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The Addition Formula for Little q-Legendre Polynomials and the $SU(2)$ Quantum Group

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From the interpretation of little q-Jacobi polynomials as matrix elements of the irreducible unitary representations of the SU(2) quantum group an addition formula is derived for the little q-Legendre polynomials. It involves an expansion in terms of Wall polynomials. A product formula for little q-Legendre polynomials follows by q-integration.

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1. Introduction

Quantum groups were recently introduced by Drinfeld [5] and Woronowicz [17]. Interesting examples are provided by deformation of some suitable commutative Hopf algebra of functions on a specific group into a non-commutative Hopf algebra. The most elementary nontrivial example comes from deformation of the algebra of polynomials on $SU(2)$, cf. Woronowicz [18]. We denote the resulting quantum group by $SU_\mu(2)$.

It was proved by Vaksman and Soibelman [15], Masuda e.a. [11], [12], and the author [10], that the matrix elements of the irreducible unitary representations of the quantum group $SU_\mu(2)$ can be expressed in terms of the little q-Jacobi polynomials. In the present paper we use this interpretation in order to derive an addition formula for the little q-Legendre polynomials, i.e., for the q-analogues of the Legendre polynomials within the class of little q-Jacobi polynomials. The derivation of this formula is straightforward and analogous to a proof of the addition formula for Legendre polynomials using irreducible representations of $SU(2)$, cf. Vilenkin [16, Ch. 3]. However, the resulting formula involves non-commuting variables. It is less easy to rewrite it equivalently as a formula only involving commuting variables. We can do this by using an infinite-dimensional irreducible $*$-representation of the Hopf algebra considered as a $*$-algebra. The result (Theorem 4.1) gives an expansion in terms of Wall polynomials (little q-Jacobi polynomials with the second parameter equal to 0).

Our addition formula is somewhat resemblant to an addition formula for (continuous) q-ultraspherical polynomials derived by Rahman and Verma [14], but for their formula a (quantum) group theoretic interpretation is not yet known. It would have been hard to find our formula without the guidance from the quantum group. Indeed, it is a good demonstration of the power and depth of the quantum group theoretic interpretation of special functions that it is possible to obtain such a formula. It turns out to be highly nontrivial to prove this formula analytically or to show that its limit case for $q \uparrow 1$ is the

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addition formula for Legendre polynomials. The analytic proof was done by Rahman [13] and the limit result is proved in joint work with Van Assche [2].

The contents of this paper are as follows. In sections 2 and 3 the preliminaries about q-hypergeometric orthogonal polynomials and quantum groups, respectively, are presented. The derivation of the addition formula is given in section 4. Finally, a product formula for little q-Legendre polynomials is derived from the addition formula in section 5.

2. Some q-hypergeometric orthogonal polynomials

Let $1 \neq q \in \mathbb{C}$. We use the familiar definitions and notations for q-shifted factorials and q-hypergeometric functions, cf. [6, Chapter 1] and [10, §2]. The *little q-Jacobi polynomials*

$$p_n(x;a,b \mid q) := {}_2\phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}; q, qx\right) \tag{2.1}$$

occurred as part of a classification by Hahn [7] of orthogonal polynomials satisfying q-difference equations. Their detailed properties as orthogonal polynomials were given by Andrews and Askey [1]. For $a = b = 1$ we will call these polynomials *little q-Legendre polynomials.*

The special little q-Jacobi polynomials obtained by putting $b := 0$ in (2.1) are known as *Wall polynomials*, cf. Chihara [3, §5, Case I], [4, Ch. 6, §11]:

$$p_n(x;a,0 \mid q) = {}_2\phi_1(q^{-n}, 0, aq; q, qx). \tag{2.2}$$

(Chihara uses another notation.) These polynomials can be viewed as one of the many q-analogues of the Laguerre polynomials in view of the limit formula

$$\lim_{q\uparrow 1} p_n\big((1-q)x; q^\alpha, 0 \mid q\big) = L_n^\alpha(x)/L_n^\alpha(0).$$

By specialization of the orthogonality relations for the little q-Jacobi polynomials we obtain the orthogonality relations for the Wall polynomials:

$$\begin{aligned}\frac{(q^{\alpha+1};q)_\infty}{(1-q)\,(q;q)_\infty} \int_0^1 p_n(t;q^\alpha,0 \mid q)\, p_m(t;q^\alpha,0 \mid q)\, t^\alpha\, (qt;q)_\infty\, d_qt \\ = \delta_{n,m}\, \frac{q^{n(\alpha+1)}\,(q;q)_n}{(q^{\alpha+1};q)_n}\,,\end{aligned} \tag{2.3}$$

where the q-integral is defined by

$$\int_0^1 f(t)\, d_qt := \sum_{k=0}^\infty f(q^k)\,(q^k - q^{k+1})$$

and where we supposed that $0 < q < 1$ and $\alpha > -1$. From [3, (5.1), (5.2)] together with (2.2) we obtain the three-term recurrence relation

$$\begin{aligned} x\, p_n(x;a,0 \mid q) &= -q^n\,(1-aq^{n+1})\, p_{n+1}(x;a,0 \mid q) \\ &\quad + q^n\,(1+a-aq^n-aq^{n+1})\, p_n(x;a,0 \mid q) - q^n\,(a-aq^n)\, p_{n-1}(x;a,0 \mid q), \\ p_{-1}(x;a,0 \mid q) &= 0, \qquad p_0(x;a,0 \mid q) = 1. \end{aligned} \tag{2.4}$$

Put

$$P_n(q^k;q^\alpha \mid q) := \left(\frac{(q^{\alpha+1};q)_\infty\,(q^{k+1};q)_\infty\,(q^{\alpha+1};q)_n}{(q;q)_\infty\,(q;q)_n}\right)^{1/2} \times (-1)^n\, q^{(k-n)(\alpha+1)/2}\, p_n(q^k;q^\alpha,0 \mid q). \tag{2.5}$$

Then (2.3) can be rewritten as

$$\sum_{k=0}^{\infty} P_n(q^k;q^\alpha \mid q)\, P_m(q^k;q^\alpha \mid q) = \delta_{n,m}\,, \qquad n,m = 0,1,2,\ldots. \tag{2.6}$$

Since the orthogonality measure in (2.3) has compact support, the orthonormal system

$$\{\, P_n(q^k;q^\alpha \mid q)\,\}_{k=0,1,2,\ldots}\,, \qquad n = 0,1,2,\ldots\,,$$

is complete in the Hilbert space l^2. Hence, we have also the dual orthogonality relations

$$\sum_{n=0}^{\infty} P_n(q^k;q^\alpha \mid q)\, P_n(q^l;q^\alpha \mid q) = \delta_{k,l}\,, \qquad k,l = 0,1,2,\ldots. \tag{2.7}$$

We conclude this section with an expression of Wall polynomials in terms of a ${}_3\phi_2$:

$$p_n(x;q^\alpha,0 \mid q) = \frac{(-1)^n\, q^{n(n+2\alpha+1)/2}\, x^n}{(q^{\alpha+1};q)_n}\; {}_3\phi_2\!\left(\begin{matrix} q^{-n}, q^{-n-\alpha}, x^{-1} \\ 0,0 \end{matrix}\, ; q, q\right). \tag{2.8}$$

This follows by putting $b := 0$ in

$$\begin{aligned} &{}_2\phi_1\!\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}\, ; q, qx\right) \\ &= (q^{-n+1}x;q)_n\, {}_2\phi_2\!\left(\begin{matrix} q^{-n}, q^{-n}b^{-1} \\ qa, q^{-n+1}x \end{matrix}\, ; q, q^{n+2}abx\right) \\ &= \frac{(qb;q)_n\, q^{n(n-1)/2}\,(-aqx)^n}{(qa;q)_n}\; {}_3\phi_2\!\left(\begin{matrix} q^{-n}, q^{-n}a^{-1}, x^{-1} \\ qb, 0 \end{matrix}\, ; q, q\right). \end{aligned}$$

Here we used a transformation formula for ${}_2\phi_1$ (cf. [6, Chapter 1]) in the first equality and reversion of summation order in the second equality. See also [8].

3. The quantum group $SU_\mu(2)$

In the rest of this paper we fix $0 \neq \mu \in (-1,1)$. Let $\mathcal{A}$ be the unital $*$-algebra generated by the two elements α and γ satisfying the relations

$$\begin{gathered} \alpha\gamma = \mu\gamma\alpha, \qquad \alpha\gamma^* = \mu\gamma^*\alpha, \qquad \gamma\gamma^* = \gamma^*\gamma, \\ \alpha^*\alpha + \gamma\gamma^* = I, \qquad \alpha\alpha^* + \mu^2\gamma\gamma^* = I. \end{gathered}$$

Let $\Phi : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ be the unital $*$-homomorphism such that

$$\Phi(\alpha) = \alpha \otimes \alpha - \mu \gamma^* \otimes \gamma, \qquad \Phi(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

Then Φ acts as a comultiplication and $\mathcal{A}$ thus becomes a Hopf algebra with involution which we say to be *associated with the compact matrix quantum group* $SU_\mu(2)$. In the limit for $\mu \uparrow 1$, $\mathcal{A}$ becomes the algebra of polynomials in the matrix elements of the natural representation of $SU(2)$ and the comultiplication is then induced by the group structure of $SU(2)$.

It is possible to embed the $*$-algebra $\mathcal{A}$ as a dense $*$-subalgebra of a C*-algebra by a universal construction. The C*-algebra approach is in particular emphasized by Woronowicz [17], [18]. In the following one might work with this C*-algebra, but only elements of the dense $*$-subalgebra $\mathcal{A}$ will be needed, so we will use this latter algebra. Other references for $SU_\mu(2)$, beside [17], [18], are [5], [15], [11], [12].

The irreducible unitary corepresentations of $\mathcal{A}$ (which are called irreducible unitary representations of $SU_\mu(2)$ in [18], [10]) have been completely classified, cf. [18], [9], [15], [11], [12], [10]. Up to equivalence, there is one such corepresentation for each finite dimension. We will denote the corepresentation of dimension $2l+1$ by $t^{l,\mu}$ ($l = 0, 1/2, 1, \ldots$), and its matrix elements with respect to a suitable orthonormal basis corresponding to the quantum subgroup $U(1)$ by $t^{l,\mu}_{n,m}$ ($n, m = -l, -l+1, \ldots, l$). Then the corepresentation property of $t^{l,\mu}$ is expressed by

(3.1) $$\Phi(t^{l,\mu}_{n,m}) = \sum_{k=-l}^{l} t^{l,\mu}_{n,k} \otimes t^{l,\mu}_{k,m}\,.$$

The $t^{l,\mu}_{n,m}$ have been computed explicitly in terms of little q-Jacobi polynomials, cf. [15], [11], [12], [10]. Here we will only need the cases that $l = 0, 1, 2, \ldots$ and m or $n = 0$. Put

(3.2) $$p^{\mu}_{l,k}(x) := \begin{bmatrix} l \\ k \end{bmatrix}^{1/2}_{\mu^2} \begin{bmatrix} l+k \\ k \end{bmatrix}^{1/2}_{\mu^2} \mu^{-k(l-k)}\, p_{l-k}(x; \mu^{2k}, \mu^{2k} \mid \mu^2),$$

and

(3.3) $$p^{\mu}_{l}(x) := p^{\mu}_{l,0}(x) = p_l(x; 1, 1 \mid \mu^2).$$

Here the notation (2.1) for the little q-Jacobi polynomials is used and (3.3) gives a little q-Legendre polynomial. Then (cf. [10, Theorem 5.3]), for $k = 0, 1, \ldots, l$:

(3.4) $$\begin{aligned} t^{l,\mu}_{k,0} &= (\alpha^*)^k\, p^{\mu}_{l,k}(\gamma\gamma^*)\, \gamma^k, \\ t^{l,\mu}_{0,k} &= (\alpha^*)^k\, p^{\mu}_{l,k}(\gamma\gamma^*)\, (-\mu\gamma^*)^k, \\ t^{l,\mu}_{-k,0} &= (-\mu\gamma^*)^k\, p^{\mu}_{l,k}(\gamma\gamma^*)\, \alpha^k, \\ t^{l,\mu}_{0,-k} &= \gamma^k\, p^{\mu}_{l,k}(\gamma\gamma^*)\, \alpha^k. \end{aligned}$$

Hence

$$t^{l,\mu}_{0,0} = p^\mu_l(\gamma\gamma^*). \tag{3.5}$$

The irreducible $*$-representations of the $*$-algebra $\mathcal{A}$ on a Hilbert space were classified in [15, Theorem 3.2]. There is a family of one-dimensional and a family of infinite-dimensional representations, both parametrized by the unit circle. We pick one of these infinite-dimensional representations: Let $\mathcal{H}$ be a Hilbert space with orthonormal basis e_0, $e_1, \ldots$. Put $e_{-1}, e_{-2}, \ldots := 0$. We define a $*$-representation τ of $\mathcal{A}$ on $\mathcal{H}$ by specifying the action of the generators of $\mathcal{A}$:

$$\begin{aligned}
\tau(\alpha)\, e_n &:= (1-\mu^{2n})^{1/2}\, e_{n-1}\,,\\
\tau(\alpha^*)\, e_n &:= (1-\mu^{2n+2})^{1/2}\, e_{n+1}\,,\\
\tau(\gamma)\, e_n &:= \mu^n\, e_n\,,\\
\tau(\gamma^*)\, e_n &:= \mu^n\, e_n\,.
\end{aligned} \tag{3.6}$$

4. Proof of the addition formula

Let $l = 0, 1, 2, \ldots$. A special case of (3.1) is

$$\Phi(t^{l,\mu}_{0,0}) = \sum_{k=-l}^{l} t^{l,\mu}_{0,k} \otimes t^{l,\mu}_{k,0}\,.$$

Hence, by (3.4) and (3.5):

$$\begin{aligned}
p^\mu_l\,(\Phi(\gamma\gamma^*)) &= p^\mu_l(\gamma\gamma^*) \otimes p^\mu_l(\gamma\gamma^*)\\
&\quad + \sum_{k=1}^{l} (\alpha^*)^k\, p^\mu_{l,k}(\gamma\gamma^*)\,(-\mu\gamma^*)^k \otimes (\alpha^*)^k\, p^\mu_{l,k}(\gamma\gamma^*)\,\gamma^k\\
&\quad + \sum_{k=1}^{l} \gamma^k\, p^\mu_{l,k}(\gamma\gamma^*)\,\alpha^k \otimes (-\mu\gamma^*)^k\, p^\mu_{l,k}(\gamma\gamma^*)\,\alpha^k.
\end{aligned} \tag{4.1}$$

This formula might already be called an addition formula for little q-Legendre polynomials p^μ_l. It involves noncommuting variables. In the limit, for $\mu \uparrow 1$, the variables will commute and (4.1) becomes the classical addition formula for Legendre polynomials. In three steps we will rewrite (4.1) into a formula involving commuting variables: First we represent (4.1) as an operator identity on the Hilbert space $\mathcal{H} \otimes \mathcal{H}$ by using the representation $\tau \otimes \tau$ (cf. (3.6)). Second, we let these operators act on the standard basis of $\mathcal{H} \otimes \mathcal{H}$. Thus we obtain a family of vector identities in $\mathcal{H} \otimes \mathcal{H}$. Third, we take inner products with respect to another suitable orthonormal basis of $\mathcal{H} \otimes \mathcal{H}$. This will yield a family of scalar identities.

Apply $\tau\otimes\tau$ to both sides of (4.1) and let both sides of the resulting operator equality act on $e_{x+y}\otimes e_y$. Then

$$
\begin{aligned}
p_l^\mu\big((\tau\otimes\tau)\Phi(\gamma\gamma^*)\big)\,e_{x+y}\otimes e_y &= p_l^\mu(\mu^{2x+2y})\,p_l^\mu(\mu^{2y})\,e_{x+y}\otimes e_y \\
&+\sum_{k=1}^{l}(-1)^k\,\mu^{k(x+2y+1)}\,(\mu^{2(x+y+1)};\mu^2)_k^{1/2}\,(\mu^{2(y+1)};\mu^2)_k^{1/2} \\
&\qquad\times p_{l,k}^\mu(\mu^{2x+2y})\,p_{l,k}^\mu(\mu^{2y})\,e_{x+y+k}\otimes e_{y+k} \\
&+\sum_{k=1}^{l}(-1)^k\,\mu^{k(x+2y-2k+1)}\,(\mu^{2(x+y)};\mu^{-2})_k^{1/2}\,(\mu^{2y};\mu^{-2})_k^{1/2} \\
&\qquad\times p_{l,k}^\mu(\mu^{2x+2y-2k})\,p_{l,k}^\mu(\mu^{2y-2k})\,e_{x+y-k}\otimes e_{y-k}\,.
\end{aligned}
\tag{4.2}
$$

(Remember the convention that $e_n=0$ for $n<0$.)

In order to say more about the left hand side of (4.2) we consider the action of

$$\Phi(\gamma\gamma^*)=(\gamma\otimes\alpha+\alpha^*\otimes\gamma)\,(\gamma^*\otimes\alpha^*+\alpha\otimes\gamma^*)$$

on $e_{x+y}\otimes e_y$. We obtain

$$
\begin{aligned}
&(\tau\otimes\tau)\big(\Phi(\gamma\gamma^*)\big)\,e_{x+y}\otimes e_y \\
&=\mu^{x+2y+1}\,(1-\mu^{2x+2y+2})^{1/2}\,(1-\mu^{2y+2})^{1/2}\,e_{x+y+1}\otimes e_{y+1} \\
&\quad+(\mu^{2x+2y}+\mu^{2y}-\mu^{2x+4y}-\mu^{2x+4y+2})\,e_{x+y}\otimes e_y \\
&\quad+\mu^{x+2y-1}\,(1-\mu^{2x+2y})^{1/2}\,(1-\mu^{2y})^{1/2}\,e_{x+y-1}\otimes e_{y-1}\,.
\end{aligned}
$$

Hence, if

$$f:=\sum_{y=0}^{\infty}c_y\,e_{x+y}\otimes e_y$$

belongs to $\mathcal{H}\otimes\mathcal{H}$, then

$$
\begin{aligned}
(\tau\otimes\tau)\big(\Phi(\gamma\gamma^*)\big)\,f=\sum_{y=0}^{\infty}\Big[&c_{y-1}\,\mu^{x+2y-1}\,(1-\mu^{2x+2y})^{1/2}\,(1-\mu^{2y})^{1/2} \\
&+c_y\,(\mu^{2x+2y}+\mu^{2y}-\mu^{2x+4y}-\mu^{2x+4y+2}) \\
&+c_{y+1}\,\mu^{x+2y+1}\,(1-\mu^{2x+2y+2})^{1/2}\,(1-\mu^{2y+2})^{1/2}\Big]\,e_{x+y}\otimes e_y\,.
\end{aligned}
\tag{4.3}
$$

Now choose

$$c_y:=P_y(\mu^{2z};\mu^{2x}\mid\mu^2),$$

where P_y is defined in terms of Wall polynomials by (2.5). Then $f\in\mathcal{H}\otimes\mathcal{H}$ and, by (2.4), the expression in square brackets at the right hand side of (4.3) is equal to $\mu^{2z}c_y$. Define

$$f_z^x:=\sum_{y=0}^{\infty}P_y(\mu^{2z};\mu^{2x}\mid\mu^2)\,e_{x+y}\otimes e_y\,. \tag{4.4}$$

Then, by the orthogonality relations (2.7), the vectors $\{f_z^x\}_{z=0,1,2,\ldots}$ form an orthonormal basis of

$$\bigoplus_{y=0}^{\infty} \mathbb{C}\, e_{x+y} \otimes e_y \,. \tag{4.5}$$

We also have

$$(\tau \otimes \tau)\big(\Phi(\gamma\gamma^*)\big)\, f_z^x = \mu^{2z}\, f_z^x \,. \tag{4.6}$$

Now take the inner product of both sides of (4.2) with respect to f_z^x and apply (4.4), (4.6) and the selfadjointness of $\Phi(\gamma\gamma^*)$ acting on $\mathcal{H} \otimes \mathcal{H}$. Then we obtain:

$$\begin{aligned}
p_l^{\mu}(\mu^{2z})\, P_y(\mu^{2z};\mu^{2x} \mid \mu^2) &= p_l^{\mu}(\mu^{2x+2y})\, p_l^{\mu}(\mu^{2y})\, P_y(\mu^{2z};\mu^{2x} \mid \mu^2) \\
&+ \sum_{k=1}^{l} (-1)^k \mu^{k(x+2y+1)} (\mu^{2(x+y+1)};\mu^2)_k^{1/2} (\mu^{2(y+1)};\mu^2)_k^{1/2} \\
&\qquad \times p_{l,k}^{\mu}(\mu^{2x+2y})\, p_{l,k}^{\mu}(\mu^{2y})\, P_{y+k}(\mu^{2z};\mu^{2x} \mid \mu^2) \\
&+ \sum_{k=1}^{l} (-1)^k \mu^{k(x+2y-2k+1)} (\mu^{2(x+y)};\mu^{-2})_k^{1/2} (\mu^{2y};\mu^{-2})_k^{1/2} \\
&\qquad \times p_{l,k}^{\mu}(\mu^{2x+2y-2k})\, p_{l,k}^{\mu}(\mu^{2y-2k})\, P_{y-k}(\mu^{2z};\mu^{2x} \mid \mu^2).
\end{aligned} \tag{4.7}$$

Finally substitute (3.2), (3.3) and (2.5) in (4.7) and replace μ^2 by q. Then we obtain:

Theorem 4.1 (*addition formula for little q-Legendre polynomials*).
For x, y, $z = 0, 1, 2, \ldots$ we have:

$$\begin{aligned}
p_l(q^z;1,1 \mid q)\, p_y(q^z;q^x,0 \mid q) &= p_l(q^{x+y};1,1 \mid q)\, p_l(q^y;1,1 \mid q)\, p_y(q^z;q^x,0 \mid q) \\
&+ \sum_{k=1}^{l} \frac{(q;q)_{x+y+k}\,(q;q)_{l+k}\, q^{k(y-l+k)}}{(q;q)_{x+y}\,(q;q)_{l-k}\,(q;q)_k^2} \\
&\quad \times p_{l-k}(q^{x+y};q^k,q^k \mid q)\, p_{l-k}(q^y;q^k,q^k \mid q)\, p_{y+k}(q^z;q^x,0 \mid q) \\
&+ \sum_{k=1}^{l} \frac{(q;q)_y\,(q;q)_{l+k}\, q^{k(x+y-l+1)}}{(q;q)_{y-k}\,(q;q)_{l-k}\,(q;q)_k^2} \\
&\quad \times p_{l-k}(q^{x+y-k};q^k,q^k \mid q)\, p_{l-k}(q^{y-k};q^k,q^k \mid q)\, p_{y-k}(q^z;q^x,0 \mid q).
\end{aligned} \tag{4.8}$$

Remark 4.2. It makes no difference in the final result (4.8) when we would have derived it from (4.1) by using one of the other members of the series of infinite-dimensional irreducible $*$-representations of $\mathcal{A}$ as given by [15, Theorem 3.2]. The same result would also have been obtained by use of the faithful $*$-representation of $\mathcal{A}$ given in [18, Theorem 1.2]. Furthermore, it can be shown that formula (4.8) taken for all x, y, $z = 0, 1, 2, \ldots$ and with $q = \mu^2$ is equivalent to (4.1).

Remark 4.3. It is possible to give a more conceptual interpretation of the occurrence of Wall polynomials in the addition formula. Namely, it can be shown that Wall polynomials have an interpretation as Clebsch-Gordan coefficients for the decomposition of the $*$-representation $(\tau \otimes \tau) \circ \Phi$ of $\mathcal{A}$ as a direct integral of irreducible $*$-representations of $\mathcal{A}$.

5. The product formula for little q-Legendre polynomials

If we multiply both sides of (4.7) with $P_y(\mu^{2z}; \mu^{2x} \mid \mu^2)$ and sum over z then, by (2.6), we obtain the product formula

$$p_l^\mu(\mu^{2x+2y})\, p_l^\mu(\mu^{2y}) = \sum_{z=0}^{\infty} p_l^\mu(\mu^{2z}) \big(P_y(\mu^{2z}; \mu^{2x} \mid \mu^2)\big)^2.$$

After substitution of (3.3), (2.5) and (2.8) and after replacing x by $x-y$ and μ^2 by q this becomes the desired product formula:

Theorem 5.1. For x, $y = 0, 1, 2, \ldots$ we have

$$p_l(q^x; 1, 1 \mid q)\, p_l(q^y; 1, 1 \mid q) = (1-q) \sum_{z=0}^{\infty} p_l(q^z; 1, 1 \mid q)\, K(q^x, q^y, q^z \mid q)\, q^z$$

with

$$K(q^x, q^y, q^z \mid q) := \frac{(q^{x+1}; q)_\infty\, (q^{y+1}; q)_\infty\, (q^{z+1}; q)_\infty\, q^{xy+xz+yz}}{(q; q)_\infty^2\, (1-q)} \times \left\{ {}_3\phi_2 \left(\begin{matrix} q^{-x}, q^{-y}, q^{-z} \\ 0, 0 \end{matrix} ; q, q \right) \right\}^2.$$

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