representations; for instance, equivalence between AR and pencil representations could be proved by reducing both to their associated behaviors. For systems in continuous time, however, the representation of a behavior in terms of itself is much less manageable, and one is forced to work with representations in terms of equations. The formal definition of a continuous-time behavior requires the specification of a function class to which the trajectories should belong. We shall denote the function class to which the (components of the) trajectories of the external variables belong by $\mathscr{F}$; the class from which the components of the trajectories of internal (auxiliary) variables are taken will be denoted by $\mathscr{D}$. We shall assume that $\mathscr{D}$ is a linear function space that is closed under differentiation and that contains $\mathscr{F}$; differential equations will always be considered in the sense of $\mathscr{D}$. All properties used below will be valid when $\mathscr{F} = \mathscr{D} = C^\infty(\mathbb{R})$ (see for instance [13]), but other choices are also possible — however, we shall not go into the axiomatics here. Confer also the discussion in [2, Ch. 4, 5]. The development below may also be applied to systems in discrete time, although the approach of the preceding two sections would seem to be preferable for its intuitive appeal.

To start with, we need the following lemma which relates 'ARMA' to 'AR' representations. In the context of polynomial matrices, we use the term 'right unimodular' for 'having a polynomial right inverse'.

LEMMA 4.1 *Let a behavior $\mathscr{B}$ be given by the ARMA representation*

$$\mathscr{B} = \{w \in \mathscr{F}^q \mid \exists \xi \in \mathscr{D}^n \text{ s.t. } P(\sigma)w = Q(\sigma)\xi\} \tag{4.1}$$

where σ denotes differentiation, $P(s)$ is a polynomial matrix of size $k \times q$, and $Q(s)$ is a polynomial matrix of size $k \times n$. Let the rank of $Q(s)$ be r; then there exists a right unimodular matrix $V(s)$ of size $(k-r)\times k$ such that $V(s)Q(s) = 0$. If $V(s)$ is any such matrix and if we define $R(s)$ by $R(s) = V(s)P(s)$, then $R(s)$ is an AR matrix for $\mathscr{B}$ — that is,

$$\mathscr{B} = \{w \in \mathscr{F}^q \mid R(\sigma)w = 0\}. \tag{4.2}$$

PROOF For instance by reduction to Hermite form [7, p. 375], one can find a unimodular matrix $[T(s)^{\mathrm{T}} \quad V(s)^{\mathrm{T}}]^{\mathrm{T}}$ such that

$$\begin{bmatrix} T(s) \\ V(s) \end{bmatrix} Q(s) = \begin{bmatrix} N(s) \\ 0 \end{bmatrix} \tag{4.3}$$

where $N(s)$ has full row rank. Clearly, the number of rows of $T(s)$ must be equal to r, so that the size of $V(s)$ is $(k-r)\times k$. Furthermore, $V(s)$ consists of a number of rows of a unimodular matrix, and so it is right unimodular.

Now, let $V(s)$ be a matrix that satisfies the conditions of the theorem; then $V(s)$ may be completed to a unimodular matrix $[T(s)^{\mathrm{T}} \quad V(s)^{\mathrm{T}}]^{\mathrm{T}}$. The equation $P(\sigma)w = Q(\sigma)\xi$ is equivalent to the set of two equations

$$T(\sigma)P(\sigma)w = T(\sigma)Q(\sigma)\xi \tag{4.4}$$

$$V(\sigma)P(\sigma)w = 0. \tag{4.5}$$

Because $T(s)Q(s)$ has full row rank, the equation (4.4) puts no constraint on w (cf. [15, Prop. 3.3]), which leaves (4.5) as the defining equation.

We can now formulate a criterion for equivalence between a pencil representation and an AR representation. The criterion will be formulated only for pencil representations of the form (1.5) that are nonredundant in the sense that the matrix $sG - F$ has full row rank and $[sG^{\mathrm{T}} - F^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$ has full column rank. In applications, these requirements will often be met by virtue of the stronger conditions (i) and (ii) of Prop. 1.1. It is shown in [13] that every pencil representation can be reduced to an equivalent pencil representation that satisfies these conditions.

LEMMA 4.2 *Consider the pencil representation (1.5) with $G, F \in \mathbb{R}^{n\times(n+m)}$, $H \in \mathbb{R}^{q\times(n+m)}$, under the assumptions that $sG - F$ has full row rank and $[sG^{\mathrm{T}} - F^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$ has full column rank. This representation is equivalent to the AR representation*

$$R(\sigma)w = 0 \tag{4.6}$$

with $R(s) \in \mathbb{R}^{k\times q}[s]$ of full row rank if and only if $k = q - m$ and there exists a matrix $V(s) \in \mathbb{R}^{k\times n}[s]$ that *is left coprime with $R(s)$ and that satisfies*

$$V(s)(sG - F) = R(s)H. \tag{4.7}$$

PROOF The representation (1.5) can, of course, also be written in ARMA form:

$$\begin{bmatrix} 0 \\ I \end{bmatrix} w = \begin{bmatrix} \sigma G - F \\ H \end{bmatrix} z. \tag{4.8}$$

As in the proof of the preceding lemma, one can find a right unimodular matrix $[V_1(s) \quad V_2(s)]$ such that

$$[V_1(s) \quad V_2(s)] \begin{bmatrix} sG - F \\ H \end{bmatrix} = 0. \tag{4.9}$$

An AR matrix for the system given by (1.5) is then found as

$$[V_1(s) \quad V_2(s)] \begin{bmatrix} 0 \\ I \end{bmatrix} = V_2(s). \tag{4.10}$$

Note that the matrix $V_2(s)$ must be of full row rank, because by (4.9) we can write $[V_1(s) \quad V_2(s)] = V_2(s)[-HN(s) \quad I]$ where $N(s)$ is a right inverse of $sG-F$. Now, if the representation (4.6) is equivalent to (1.5), then, by the uniqueness theorem for AR representations (see [16, Section 4] and [13, Cor. 2.5]), there must be a unimodular matrix $U(s)$ such that $U(s)V_2(s) = R(s)$. The conditions in the statement of the lemma are then satisfied by taking $V(s) = U(s)V_1(s)$. Also, we have $k = q - m$ because the number of rows of $V_2(s)$ is equal to $n + q - (n + m) = q - m$.

Conversely, if there exists a polynomial matrix $V(s)$ as stated in the lemma, then (4.9) will be satisfied by taking $V_1(s) = -V(s)$ and $V_2(s) = R(s)$. If we also assume that $k = q - m$ then the number of rows of $[V_1(s) \quad V_2(s)]$ is equal to the number of rows of $[sG^{\mathrm{T}} - F^{\mathrm{T}} \quad H^{\mathrm{T}}]^{\mathrm{T}}$ minus the rank of this matrix, and consequently the representations (1.5) and (4.6) are equivalent by Lemma 4.1.

In the discrete-time context, we used quotients of sequence spaces to construct the vector spaces that are needed in a pencil representation. It should be noted that the end result would have been the same if we would have replaced the sequence spaces by corresponding spaces of rational vector functions; in particular, the space $W(\lambda)$ of rational functions with values in W may be substituted for ΛW, and $\lambda^{-1}W[[\lambda^{-1}]]$ (the space of strictly proper rational W-valued functions) for ΩW. For continuous-time systems, the use of sequence spaces is less natural, and we shall use the rational setting. This will also facilitate comparison with the results of Fuhrmann (see, e.g., [5]). The symbol π_- will be used now for the natural projection of $X(\lambda)$ (where X is any vector space) onto $\lambda^{-1}X[[\lambda^{-1}]]$. We use λ as a formal parameter and s as a complex parameter. For an element $w(\lambda)$ of $\lambda^{-1}W[[\lambda^{-1}]]$, the value of $sw(s)$ at infinity will be denoted by w_{-1} in accordance with the notation of [5], rather than by w_0 as would be suggested by (3.2).

The next theorem is the main result of this section. Essentially, it shows how to solve the equations that one obtains by requiring that Diagram 2 commutes.

THEOREM 4.3 *Let a system be given in AR form (4.6), with $R(s) \in \mathbb{R}^{k \times q}(s)$ of full row rank. Consider the following spaces of rational vector functions in a formal parameter λ:*

$$X^R = \{w(\lambda) \in \lambda^{-1}W[[\lambda^{-1}]] \mid \pi_- R(\lambda)w(\lambda) = 0\} \tag{4.11}$$

$$X_R = \{p(\lambda) \in \mathbb{R}^k[\lambda] \mid \exists w(\lambda) \in \lambda^{-1}W[[\lambda^{-1}]] \text{ s.t. } p(\lambda) = R(\lambda)w(\lambda)\} \tag{4.12}$$

$$N^R = \{w(\lambda) \in \lambda^{-1}W[[\lambda^{-1}]] \mid R(\lambda)w(\lambda) = 0\}. \tag{4.13}$$

The following mappings (G and F from $X^R/\lambda^{-1}N^R$ to X_R, H from $X^R/\lambda^{-1}N^R$ to W) are well-defined:

$$G: w(\lambda) \bmod \lambda^{-1}N^R \mapsto R(\lambda)w(\lambda) \tag{4.14}$$

$$F: w(\lambda) \bmod \lambda^{-1}N^R \mapsto R(\lambda)\pi_-(\lambda w(\lambda)) \tag{4.15}$$

$$H: w(\lambda) \bmod \lambda^{-1}N^R \mapsto w_{-1}. \tag{4.16}$$

With these definitions, $(X^R/\lambda^{-1}N^R, X_R, W; F, G, H)$ is a minimal pencil representation of the behavior given by (4.6).

PROOF The well-definedness of the mappings F, G and H is easily verified. By the preceding lemma, the assertion that the pencil representation constructed in the theorem statement is equivalent to the AR representation (4.6) will be proved if we can find a polynomial mapping $V(s)\colon X_R \to \mathbb{C}^k$ such that (4.7) is satisfied, and such that $V(s)$ is left coprime with $R(s)$. We claim that such a polynomial mapping is given by the 'evaluation map' which replaces the formal parameter λ by the complex number s:

$$V(s)\colon X_R \ni p(\lambda) \mapsto p(s) \in \mathbb{C}^k. \tag{4.17}$$

This map is polynomial because X_R consists of polynomial vectors; this is evident when one writes a matrix representation of $V(s)$. To verify that (4.7) holds, we compute, for $w(\lambda) \in X^R$:

$$\begin{aligned} V(s)(sG - F)w(\lambda) &= V(s)[sR(\lambda)w(\lambda) - R(\lambda)(\lambda w(\lambda) - w_{-1})] = \\ &= sR(s)w(s) - R(s)(sw(s) - w_{-1}) = \\ &= R(s)w_{-1} = R(s)Hw(\lambda). \end{aligned} \tag{4.18}$$

Next, we have to show that $V(s)$ and $R(s)$ are left coprime; for this purpose, it suffices to produce polynomial mappings $Q_1(s)$ and $Q_2(s)$ such that

$$V(s)Q_1(s) + R(s)Q_2(s) = I. \tag{4.19}$$

By assumption, $R(s)$ has full row rank, so it has a rational right inverse, say $T(s)$. We split $T(s)$ in a polynomial and a strictly proper part, denoted respectively by $T_+(s)$ and $T_-(s)$. Obviously, we have

$$R(s)T_-(s) = I - R(s)T_+(s), \tag{4.20}$$

where the right hand side is polynomial. It follows that the columns of $R(\lambda)T_-(\lambda)$ belong to X_R. Consequently, there exists a constant matrix Q_1 such that

$$R(s)T_-(s) = V(s)Q_1. \tag{4.21}$$

Writing $T_+(s)$ as $Q_2(s)$, we get

$$V(s)Q_1 + R(s)Q_2(s) = R(s)T_-(s) + R(s)T_+(s) = R(s)T(s) = I. \tag{4.22}$$

It remains to show that the realization obtained above is minimal. For this, we use the criterion given in Prop. 1.1. Because $\lambda^{-1}N^R$ is contained in N^R, it is obvious from the definition (4.14) that G is surjective. If, for some $w(\lambda) \in X^R$, both $w_{-1} = 0$ and $R(\lambda)w(\lambda) = 0$, then $\lambda w(\lambda)$ belongs to N^R so $w(\lambda)$ belongs to $\lambda^{-1}N^R$. This shows that the mapping $[G^\top \quad H^\top]^\top$ is injective. Finally, suppose that $s \in \mathbb{C}$ and $w(\lambda) \in X^R$ are such that one has

$$sR(\lambda)w(\lambda) - R(\lambda)\pi_-(\lambda w(\lambda)) = 0 \tag{4.23}$$

$$w_{-1} = 0. \tag{4.24}$$

Because of (4.24), $\pi_-(\lambda w(\lambda))$ is equal to $\lambda w(\lambda)$, and (4.23) may be rewritten as

$$(s - \lambda)R(\lambda)w(\lambda) = 0. \tag{4.25}$$

Of course, this implies that $R(\lambda)w(\lambda) = 0$. Because we also have (4.24), it follows that $w(\lambda) \in \lambda^{-1}N^R$. By the definitions, this shows that $[sG^\top - F^\top \quad H^\top]^\top$ is injective for all $s \in \mathbb{C}$, and the proof is complete.

5. REALIZATION WITH A CAUSAL INPUT/OUTPUT STRUCTURE

In the realization procedure of the previous section, we could replace the quotient space X^R/N^R by the space of polynomials X_R, because we had a natural isomorphism available between these two spaces, given essentially by multiplication by $R(\lambda)$. The other space that we used, $X^R/\lambda^{-1}N^R$, is isomorphic to the direct sum $X_R \oplus W^0$, where W^0 is the subspace of W defined by

$$W^0 = \{w \in W \mid \exists w(\lambda) \in N^R \text{ s. t. } w = w_{-1}\}. \tag{5.1}$$

(In other words, $W^0 = HN^R$, so that this space is indeed the same as the one introduced earlier under the same name for discrete-time systems.) Indeed, one has

$$X^R/\lambda^{-1}N^R \simeq X^R/N^R \oplus N^R/\lambda^{-1}N^R \simeq X_R \oplus W^0. \tag{5.2}$$

Unfortunately, the first isomorphism in the formula above must be established by selecting a complement to $N^R/\lambda^{-1}N^R$ in $X^R/\lambda^{-1}N^R$, and so we do not have a *natural* isomorphism available. This is also reflected in the non-uniqueness of 'driving-variable' representations as described in [15, Thm. 7.1]. It should be noted that the space W^0 itself is canonically given (i. e., it is an invariant under external equivalence), and this space will play an important role below.

Now, suppose that we add more structure by dividing the external variables in *inputs* and *outputs*. Such a division is given by a decomposition of the external variable space W as the direct sum of two subspaces Y and U, corresponding to a splitting of the defining AR matrix $R(s)$ as

$$R(s) = [R_1(s) \quad R_2(s)]. \tag{5.3}$$

The projection onto U along Y will be denoted by π_U, the complementary projection by π_Y. We shall first consider the 'causal' situation as described in the following lemma, which is a formalization of remarks in [15, §6]. General input/output structures will be discussed in the next section.

LEMMA 5.1 *With the notations introduced above, the following statements are equivalent:*

(i) *$R_1(s)$ is invertible as a rational matrix, and $R_1^{-1}(s)R_2(s)$ is proper rational;*

(ii) *the projection π_U, taken as a mapping from W^0 to U, is an isomorphism;*

(iii) *there exists a mapping $D: U \mapsto Y$ such that*

$$W^0 = \{ \begin{bmatrix} Du \\ u \end{bmatrix} \mid u \in U\} \tag{5.4}$$

where the vector notation is adapted to the decomposition of W as $Y \oplus U$;

(iv) *Y is a complement of W^0 in W.*

PROOF The equivalence between statements (ii), (iii), and (iv) is a matter of straightforward linear algebra. To prove that (i) implies (iii), define

$$D = [-R_1^{-1}(s)R_2(s)]_{s=\infty}. \tag{5.5}$$

Take $w \in W^0$, and let $w(\lambda) \in N^R$ be such that $w_{-1} = w$. From $R(\lambda)w(\lambda) = 0$, we have

$$\pi_Y w(\lambda) + R_1^{-1}(\lambda)R_2(\lambda)\pi_U w(\lambda) = 0, \tag{5.6}$$

and this implies

$$\pi_Y w_{-1} = D\pi_U w_{-1}. \tag{5.7}$$

Conversely, suppose that $w \in W$ is of the form

$$w = \begin{bmatrix} Du \\ u \end{bmatrix}. \tag{5.8}$$

Define $w(\lambda)$ by

$$w(\lambda) = \lambda^{-1} \begin{bmatrix} R_1^{-1}(\lambda)R_2(\lambda)u \\ u \end{bmatrix}; \tag{5.9}$$

then $w(\lambda) \in N^R$ and $w_{-1} = w$, so that $w \in W^0$.

Now, assume that (ii)-(iv) hold. Let $N(\lambda)$ be a basis matrix for the rational vector space $\ker R(\lambda)$; we may assume that $N(\lambda)$ is proper rational, and that its leading coefficient matrix $N_0 = [N(s)]_{s=\infty}$ has full column rank. (To see this, note that by reducing $R(\lambda)$ to row reduced form one actually writes $R(\lambda) = [S(\lambda) \quad 0]B(\lambda)$ where $S(\lambda)$ is a nonsingular polynomial matrix, and $B(\lambda)$ is bicausal. One may then take $N(\lambda) = B^{-1}(\lambda)[0 \quad I]^{\mathrm{T}}$.) Under these conditions, N_0 is a basis matrix for W^0 and it follows that $\dim U = \dim W^0 = q - k$ where k is the number of rows of $R(\lambda)$. So, $\dim Y = k$ and it is seen that the matrix $R_1(\lambda)$ is square. To prove that $R_1(\lambda)$ is invertible, suppose that $R_1(\lambda)y(\lambda) = 0$ for some $y(\lambda) \in Y(\lambda)$ not equal to zero. It is no restriction of the generality to assume that $y(\lambda)$ is strictly proper with a nonzero leading term y_{-1}; but then the vector $[y_{-1}^{\mathrm{T}} \quad 0]^{\mathrm{T}}$ belongs to $Y \cap W^0$ and so should be zero according to (iv). Finally, note that by definition we have

$$R_1(\lambda)\pi_Y N(\lambda) + R_2(\lambda)\pi_U N(\lambda) = 0. \tag{5.10}$$

Moreover, the rational matrix $\pi_U N(\lambda)$ is proper with an invertible leading coefficient matrix, as is seen from (ii), and this implies that

$$R_1^{-1}(\lambda)R_2(\lambda) = -\pi_Y N(\lambda)(\pi_U N(\lambda))^{-1} \tag{5.11}$$

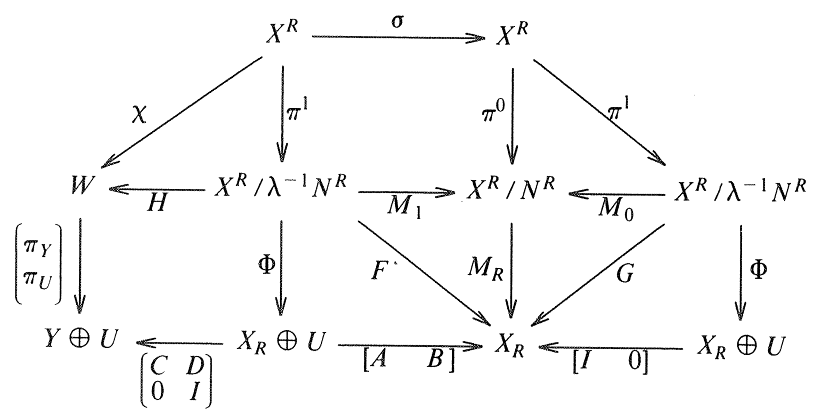
is proper rational.

Define a mapping Φ from $X^R/\lambda^{-1}N^R$ to $X_R \oplus U$ by

$$\Phi\colon w(\lambda) \bmod \lambda^{-1}N^R \mapsto \begin{bmatrix} R(\lambda)w(\lambda) \\ \pi_U w_{-1} \end{bmatrix} \tag{5.12}$$

(it is easily seen that this is well-defined). To prove that Φ is injective, let $w(\lambda) \in X^R$ be such that $R(\lambda)w(\lambda) = 0$ and $\pi_U w_{-1} = 0$. For such a $w(\lambda)$, we get $w(\lambda) \in N^R$ so $w_{-1} \in W^0$. The condition $\pi_U w_{-1} = 0$ implies $w_{-1} \in Y$, so that $w_{-1} \in Y \cap W^0 = \{0\}$, which proves that $w(\lambda) \in \lambda^{-1}N^R$. This shows that Φ is injective; the fact that Φ is actually an isomorphism then follows easily by a dimension argument.

Using the obvious facts $[I \quad 0]\Phi = G$ and $[0 \quad I]\Phi = \pi_U H$, we can now write down the diagram below which we use to define the mappings A, B, C, and D that will appear in an input/state/output representation of the given behavior.



— Diagram 3 —

To give more explicit expressions for the four mappings defined by requiring that Diagram 3 commutes, we use the form (5.3) where $R_1(s)$ is invertible. Note that $R_1^{-1}(s)p(s)$ is strictly proper if $p(\lambda) \in X_R$; indeed, suppose that $p(s) = R_1(s)\pi_Y w(s) + R_2(s)\pi_U w(s)$ for $w(\lambda) \in X^R$, then

$$R_1^{-1}(s)p(s) = \pi_Y w(s) + R_1^{-1}(s)R_2(s)\pi_U(s)w(s) \tag{5.13}$$

and this is obviously strictly proper. With this information, it is easily seen that the inverse of the isomorphism Φ may be given as follows:

$$\Phi^{-1}: X_R \oplus U \ni \begin{bmatrix} p(\lambda) \\ u \end{bmatrix} \mapsto \begin{bmatrix} R_1^{-1}(\lambda)p(\lambda) - \lambda^{-1}R_1^{-1}(\lambda)R_2(\lambda)u \\ \lambda^{-1}u \end{bmatrix} \bmod \lambda^{-1}N^R. \tag{5.14}$$

The mapping $[A \quad B]$ can now be computed as $M_R M_1 \Phi^{-1}$. Explicitly, this gives:

$$[A \quad B] \begin{bmatrix} p(\lambda) \\ u \end{bmatrix} = [R_1(\lambda) \quad R_2(\lambda)]\pi_- \lambda \begin{bmatrix} R_1^{-1}(\lambda)p(\lambda) - \lambda^{-1}R_1^{-1}(\lambda)R_2(\lambda)u \\ \lambda^{-1}u \end{bmatrix} =$$

$$= R_1(\lambda)\pi_- \lambda R_1^{-1}(\lambda)(p(\lambda) - R_2(\lambda)u) + R_2(\lambda)\pi_- u =$$

$$= \pi_{R_1}\lambda p(\lambda) - \pi_{R_1}R_2(\lambda)u, \tag{5.15}$$

where the notation π_{R_1} is used, following [5], for the projection on X_R given by

$$\pi_{R_1}: p(\lambda) \mapsto R_1(\lambda)\pi_- R_1^{-1}(\lambda)p(\lambda). \tag{5.16}$$

In particular, we find

$$A: p(\lambda) \mapsto \pi_{R_1}\lambda p(\lambda) = \lambda p(\lambda) - R_1(\lambda)[R_1^{-1}(\lambda)p(\lambda)]_{-1} \tag{5.17}$$

and

$$B: u \mapsto -\pi_{R_1}R_2(\lambda)u. \tag{5.18}$$

The expression for B may also be written in a different way if we introduce a constant matrix D_∞ by

$$D_\infty = [R_1^{-1}(s)R_2(s)]_{s=\infty}; \tag{5.19}$$

namely,

$$B: u \mapsto -R_2(\lambda)u + R_1(\lambda)D_\infty u. \tag{5.20}$$

Quite similarly, we obtain explicit expressions for the mappings C and D from the formula $[C \quad D] = \pi_Y H\Phi^{-1}$. We find

$$C: p(\lambda) \mapsto [R_1^{-1}(\lambda)p(\lambda)]_{-1} \tag{5.21}$$

and

$$D: u \mapsto -D_\infty u. \tag{5.22}$$

So, in this way we recover Fuhrmann's realization of a transfer matrix $-R_1^{-1}(s)R_2(s)$ in left matrix fractional representation. Notice that actually we proved more: it is known from Fuhrmann's work that the realization is minimal under transfer equivalence if and only if the fractional representation is coprime, whereas we have shown here the more general statement that the realization is *always* minimal under *external* equivalence.

It is also possible to set up diagrams to define single mappings from the quadruple (A, B, C, D). For instance, by transforming Diagram 3 one obtains the diagram below, which can be used to define the mapping A. This clearly displays A as a version of the shift.

$$
\begin{array}{ccc}
X^R & \xrightarrow{\quad \sigma \quad} & X^R \\
\Big\downarrow \pi^0 & & \Big\downarrow \pi^0 \\
X^R/N^R & \xrightarrow{\quad \bar{\sigma} \quad} & X^R/N^R \\
\Big\downarrow M_R & & \Big\downarrow M_R \\
X_R & \xrightarrow{\quad A \quad} & X_R
\end{array}
$$

— Diagram 4 —

6. Realization with a general input/output structure

In case we have given a not necessarily causal input/output description, it is possible to obtain a representation in descriptor form. To arrive at this representation, it turns out to be advantageous to use the pencil form as an intermediate step. We shall present a construction of a descriptor representation from a pencil representation of the general form (1.5), with the only requirement that the mapping G should be surjective. The construction will be based on the selection of suitable subspaces in Z and U, and we shall comment on the interpretation of these subspaces after the construction is completed.

So, we start from a pencil representation $(Z, X, W; F, G, H)$ in which G is surjective; also, a decomposition $W = Y \oplus U$ is given, with associated projections π_Y and π_U. No further assumptions are made. We first construct a direct sum decomposition of Z based on the subspaces $\ker G$ and $\ker \pi_U H$. Write $Z_1 = \ker G \cap \ker \pi_U H$. Let Z_{01} be a complement of Z_1 in $\ker \pi_U H$; let Z_2 be a complement of Z_1 in $\ker G$; and let Z_{02} be a complement of $\ker G + \ker \pi_U H$ in Z. Finally, write $Z_0 = Z_{01} \oplus Z_{02}$. We then have the following properties:

(i) $Z_0 \oplus Z_1 \oplus Z_2 = Z$

(ii) $Z_1 \oplus Z_2 = \ker G$

(iii) $Z_0 \cap (\ker G + \ker \pi_U H) \subset \ker \pi_U H$.

Next, decompose U as the direct sum of two subspaces U_1 and U_2, in such a way that the following conditions hold:

(i) $U_2 = \pi_U H Z_2 \; (= \pi_U H[\ker G])$

(ii) $U_1 \supset \pi_U H Z_0$.

For such a decomposition to be possible, the subspaces $\pi_U H Z_0$ and $\pi_U H Z_2$ should be independent. Suppose that u belongs to both subspaces; then $u = \pi_U H z_0 = \pi_U H z_2$ for some $z_0 \in Z_0$ and $z_2 \in Z_2$. Obviously, one has $z_0 - z_2 \in \ker \pi_U H$, so that

$$z_0 \in Z_0 \cap (\ker G + \ker \pi_U H) \subset \ker \pi_U H. \tag{6.1}$$

From this, we have $u = \pi_U H z_0 = 0$, which proves that the two subspaces are indeed independent.

With respect to the above decompositions, one gets the following partitioned representations for the system mappings F, G, and H.

$$G = [G_0 \quad 0 \quad 0] \tag{6.2}$$

$$F = [F_0 \quad F_1 \quad F_2] \tag{6.3}$$

$$\pi_Y H = [H_{00} \quad H_{01} \quad H_{02}] \tag{6.4}$$

$$\pi_U H = \begin{bmatrix} H_{10} & 0 & H_{12} \\ 0 & 0 & H_{22} \end{bmatrix} \tag{6.5}$$

Here, the matrices G_0 and H_{22} are invertible. The system equations take the form (in obvious notation)

$$\sigma G_0 z_0 = F_0 z_0 + F_1 z_1 + F_2 z_2 \tag{6.6}$$

$$y = H_{00} z_0 + H_{01} z_1 + H_{02} z_2 \tag{6.7}$$

$$u_1 = H_{10} z_0 + H_{12} z_2 \tag{6.8}$$

$$u_2 = H_{22} z_2. \tag{6.9}$$

This may be rewritten in descriptor form:

$$\sigma \begin{bmatrix} G_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} F_0 & F_1 \\ H_{10} & 0 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} + \begin{bmatrix} 0 & F_2 H_{22}^{-1} \\ -I & H_{12} H_{22}^{-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{6.10}$$

$$y = [H_{00} \quad H_{01}] \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} + [0 \quad H_{02} H_{22}^{-1}] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{6.11}$$

To interpret this construction, first note that the requirement that G should be surjective implies that z mod (ker G) can be interpreted as a 'state', with the remaining z-variables serving as 'driving variables'. The subspace Z_0 is a complement of ker G and so is isomorphic to $Z/\ker G$; we interpret the corresponding z-variables as states. The subspaces Z_1 and Z_2 correspond to driving variables; Z_1 gives the variables that cannot be identified with u-variables (nominal inputs), whereas the component z_2 is replaced by u_2 in the descriptor representation. From the point of view of the inputs, U_2 contains the u-variables that can also be seen as driving variables (so these are inputs in the standard sense), whereas U_1 gives the complementary component which has to be taken into the descriptor representation through the algebraic equations in (6.10).

The above construction has been carried out only under the assumption that the matrix G is surjective. It will be shown in the next section that the construction leads to a *minimal* descriptor representation (in an appropriate sense) if the pencil representation that we start with is minimal, so if all conditions of Prop. 1.1 are satisfied. The following lemma will be needed there. The lemma also throws some light on the role of the subspace U_2, which arises as the image under π_U of the subspace $H[\ker G]$ of W; cf. Lemma 5.1.

LEMMA 6.1 *Consider a pencil representation (1.5) and an equivalent AR representation (4.6); assume that G is surjective and that* $[G^{\mathsf T} \quad H^{\mathsf T}]^{\mathsf T}$ *is injective. Let the subspace* W^0 *of W be defined by (5.1). We then have*

$$W^0 = H[\ker G]. \tag{6.12}$$

PROOF Let $V(\lambda) \in \mathbb{R}^{k \times n}[\lambda]$ be as in Lemma 4.2. Take $w(\lambda) \in N^R$, so that $R(\lambda) w(\lambda) = 0$. Obviously, one has

$$[V(\lambda) \quad -R(\lambda)] \begin{bmatrix} 0 \\ w(\lambda) \end{bmatrix} = 0. \tag{6.13}$$

It follows from Lemma 4.2 that the matrix $[\lambda G^{\mathsf T} - F^{\mathsf T} \quad H^{\mathsf T}]^{\mathsf T}$ is a basis matrix for the kernel of $[V(\lambda) \quad R(\lambda)]$, considered as a mapping between rational vector spaces. As a consequence, there must exist a rational vector $z(\lambda) \in \mathbb{R}^{n+m}(\lambda)$ such that

$$\begin{bmatrix} 0 \\ w(\lambda) \end{bmatrix} = \begin{bmatrix} \lambda G - F \\ H \end{bmatrix} z(\lambda). \tag{6.14}$$

Write

$$z(\lambda) = z_j \lambda^j + \cdots + z_0 + z_{-1} \lambda^{-1} + \cdots. \tag{6.15}$$

Because $[G^{\mathsf T} \quad H^{\mathsf T}]^{\mathsf T}$ is injective, the matrix $[\lambda G^{\mathsf T} - F^{\mathsf T} \quad H^{\mathsf T}]^{\mathsf T}$ has a proper rational left inverse (see Lemma 1.3), and so $z(\lambda)$ is strictly proper. In particular, we have $G z_{-1} = F z_0 = 0$ and $H z_{-1} = w_{-1}$, so that $w_{-1} \in H[\ker G]$.

We have just shown that $W^0 \subset H[\ker G]$. Now, note that (cf. the proof of Lemma 5.1)

$$\dim W^0 = \dim_{\mathbb{R}(s)} \ker R(s) = q - k, \tag{6.16}$$

while at the same time, since $\ker H \cap \ker G = \{0\}$,

$$\dim H[\ker G] = \dim \ker G = q - k \tag{6.17}$$

(see Lemma 4.2). This completes the proof.

The above lemma gives a characterization of W^0 in pencil terms. For *minimal* pencil representations, this characterization can also be derived immediately from the realization in Section 4. In the causal case (see Lemma 5.1), both U_1 and Z_1 reduce to zero and the construction above leads to a standard state space representation, as expected.

7. Indices and minimality

In this section, we shall discuss the minimality of descriptor representations. While for standard state space systems there is only one index that plays a role to determine the minimality (viz., the dimension of the state space), there are three such indices for descriptor systems: the *rank* of E, the *column defect* of E ($\dim \ker E$ = the number of columns minus the rank), and the *row defect* of E ($\operatorname{codim} \operatorname{im} E$ = the number of rows minus the rank). A *minimal* descriptor representation is, by definition, one in which each of these three indices is minimal within the set of descriptor representations for a given behavior. Note that, with this definition, even the existence of a minimal representation is not trivial. Our strategy will be to establish first lower bounds for each of the three indices separately, and to show next that these minima can be achieved simultaneously. Note that, by minimizing the three indices above, one also automatically minimizes the number of descriptor variables (= the number of columns of E = rank + column defect) and the number of equations (= the number of rows of E = rank + row defect).

A lower bound for the rank of E is easily established.

Proposition 7.1 *Let an input/output behavior be given by autoregressive equations*

$$[R_1(\sigma) \quad R_2(\sigma)] \begin{bmatrix} y \\ u \end{bmatrix} = 0. \tag{7.1}$$

Write n for the sum of the minimal row indices of $R(s)$ (stated in other terms, n is the maximal degree of the full-size minors of $R(s)$). Suppose that a descriptor representation of the behavior determined by (7.1) is given by

$$\sigma E\xi = A\xi + Bu \tag{7.2}$$

$$y = C\xi + Du. \tag{7.3}$$

Under these conditions, the rank of E is at least equal to n.

Proof By a suitable choice of coordinates and introduction of new variables, the descriptor equations (7.2-7.3) may be written as follows:

$$\sigma\xi_1 = A_{11}\xi_1 + A_{12}\xi_2 + B_1\eta \tag{7.4}$$

$$0 = A_{21}\xi_1 + A_{22}\xi_2 + B_2\eta \tag{7.5}$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}. \tag{7.6}$$

The algorithm of [13] may be used to reduce this to state space (driving-variable) form; the dimension of the state space will be at most equal to the length of the vector ξ_1, which in turn is equal to the rank of E. On the other hand, it is well-known (see [16, Thm. 6]) that the dimension of the state space must be at least equal to the sum of the minimal row indices of $R(s)$. The stated result follows.

Some preparations will be needed before we can arrive at lower bounds for the column deficit and the row deficit of E. The following is a technical lemma which connects properties of polynomial matrices in the operator σ to properties of the same matrices, with σ replaced by the formal variable λ, as mappings on rational vector spaces.

LEMMA 7.2 *Consider a behavior given by the sets of equations*

$$P(\sigma)\xi = 0 \tag{7.7}$$

$$Q(\sigma)\xi = w, \tag{7.8}$$

where both $P(s)$ *and* $Q(s)$ *are polynomial matrices. If*

$$R(\sigma)w = 0 \tag{7.9}$$

is an AR *representation for the same behavior, then one has*

$$\ker R(\lambda) = Q(\lambda)[\ker P(\lambda)] \tag{7.10}$$

where all matrices are interpreted as mappings between vector spaces over the field of rational functions.

PROOF Let $U(s)$ be a unimodular matrix such that

$$\begin{bmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{bmatrix} \begin{bmatrix} P(s) \\ Q(s) \end{bmatrix} = \begin{bmatrix} M(s) \\ 0 \end{bmatrix} \tag{7.11}$$

where $M(s)$ has full row rank. Under these conditions, the matrix $U_{22}(s)$ provides an AR description of the system (7.7-7.8) (see [13, Cor. 2.3]) and this implies that $\ker U_{22}(\lambda) = \ker R(\lambda)$ [13, Cor. 2.5]. So we have to prove that $\ker U_{22}(\lambda) = Q(\lambda)[\ker P(\lambda)]$.

First, take $\eta(\lambda) \in Q(\lambda)[\ker P(\lambda)]$; let $\eta(\lambda) = Q(\lambda)\zeta(\lambda)$ with $P(\lambda)\zeta(\lambda) = 0$. Because of the equality (7.11) above, we have

$$[U_{21}(\lambda) \quad U_{22}(\lambda)] \begin{bmatrix} P(\lambda) \\ Q(\lambda) \end{bmatrix} \zeta(\lambda) = 0 \tag{7.12}$$

which implies that $U_{22}(\lambda)\eta(\lambda) = 0$.

Conversely, take $\eta(\lambda) \in \ker U_{22}(\lambda)$; we then have

$$[U_{21}(\lambda) \quad U_{22}(\lambda)] \begin{bmatrix} 0 \\ \eta(\lambda) \end{bmatrix} = 0. \tag{7.13}$$

By definition, the columns of $[P(\lambda)^{\mathrm{T}} \quad Q(\lambda)^{\mathrm{T}}]^{\mathrm{T}}$ span the rational vector space $\ker[U_{21}(\lambda) \quad U_{22}(\lambda)]$. Therefore, there must exist a rational vector $\zeta(\lambda)$ such that

$$\begin{bmatrix} 0 \\ \eta(\lambda) \end{bmatrix} = \begin{bmatrix} P(\lambda) \\ Q(\lambda) \end{bmatrix} \zeta(\lambda) \tag{7.14}$$

which implies that $\eta(\lambda) \in Q(\lambda)[\ker P(\lambda)]$.

Next, we prove two lemmas that give necessary conditions for minimality of a descriptor representation. These lemmas are related to 'observability and controllability at infinity' (compare Prop. 1.2).

LEMMA 7.3 *A necessary condition for (7.2-7.3) to be a minimal descriptor representation is that the matrix* $[E^{\mathrm{T}}, \ C^{\mathrm{T}}]^{\mathrm{T}}$ *is injective.*

PROOF Suppose that the condition of the lemma is not satisfied, so that $\ker E$ and $\ker C$ have a nontrivial intersection. By a suitable choice of coordinates, we may then write

$$E = [E_1 \quad 0], \qquad C = [C_1 \quad 0] \tag{7.15}$$

where the number of the columns in the zero matrices is equal to $\dim(\ker E \cap \ker C)$. The equations (7.2-7.3) will then appear in the form

$$\sigma E_1 \xi_1 = A_{11}\xi_1 + A_{12}\xi_2 + Bu \tag{7.16}$$

$$y = C_1 \xi_1 + Du. \tag{7.17}$$

Denote the 'equation space' (the space into which E maps) by X_e. Let X_e' and $T: X_e \rightarrow X_e'$ be such that T is surjective and satisfies $\ker T = \operatorname{im} A_{12}$. The equations (7.16-7.17) are equivalent to

$$\sigma T E_1 \xi_1 = T A_{11} \xi_1 + TBu \tag{7.18}$$

$$y = C_1 \xi_1 + Du. \tag{7.19}$$

We want to show that this system precedes the original system in the partial ordering determined by the three indices (rank, column defect, row defect) introduced above. That is, we want to show that the following inequalities hold, with strict inequality in at least one case:

$$\operatorname{rank} TE_1 \leqslant \operatorname{rank} E \tag{7.20}$$

$$\dim\ker TE_1 \leqslant \dim\ker E \tag{7.21}$$

$$\operatorname{codim}\operatorname{im} TE_1 \leqslant \operatorname{codim}\operatorname{im} E. \tag{7.22}$$

As to (7.20), we have

$$\begin{aligned}\dim\operatorname{im} TE_1 &= \dim\operatorname{im} E_1 - \dim(\ker T \cap \operatorname{im} E_1) \leqslant \\ &\leqslant \dim\operatorname{im} E_1 = \dim\operatorname{im} E\end{aligned} \tag{7.23}$$

with equality if and only if

$$\operatorname{im} A_{12} \cap \operatorname{im} E_1 = \{0\}. \tag{7.24}$$

We next consider (7.21):

$$\begin{aligned}\dim\ker TE_1 &= \dim\ker E_1 + \dim(\operatorname{im} E_1 \cap \operatorname{im} A_{12}) \leqslant \\ &\leqslant \dim\ker E_1 + \dim(\ker E \cap \ker C) = \dim\ker E\end{aligned} \tag{7.25}$$

where we used the fact that the number of columns of A_{12} is equal to $\dim(\ker E \cap \ker C)$. Here, equality holds if and only A_{12} has full column rank and

$$\operatorname{im} A_{12} \subset \operatorname{im} E_1. \tag{7.26}$$

Finally, we verify (7.22):

$$\operatorname{codim}\operatorname{im} TE_1 = \operatorname{codim} T[\operatorname{im} E_1] \leqslant \operatorname{codim}\operatorname{im} E_1 = \operatorname{codim}\operatorname{im} E \tag{7.27}$$

with equality if and only if $\ker T \subset \operatorname{im} E_1$, that is, if and only if (7.26) holds. (We use here the following easily verified fact from linear algebra: if A is a surjective mapping from a space X to a space Y, and X_0 is a subspace of X, then $\operatorname{codim} AX_0 \leqslant \operatorname{codim} X_0$; equality holds if and only if $\ker A \subset X_0$.)

Now, assume that equality would hold in all three cases. The matrix A_{12} should then have full column rank, so that the rank of A should equal the number of columns of A_{12}, which in its turn is equal to $\dim(\ker C \cap \ker E)$. On the other hand, it follows from (7.24) and (7.26) that $A_{12} = 0$, so that it would follow that $\dim(\ker C \cap \ker E) = 0$, which contradicts our assumption that the subpaces $\ker C$ and $\ker E$ intersect nontrivially. This completes the proof.

LEMMA 7.4 *A necessary condition for (7.2-7.3) to be a minimal descriptor representation is that the matrix* $[E \quad B]$ *is surjective.*

PROOF The proof is quite similar to the proof of the previous lemma, and we shall not work out all details. Suppose that $[E \quad B]$ is not surjective; then, by a suitable choice of coordinates, we can write

$$E = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \tag{7.28}$$

where $[E_1 \quad B_1]$ is surjective, and the number of zero rows is equal to $\operatorname{codim}[E \quad B]$. With this choice of coordinates, the equations (7.2-7.3) can be written as follows:

$$\sigma E_1 \xi = A_1 \xi + B_1 u \tag{7.29}$$

$$0 = A_2 \xi \tag{7.30}$$

$$y = C\xi + Du. \tag{7.31}$$

Let S be an injective mapping such that $\operatorname{im} S = \ker A_2$. The above equations are equivalent to:

$$\sigma E_1 S\tilde{\xi} = A_1 S\tilde{\xi} + B_1 u \tag{7.32}$$

$$y = CS\tilde{\xi} + Du. \tag{7.33}$$

To prove the lemma, we need to show that the following three inequalities hold, with strict inequality in at least one case:

$$\operatorname{codim} \operatorname{im} E_1 S \leqslant \operatorname{codim} \operatorname{im} E \tag{7.34}$$

$$\dim \ker E_1 S \leqslant \dim \ker E \tag{7.35}$$

$$\operatorname{rank} E_1 S \leqslant \operatorname{rank} E. \tag{7.36}$$

This proof can be conducted as above (or the statement can be derived from the one in the previous lemma by duality).

PROPOSITION 7.5 *Let (7.2-7.3) be a descriptor representation for the behavior described by (7.1), and define* W^0 *as in (5.1). Under these conditions, the following inequalities hold:*

$$\dim \ker E \geqslant \dim (Y \cap W^0) \tag{7.37}$$

$$\operatorname{codim} \operatorname{im} E \geqslant \operatorname{codim}(Y + W^0). \tag{7.38}$$

PROOF It follows from the lemmas we just proved that we may suppose that the matrix $[E^{\mathrm{T}} \quad C^{\mathrm{T}}]^{\mathrm{T}}$ is injective and that the matrix $[E \quad B]$ is surjective. Note that the descriptor equations (7.2-7.3) may also be written in the following form:

$$[\sigma E - A \quad -B] \begin{bmatrix} \xi \\ \eta \end{bmatrix} = 0 \tag{7.39}$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}. \tag{7.40}$$

Take $w \in W^0$; then there exists a proper rational W-valued function $w(\lambda)$ satisfying $w_0 = w$ and $R(\lambda)w(\lambda) = 0$. By lemma 7.2 above, there must exist rational vector functions $\xi(\lambda)$ and $\eta(\lambda)$ such that

$$[\lambda E - A \quad -B] \begin{bmatrix} \xi(\lambda) \\ \eta(\lambda) \end{bmatrix} = 0 \tag{7.41}$$

$$w(\lambda) = \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \begin{bmatrix} \xi(\lambda) \\ \eta(\lambda) \end{bmatrix}. \tag{7.42}$$

These equations may also be written as follows:

$$\begin{bmatrix} \lambda E - A \\ C \end{bmatrix} \xi(\lambda) = \begin{bmatrix} 0 & B \\ I & -D \end{bmatrix} w(\lambda). \tag{7.43}$$

Because $[E^{\mathrm{T}} \quad C^{\mathrm{T}}]^{\mathrm{T}}$ is injective, the rational matrix $[(\lambda E - A)^{\mathrm{T}} \quad C^{\mathrm{T}}]^{\mathrm{T}}$ has a proper rational left inverse (see Lemma 1.3). Therefore, $\xi(\lambda)$ in the above equation must be proper rational. From this, it also follows that ξ_0, the constant term in the power series development of $\xi(\lambda)$, must satisfy $E\xi_0 = 0$. Now, suppose that $w \in Y \cap W^0$. Then, again from (7.43), it follows that $w = C\xi_0$; so $w \in C[\ker E]$. Therefore,

$$\dim(Y \cap W^0) \leqslant \dim C[\ker E] = \dim \ker E. \tag{7.44}$$

For the proof of the second part, we note that it suffices to show that

$$\{u \in U \mid Bu \in \operatorname{im} E\} \subset \pi_U W^0. \tag{7.45}$$

Indeed, one verifies easily that $\operatorname{codim} \pi_U W^0$ (with $\pi_U W^0$ considered as a subspace of U) is equal to $\operatorname{codim}(Y + W^0)$, and one can apply the following rule which holds generally for mappings A between vector spaces X and Y: $\operatorname{codim} A^{-1} Y_0 \leqslant \operatorname{codim} Y_0$ (Y_0 a subspace of Y). To show (7.45), let $u \in U$ be such that $Bu \in \operatorname{im} E$. The desired conclusion will follow if we can exhibit proper rational functions $\xi(\lambda)$ and $u(\lambda)$ such that $u_0 = u$ and

$$(\lambda E - A)\xi(\lambda) = Bu(\lambda). \tag{7.46}$$

If we define $y(\lambda) = C\xi(\lambda) + Du(\lambda)$, then $y(\lambda)$ is proper rational and

$$\begin{bmatrix} \lambda E - A \\ C \end{bmatrix} \xi(\lambda) = \begin{bmatrix} 0 & B \\ I & -D \end{bmatrix} \begin{bmatrix} y(\lambda) \\ u(\lambda) \end{bmatrix} \tag{7.47}$$

so that

$$u = \pi_U \begin{bmatrix} y_0 \\ u_0 \end{bmatrix} \in \pi_U W^0. \tag{7.48}$$

Writing $u(\lambda) = u_0 + \eta(\lambda)$, we see that it will be sufficient to find a *strictly* proper solution $[\xi(\lambda)^T \quad \eta(\lambda)^T]^T$ of the equation

$$[\lambda E - A \quad -B] \begin{bmatrix} \xi(\lambda) \\ \eta(\lambda) \end{bmatrix} = Bu. \tag{7.49}$$

Equivalently, we are looking for a *proper* solution of the same equation with Bu replaced by λBu. It follows from Thm. 6.3-12 in [7] that such a solution does indeed exist.

Actually, it is not difficult to display an explicit strictly proper solution to (7.49), if we rewrite this equation by a change of variables as

$$\begin{bmatrix} \lambda I - A_{11} & -A_{12} & -B_{11} & -B_{12} \\ -A_{21} & -A_{22} & I & -B_{22} \end{bmatrix} \begin{bmatrix} \xi_1(\lambda) \\ \xi_2(\lambda) \\ \eta_1(\lambda) \\ \eta_2(\lambda) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}. \tag{7.50}$$

(The identity matrix in the (2,3) position is allowed by the assumption that $[E \quad B]$ is surjective.) A strictly proper solution is

$$\begin{bmatrix} \xi_1(\lambda) \\ \xi_2(\lambda) \\ \eta_1(\lambda) \\ \eta_2(\lambda) \end{bmatrix} = \begin{bmatrix} (\lambda I - A_{11} - B_{11}A_{21})^{-1} x_0 \\ 0 \\ A_{21}(\lambda I - A_{11} - B_{11}A_{21})^{-1} x_0 \\ 0 \end{bmatrix}, \tag{7.51}$$

as can be verified immediately.

THEOREM 7.6 *Let an input/output behavior be given by autoregressive equations (7.1). Denote the sum of the minimal indices of $R(s)$ by n, and define W^0 by (5.1). There exists an externally equivalent descriptor representation (7.2-7.3) satisfying the following requirements:*

$$\operatorname{rank} E = n \tag{7.52}$$

$$\dim \ker E = \dim(Y \cap W^0) \tag{7.53}$$

$$\operatorname{codim} \operatorname{im} E = \operatorname{codim}(Y + W^0). \tag{7.54}$$

Moreover, a descriptor representation of the behavior given by (7.1) is minimal if and only if the above three equalities hold.

PROOF In view of the previous results in this section, it only remains to show that a descriptor representation satisfying (7.52-7.54) exists. We claim that the representation obtained in the previous section satisfies all requirements, supposing that this representation is formed from a minimal pencil representation (see Prop. 1.1). Using the notation of section 6, one has indeed:

$$\operatorname{rank} E = \dim Z_0 = \dim \operatorname{im} G = \dim X_R = n \tag{7.55}$$

$$\begin{aligned} Y \cap W^0 &= \ker \pi_U \cap H[\ker G] = \\ &= \{w \in W \mid \exists z \in Z:\ Gz = 0,\ w = Hz,\ \pi_U w = 0\} = \\ &= H[\ker G \cap \ker \pi_U H] = HZ_1 \end{aligned} \tag{7.56}$$

$$\dim \ker E = \dim Z_1 = \dim(\ker \pi_U H \cap \ker G) = \dim(Y \cap W^0) \tag{7.57}$$

(because $\ker G \cap \ker H = \{0\}$, so that the restriction of H to Z_1 is injective), and

$$\operatorname{codim} \operatorname{im} E = \dim U_1 = \operatorname{codim} \pi_U W^0 = \operatorname{codim}(Y + W^0). \tag{7.58}$$

REMARK 7.7 By unimodular operations, we can take the given polynomial matrix $R(s)$ to row proper form (see [7, p. 386]); so, we may assume that $R(s)$ is row proper to start with. This means that we can write

$$R(s) = \Delta(s)B(s) \tag{7.59}$$

where $B(s)$ is right bicausal, and

$$\Delta(s) = \operatorname{diag}(s^{\kappa_1}, \cdots, s^{\kappa_k}). \tag{7.60}$$

It is not difficult to verify that the subspace W^0 is characterized in these terms as

$$W^0 = \ker B(\infty). \tag{7.61}$$

Note that $B(\infty)$ is nothing but the 'leading row coefficient matrix' of $R(s)$. The partitioning of $R(s)$ as $[R_1(s) \quad R_2(s)]$ induces a similar partitioning of $B(\infty)$:

$$B(\infty) = [B_1(\infty) \quad B_2(\infty)]. \tag{7.62}$$

Using standard manipulations, we find the following expressions for $\dim(Y \cap W^0)$ and $\operatorname{codim}(Y + W^0)$:

$$\dim(Y \cap W^0) = \dim \ker B_1(\infty) \tag{7.63}$$

$$\operatorname{codim}(Y + W^0) = \operatorname{codim} \operatorname{im} B_1(\infty). \tag{7.64}$$

So, one has easy criteria for minimality of descriptor representations of a behavior given by a row proper AR matrix: the rank of E should be equal to the sum of the row indices, and the row and column defects of E should be equal to the corresponding indices of $B_1(\infty)$.

8. COMPUTATION

In this section, we will show how to obtain concrete matrix representations in pencil form and in descriptor form, starting from autoregressive equations determined by a $k \times q$ polynomial matrix $R(s)$. For this purpose, we shall construct specific bases for the spaces that appear in the abstract realization of Section 4.

The first step is to take the given polynomial matrix $R(s)$ to row proper form [7, p. 386]. To alleviate the notation, the resulting equivalent AR matrix will still be denoted by $R(s)$. So we have

$$R(s) = \Delta(s)B(s) \tag{8.1}$$

where $B(s)$ is right bicausal, and

$$\Delta(s) = \text{diag}\,(s^{\kappa_1}, \cdots, s^{\kappa_k}). \tag{8.2}$$

Now, let $\tilde{B}(s)$ be any matrix such that $\hat{B}(s) = [B^{\mathrm{T}}(s) \quad \tilde{B}^{\mathrm{T}}(s)]^{\mathrm{T}}$ is bicausal. It will be discussed later how to make a suitable choice for $\tilde{B}(s)$. We can write $R(s) = [\Delta(s) \quad 0]\hat{B}(s)$, and it is seen from this that a basis for $X^R/\lambda^{-1}N^R$ is given by the equivalence classes modulo $\lambda^{-1}N^R$ of the columns of the following matrix of size $q \times (n+q-k)$:

$$\hat{B}^{-1}(\lambda)\begin{bmatrix} \lambda^{-1} \cdot \cdot \lambda^{-\kappa_1} & & & & \\ & \cdots & & & \\ & & \lambda^{-1} \cdot \cdot \lambda^{-\kappa_k} & & \\ & & & \lambda^{-1} & \\ & & & & \cdot \\ & & & & & \lambda^{-1} \end{bmatrix}. \tag{8.3}$$

A basis matrix for X_R is given by the following matrix of size $k \times n$:

$$\begin{bmatrix} \lambda^{\kappa_1-1} \cdot \cdot \lambda \;\; 1 & 0 & & \\ 0 & \lambda^{\kappa_2-1} \cdot \cdot & & \\ \cdot \quad / & 0 & & \\ \cdot & \cdot & & \\ 0 & \cdot & \lambda^{\kappa_k-1} \cdot \cdot \lambda \;\; 1 \end{bmatrix}. \tag{8.4}$$

With respect to these bases, we now compute the matrix forms of F, G and H. It is easily seen that G will take the form $[I \quad 0]$. Because $\hat{B}(\lambda)$ is bicausal, the matrix of H will have the form

$$H = \hat{B}(\infty)^{-1}\begin{bmatrix} 1 \cdot \cdot 0 & & & & \\ & \cdots & & & \\ & & 1 \cdot \cdot 0 & & \\ & & & 1 & \\ & & & & \cdot \\ & & & & & 1 \end{bmatrix}. \tag{8.5}$$

Here, we see that we will need the inverse of $\hat{B}(\infty)$. Finally, if we let $G(\lambda)$ denote the matrix whose columns are the images under G in $\mathbb{R}^k[\lambda]$ of the basis elements for $X^R/\lambda^{-1}N^R$ displayed above, then we can compute a similar matrix for F by the formula

$$F(\lambda) = \lambda G(\lambda) - R(\lambda)H, \tag{8.6}$$

which follows from the definitions of F, G, and H. This is easily transformed into a matrix expression for F because of the simple basis we chose for X_R.

EXAMPLE 8.1 Let $R(s)$ be given by

$$R(s) = \begin{bmatrix} s^2 & s^2+1 & 0 \\ 1 & s+2 & 3 \end{bmatrix}. \tag{8.7}$$

The leading row coefficient matrix of this is

$$B(\infty) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{8.8}$$

which has full row rank, so $R(s)$ is already row proper. The row degrees are 2 and 1, so a polynomial basis matrix for X_R is given by

$$\begin{bmatrix} \lambda & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{8.9}$$

We get $G = [I \quad 0] \in \mathbb{R}^{3\times 4}$. We now have to choose $\tilde{B}$ to complete $B(\infty)$ to an invertible matrix; we can take $\tilde{B} = [0 \quad 0 \quad 1]$, which gives

$$\hat{B}(\infty) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8.10}$$

so that

$$\hat{B}(\infty)^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{8.11}$$

Therefore,

$$H = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{8.12}$$

Finally,

$$F(\lambda) = \begin{bmatrix} \lambda^2 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{bmatrix} - \begin{bmatrix} \lambda^2 & \lambda^2+1 & 0 \\ 1 & \lambda+2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & \lambda & -1 & 0 \\ -1 & 0 & -1 & -3 \end{bmatrix}. \tag{8.13}$$

The matrix of F is, therefore,

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & -3 \end{bmatrix}. \tag{8.14}$$

Now, suppose that a division of the external variables into inputs and outputs has been given, and that we want to obtain a representation in descriptor form. We start from the autoregressive equations, which appear in partitioned form:

$$[R_1(\sigma) \quad R_2(\sigma)] \begin{bmatrix} y \\ u \end{bmatrix} = 0. \tag{8.15}$$

Taking $R(s)$ to row proper form as before, we get a corresponding partitioning of the right bicausal matrix $B(s)$:

$$[R_1(s) \quad R_2(s)] = \Delta(s)[B_1(s) \quad B_2(s)]. \tag{8.16}$$

By renumbering the inputs if necessary, we may assume that

$$B_2(\infty) = [B_2^1(\infty) \quad B_2^2(\infty)] \tag{8.17}$$

where $B_2^1(\infty)$ has full column rank, and the columns of $B_2^2(\infty)$ depend linearly on those of $[B_1(\infty) \quad B_2^1(\infty)]$. Let $B_2^2(\infty)$ have m_2 columns. It is easily verified that a matrix $\tilde{B}$ which completes $B(\infty)$ to an invertible matrix may be found whose last m_2 rows are in the form $[0 \quad I]$. With such a choice of $\tilde{B}$, the matrix $\hat{B}(\infty)^{-1}$ will also have its last m_2 rows in this form, and, as a consequence, the matrix of H will have the same property. The computational procedure that we just discussed will therefore lead to equations of the following form:

$$\sigma z_0 = A_0 z_0 + B_1 z_1 + B_2 z_2 \tag{8.18}$$

$$y = H_{00}z_0 + H_{01}z_1 + H_{02}z_2 \tag{8.19}$$

$$u_1 = H_{10}z_0 + H_{11}z_1 + H_{12}z_2 \tag{8.20}$$

$$u_2 = \qquad z_2. \tag{8.21}$$

This can obviously be rewritten as

$$\sigma\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} A_0 & B_1 \\ H_{10} & H_{11} \end{bmatrix}\begin{bmatrix} z_0 \\ z_1 \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ -I & H_{12} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{8.22}$$

$$y = [H_{00} \quad H_{01}]\begin{bmatrix} z_0 \\ z_1 \end{bmatrix} + [0 \quad H_{02}]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{8.23}$$

We now have a representation in descriptor form; as can be verified by checking the dimensions (using Remark 7.7), it is in fact a minimal representation.

EXAMPLE 8.2 Take

$$R(s) = \begin{bmatrix} s+1 & 0 & s^2 & 2 \\ s+2 & 2s & 1 & s-1 \end{bmatrix} \tag{8.24}$$

and let the first two external variables be outputs, and the other two inputs. The leading row coefficient matrix

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

has full row rank, so that the given matrix $R(s)$ is already row reduced; also, $m_2 = 1$ and the inputs need not be renumbered. We see that the sum of the row indices of $R(s)$ is 3 and that the row and the column defects of $B_1(\infty)$ (formed by the first two columns of the matrix above) are both equal to 1; so, a descriptor representation (E, A, B, C, D) will be minimal if and only if the matrix E has size 4×4 and rank 3.

We can take

$$\tilde{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8.25}$$

which leads to

$$\hat{B}(\infty)^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & ½ & -½ & -½ \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{8.26}$$

Consequently, we get

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & ½ & -½ & -½ \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{8.27}$$

The matrix of F is computed from

$$F(\lambda) = \begin{bmatrix} \lambda^2 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \end{bmatrix} - \begin{bmatrix} \lambda+1 & 0 & \lambda^2 & 2 \\ \lambda+2 & 2\lambda & 1 & \lambda-1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & ½ & -½ & -½ \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & \lambda & 0 & -\lambda - 1 & -2 \\ -1 & 0 & 0 & 2 & 1 \end{bmatrix}. \tag{8.28}$$

This gives

$$F = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & -2 \\ -1 & 0 & 0 & 2 & 1 \end{bmatrix}. \tag{8.29}$$

Of course, $G = [I_3 \quad 0]$. Re-organizing the pencil equations as described above, one obtains

$$\sigma \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -2 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{8.30}$$

$$y = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & ½ & -½ \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -½ \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{8.31}$$

9. Conclusions

In this paper, we have shown a procedure which leads from a representation in autoregressive form (and in particular, from a left polynomial factorization) to a minimal descriptor representation. This procedure does not require the separation of finite and infinite frequencies. In fact, the transfer matrix is never computed, and the heaviest computational load in the algorithm consists of the inversion of one constant matrix. The basic tool that we used is the pencil representation, which appears as a natural form that can be derived from autoregressive equations by a very simple formula. This formula also provides the link between the realization theory of J. C. Willems and that of P. A. Fuhrmann. The direct connection between autoregressive representations and descriptor representations which has now been established enables one to study more closely the relations between the two representations. Some of these relations have already been mentioned in this paper in connection with the minimality issue; other ones will be reported on in future work.

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